

# Generic Market Models<sup>1</sup>

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**Abstract.** Currently, there are two market models for valuation and risk management of interest rate derivatives, the LIBOR and swap market models. In this paper, we introduce arbitrage-free constant maturity swap (CMS) market models and generic market models featuring forward rates that span periods other than the classical LIBOR and swap periods. We develop generic expressions for the drift terms occurring in the stochastic differential equation driving the forward rates under a single pricing measure. The generic market model is particularly apt for pricing of Bermudan CMS swaptions, fixed-maturity Bermudan swaptions, and callable hybrid coupon swaps.

**Key words:** market model, generic market models, generic drift terms, hybrid products, BGM model

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# 1 Introduction

Currently, there are two types of market models for valuation and risk management of interest rate derivatives, which are the LIBOR and swap market models of Brace, Gątarek & Musiela (1997), Jamshidian (1997), Musiela & Rutkowski (1997) and Miltersen, Sandmann & Sondermann (1997). In this paper, we introduce generic market models featuring forward rates that span periods other than the classical LIBOR and swap periods. The generic market model generalizes the LIBOR and swap market models. We derive necessary and sufficient conditions for the structure of the forward rates to span an arbitrage-free economy in terms of relative discount bond prices, at all times. We develop generic expressions for the drift terms occurring in the stochastic differential equation (SDE) driving the forward rates under a single pricing measure. We show how the instantaneous correlation of the generic forward rates can be calculated from the single instantaneous correlation matrix of forward LIBOR rates. These results are sufficient for implementation of calibration and pricing algorithms for generic market models.

Generic market models are specifically designed for the pricing of certain types of swaps. In particular, we will consider constant maturity swaps (CMS) and hybrid coupon swaps. An interest rate swap is an agreement to exchange, over a specified period, interest rate payments, at a specified frequency, over a specified underlying notional that is not exchanged. In a plain-vanilla swap, the floating interest rate is the LIBOR rate. A *constant maturity swap* pays not the LIBOR rate but instead a swap rate with specified tenor, fixed for all payments in the CMS swap. The payment frequency remains unchanged however. A *hybrid coupon swap* is a swap that features a floating payment schedule, designating the nature of each of the floating payments. The nature of the floating payment can be that it is determined by either a LIBOR rate with varying maturity or a swap rate with varying tenor. An example of such a payment schedule has been given in Table 1. Additionally, the function that transforms the LIBOR or swap rate into a cash flow may even be not entirely linear, for example, capped, floored or inverse.

The above swaps may have the feature that the swaps can be cancelled. Such versions are deemed *cancellable swaps*. To hold a cancellable swap is equal to holding a swap and an option to enter into the very same swap but with reversed cash flows<sup>2</sup>. The latter option is called a *callable swap*. In this paper we will also be concerned with the pricing of callable and cancellable CMS and hybrid coupon swaps. There are two types of callable swaptions: *fixed-maturity* or *co-terminal*. A co-terminal option allows to enter into an underlying swap at several exercise opportunities, where each swap ends at the same contractually determined end date. The swap maturity becomes shorter as exercise is delayed. In contrast, for the fixed-maturity version, each underlying swap has the same contractually specified maturity and the respective end dates then differ.

The main outset of the paper is that a model is deemed to be proper for valu-

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<sup>2</sup>Some readers might not be familiar with ‘callable’ and ‘cancellable’ swaps and might prefer to think of swaps and options thereon.

Table 1: Example of a hybrid coupon swap payment structure for the floating side. Date roll is modified following and day count is actual over 365.

Fixing date	Day count fraction	Payment date	Rate
11-Jun-04	1.005479	13-Jun-05	1Y LIBOR
13-Jun-05	0.997260	12-Jun-06	2Y swap rate
12-Jun-06	0.997260	11-Jun-07	4Y swap rate
11-Jun-07	1.002740	11-Jun-08	1Y LIBOR
11-Jun-08	1.000000	11-Jun-09	2Y swap rate
11-Jun-09	1.000000	11-Jun-10	1Y LIBOR

ing a certain callable or cancellable swap, if the volatility of a rate that appears in the contract payoff has been calibrated correctly to the market volatility. The concept is best illustrated by example. In the case of the hybrid coupon swap of Table 1 at the valuation date 11 June 2004, we would want to calibrate exactly to the volatilities of the  $1Y \times 2Y$  swaption,  $2Y \times 4Y$  swaption,  $3Y$  caplet,  $4Y \times 2Y$  swaption and  $5Y$  caplet. In contrast, for a cap one would calibrate to the volatilities of the  $1Y$ ,  $2Y$ ,  $3Y$ ,  $4Y$  and  $5Y$  caplets. For a co-terminal Bermudan swaption, to the volatilities of the  $1Y \times 5Y$ ,  $2Y \times 4Y$ ,  $3Y \times 3Y$ ,  $4Y \times 2Y$  and  $5Y \times 1Y$  swaptions. When employing a LIBOR market model to value a cap, the model would feature the following  $1Y$  forward LIBOR rates:  $1Y$ ,  $2Y$ ,  $3Y$ ,  $4Y$  and  $5Y$ . If a swap market model would be used to value the Bermudan swaption, it would feature the  $1Y \times 5Y$ ,  $2Y \times 4Y$ ,  $3Y \times 3Y$ ,  $4Y \times 2Y$  and  $5Y \times 1Y$  forward swap rates. For both LIBOR and swap market models, the canonical interest rates are simply equipped with the corresponding canonical volatilities, allowing for an efficient and straightforward calibration. Obviously, to straightforwardly calibrate a market model for the hybrid coupon swap of Table 1, and callable or cancellable versions thereof, the model would have to feature the forward swap rates  $1Y \times 2Y$ ,  $2Y \times 4Y$ ,  $4Y \times 2Y$ , and the  $1Y$  forward LIBOR rates at  $3Y$  and  $5Y$ . Up to now, whether a model containing such rates would be arbitrage-free is not well-known. To our knowledge, generic methods for deriving the arbitrage-free drift terms for the SDE driving the various forward rates have not been developed yet. In this paper, we develop such generic theory.

In terms of practical relevance, the generic market model technology is valuable to financial institutions that aim to trade in CMS Bermudan swaptions or callable hybrid coupon swaps. As such, their costumers might require any sequence of various maturity LIBOR or swap rate payments in the tailored exotic derivatives that they demand for their business. In this paper, we show that a generic implementation of the resulting drift terms is feasible in practice, thereby enabling proper pricing and hedging of such hybrid coupon swaps.

A further motivation for the theory in this paper is that the idea of generic market models is not new to the finance literature, since it has already been

suggested by Galluccio, Huang, Ly & Scaillet (2004). These authors discuss what they call the *co-sliding* (commonly referred to as ‘LIBOR’) and *co-terminal* (commonly referred to as ‘swap’) market models. The class of co-sliding market models corresponds to our class of CMS market models, but ours is defined differently. Galluccio et al. (2004) show that the only admissible co-sliding model is the LIBOR market model. Interestingly, we show that there are  $n$  arbitrage-free CMS market models associated with a tenor structure with  $n$  fixings, and the LIBOR and swap models are two special cases of these CMS models. In addition to the  $n$  CMS models, we introduce generic market models, extending the number of arbitrage-free market models to  $n!$ . Also, Galluccio et al. (2004) discuss the *co-initial* market model, but this model does not fit into our dynamic market model framework. Moreover, in contrast to Galluccio et al. (2004), we derive generic expressions for the drift terms of the forward rates, for all  $n!$  models (thus for LIBOR, swap, CMS and generic models).

An alternative way of calibrating a model to the relevant volatility levels, is to take a LIBOR market model, and derive generic approximate expressions for the volatility of various forward rates. Such a procedure, for the specific case of calibration of the LIBOR model to swaption volatility, has been investigated in Jäckel & Rebonato (2003), Joshi & Theis (2002), Hull & White (2000) and Pietersz & Pelsser (2004). The advantage of the generic market model specification is that the relevant volatility functions can be directly specified. Moreover, the development of the theory of generic market models is justified already by the additional insight into the workings of LIBOR and swap market models.

We mention three areas of market model theory to which the generic market model approach extends. First, generic models may also be used in multi-currency market models, see Schlögl (2002). Second, a numerical implementation of a generic model may utilize drift approximations, see, for example, Hunter, Jäckel & Joshi (2001) and Pietersz, Pelsser & van Regenmortel (2004*a*, *b*). Third, generic models may be equipped with smile dynamics. The volatility smile is the phenomenon that for European options different Black (1976) implied volatilities are quoted in the market when the strike of the option is varied. The derivation of generic market models in this paper does not make any assumptions on the instantaneous volatility. As a result, smile-incorporating models, such as the displaced diffusion (Rubinstein 1983), and constant elasticity of variance (CEV) (Cox & Ross 1976) models, can be readily applied to the generic market model framework. An application of the CEV specification to the LIBOR market model can be found in Andersen & Andreasen (2000).

Finally, for an in-depth overview of pricing models for interest rate derivatives, the reader is referred to Rebonato (2004).

An outline of the paper is as follows. First, preliminaries are introduced. Second, necessary and sufficient no-arbitrage conditions on the structure and values of the forward rates are derived. Third, generic arbitrage-free drift terms for the forward rates are derived under a change of measure in a market model setting. Fourth, the numeric efficiency of the generic drift term calculations is discussed. Fifth, the issue of calibrating generic market models to correlation is addressed. Sixth, we end with conclusions.

## 2 Preliminaries

Consider *tenor times* or a *tenor structure*  $0 =: t_1 < \dots < t_{n+1}$  and *day count fractions*  $\alpha_i$ , over the period  $[t_i, t_{i+1}]$ , for  $i = 1, \dots, n$ . Suppose traded in the market is a set of  $m$  forward LIBOR or swap rate agreements that are associated with that tenor structure<sup>3</sup>. Initially,  $m$  may be different from  $n$ , but in Theorem 1 we show that it makes sense, from an economic point of view, to consider only  $m = n$ . The set of associated forward swap agreements is administered by a set of pairs

$$\mathcal{E} = \left\{ \epsilon_j = (s(j), e(j)) ; j = 1, \dots, m ; s(j), e(j) \text{ integers ;} \right. \\ \left. 1 \leq s(j) < e(j) \leq n + 1 \right\}. \quad (1)$$

Here  $s(j)$  and  $e(j)$  denote *start* and *end* of the forward swap agreement. The above set expression for  $\mathcal{E}$  simply designates that there are  $m$  associated forward swap agreements, that each forward swap agreement starts and ends on one of the tenor times and that a start is strictly before an end. If the start  $s$  and end  $e$  of two forward swap agreements  $\epsilon^{(1)}$ ,  $\epsilon^{(2)}$  are equal, then  $\epsilon^{(1)}$  and  $\epsilon^{(2)}$  are considered equal, thereby a priori excluding the possibility of different forward rates for the same forward swap agreement. Note also that different payment frequencies for a given swap period are not allowed. The value of the forward rate associated with  $\epsilon_j$  is denoted by  $f_j$ . Forward rate  $f_j$  may, and shall, in the course of our paper, depend on time,  $f_j = f_j(t)$ . The associated forward swap agreement is defined as follows. At times  $t_{s(j)}$  and  $t_{e(j)}$  the agreement starts and ends, respectively. The agreement is partitioned by the  $e(j) - s(j)$  accrual periods  $[t_{s(j)}, t_{s(j)+1}]$ ,  $\dots$ ,  $[t_{e(j)-1}, t_{e(j)}]$ . The LIBOR rate is recorded at the start of each accrual period. If the accrual periods are indexed by  $i = s(j), \dots, e(j) - 1$ , then the LIBOR-observation time is  $t_i$ , the maturity of the LIBOR deposit is  $t_{i+1} - t_i$ , and the observed LIBOR rate is denoted by  $\ell(t_i)$ . If forward swap agreement  $j$  has been entered into at time  $t^*$  at rate  $f_j(t^*)$ , then the fixed and floating payments are  $\alpha_i f_j(t^*)$  and  $\alpha_i \ell(t_i)$ , respectively. We assume liquid trading in the market at times  $t^* = t_1, \dots, t_n$  of those forward swap agreements  $\epsilon \in \mathcal{E}$  for which  $t_{s(j)} \geq t^*$ . In other words, there is trading in a forward swap agreement if the agreement has not yet started or is about to start. We assume the cost of entering into any forward swap agreement at any tenor time to be zero.

The forward swap agreement structures of the LIBOR and swap market models fit into the framework of (1). For the LIBOR market model (LMM),  $\mathcal{E}_{\text{LMM}} = \{(1, 2), (2, 3), \dots, (n, n + 1)\}$ . For the swap market model (SMM),  $\mathcal{E}_{\text{SMM}} = \{(1, n + 1), (2, n + 1), \dots, (n, n + 1)\}$ . We introduce here a third kind of market model, associated with the  $q$ -period CMS rates. We name it the *CMS( $q$ ) market model*, for  $q = 1, \dots, n$ , and it is defined by  $\mathcal{E}_{\text{CMS}(q)} = \{(1, 1 + q), (2, 2 +$

<sup>3</sup>The frequency of the floating payments is restricted to one payment per fixed-payment period, but this is only for ease of exposition. In practice, this assumption may be relaxed and the theory follows through unchanged for any positive whole number of floating payments per fixed-payment period.

$q), \dots, (n - q + 1, n + 1), (n - q + 2, n + 1), \dots, (n, n + 1)\}$ . Note that for  $q = 1$  and  $q = n$  we retain the LIBOR and swap market models, respectively.

The structure of these market models can be specified equivalently as follows, too. There exists an enumeration  $\epsilon_j = (s(j), e(j))$ , such that, for the LIBOR model,  $s(j) = j$ ,  $e(j) = j + 1$ . For the swap model,  $s(j) = j$ ,  $e(j) = n + 1$ . For the CMS( $q$ ) model,  $s(j) = j$ ,

$$e(j) = j + q \quad (j = 1, \dots, n - q + 1), \quad e(j) = n + 1 \quad (j = n - q + 2, \dots, n). \quad (2)$$

## 2.1 Absence of Arbitrage

Associated with the tenor structure we also consider *discount bonds*. A discount bond is a hypothetical security that pays one unit of currency at its maturity. The price at time  $t$  of a discount bond maturing at time  $t_i$  is denoted by  $b_i(t)$ . Note that there are  $n + 1$  discount bonds and that we necessarily have  $b_i(t_i) = 1$  for  $i = 1, \dots, n + 1$ . The latter is just saying that the cost of immediately receiving one unit of currency is one unit of currency. The time- $t_1$  discount bond prices are sometimes simply denoted by  $b_i$  rather than by  $b_i(t_1)$ .

In terms of price consistency among the discount bonds, forward swap agreements, and LIBOR deposits, we require some form of absence of arbitrage. We follow Musiela & Rutkowski (1997), in which two forms of no-arbitrage are introduced. First, a weaker notion of no-arbitrage is the usual no-arbitrage condition in a *pure bond market*. Second, a stronger notion of no-arbitrage assumes, in addition, that *cash* is also available in the market, which means that money, not stored in a money market account, can be carried over at zero cost. The stronger form of no-arbitrage excludes a number of situations allowed by the weaker form. For example, discount bond prices greater than 1 (negative interest rates) are excluded by the strong form, but not by the weak form. More generally, the discount bond prices are required, by the strong form, but not by the weak form, to not increase with increasing maturity, as shown by Musiela & Rutkowski (1997, page 267, below Equation (13)). In the next section, it will be shown that the generic market models guarantee the weak form of no-arbitrage. Conditions guaranteeing the stronger form of no-arbitrage are more difficult to derive. Therefore, hereafter we only consider the weak form of no-arbitrage, and any mentioning of ‘no-arbitrage’ will refer to the weak form. Note that the weak form of absence of arbitrage is guaranteed when all discount bond prices are positive, since a set of positive future cash flows implies a portfolio that holds non-negative amounts of discount bonds, of which at least one position is positive. Since all discount bond prices are positive by assumption, we have that the price of such a portfolio is positive, thereby excluding arbitrage.

Valuation of non-European interest rate derivatives requires a dynamic model, that is, a model that generates unique arbitrage-free discount bond prices at all future time points. Examples of such dynamic models are the LIBOR and swap market models. An example of a non-dynamic model is the co-initial market model, as defined by Galluccio & Hunter (2004). The co-initial model features forward swap rates that span the periods  $(1, 2), (1, 3), \dots, (1, n + 1)$ , that is, all swap rates start at time  $t_1$  but end consecutively at times  $t_2, \dots, t_{n+1}$ .

The co-initial specification is non-dynamic since at time  $t_2$ , all forward swap agreements have expired. From a practical point of view, non-dynamic models are less useful than dynamic models, since non-dynamic models can only be used for European-style options. For the dynamic case, arbitrary specification of forward rates at not only  $t_1$ , but at all time points  $t_1, \dots, t_n$ , is required to lead to unique discount bond prices.

Given an arbitrary set  $\mathcal{E}$  of forward rates and their values  $\{f_j(t_i)\}_{i,j}$ , there are two mutually exclusive possibilities, that are given in the following definition.

**Definition 1**

- Condition A. *At each of the times  $t_1, \dots, t_n$ , there is a unique system of prices for the discount bonds, such that the resulting aggregate trade system of discount bonds, forward swap agreements, and LIBOR deposits, is arbitrage-free.*
- Condition B. *At least at one of the times  $t_1, \dots, t_n$ , either there exists no system or there are more than one different systems of prices for the discount bonds, such that the resulting aggregate trade system of discount bonds, forward swap agreements, and LIBOR deposits, is arbitrage-free.*

Obviously, we would want condition A to hold in financial models, and, in particular, in generic market models. In this paper, we will derive necessary and sufficient conditions on  $\mathcal{E}$  and the values  $\{f_j(t_i)\}$ , for condition A to hold. In particular, given a number of  $n + 1$  tenor times, we will show that there are exactly  $n!$  possibilities of choosing  $\mathcal{E}$ . The CMS market model (with LIBOR and swap market models as special cases) only accounts for  $n$  of these possibilities. An example for  $n = 6$  with market models of LIBOR, CMS(3), swap, co-initial, and the hybrid swap of Table 1 (viewed from the valuation date 11 June 2003), is given in Figures 1 and 2.

**Remark 1** (*Forward LIBOR versus swaption frequencies*) In this remark we point out a silent assumption that is sometimes made when calibrating a market model to parts of the swaption volatility matrix. For concreteness, we consider the EUR market, for which market traded swaps have annual fixed payments and semi-annual floating LIBOR payments. If a market model with semi-annual fixed payments is calibrated to a swaption volatility, then silently it has been assumed that there is no significant difference between semi-annual fixed versus semi-annual floating swaption volatility and annual fixed versus semi-annual floating swaption volatility.

### 3 Necessary and Sufficient Conditions on the Forward Swap Agreements Structure for Guaranteed No-Arbitrage

In this section we derive the necessary and sufficient conditions for a set of forward rates to specify unique arbitrage-free discount bond prices. The program to achieve that goal is as follows. First, we value the forward swap agreements

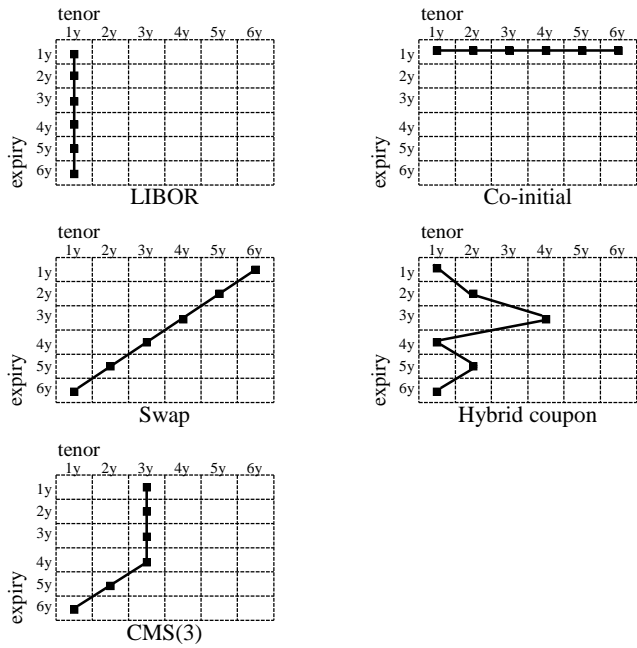


Figure 1: The swaptions from the swaption matrix to which various market models are calibrated.

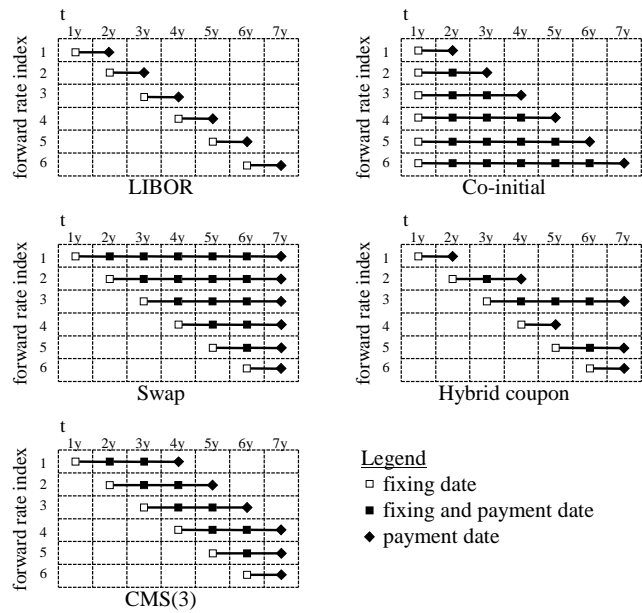


Figure 2: An overview of the forward swap agreements for various market models.



in terms of discount bond prices. Second, the conditions on the forward swap agreements are translated into conditions on the discount bond prices.

A forward swap agreement is valued by valuation of its floating and fixed payments in turn. The collections of floating and fixed payments of a forward swap agreement are called *floating* and *fixed legs*, respectively. The value  $\pi_{\text{ftt}}(\epsilon)$  of the floating leg of a forward swap agreement  $\epsilon = (s, e)$  is<sup>4</sup>

$$\pi_{\text{ftt}}(\epsilon) = b_s - b_e.$$

This equation can be seen to hold by considering a portfolio in the discount bonds that will have the exact same cash flows as the floating leg, to wit, long a discount bond maturing at time  $t_s$  and short a bond maturing at time  $t_e$ . At time  $t_s$ , we invest the proceeds of the long position in the discount bond into the LIBOR deposit. At each LIBOR payment, we re-invest the notional into the LIBOR deposit. At the end of the floating leg, the notional cancels against the short position in the discount bond. It is not hard to see that such procedure provides the exact same cash flows as a floating leg.

The value  $\pi_{\text{fxd}}(\epsilon, f)$  of a fixed leg with forward rate  $f$  can be obtained by simply discounting back the known future cash flows<sup>5</sup>,

$$\pi_{\text{fxd}}(\epsilon, f) = f \underbrace{\sum_{i=s}^{e-1} \alpha_i b_{i+1}}.$$

The under-braced expression is also called *present value of a basis point (PVBP* in short), and is denoted by  $p_{s:e}$ .

The conditions on the forward rates are governed by the forward swap agreements to have zero value, that is,  $\pi_{\text{ftt}}(\epsilon) - \pi_{\text{fxd}}(\epsilon, f) = 0$ . In fact, there exists a unique system of prices for the discount bonds consistent with the forward rates if and only if the system of  $m$  linear equations in the  $n$  unknown variables  $b_2, \dots, b_{n+1}$  given by

$$\left\{ b_{s(j)} - b_{e(j)} - \sum_{i=s(j)}^{e(j)-1} f_j \alpha_i b_{i+1} = 0 \right\}_{j=1}^m, \quad (3)$$

with  $b_1 = 1$ , has a unique solution. The latter is already a precisely specified and tractable necessary and sufficient condition for existence of unique discount bond prices that are consistent with the forward rates. This condition can be validated by numerically checking invertibility of linear equation (3). In the sequel, we will develop conditions and implications that are more straightforward to verify and that a priori guarantee invertibility of (3), and we will sketch scenarios in which these implications will hold. It will be shown that invertibility of (3) is

<sup>4</sup>Here we assume equality of the forecast and discount curves and of the payment and index day count fractions.

<sup>5</sup>Note that we assume, for notational simplicity only, that the fixed payment frequency equals the floating payment frequency.

guaranteed in typical finance scenarios, and that invertibility can be violated only under extreme situations, that are fully irrelevant to a finance setting.

If  $m < n$  then if a solution exists, it is bound to exhibit non-uniqueness. If  $m > n$ , then the system is in general over-determined. Only for a very particular choice of forward rates  $f_j$ , the system could then be degenerate, thereby still allowing for a unique solution. Given arbitrarily specified forward rates however, the degeneracy will occur, if at all, only occasionally. Generally specified forward rates span a non-degenerate set of equations, thereby implying that, when  $m > n$ , in most cases the model does not have unique discount bond prices. In other words, two different subsets of  $n$  forward rates determine, via (3), two sets of discount bond prices that are different and thus inconsistent with each other. The model should have the property that there exist unique discount bond prices regardless of how the forward rates are specified. The possibility of degeneracy is excluded by the following assumption on the values that the forward rates can attain.

**Assumption 1** *A forward rate  $f$  can only attain any non-negative value, that is, we must have*

$$f \geq 0. \tag{4}$$

Assumption 1 will be satisfied almost always in any interest rate market. Only in very rare occasions have negative interest rates been observed. An example of negative interest rates in Japan at the start of November 1998 is given in Ostrom (1998). These interest rates reached -3 to -6 basis points (bp) (-.03% to -.06%). Moreover, the popular displaced diffusion smile model of Rubinstein (1983) can generate negative forward rates with positive probability, if the displacement parameter is negative. However, violation of Assumption 1 does not necessarily imply that the system of forward rates admits arbitrage of the weak form. In fact, we make plausible that slightly negative interest rates still allow for unique discount bond prices that are arbitrage-free in the weak sense, by considering a simple numerical example. Consider a single forward rate, two tenor times  $\{t_1 = 0, t_2\}$  market model. The price of the discount bond for maturity at time  $t_2$  is given by  $1/(1 + \alpha f)$ . The rate  $f$  should thus satisfy  $f > -1/\alpha$ , to ensure a positive and finite price for the discount bond. For annual payments, for which  $\alpha \approx 1$ , we have  $-1/\alpha \approx -100\%$ . In fact, for more frequent payments than annual, the arbitrage-defying rate is even more negative than  $-100\%$ . These considerations lead us to conclude that arbitrage of the weak form in a forward swap agreement market can occur only in situations that are considered financially extreme. Essential to no-arbitrage is thus the structure of the forward swap agreements.

### 3.1 Main Result

The main result can now be formulated. The theorem below states that, for dynamic market models, (i) if a tenor structure has  $n$  fixing times  $t_1, \dots, t_n$ , then we require  $n$  forward swap agreements, and (ii) for each fixing time  $t_i$ , there is exactly one forward swap agreement that starts at that fixing time  $t_i$ ,

$i = 1, \dots, n$ . Note that the co-initial model does not fit the requirements below, though it is a perfectly sensible arbitrage-free model. The reason that the co-initial model is not incorporated is the requirement that a model be *dynamic*, see the discussion in Section 2.1.

**Theorem 1** *Let  $\{t_1, \dots, t_{n+1}\}$  be a set of tenor times. Let  $\mathcal{E} = \{\epsilon_j\}_{j=1}^m$  and  $f_j$  be a set of forward swap agreements and forward rates, respectively, associated with the tenor times. Then, at each of the times  $t_1, \dots, t_n$ , for all forward rates  $\{f_j\}_{j=1}^m$  satisfying Assumption 1, there exists a unique weak-form arbitrage-free solution to the system of linear equations (3) in the discount bond prices, if and only if  $m = n$  and there exists an ordering of the  $n$  forward swap agreements  $\epsilon_j = (s(j), e(j))$ ,  $j = 1, \dots, m$  such that  $s(j) = j$ .*

*Proof:* The proof is split into two parts. First, we prove that the described structure of forward rates leads to arbitrage-free invertibility of system (3) for all forward rates satisfying Assumption 1. Second, the reverse implication is proven.

Suppose that the structure  $\mathcal{E}$  of forward swap agreements is such that  $m = n$  and that there exists an ordering of the  $n$  forward swap agreements  $\epsilon_j = (s(j), e(j))$ ,  $j = 1, \dots, m$  such that  $s(j) = j$ . The existence of unique arbitrage-free discount bond prices is guaranteed if we show there exists unique discount bond prices that are all positive. To that order, consider system (3) in terms of the deflated discount bond prices,  $\hat{b}_i \equiv b_i/b_{n+1}$ , and substitute  $s(j) = j$ ,

$$\left\{ \hat{b}_j - \hat{b}_{e(j)} - \sum_{i=j}^{e(j)-1} f_j \alpha_i \hat{b}_{i+1} = 0 \right\}_{j=1}^n, \quad \{\hat{b}_{n+1} = 1\}. \quad (5)$$

Note that the  $(n+1) \times (n+1)$  matrix  $\mathbf{U} = \mathbf{U}(\mathbf{f})$  associated with this system is unit upper-triangular, which means that the diagonal contains ones and that the lower-triangular part of the matrix contains zeros. It follows that this matrix is invertible. We thus have

$$\mathbf{U}(\mathbf{f})\hat{\mathbf{b}} = \mathbf{c}, \quad \hat{\mathbf{b}} = \mathbf{U}(\mathbf{f})^{-1}\mathbf{c}, \quad \mathbf{c} = (0 \ \dots \ 0 \ 1)^T \in \mathbb{R}^{n+1}.$$

An efficient method for calculating the inverse of a unit upper-triangular matrix is *back substitution*, see for example Golub & van Loan (1996, Algorithm 3.1.2). Back substitution will aid in the proof, therefore it has been displayed in Algorithm 1. We show by induction for  $i = n+1, n, \dots, 1$  that  $\hat{b}_i \geq 1$ . For  $i = n+1$ ,  $\hat{b}_i = \hat{b}_{n+1} = 1$ , by line 1 of Algorithm 1, which states that  $\hat{b}_{n+1} = c_{n+1} = 1$ . Suppose, then, that  $\hat{b}_j \geq 1$  for  $j = i+1, \dots, n+1$ . We have, by line 3 of Algorithm 1, that  $\hat{b}_i = c_i - \sum_{j=i+1}^{n+1} u_{ij} \hat{b}_j = -\sum_{j=i+1}^{n+1} u_{ij} \hat{b}_j$ . Note that, for  $j > i$ ,  $u_{ij}$  is either  $-\alpha_j f_i$ ,  $-1 - \alpha_j f_i$ , or 0. It follows that

$$\hat{b}_i = f_i \underbrace{\sum_{j=i}^{e(i)-1} \alpha_j \hat{b}_{j+1}}_{\geq 0} + \underbrace{\hat{b}_{e(i)}}_{\geq 1} \geq 1,$$

---

**Algorithm 1** Back substitution.

---

Input:  $n, \mathbf{U}$  ( $(n+1) \times (n+1)$  unit upper-triangular),  $\mathbf{c} \in \mathbb{R}^{n+1}$ .Output:  $\hat{\mathbf{b}} = \mathbf{U}^{-1}\mathbf{c} \in \mathbb{R}^{n+1}$ .

- 1: Set  $\hat{b}_{n+1} \leftarrow c_{n+1}$ .
  - 2: **for**  $i = n, \dots, 1$  **do**
  - 3:    $\hat{b}_i \leftarrow c_i - \sum_{j=i+1}^{n+1} u_{ij}\hat{b}_j$ .
  - 4: **end for**
- 

which concludes the induction proof. The unique solution for the *undeflated* discount bond prices is then given by  $b_i \equiv \hat{b}_i/\hat{b}_1$ , which is defined and positive since  $\hat{\mathbf{b}} = (\hat{b}_1, \dots, \hat{b}_{n+1}) \geq 1$ .

Note that the above proof is independent of the number of tenor times. Therefore the forward swap agreements structure  $n = m$  and  $\{s(j) = j\}$  guarantees existence of unique arbitrage-free discount bond prices for *all* forward rates satisfying Assumption 1 at all tenor times  $t_1, \dots, t_n$ , which was to be shown.

The reverse implication is proven by induction on  $n$ . For  $n = 1$ , the result is immediate. Now, assume the result is true for  $i = 1$  to  $n - 1$ . We want to prove it is true for  $n$ . The model viewed from  $t_2$  has  $n$  tenor points, so by the induction hypothesis we must have that: (i)  $m \geq n - 1$ , (ii) there are exactly  $n - 1$  forward swap agreements that start at  $t_2$  or later, (iii) for these  $n - 1$  forward swap agreements, there is an enumeration  $j = 2, \dots, n$ , such that  $s(j) = j$ . There are three possibilities:  $m = n - 1$ ,  $m > n$  or  $m = n$ . We show that the cases  $m = n - 1$  and  $m > n$  lead to non-uniqueness or non-invertibility of (3) for some of the forward rates  $f$  that satisfy Assumption 1.

If  $m = n - 1$ , there are less equations than unknown variables in (3), and it follows that, if there is a solution at all, it will be non-unique.

If  $m > n$ , then we may form a sub-model with  $n$  forward swap agreements such that  $s(j) = j$  for  $j = 1, \dots, n$ . We have already proven that such a structure with  $n$  forward rates leads to unique positive discount bond prices. For a left out forward swap agreement, say  $\epsilon = (s, e)$ , the associated forward rate  $f$  should then satisfy

$$f = \frac{b_s - b_e}{\sum_{i=s}^{e-1} \alpha_i b_{i+1}}. \quad (6)$$

We conclude then that there are forward rates satisfying Assumption 1 for which there do not exist discount bond prices.

Thus we must have  $m = n$  and for remaining forward swap agreement 1 we have  $s(1) = 1$  from which the result follows.  $\square$

As a corollary, we can count the dynamic market model structures given the number of tenor times  $n + 1$ . For forward rate 1, we can chose from  $n$  end times  $t_2, \dots, t_{n+1}$ , for forward rate 2, from  $n - 1$  end times  $t_3, \dots, t_{n+1}$ , etcetera.

**Corollary 1** (Counting dynamic market model structures) *Consider market models with  $n + 1$  tenor times. Then there are  $n!$  ways of selecting forward swap agreements such that, for all forward rates satisfying Assumption 1, and*

at all tenor times  $t_1, \dots, t_n$ , there exist unique weak-form arbitrage-free discount bond prices satisfying (3).

Note that Theorem 1 rules out the applicability of generic market models to Bermudan-callable spread options, in the sense that we cannot define two rates, fixing at the same time, as state variables.

## 4 Generic Expressions for No-Arbitrage Drift Terms

In this section, generic expressions are derived for the arbitrage-free drift terms of generic market models, that are so characteristic for the LIBOR and swap market models. We assume given a dynamic market model, therefore the forward swap agreements are of the form  $\epsilon_i = (i, e(i))$ . If dependency of the end index is clear we simply write  $e(i)$  as  $e$ . The forward rate  $f_{i:e}$  has start date  $t_i$  and end date  $t_e$ . Forward rate  $f_{i:e}$  is modelled under its forward measure, which is associated with the PVBP  $p_{i:e}$  as numeraire. Forward rate  $f_{i:e}$  is modelled as

$$\frac{df_{i:e}(t)}{f_{i:e}(t)} = \boldsymbol{\sigma}_{i:e}(t) \cdot d\mathbf{w}^{(i:e)}(t), \quad (7)$$

with  $\boldsymbol{\sigma}_{i:e}$  denoting a  $d$ -dimensional volatility vector, and with  $\mathbf{w}^{(i:e)}$  denoting a  $d$ -dimensional Brownian motion under the forward measure  $\mathbb{Q}_{i:e}$  associated with  $p_{i:e}$  as numeraire. The positive integer  $d$  is deemed the *number of factors* of the model. The volatility vector  $\boldsymbol{\sigma}_{i:e}(t) = \boldsymbol{\sigma}_{i:e}(t, \omega)$  can be state dependent to allow for smile modelling.

For pricing of non-standard interest rate derivatives, it is necessary to jointly implement the above scheme (7) for all forward rates simultaneously. Therefore we must work out the SDE for the forward rates under a single pricing measure. We can work either with the terminal or spot measure. Each is treated below consecutively.

### 4.1 Terminal Measure

In this subsection, we work with the terminal measure  $\mathbb{Q}_{n+1}$ , that is the measure associated with the terminal discount bond  $b_{n+1}$  as numeraire.

Without loss of generality, the presentation is given as if all forward rates have not yet expired. We work with the numeraire-deflated discount bond prices. The quantity  $\hat{p}_{i:e}$  denotes the deflated PVBP,  $\hat{p}_{i:e} \equiv p_{i:e}/b_{n+1}$ . The deflated PVBPs can be calculated, in turn, when the deflated discount bond prices  $\hat{b}_i \equiv b_i/b_{n+1}$  are known. The deflated discount bond prices are given by (5). Recall that (5) can be written in matrix form as  $\mathbf{U}\mathbf{b} = \mathbf{c}$ , with  $\mathbf{c} = (0 \dots 0 1)^T$ , and  $\mathbf{U} = \mathbf{U}(\mathbf{f})$  an  $(n+1) \times (n+1)$  unit upper-triangular matrix, given by

$$u_{ij} = \begin{cases} 0 & \text{if } i > j \text{ or } (i < j \text{ and } j > e(i)), \\ 1 & \text{if } i = j, \\ -\alpha_{j-1} f_{i:e(i)} & \text{if } i < j \text{ and } j < e(i), \\ -\alpha_{j-1} f_{i:e(i)} - 1 & \text{if } i < j \text{ and } j = e(i). \end{cases}$$

Thus  $\hat{\mathbf{b}} = \mathbf{U}(\mathbf{f})^{-1}\mathbf{c}$ . We may write  $\hat{\mathbf{p}}$  as a function of the forward rates,  $\hat{\mathbf{p}} = \hat{\mathbf{p}}(\mathbf{f})$ . In fact,

$$\hat{\mathbf{p}} = \mathbf{A}\hat{\mathbf{b}}, \quad \mathbf{A} \equiv \begin{pmatrix} 0 & (\alpha_1 \cdots \alpha_{e(1)-1} 0 \cdots 0) \\ 0 & 0 & (\alpha_2 \cdots \alpha_{e(2)-1} 0 \cdots 0) \\ 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & (\alpha_n) \end{pmatrix},$$

for the  $n \times (n+1)$  matrix  $\mathbf{A}$ . Thus,  $\hat{\mathbf{p}} = \mathbf{A}\mathbf{U}(\mathbf{f})^{-1}\mathbf{c}$ . Subsequently, we define the Radon-Nikodým density

$$z_{i:e,n+1}(t) \equiv \frac{p_{i:e}(t)/b_{n+1}(t)}{p_{i:e}(0)/b_{n+1}(0)} = \frac{\hat{p}_{i:e}(t)}{\hat{p}_{i:e}(0)}. \quad (8)$$

Note that  $z_{i:e,n+1}(t)$  is a martingale under the terminal measure  $\mathbb{Q}_{n+1}$ . This implies that

$$\frac{dz_{i:e,n+1}(t)}{z_{i:e,n+1}(t)} = \frac{d\hat{p}_{i:e}(t)}{\hat{p}_{i:e}(t)} = \boldsymbol{\theta}_{i:e,n+1}(t) \cdot \mathbf{w}^{(n+1)}(t), \quad (9)$$

with the  $d$ -dimensional vector  $\boldsymbol{\theta}$  given by

$$\boldsymbol{\theta}_{i:e,n+1}(t) = \frac{1}{\hat{p}_{i:e}(t)} \sum_{k=i+1}^n \frac{\partial \hat{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) \boldsymbol{\sigma}_{k:e(k)}(t). \quad (10)$$

The summation is required only from  $i+1$  to  $n$  since  $\hat{p}_{i:e}$  is dependent on  $f_{k:e(k)}$  only for  $k > i$ . Finally we apply Girsanov's theorem to obtain the required expression for  $d\mathbf{w}^{(i:e)}(t) - d\mathbf{w}^{(n+1)}(t)$ ,

$$d\mathbf{w}^{(i:e)}(t) - d\mathbf{w}^{(n+1)}(t) = -\boldsymbol{\theta}_{i:e,n+1}(t)dt. \quad (11)$$

Thus,

$$\begin{aligned} \frac{df_{i:e}(t)}{f_{i:e}(t)} &= -\frac{1}{\hat{p}_{i:e}(t)} \sum_{k=i+1}^n \frac{\partial \hat{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) |\boldsymbol{\sigma}_{k:e(k)}(t)| |\boldsymbol{\sigma}_{i:e}(t)| \rho_{k:e(k),i:e}(t) dt \\ &\quad + \boldsymbol{\sigma}_{i:e}(t) \cdot d\mathbf{w}^{(n+1)}(t), \end{aligned} \quad (12)$$

where the scalar  $\rho_{k:e(k),i:e}$  has been defined as

$$\rho_{k:e(k),i:e}(t) = \frac{\boldsymbol{\sigma}_{k:e(k)}(t) \cdot \boldsymbol{\sigma}_{i:e}(t)}{|\boldsymbol{\sigma}_{k:e(k)}(t)| |\boldsymbol{\sigma}_{i:e}(t)|},$$

and has the interpretation of instantaneous correlation.

An expression is given for  $\partial \hat{\mathbf{p}} / \partial f_{k:e(k)}$ . Note that  $\partial \mathbf{U} / \partial f_{k:e(k)}$  is a matrix that is zero bar a single row, the  $k^{\text{th}}$  row, and that the derivative is independent of  $f$ , since all  $f$  terms occur linearly in the matrix  $\mathbf{U}$ . The  $k^{\text{th}}$  row is filled, from entry  $(k, k+1)$ , with the row vector  $(-\alpha_k \cdots -\alpha_{e(k)-1} 0 \cdots 0)$ . We have that

$$\frac{\partial \hat{\mathbf{p}}}{\partial f_{k:e(k)}} = -\mathbf{A}\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \mathbf{U}^{-1}\mathbf{c} = -\mathbf{A}\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \hat{\mathbf{b}} = \mathbf{A}\mathbf{U}^{-1} \mathbf{c}_k \hat{p}_{k:e(k)}, \quad (13)$$

where  $\mathbf{c}_k \in \mathbb{R}^{n+1}$  denotes the standard basis vector with unit  $k^{\text{th}}$  coordinate, and zero coordinates otherwise. We define  $\tilde{b}_i^{(k)}$  by

$$\tilde{b}_i^{(k)} = (\mathbf{U}^{-1}\mathbf{c}_k)_i, \quad i = 1, \dots, n, \quad k = 1, \dots, n. \quad (14)$$

Substituting (14) into (13) yields

$$\frac{\partial \hat{p}_{i:e}}{\partial f_{k:e(k)}} = 1_{\{k \geq i+1\}} \hat{p}_{k:e(k)} \left( \sum_{j=i}^{\min(e(i)-1, k-1)} \alpha_j \tilde{b}_{j+1}^{(k)} \right). \quad (15)$$

Define  $\mu(i, k) \equiv \min(e(i)-1, k-1)$ . Substituting (15) into (12), suppressing the dependency of time, and using  $\hat{p}_{k:e(k)} f_{k:e(k)} = \hat{b}_k - \hat{b}_{e(k)}$ , we obtain the generic market model SDE under the terminal measure:

$$\frac{df_{i:e}}{f_{i:e}} = -\frac{1}{\hat{p}_{i:e}} \sum_{k=i+1}^n (\hat{b}_k - \hat{b}_{e(k)}) \left( \sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} \right) \boldsymbol{\sigma}_{k:e(k)} \cdot \boldsymbol{\sigma}_{i:e} dt + \boldsymbol{\sigma}_{i:e} \cdot d\mathbf{w}^{(n+1)}. \quad (16)$$

## 4.2 Spot Measure

In this subsection, we work with the spot measure  $\mathbb{Q}_{\text{Spot}}$ , that is the measure associated with the spot LIBOR numeraire, defined as follows. The account starts out with one unit of currency. Subsequently, this amount is invested in the spot LIBOR account. After the first accrual period, the proceeds are re-invested in the then spot LIBOR account. This procedure is repeated. For the spot measure it is convenient to define the *spot index*  $i(t)$ , defined by  $i(t) = \min\{\text{integer } i; t < t_i\}$ .

For the spot measure, we work with discount bond prices, deflated by the spot discount bond  $b_{i(t)}$ . The quantities  $\bar{\mathbf{p}}$  and  $\bar{\mathbf{b}}$  denote the vectors of  $b_{i(t)}$ -deflated PVBPs and discount bond prices, respectively. We have  $\bar{\mathbf{p}} = \mathbf{A}\bar{\mathbf{b}}$  and

$$\bar{\mathbf{b}} = \frac{1}{\tilde{b}_{i(t)}} \hat{\mathbf{b}} = \frac{1}{(\mathbf{U}^{-1}\mathbf{c})_{i(t)}} \mathbf{U}^{-1}\mathbf{c}.$$

The Radon-Nikodým density  $z_{i:e,i(t)}(t)$  is defined similarly to (8). A martingale SDE for the Radon-Nikodým density holds,

$$\frac{dz_{i:e,i(t)}(t)}{z_{i:e,i(t)}(t)} = \frac{d\bar{p}_{i:e,i(t)}(t)}{\bar{p}_{i:e,i(t)}(t)} = \boldsymbol{\theta}_{i:e,i(t)}(t) \cdot d\mathbf{w}^{(i(t))},$$

similar to (9), with  $d$ -dimensional volatility vector equal to

$$\boldsymbol{\theta}_{i:e,i(t)}(t) = \frac{1}{\bar{p}_{i:e}(t)} \sum_{k=i(t)}^n \frac{\partial \bar{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) \boldsymbol{\sigma}_{k:e(k)}(t). \quad (17)$$

If we compare (17) to (10), we find that, for the spot measure, we sum over all available forward rates from  $i(t)$  to  $n$ , since  $\bar{p}_{i:e}$  depends on all those forward rates. Recall that, for the terminal measure, we need only sum from  $i+1$  to  $n$ .

Similar to (11), we have  $d\mathbf{w}^{(i:e)} - d\mathbf{w}^{(i(t))} = -\boldsymbol{\theta}_{i:e,i(t)} dt$ . Thus we obtain the equivalent of (12),

$$\frac{df_{i:e}(t)}{f_{i:e}(t)} = -\frac{1}{\bar{p}_{i:e}(t)} \sum_{k=i(t)}^n \frac{\partial \bar{p}_{i:e}}{\partial f_{k:e(k)}}(t) f_{k:e(k)}(t) \|\boldsymbol{\sigma}_{k:e(k)}(t)\| \boldsymbol{\sigma}_{i:e}(t) \|\rho_{k:e(k),i:e}(t) dt + \boldsymbol{\sigma}_{i:e}(t) \cdot d\mathbf{w}^{(i(t))}(t). \quad (18)$$

An expression for  $\partial \bar{\mathbf{p}} / \partial f_{k:e(k)}$  is given by

$$\frac{\partial \bar{\mathbf{p}}}{\partial f_{k:e(k)}} = \frac{1}{\hat{b}_{i(t)}} \frac{\partial \hat{\mathbf{p}}}{\partial f_{k:e(k)}} + \frac{1}{\hat{b}_{i(t)}} \underbrace{\left( \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \mathbf{U}^{-1} \mathbf{c} \right)_{i(t)}}_{=\hat{p}_{k:e(k)} \tilde{b}_{i(t)}^{(k)}} \bar{\mathbf{p}}. \quad (19)$$

Similar as in (13) and (15) for the terminal measure, we find for the spot measure:

$$\frac{\partial \bar{p}_{i:e}}{\partial f_{k:e(k)}} = 1_{\{k \geq i+1\}} \bar{p}_{k:e(k)} \sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} - \bar{p}_{k:e(k)} \bar{p}_{i:e} \tilde{b}_{i(t)}^{(k)}. \quad (20)$$

Substituting (20) into (18), suppressing the dependency of time, and using  $\bar{p}_{k:e(k)} f_{k:e(k)} = \bar{b}_k - \bar{b}_{e(k)}$ , we obtain the generic market model SDE under the spot measure:

$$\frac{df_{i:e}}{f_{i:e}} = -\frac{1}{\bar{p}_{i:e}} \sum_{k=i(t)}^n (\bar{b}_k - \bar{b}_{e(k)}) \left( 1_{\{k \geq i+1\}} \sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} - \bar{p}_{i:e} \tilde{b}_{i(t)}^{(k)} \right) \boldsymbol{\sigma}_{k:e(k)} \cdot \boldsymbol{\sigma}_{i:e} dt + \boldsymbol{\sigma}_{i:e} \cdot d\mathbf{w}^{(i(t))}. \quad (21)$$

### 4.3 An example: The LIBOR Market Model

For illustration, in this section the LIBOR drift terms are calculated starting from the generic market model framework. We stress here that the explicit calculation in this section of the generic expressions of the previous section is *not* required for implementation of the generic market model framework, but is merely performed for illustration only.

First, we derive the LIBOR SDE for the terminal measure, by applying (16). In the LIBOR market model, a forward rate  $f_{k:e(k)}$  is denoted by  $f_k$ . Note that:

- (i)  $\hat{p}_{i:e(i)} = \hat{p}_{i:i+1} = \alpha_i \hat{b}_{i+1}$ ,
- (ii)  $\mu(i, k) = \min(e(i) - 1, k - 1) = \min(i, k - 1) = i$ , for  $k = i + 1, \dots, n$ ,
- (iii)  $\tilde{b}_j^{(k)} = \frac{\hat{b}_j}{\bar{b}_k} 1_{\{j \leq k\}} = \frac{\bar{b}_j}{\bar{b}_k} 1_{\{j \leq k\}}$ ,
- (iv)  $\frac{\hat{b}_k - \hat{b}_{k+1}}{\hat{b}_k} = \frac{\bar{b}_k - \bar{b}_{k+1}}{\bar{b}_k} = 1 - \frac{1}{1 + \alpha_k f_k} = \frac{\alpha_k f_k}{1 + \alpha_k f_k}$ ,
- (v)  $\sum_{j=i}^{\mu(i,k)} \alpha_j \tilde{b}_{j+1}^{(k)} = \frac{\hat{p}_{i:e(i)}}{\hat{b}_k} = \frac{\bar{p}_{i:e(i)}}{\bar{b}_k}$ .



Substituting (i)–(v) into (16), we obtain,

$$\frac{df_i}{f_i} = - \sum_{k=i+1}^n \frac{\alpha_k f_k}{1 + \alpha_k f_k} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_i dt + \boldsymbol{\sigma}_i \cdot d\mathbf{w}^{(n+1)},$$

which is the familiar expression for the SDE of the LIBOR market model under the terminal measure.

Second, we derive the LIBOR SDE for the spot measure. If we substitute (i)–(v) into (21), we see that for  $k \geq i+1$ ,  $\sum_{j=i}^k \alpha_j \tilde{b}_{j+1}^{(k)}$  cancels against  $\bar{p}_{i:i+1} \tilde{b}_{i(t)}^{(k)}$ , and for  $k \leq i$ , we are left with  $-\bar{p}_{i:i+1} \tilde{b}_{i(t)}^{(k)}$ , therefore:

$$\frac{df_i}{f_i} = \sum_{k=i(t)}^i \frac{\alpha_k f_k}{1 + \alpha_k f_k} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_i dt + \boldsymbol{\sigma}_i \cdot d\mathbf{w}^{(i(t))},$$

which is the familiar expression for the SDE of the LIBOR market model under the spot measure.

## 5 Complexity

We study the complexity of the drift calculation over a single time step in a numerical implementation. For generic market models, we show that the complexity is, at worse, of order  $\mathcal{O}(n^3)$ . For specific market models, such as the LIBOR, swap, and CMS market models, we show that a more efficient implementation is available that renders the order to  $\mathcal{O}(nd)$ . For CMS market models, this more efficient implementation is approximate.

For generic market models, the results are derived for the terminal measure, but can equally well be derived for the spot measure. Recall (13) that occurs in the drift calculation,

$$\frac{\partial \hat{\mathbf{p}}}{\partial f_{k:e(k)}} = -\mathbf{A}\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial f_{k:e(k)}} \mathbf{U}^{-1} \mathbf{c}.$$

The inverse of  $\mathbf{U}$  can be calculated in  $\mathcal{O}(n^3)$  operations. Subsequently, the 4 consecutive matrix multiplications with a vector require  $\mathcal{O}(n^2)$  operations, for each forward rate  $k$ , thus in total  $\mathcal{O}(n^3)$  operations. Therefore a generic market model has at worse a complexity of  $\mathcal{O}(n^3)$ .

The LIBOR market model has a special structure that renders the complexity to  $\mathcal{O}(nd)$ , which has been shown by Joshi (2003). In Algorithm 2 such an  $\mathcal{O}(nd)$  algorithm has been displayed that calculates the forward LIBOR rates for a time step under the terminal measure. An algorithm for the spot measure can be defined analogously, by summing from 1 to  $n$  and by incrementing  $\gamma_j$  (rather than decrementing) before updating  $\phi_i^{(2)}$ .

We show that a similar approximate algorithm can be defined for CMS( $q$ ) market models, for the terminal measure. The algorithm is shown to be exact

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**Algorithm 2** An  $\mathcal{O}(nd)$ -algorithm for calculating the forward LIBOR rates for a time step in the LIBOR market model. The number of factors is denoted by  $d$ . The log forward rates,  $\log \mathbf{f}(t) = (\log f_{i(t)}(t), \dots, \log f_n(t))$  at time  $t$ , and  $\log \mathbf{f}(t + \Delta t)$  at time  $t + \Delta t$ , are denoted by  $\boldsymbol{\phi}^{(1)}$  and  $\boldsymbol{\phi}^{(2)}$ , respectively. Here  $\boldsymbol{\Sigma} = (\sigma_{ij})$  governs the volatility, with  $\sigma_{ij}$  the time- $t$  volatility of forward rate  $f_i$  with respect to factor  $j$  in the model.  $\Delta \mathbf{w}$  should be sampled from a  $\mathcal{N}(\mathbf{0}, \sqrt{\Delta t} \mathbf{I}_d)$  distribution.

---

Input:  $n$ ;  $d$  ( $1 \leq d \leq n$ );  $\boldsymbol{\phi}^{(1)}, \boldsymbol{\alpha} \in \mathbb{R}^n$ ;  $\Delta \mathbf{w} \in \mathbb{R}^d$ ;  $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times d}$ ;  $\Delta t$ .

Output:  $\boldsymbol{\phi}^{(2)} \in \mathbb{R}^n$ .

- 1: Set  $\boldsymbol{\gamma} \leftarrow \mathbf{0}$  with  $\boldsymbol{\gamma} \in \mathbb{R}^d$ .
  - 2: **for**  $i = n, \dots, i(t)$  **do**
  - 3:    $\phi_i^{(2)} \leftarrow \phi_i^{(1)}$ .
  - 4:   **for**  $j = 1, \dots, d$  **do**
  - 5:      $\phi_i^{(2)} \leftarrow \phi_i^{(2)} + (\gamma_j - \frac{1}{2} \sigma_{ij}) \sigma_{ij} \Delta t + \sigma_{ij} \Delta w_j$ .
  - 6:      $\gamma_j \leftarrow \gamma_j - \frac{\alpha_i \exp(\phi_i^{(1)})}{1 + \alpha_i \exp(\phi_i^{(1)})} \sigma_{ij}$ .
  - 7:   **end for**
  - 8: **end for**
- 

for the swap market model ( $q = n$ ). The following quantity that occurs in the drift term is approximated:

$$\tilde{p}_{i:\mu(i,k)+1}^{(k)} = \tilde{p}_{i:\min(k,i+q)}^{(k)} = \sum_{j=i}^{\min(k,i+q)} \alpha_j \tilde{b}_{j+1}^{(k)} \quad (i < k). \quad (22)$$

The approximation is based on the assumption that  $\alpha_i$  is close to  $\alpha_{i+q}$ , for  $i = 1, \dots, n - q$ . Note that this assumption is used only to efficiently approximate (22) for calculation of drift terms, and this assumption is *not* used in the calculation of contract payoffs. Moreover, if needs be, the drift terms can be calculated exactly by exact calculation of (22).

**Approximation 1** *Approximately, by assumption of  $\alpha_i \approx \alpha_{i+q}$  ( $i = 1, \dots, n - q$ ), we have, for  $\tilde{p}_{i:\mu(i,k)+1}^{(k)}$  defined in (22),*

$$\tilde{p}_{i:\mu(i,k)+1}^{(k)} \approx \alpha_{k-1} \prod_{m=i}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}) \quad (i < k). \quad (23)$$

*Here, an empty product denotes 1. Formula (23) is exact for  $i > k - q - 1$ . In particular, (23) is exact for any  $i$  in the swap market model ( $q = n$ ).*

The rationale for Approximation 1, as well as the proof of exactness when  $i > k - q - 1$ , are given in Appendix A. Note that accumulating errors in (23) are likely to cancel, since in practice the difference  $\alpha_i - \alpha_{i+q}$  is both negative and

Table 2: Deal description for the test of exact versus approximate drift terms in CMS( $q$ ) models.

Currency:	USD
Market data:	Swap rates and at-the-money swaption volatility
Valuation date:	18 July 2003
Deal:	30 year fixed-maturity Bermudan swaption
Start date:	16 June 2004
Frequency:	Annual
Day count:	ACT/365
Date roll:	Modified following
Fixed coupon:	3.2%

positive. From (16) and Approximation 1, we obtain,

$$\frac{df_{i:e}}{f_{i:e}} \approx -\frac{1}{\hat{p}_{i:e}} \sum_{k=i+1}^n (\hat{b}_k - \hat{b}_{e(k)}) \alpha_{k-1} \prod_{m=i}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}) \boldsymbol{\sigma}_{k:e(k)} \cdot \boldsymbol{\sigma}_{i:e} dt + \boldsymbol{\sigma}_{i:e} \cdot d\mathbf{w}^{(n+1)}. \quad (24)$$

Define

$$\mathbf{v}_i = \sum_{k=i+1}^n (\hat{b}_k - \hat{b}_{e(k)}) \alpha_{k-1} \prod_{m=i}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}) \boldsymbol{\sigma}_{k:e(k)}. \quad (25)$$

The proof of the following lemma has been deferred to Appendix B.

**Lemma 1** *The quantity  $\mathbf{v}_i$  defined in (25) satisfies the following recursive formulas:*

- $\mathbf{v}_n = \mathbf{0}$ ,
- $\mathbf{v}_i = (1 + \alpha_i f_{i+1:e(i+1)}) \mathbf{v}_{i+1} + \alpha_i (\hat{b}_{i+1} - \hat{b}_{e(i+1)}) \boldsymbol{\sigma}_{i+1:e(i+1)}$ .

In Algorithm 3 an  $\mathcal{O}(nd)$  algorithm, based on Lemma 1, has been displayed that approximately calculates the forward swap rates for a time step under the terminal measure, for the CMS( $q$ ) market model. This algorithm is exact for the swap market model ( $q = n$ ).

To benchmark the accuracy of Algorithm 3, various fixed-maturity Bermudan swaptions are priced in their corresponding CMS( $q$ ) market models, with both exact SDE (16) and approximate SDE (24). The deal specification is given in Table 2. The swap tenor is  $q$  years, with  $31 - q$  exercise opportunities, at (16 June 2004 +  $i$  years),  $i = 0, \dots, 30 - q$ , for  $q = 1, \dots, 30$ . The difference between the minimum (0.996) and maximum (1.007) attained day count fractions is 0.011. To price fixed-maturity Bermudan swaptions in Monte Carlo, we use the algorithm of Longstaff & Schwartz (2001), with the swap value as explanatory variable

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**Algorithm 3** An  $\mathcal{O}(nd)$ -algorithm for approximately calculating the forward swap rates for a time step in the CMS( $q$ ) market model (exact when  $q = n$ ), under the terminal measure. The number of factors is denoted by  $d$ . The log forward rates,  $\log \mathbf{f}(t) = (\log f_{i(t):e(i(t))}(t), \dots, \log f_{n:e(n)}(t))$  at time  $t$ , and  $\log \mathbf{f}(t + \Delta t)$  at time  $t + \Delta t$ , are denoted by  $\phi^{(1)}$  and  $\phi^{(2)}$ , respectively. Here  $\Sigma = (\sigma_{ij})$  governs the volatility, with  $\sigma_{ij}$  the time- $t$  volatility of forward rate  $f_{i:e(i)}$  with respect to factor  $j$ . Here,  $e(\cdot)$  is defined in (2).  $\Delta \mathbf{w}$  should be sampled from a  $\mathcal{N}(\mathbf{0}, \sqrt{\Delta t} \mathbf{I}_d)$  distribution.

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Input:  $n; d, q$  ( $1 \leq d, q \leq n$ );  $\phi^{(1)}, \alpha \in \mathbb{R}^n; \Delta \mathbf{w} \in \mathbb{R}^d; \Sigma \in \mathbb{R}^{n \times d}; \Delta t$ .

Output:  $\phi^{(2)} \in \mathbb{R}^n$ .

- 1:  $\beta_{n+1} \leftarrow 1. \varpi_{n+1} \leftarrow 0.$
- 2: **for**  $i = n, \dots, i(t)$  **do**
- 3:  $\varpi_i \leftarrow \varpi_{i+1} + \alpha_i \beta_{i+1} - \mathbf{1}_{\{i < n \ \& \ e(i) = e(i+1) - 1\}} \alpha_{e(i+1) - 1} \beta_{e(i+1)}.$
- 4:  $f_i^{(1)} \leftarrow \exp(\phi_i^{(1)}).$
- 5:  $\beta_i \leftarrow \varpi_i f_i^{(1)} + \beta_{e(i)}.$
- 6: **If**  $i = n$ , set  $\mathbf{v}_n \leftarrow \mathbf{0} \in \mathbb{R}^d$ , **else** ( $i < n$ ), set

$$\mathbf{v}_i \leftarrow (1 + \alpha_i f_{i+1}^{(1)}) \mathbf{v}_{i+1} + \alpha_i (\beta_{i+1} - \beta_{e(i+1)}) \boldsymbol{\sigma}_{i+1}.$$

- 7:  $\phi_i^{(2)} \leftarrow \phi_i^{(1)} + (-\frac{1}{\varpi_i} \mathbf{v}_i - \frac{1}{2} \boldsymbol{\sigma}_i) \cdot \boldsymbol{\sigma}_i \Delta t + \boldsymbol{\sigma}_i \cdot \Delta \mathbf{w}.$
  - 8: **end for**
- 

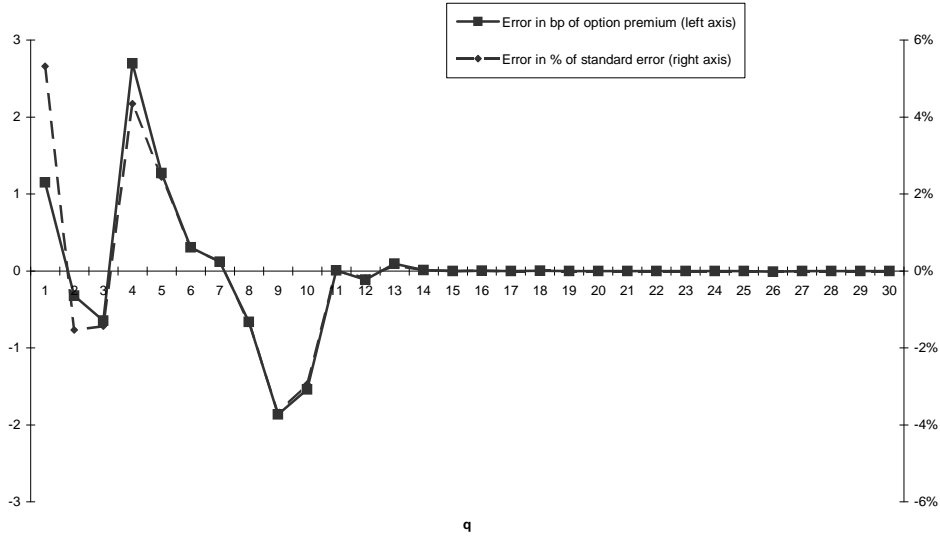


Figure 3: Results of the test of exact versus approximate drift terms in CMS( $q$ ) models.

$x$ , and basis functions 1,  $x$  and  $x^2$ . An 8 factor model is used ( $d = 8$ ), with the correlation of the forward CMS( $q$ ) rates given by the parametrization of Rebonato (1998, Equation (4.5), page 83),  $\exp(-\beta|t_i - t_j|)$ , for rates  $f_{i:e(i)}$  and  $f_{j:e(j)}$ , with  $\beta = 3\%$ . The differences between the prices obtained with exact and approximate drift terms have been displayed in Figure 3. We note that for  $q = n$ , equal prices are obtained up to all digits. The results show that the error is small, up to only 3 bp of the option premium, and up to only 6% of the simulation standard error. Moreover, the error fluctuates robustly around 0, since the difference  $\alpha_i - \alpha_{i+q}$  is both negative and positive, in practice.

A significant reduction of computational time can thus be attained by selecting a low number of factors  $d$ . A result of using a low number of factors is that the instantaneous correlation matrix ( $\rho_{ij}$ ) cannot be exactly fit to the historically estimated or market implied correlation matrix. The procedure for fitting a generic market model to correlation is exactly the same as for the LIBOR market model. For fitting a low-factor LIBOR market model to correlation, the reader is referred to Pietersz & Groenen (2004a, b), Grubišić & Pietersz (2005), Wu (2003) and Rebonato (2002, Section 9) or Brigo (2002).

## 6 Generic Calibration to Correlation

When each particular interest rate derivative has its own generic market model that is used for its valuation and risk management, then the associated input correlation to those models involves different interest rates. There is a relationship between these correlations, and this relationship allows for netting correlation risk or reserves. Moreover, utilizing the relationship between the correlations means that correlation is determined consistently across different products. In general all interest rate correlations stem from the correlations between different segments of the yield curve. In this section we show how forward LIBOR correlations can be used to determine subsequently the correlations for any of the generic market models specific to certain interest rate products.

The advantage of considering all correlations in this way comes from the fact that one can treat correlation risk (or reserves) in a consistent fashion across all interest rate products. Netting of correlation reserves will subsequently occur naturally. Furthermore only instantaneous forward LIBOR correlations have to be determined and administered.

The key to the method is the well-known fact that, within the LIBOR market model, the instantaneous volatility vector  $\sigma_{s:e}(t)$  of a forward swap rate  $f_{s:e}$  can be expressed as weighted averages of instantaneous volatility vectors  $\sigma_i(t)$  of forward LIBORs,

$$\sigma_{s:e}(t) = \sum_{i=s}^{e-1} w_i^{s:e}(t) \sigma_i(t).$$

An expression for the weights  $w_i^{s:e}$  may be found, for example, in Hull & White (2000, page 53). The weights  $w_i^{s:e}$  are state dependent. A highly accurate deterministic approximation  $\bar{\sigma}_{s:e}(t)$  for the instantaneous volatility can however

be obtained by evaluating the weights at time zero,

$$\tilde{\sigma}_{s:e}(t) = \sum_{i=s}^{e-1} w_i^{s:e}(0) \sigma_i(t).$$

From the preceding considerations it should be clear that the instantaneous forward rate correlation  $\rho_{s(1):e(1),s(2):e(2)}(t)$  can be approximately expressed as a function of the instantaneous forward LIBOR correlations  $\rho_{ij}(t)$ ,

$$\begin{aligned} \rho_{s(1):e(1),s(2):e(2)}(t) &= \rho\left(\frac{df_{s(1):e(1)}(t)}{f_{s(1):e(1)}(t)}, \frac{df_{s(2):e(2)}(t)}{f_{s(2):e(2)}(t)}\right) \\ &= \frac{\sigma_{s(1):e(1)}^T(t) \sigma_{s(2):e(2)}(t)}{\sqrt{\sigma_{s(1):e(1)}^T(t) \sigma_{s(1):e(1)}(t) \sigma_{s(2):e(2)}^T(t) \sigma_{s(2):e(2)}(t)}}, \end{aligned}$$

where

$$\begin{aligned} \sigma_{i:j}^T(t) \sigma_{k:l}(t) &\approx \tilde{\sigma}_{i:j}^T(t) \tilde{\sigma}_{k:l}(t) \\ &= \sum_{m_1=i}^{j-1} \sum_{m_2=k}^{l-1} w_{m_1}^{i:j}(0) w_{m_2}^{k:l}(0) |\sigma_{m_1}(t)| |\sigma_{m_2}(t)| \rho_{m_1 m_2}(t). \end{aligned}$$

## 7 Conclusions

In this paper, a generalization of market models has been studied, whereby arbitrary forward rates are allowed to span the tenor structure relevant to an interest rate derivative. The benefit of such generalization is that straightforward volatility-calibration can be achieved for the fixings of LIBOR or swap rates relevant to the interest rate derivative. Generic market models are therefore particularly apt for pricing and risk management of CMS and hybrid coupon swaps, and callable and cancellable versions thereof, in particular, Bermudan CMS swaptions and fixed-maturity Bermudan swaptions. We showed that the LIBOR and swap market models are special cases of the generic market model framework. The need for a generic specification of market models has been illustrated by counting the admissible market model structures with  $n + 1$  tenor times. We found  $n!$  possible market models. For example, already only for an annual-paying deal of 10 years, there are  $10! = 3,628,800$  market models, thereby establishing the need for a generic specification. Necessary and sufficient conditions were derived for a set of forward swap agreements to provide a unique solution for discount bond prices, essentially regardless of the scenario of attained forward rates. The major novelty of this paper is the derivation of generic expressions for no-arbitrage drift terms in generic market models. We developed a novel algorithm of order  $\mathcal{O}(nd)$  for approximate drift calculations in CMS market models under the terminal measure.

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## A Rationale for Approximation 1

We proceed by induction on  $i = k - 1, \dots, 1$ .

- For  $i = k - 1$ :  $\tilde{p}_{k-1:\min(k,k-1+q)}^{(k)} = \alpha_{k-1}\tilde{b}_k^{(k)} = \alpha_{k-1}$ .
- For  $i = k - q, \dots, k - 2$ , we have  $\min(k, i + q) = k$ . The quantity  $\tilde{b}_{i+1}^{(k)}$  satisfies, from the definition  $\tilde{b}_{i+1}^{(k)} = (\mathbf{U}^{-1}\mathbf{c}_k)_{i+1}$ ,  $\tilde{b}_{i+1}^{(k)} = f_{i+1:e(i+1)}\tilde{p}_{i+1:k}^{(k)}$ , since  $k < i + q + 1$ . It follows,

$$\begin{aligned}\tilde{p}_{i:k}^{(k)} &= \tilde{p}_{i+1:k}^{(k)} + \alpha_i\tilde{b}_{i+1}^{(k)} = \tilde{p}_{i+1:k}^{(k)}(1 + \alpha_i f_{i+1:e(i+1)}) \\ &\stackrel{(*)}{=} \alpha_{k-1} \prod_{m=i+1}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}) (1 + \alpha_i f_{i+1:e(i+1)}) \\ &= \alpha_{k-1} \prod_{m=i}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}),\end{aligned}$$

where equality (\*) follows from the induction hypothesis.

- For  $i = i(t), \dots, k - q - 1$ , we have  $\min(k, i + q) = i + q$  and  $\tilde{b}_{i+1}^{(k)} = f_{i+1:e(i+1)}\tilde{p}_{i+1:i+q+1}^{(k)} + \tilde{b}_{i+q+1}^{(k)}$ . Therefore,

$$\begin{aligned}\tilde{p}_{i:k}^{(k)} &= \alpha_i\tilde{b}_{i+1}^{(k)} - \alpha_{i+q}\tilde{b}_{i+q+1}^{(k)} + \tilde{p}_{i+1:i+q+1}^k \\ &= \alpha_i (f_{i+1:e(i+1)}\tilde{p}_{i+1:i+q+1}^{(k)} + \tilde{b}_{i+q+1}^{(k)}) - \alpha_{i+q}\tilde{b}_{i+q+1}^{(k)} + \tilde{p}_{i+1:i+q+1}^k \\ &\stackrel{(*)}{\approx} \tilde{p}_{i+1:k}^{(k)}(1 + \alpha_i f_{i+1:e(i+1)}) \\ &= \alpha_{k-1} \prod_{m=i}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}),\end{aligned}$$

where in approximation (\*), we have used  $\alpha_i \approx \alpha_{i+q}$ .  $\square$

## B Proof of Lemma 1

For  $i < n$ ,

$$\begin{aligned}\mathbf{v}_i &= \sum_{k=i+1}^n (\hat{b}_k - \hat{b}_{e(k)}) \alpha_{k-1} \prod_{m=i}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}) \boldsymbol{\sigma}_{k:e(k)} \\ &= (\hat{b}_{i+1} - \hat{b}_{e(i+1)}) \alpha_i \boldsymbol{\sigma}_{i+1:e(i+1)} + (1 + \alpha_i f_{i+1:e(i+1)}) \times \\ &\quad \left\{ \sum_{k=i+2}^n (\hat{b}_k - \hat{b}_{e(k)}) \alpha_{k-1} \prod_{m=i+1}^{k-2} (1 + \alpha_m f_{m+1:e(m+1)}) \boldsymbol{\sigma}_{k:e(k)} \right\} \\ &= (1 + \alpha_i f_{i+1:e(i+1)}) \mathbf{v}_{i+1} + \alpha_i (\hat{b}_{i+1} - \hat{b}_{e(i+1)}) \boldsymbol{\sigma}_{i+1:e(i+1)},\end{aligned}$$

which was to be shown.  $\square$