INFLATION BOND OPTION PRICING IN JARROW-YILDIRIM MODEL

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Abstract. Based on Jarrow-Yildirim model for inflation derivatives, this note propose an explicit formula for option on inflation bonds. The formula is similar to the one for coupon-bond option in the HJM model.

1. Introduction

Jarrow and Yildirim [6] introduced an inflation model based on the HJM model. The model describe the behavior of the *nominal* and *real yield curves* and the *inflation index*. In their paper they also propose a formula for *inflation index options*. There results were extended by Mercurio [7] to *zero-coupon inflation-indexed swap*, *year-on-year inflation-indexed swap* and *year-on-year inflation index cap*. In the same paper he also proposes a *market model* for inflation.

In this note, using technique similar to the one used for coupon bond option [2], the price of *option on inflation bonds* is derived. The formula obtained is explicit up to a parameter that is computed as the unique solution of a one-dimensional equation.

The bond on which the option is written is a capital-indexed bond. The description of such bonds can be found in [1, Section 2.2.1]. We denote the real amounts paid at dates t_i $(1 \leq i \leq n)$ by c_i. The amount c_i include the specific convention and frequeny of the bond and the principal at final date. At each date t_i the nominal amount $I_{t_i}/I_R c_i$, where I_R is a reference inded fixed at issuance, is paid. To simplify the notation the reference inflation index used in this document is always 1.

The discount factor linked to the real rates is denoted $P_2(t,T)$. It is the discount factor viewed from t for a payment in T. A capital indexed cash-flow c paid in T has in t the value

$$I_t c P_2(t,T).$$

The value in t_0 of the bond described above is

$$I_{t_0} \sum_{i=1}^{n} c_i P_2(t_0, t_i). \tag{1}$$

2. Model and preliminary lemmas

The Jarrow-Yildirim model describe the behaviour of the *instantaneous forward* nominal and real interest rate. The forward rates viewed from t fo the maturity T are denoted $f_i(t,T)$. For all this paper the index 1 is related to the nominal rates, the index 2 to the real rates and the index 3 to the inflation. The index is placed as a subsript or a supersript for writting easiness and not for deep mathematical reasons. In particular stochastic processes have the time as subscript and consequently the index will be placed as a superscript. The (nominal and real) short-term rate are denoted by $r_t^i = f_i(t,t)$.

The volatilities σ_i are deterministic. To simplify the writting, like in the HJM model, we use the notation

$$\nu_i(t,s) = \int_u^v \sigma_i(t,s)ds.$$

Date: First version: 17 October 2005; this version: 19 October 2005.

Key words and phrases. Inflation bond option, Jarrow-Yildirim model.

JEL classification: G13, E43.

AMS mathematics subject classification: 91B28, 91B24, 91B70.

The equations of the model are

$$df_1(t,T) = \sigma_1(t,T)\nu_1(t,T)dt + \sigma_1(t,T)dW_t^1 \tag{2}$$

$$df_2(t,T) = \sigma_2(t,T)\left(\nu_2(t,T) - \rho_{13}\sigma_3(t)\right)dt + \sigma_2(t,T)dW_t^2 \tag{3}$$

$$dI(t) = (r_t^1 - r_t^r)I_t dt + \sigma_3(t)I_t dW_t^3. \tag{4}$$

The covariation between the different Brownian motions are $[W_t^i, W_t^j] = \rho_{i,j}t$.

The cash accounts linked to the nominal and real rates are

$$N_v^i = \exp\left(\int_0^v r_s^i ds\right).$$

Remark: The equation for the *rates* have a "+" in front of the stochastic term to conform with [6]. In [2] the sign is the opposit to have the equation for the *price* which appear with a "+". The two are equivalent but the intermediary formulas may differ on the sign.

To obtain an explicit formula for the options on bonds, an extra condition on the real rate volatility is used. This is a separability condition which is satisfied by the extended Vasicek or Hull-White model [5]. This condition is used only in Theorem 1. All the preliminary lemmas and discussions are obtained without reference to the condition.

(H2): the function σ_2 satisfies $\sigma_2(t,u) = g(t)h(u)$ for some positive functions g and h.

The following technical lemmas on the cash accounts, bond prices and inflation index will be useful. The first lemma formulas are equivalent to the one for the HJM obtained in [4]. The second one is just the solution of the linear stochastic equation.

Lemma 1. *Let $0 \leq t \leq u \leq v$. In the Jarrow-Yildirim model, the real rate cash account and price of the zero-coupon bond can be written as*

$$N_u^2(N_v^2)^{-1} = P_2(u,v)\exp\left(-\int_u^v \nu_2(s,v)dW_s^1 - \int_u^v \nu_2(s,v)(\nu_2(s,v)/2 - \rho_{23}\sigma_3(s))ds\right)$$

$$\begin{aligned} P_2(u,v) &= \frac{P_2(t,v)}{P_2(t,u)}\exp\left(-\frac{1}{2}\int_t^u \nu_2^2(s,v) - \nu_2^2(s,u)ds\right. \\ &\quad \left. + \int_t^u (\nu_2(s,v) - \nu_2(s,u))\rho_{23}\sigma_3(s)ds - \int_t^u \nu_2(s,v) - \nu_2(s,u)dW_s^2\right) \end{aligned}$$

Lemma 2. *In the Jarrow-Yildirim model, the inflation index can be written as:*

$$I_t = I_0\exp\left(\int_0^t r_s^1 - r_s^2 ds - \frac{1}{2}\int_0^t \sigma_3^2(s)ds + \int_0^t \sigma_3(s)dW_s^3\right)$$

3. Option on inflation bond

The expiry of the option is t_0 and the real strike is K. The dsicussion and theorem relate to a call. Puts can be treated similarly. In t_0 the call owner can receive the bond in exchange of the payment KI_{t_0}. Using the notation $c_0 = -K$, the value of the option at expiry is then

$$\max\left(I_{t_0}\sum_{i=0}^n c_i P_2(t_0,t_i), 0\right).$$

Theorem 1. *In the Jarrow-Yildirim model with the real rate volatility satisfying the condition (H2) the value in 0 of theoption with real strike K and expiry t_0 is*

$$V_0 = I_0\sum_{i=0}^n c_i P_2(0,t_i) N\left(\frac{\kappa}{\sqrt{\tau_{11}}} - \frac{\tau_{12}}{\sqrt{\tau_{11}}} + g(t_i)\sqrt{\tau_{11}}\right).$$

where κ is the unique solution of

$$\sum_{i=0}^n c_i P_2(0,t_i)\exp\left(-\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)\kappa\right) = 0 \tag{5}$$

Proof. The following random variables will be used heavily in the sequel, so we give them a special name:

$$X_1 = \int_0^{t_0} h(s) dW_s^2 \quad X_2 = \int_0^{t_0} \sigma_3(s) dW_s^3.$$

The random variable X is normally distributed [8, Theorem 3.1] with mean 0 and variance

$$T = (\tau_{i,j}) = \begin{pmatrix} \int_0^{t_0} h^2(s)ds & \rho_{23}\int_0^{t_0} h(s)\sigma_3(s)ds \\ \rho_{23}\int_0^{t_0} h(s)\sigma_3(s)ds & \int_0^{t_0} \sigma_3^2(s)ds \end{pmatrix}.$$

The value of the option is [6]

$$V_0 = \mathrm{E}\left(\max\left(I_{t_0} \sum_{i=0}^{n} c_i P_2(t_0, t_i), 0 \right) (N_{t_0}^1)^{-1} \right).$$

The different building blocs of the problem are:

$$P_2(t_0,t_i) = \frac{P_2(0,t_i)}{P_2(0,t_0)} \exp\left(-\frac{1}{2}(g^2(t_i) - g^2(t_0))\tau_{11} + (g(t_i) - g(t_0))\tau_{12} - (g(t_i) - g(t_0))X_1 \right).$$

$$I_{t_0} = N_{t_0}^1 I_0 P_2(0,t_0) \exp\left(-\frac{1}{2} g^2(t_0)\tau_{11} + g(t_0)\tau_{12} - \frac{1}{2}\tau_{22} - g(t_0)X_1 + X_2 \right).$$

Note that we are able to split the random variable X_1 from the dependency of the coupon $g(t_i)$ thanks to the hypothesis (H2). This is the only place where the separability condition is used.

The option will be exercised when

$$\sum_{i=0}^{n} c_i P_2(0,t_i) \exp\left(-\frac{1}{2} g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)X_1 \right) > 0,$$

or when $X_1 < \kappa$. The equation 5 has a unique and non-degenerate solution, as proved in [2].

This value can be computed explicitely

$$\begin{aligned} V_0 &= \mathrm{E}\left(1\!\!1_{(X_1>\kappa)} I_0 \sum_{i=0}^{n} c_i P_2(0,t_i) \exp\left(-\frac{1}{2} g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - \frac{1}{2}\tau_{22} - g(t_i)X_1 + X_2 \right) \right) \\ &= I_0 \sum_{i=0}^{n} c_i P_2(0,t_i) \exp\left(-\frac{1}{2} g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - \frac{1}{2}\tau_{22} \right) \\ &\quad \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{|\Sigma|}} \int_{-\infty}^{\kappa} \exp(-g(t_i)x_1) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x\Sigma^{-1}x) dx_2 \, dx_1 \end{aligned}$$

Like in [3], the inside integral can be computed explicitely

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x\Sigma^{-1}x) dx_2 = \frac{\sqrt{|T|}}{\sqrt{\tau_{11}}} \exp\left(-\frac{1}{2}\frac{1}{\tau_{11}}(x_1^2 - 2\tau_{12}x_1 - |T|) \right).$$

After lengthy but straighforward calculation the result is obtained. □

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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Contents

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