Investment, Hedging, and Consumption Smoothing*

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Abstract

This paper analyzes a risk averse entrepreneur's real investment decision under incomplete markets. The entrepreneur smoothes his intertemporal consumption by investing in both a risk-free asset and a risky asset, which allows him to partially hedge against the project cash flow risk. We show that risk aversion lowers both the project value upon investment and the option value of waiting to invest through the precautionary saving effect. Furthermore, risk aversion delays investment since the project value is reduced more than the option value to invest. It is also shown that although hedging can reduce the cash flow risk, it may have a positive or negative return effect, depending on the correlation between the cash flow risk and the market. Consequently, investment timing is not monotonic with the extent of hedging opportunity. Finally, welfare implications of hedging are analyzed.

Keywords: Real options, risk aversion, incomplete markets, hedging, precautionary saving

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1 Introduction

Entrepreneurs play an important role in fostering innovation and economic growth (Schumpeter (1934)). It is often suggested that an "entrepreneur" is someone who combines upfront business investments with entrepreneurial skill to obtain the chance of earning cash flows. This notion ranges from inventors who create new products or even new industries to local business people starting restaurants and retail stores.[1] A common feature of entrepreneurs is that their business investments, consumption-saving, and portfolio selection decisions are interdependent. The aim of this paper is to provide a dynamic model to analyze this interdependence.

We model an entrepreneur's business investment decision as a real options problem. Since the seminal work of Brennan and Schwartz (1985) and McDonald and Siegel (1986), the real options approach to investment under uncertainty has become an essential part of modern economics and finance. The key insights that waiting has positive value and that the opportunity to invest in a project is analogous to an American call option on the investment opportunity have been generally accepted. This is reflected by the fact that many corporate finance textbooks devote at least a chapter to the real options approach (e.g. Brealey and Myers (2002)). Furthermore, related research has been actively published in academic journals.

Although the real options approach to investment has been developed substantially,[2] most papers in this literature either assume that markets are complete or decision makers are risk neutral. While either assumption serves as a natural starting point in order for researchers to single out and focus on the *option* value of waiting,

[1] See Gentry and Hubbard (2004) for this definition.

[2] The standard real options approach to investment has been excellently summarized in Dixit and Pindyck (1994). Recent developments include agency (Grenadier and Wang (2004), Grenadier, Miao and Wang (2004)), ambiguity (Miao and Wang (2004)), macroeconomic conditions (Guo, Miao, and Morellec (2004)), industry equilibrium (Grenadier (2002), Miao (2004)), strategic interaction (Grenadier (1996), Miltersen and Schwartz (2002)), and imperfect information (Grenadier (1999), Lambrecht and Perraudin (2003) , Berk, Green, and Naik (2004)).

both assumptions are strong and made primarily for tractability reasons. For example, under complete markets, the physical investment opportunity must be spanned by existing assets in the economy which requires that it be either freely traded or replicated by other assets or portfolios. Under this assumption, one can apply the contingent claims analysis (Black and Scholes (1973) and Merton (1973)) to determine the option value and investment timing. Although assuming risk neutrality and applying dynamic programming can deal with incomplete markets, it is not particularly relevant to the vast risk averse investors in reality.

In reality, it is often the case that risk averse entrepreneurs own investment projects and make investment decisions.[3] These projects may not be freely traded or their payoffs may not be spanned by existing assets because of liquidity restrictions or the lack of liquid markets. These capital market imperfections may be due to moral hazard, adverse selection, transactions costs, or contractual restrictions. As examples, liquid markets for projects to develop new products or R&D ventures often do not exist. Moreover, the results of these projects may be hard to predict so that the associated future cash flows may be unrelated to the risk of the existing assets. Thus these investment opportunities may have substantial idiosyncratic risks. Owning them exposes entrepreneurs to these un-diversifiable risks.[4] Consequently, entrepreneurs' lifetime well beings naturally heavily depend on the outcome of their investments subject to un-diversifiable idiosyncratic risks. Moreover, entrepreneurs' attitudes towards risk should play an important role in determining their consumption-saving, portfolio selection, and investment decisions.

[3]For example, data from the *1993 National Survey of Small Business Finances* shows that the average ownership of the entrepreneur is 81% for businesses with fewer than 500 employees. According to the estimates of the Office of Advocacy of the U.S. Small Business Administration, there were approximately 23.7 million small businesses in the United States in 2003. Here we do not focus on investment decisions for managers in corporate firms. This is because the issues of managerial compensation contracts, capital structure, and the conflict of interest between managers and shareholders may significantly complicate our analysis.

[4]See Gentry and Hubbard (2004) for empirical evidence.

This paper provides a utility-based framework to analyze a risk averse entrepreneur's investment decision under uncertainty and incomplete markets. Extending McDonald and Siegel (1986), we build a model in which the entrepreneur maximizes expected utility from consumption streams when he has a nontraded investment opportunity. We first consider a baseline model where the entrepreneur can only trade a risk-free asset to smooth consumption. We then study the case where the entrepreneur can also trade a risky asset, which can be used to hedge against the cash flow risk. This paper contributes to the literature on the real options approach to investment by providing an analysis on how risk aversion and market incompleteness affect investment timing. This paper also adds to the literature on hedging by analyzing the impact of hedging on investment timing and welfare in an incomplete-markets environment.

According to the standard real options approach under complete markets or risk neutrality, risk aversion does not play any role in real investement timing decision. By contrast, we show that risk aversion delays investment in our incomplete markets setting. The mechanism of the impact of risk aversion is manifested through the consumption smoothing (precautionary saving) effect.[5] Specifically, investment generates a stream of stochastic income and thus exposes the entrepreneur to the uninsurable cash flow risk. An increase in the degree of risk aversion raises precautionary savings, thereby reducing consumption both before and after investment. Consequently, it lowers both the project value and the option value to invest.[6] We further show that the project value is reduced more than the option value, implying that investment is delayed.

We also show that investment timing and welfare may not be monotonic with the

[5]An agent is said to be precautionary, if his marginal utility is convex. Leland (1968) provides an early contribution to precautionary saving. See Kimball (1990) for an axiomatic treatment of precautionary saving.

[6]These values are interpreted as subjective values, but not market values. They are defined using the "certainty equivalent" approach in the literature on the pricing of nontraded assets (e.g., Svensson and Werner (1993), Hall and Murphy (2000), Kahl et al. (2003), and references therein). See Section 2.2 for further discussions.

extent of hedging or the correlation between the project risk and the market. This is in sharp contrast to the conventional view that hedging reduces cash flow risk, and hence it should speed up investment and raise welfare. The reason is that in addition to the preceding risk reduction effect, for the budget constrained entrepreneur, hedging may result in losses of returns from the hedging asset, thereby reducing wealth and the net gains from investment. Depending on the degree of risk aversion, riskiness of projects, and Sharpe ratios of hedging assets, either one of the effects may dominate. This happens when the project risk is positively correlated with the market since the entrepreneur holds a short position on the hedging asset. By contrast, if the correlation is negative, then the return effect is always positive since the entrepreneur holds a long position on the hedging asset. In this case, an increase in the extent of hedging accelerates investment and raises welfare.

Our paper relates to the voluminous consumption-saving literature, pioneered by Friedman (1957). Consumption-saving models study how an individual smooths his consumption over time when he is endowed with an *exogenously* specified stochastic uninsurable income process. This paper is also related to the portfolio choice literature. Duffie et al. (1997) study hedging strategies when an investor is endowed with nontraded stochastic income and maximizes expected utility from consumption streams. Unlike these two strands of literature, in our model the stochastic income process is *endogenously* determined by the entrepreneur's investment timing decision.

The paper closest to ours is Hugonnier and Morellec (2004). In contrast to our result, Hugonnier and Morellec (2004) show that risk aversion decreases the option value to invest, thereby speeding up investment. There are three major differences between their paper and ours. First, they assume that an investor/manager maximizes expected utility from wealth at the random time of investment. They do not consider intermediate consumption and the consumption after investment. Therefore, they neglect the impact of risk aversion on the utility and consumption after investment, or on the value of the project. Second, they do not study the impact of hedging on investment timing and welfare. Finally, they consider the role of the market for corporate control in constraining management, while we abstract from this consideration.

The remainder of the paper proceeds as follows. Section 2 analyzes a baseline model in which there is no risky asset available for hedging. Section 3 analyzes a model with hedging. Section 4 concludes. Technical details are relegated to appendices.

2 A Baseline Model

This section provides a model that allows us to develop intuition on how the entrepreneur's risk aversion affects his investment decision when markets are incomplete. In order to achieve this objective in a simplest possible setting, we integrate a canonical incomplete-markets consumption-saving model with a version of irreversible investment model à la McDonald and Siegel (1986).

2.1 Setup

Time is continuous and horizon is infinite. Uncertainty is represented by a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P}\right)$, on which all stochastic processes are defined. Here $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmented filtration generated by the standard Brownian motion $(Z_t)_{t\geq 0}$.

There is a single perishable consumption good (the numeraire). Let $\mathcal{C}$ be the space of progressively measurable consumption processes C such that $\int_0^t |C_s|\, ds < \infty$ for any $t \geq 0$. The entrepreneur derives utility from a consumption process $(C_t)_{t\geq 0} \in \mathcal{C}$ according to the utility function

$$E\left[\int_0^\infty e^{-\beta t} U(C_t)\, dt\right], \tag{1}$$

where $\beta > 0$ is the discount rate and U is an increasing and concave vNM index. We consider the CARA specification $U(c) = -e^{-\gamma c}/\gamma$, $c \in \mathbb{R}$, where $\gamma > 0$ is the absolution risk aversion parameter. We choose this utility specification primarily for its technical tractability. It is well known that this utility function rules out wealth effect and hence facilitates closed form solutions.[7]

The entrepreneur has an investment project, which can be undertaken irreversibly, at a time τ chosen by him. Investment costs $I > 0$ paid at the exercising time τ. This cost is financed from the entrepreneur's own wealth. If there is shortage, it is financed from borrowing at the constant risk-free rate $r > 0$. Upon investment, the project generates continuous cash flows $(Y_t)_{t\geq\tau}$ into the future. Assume that $(Y_t)_{t\geq 0}$ is governed by an arithmetic Brownian motion process

$$dY_t = \alpha\, dt + \sigma\, dZ_t,\ Y_0 \text{ given}, \tag{2}$$

where α and σ are positive constants and Z is a standard Brownian motion. This process implies that cash flows may take negative values. We interpret negative cash flows as losses.[8]

The standard real options approach to investment tackles this type of optimization problem via one of the following two methods. One method is to assume that markets are complete in the sense that the project cash flow can be freely traded or there is another traded asset that can replicate the cash flows. Then one can appeal to the contingent claims analysis to determine the option value of investment and the option exercise time. Alternatively, it is assumed that the entrepreneur is risk neutral and thus maximizes the discounted value of cash flows. A dynamic programming approach is often used under such a setting.

Unlike the standard real options settings summarized above, the entrepreneur in our model is neither risk neutral nor faces complete markets. Instead, the en-

[7]The CARA utility has been widely adopted in the literature on consumption (Caballero (1991), Wang (2004)), asset pricing (Wang (1993)), and portfolio choice (Merton (1969), Svensson and Werner (1993), Liu (2004)).

[8]Unlike the usual geometric Brownian motion process, the specification in (2) proves more convenient within the present model. This is essentially due to the results for a class of exponential-affine models. See Duffie (2001) on introductory treatment on affine models and Wang (2004) on affine consumption models.

trepreneur only has access to one financial asset. Specifically, he may borrow or lend at a constant risk-free rate $r > 0$. In other words, saving is the only financial investment that the entrepreneur may use to smooth his consumption over time. Given that the cash flows of the investment project is stochastic, markets are naturally incomplete.

Let τ be the stopping time of investment and $\mathcal{T}$ be the set of $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times. Let $(W_t)_{t\geq 0}$ be the wealth process. Then the entrepreneur's decision problem is to choose $(\tau, C) \in \mathcal{T} \times \mathcal{C}$ so as to maximize (1) subject to the wealth dynamics

$$dW_t = (rW_t - C_t)\,dt,\ 0 \leq t < \tau,\ W_0 \text{ given,} \tag{3}$$

$$dW_t = (rW_t - C_t + Y_t)\,dt,\ \tau \leq t,\ W_\tau = W_{\tau-} - I, \tag{4}$$

and a transversality condition specified later. The first wealth dynamics (3) states that wealth is accumulated from saving assuming the entrepreneur has no other income during the period before investment $t \leq \tau$. The second wealth dynamics (4) describes the wealth accumulation after investment. At the instant of investment time τ, the entrepreneur pays investment cost I and hence wealth is lowered to $W_\tau = W_{\tau-} - I$. After investment $t \geq \tau$, the entrepreneur receives income from the investment cash flows Y_t. As usual, we interpret negative wealth as borrowing. In order to focus on the effect of market incompleteness in a simplest possible setting, we do not consider borrowing constraints or costly external financing.

2.2 Model Solution

We solve the entrepreneur's problem backward by dynamic programming. We first consider the problem after investment has been taken place. Let $J(w, y)$ be the corresponding value function. By a standard argument, $J(w, y)$ satisfies the following standard Hamilton-Jacobi-Bellman (HJB) equation:

$$\beta J(w,y) = \max_{c\in\mathbb{R}} U(c) + (rw - c + y)\,J_w(w,y) + \alpha J_y(w,y) + \frac{\sigma^2}{2} J_{yy}(w,y)\,. \tag{5}$$

The transversality condition must also be satisfied $\lim_{T\to\infty} E\left[e^{-rT}J\left(W_T, Y_T\right)\right] = 0$.

We next consider the case before investment. Let $V(w, y)$ denote the corresponding value function. Similarly, $V(w, y)$ satisfies the HJB equation

$$\beta V(w, y) = \max_{c\in\mathbb{R}} U(c) + (rw - c)\, V_w(w, y) + \alpha V_y(w, y) + \frac{\sigma^2}{2} V_{yy}(w, y)\,. \tag{6}$$

We now specify boundary conditions. First, the following no-bubble condition must be satisfied

$$\lim_{y\to-\infty} V\left(w, y\right) < \infty. \tag{7}$$

This condition states that when the investment cash flow goes to negative infinity, the entrepreneur will never exercise the option and his value function must be finite. Next, as is standard in the optimal stopping problems, at the instant of investment, the following value matching condition must hold

$$V(w, y) = J(w - I, y). \tag{8}$$

This equation implicitly determines an investment boundary $y = \overline{y}\left(w\right)$. Finally, because this boundary is chosen optimally, the following smooth-pasting condition must be satisfied[9]

$$\left.\frac{\partial V\left(w, y\right)}{\partial w}\right|_{y=\overline{y}(w)} = \left.\frac{\partial J\left(w - I, y\right)}{\partial w}\right|_{y=\overline{y}(w)}, \tag{9}$$

$$\left.\frac{\partial V\left(w, y\right)}{\partial y}\right|_{y=\overline{y}(w)} = \left.\frac{\partial J\left(w - I, y\right)}{\partial y}\right|_{y=\overline{y}(w)}. \tag{10}$$

Notice that the above problem is a mixed control and stopping problem, which is generally difficult to solve. Since our objective is to highlight the intuition on how risk aversion affects investment decision, we have intentionally chosen the CARA utility specification, because CARA utility has no wealth effect and permits a closed form solution to the value functions. The functional form of value functions implies that

[9] See, for example, Krylov (1980), Dumas (1991) and Dixit and Pindyck (1994).

wealth can be cancelled out on the two sides of equations (8)-(10). As a result, the investment boundary is flat, in that $\overline{y}(w)$ is independent of wealth w. This allows us to simplify the above optimization problem substantially from a two-dimensional free boundary problem to a one-dimensional one. We are then able to derive closed form solutions to the consumption and investment policies up to an ODE. The following proposition summarizes the solution.

Proposition 1 *Let $(g, \bar{y})$ be the solution to the free boundary problem*

$$rg(y) = \alpha g'(y) + \frac{\sigma^2}{2} g''(y) - \frac{\gamma r \sigma^2}{2} g'(y)^2, \tag{11}$$

subject to the boundary conditions

$$\begin{aligned} \lim_{y \to -\infty} g(y) &< \infty, && (12) \\ g(\bar{y}) &= f(\bar{y}) - I, && (13) \\ g'(\bar{y}) &= \frac{1}{r}, && (14) \end{aligned}$$

where $f(y)$ is given by

$$f(y) = \left(\frac{1}{r} y + \frac{\alpha}{r^2} \right) - \frac{\gamma \sigma^2}{2r^2}. \tag{15}$$

If $g(y) > f(y) - I$ for $y < \bar{y}$, then the threshold value $\bar{y}$ partitions the state space into an investment region $\{(w, y) \in \mathbb{R}^2 : y \geq \bar{y}\}$ and a waiting region $\{(w, y) \in \mathbb{R}^2 : y < \bar{y}\}$. In the waiting region, the value function $V(w, y)$ and the optimal consumption policy $\overline{c}(w, y)$ are given by

$$\begin{aligned} V(w, y) &= -\frac{1}{\gamma r} \exp \left[-\gamma r \left(w + \frac{\beta - r}{\gamma r^2} + g(y) \right) \right], && (16) \\ \overline{c}(w, y) &= r \left(w + \frac{\beta - r}{\gamma r^2} + g(y) \right). && (17) \end{aligned}$$

In the investment region, the value function $J(w, y)$ and the optimal consumption *policy* $\bar{c}(w,y)$ *are given by*

$$J(w,y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w+\frac{\beta-r}{\gamma r^2}+f(y)\right)\right], \tag{18}$$

$$\bar{c}(w,y) = r\left(w+\frac{\beta-r}{\gamma r^2}+f(y)\right). \tag{19}$$

Finally, the entrepreneur invests in the project the first time the process $(Y_t)_{t\geq 0}$ *hits the threshold* $\bar{y}$.

We first observe that equations (11)-(14) are similar to those obtained in the standard real options problems (e.g. McDonald and Siegel (1986) and Dixit and Pindyck (1994)). Specifically, one can interpret $f(y)$ as the (subjective) value of the project and $g(y)$ as the (subjective) value of the option to invest. Although under incomplete markets there is no well defined market value for the nontraded investment project, our interpretation can be justified by adopting the certainty equivalent approach in the literature on the pricing of nontraded assets. Specifically, define the (option) value of the project as the price at which the entrepreneur is indifferent between the situation where he pays this price and obtains the investment (option) cash flows and the situation where he has no investment project. It is straightforward to calculate the value function under the latter situation.[10] Thus, comparing this function with (18) and (16) delivers our preceding claim. Figure 1 plots the functions f and g, which have similar shapes to those in the standard real options model.

[Insert Figure 1 Here]

The biggest difference between our model and the standard real options model is that both the project value f and the option value g depend on not only the parameters describing the asset value such as the riskless rate r, drift α and volatility

[10] Specifically, the value function is given by $-\frac{1}{\gamma r}\exp\left[-\gamma r\left(w+\frac{\beta-r}{\gamma r^2}\right)\right]$.

σ, but also the entrepreneur's risk aversion coefficient γ. This observation is important for understanding the analysis below.

The dependence of the project value f and the option value g on risk aversion captures precisely the fact that the entrepreneur's risk aversion matters not only for consumption decisions, but also for investment decisions when markets are incomplete. We now analyze the intuition in detail. Consider first the consumption rule after investment is made. We are able to derive an explicit solution given in (19)-(15), to a large extent due to the CARA utility specification.[11] To understand this rule, we define human wealth h as the present discounted value of all investment cash flows following Friedman (1957) and Hall (1978). For our arithmetic Brownian motion income process, this gives

$$h \equiv E\left(\int_0^\infty e^{-rt} Y_t dt \,\middle|\, Y_0 = y\right) = \frac{y}{r} + \frac{\alpha}{r^2}\,. \tag{20}$$

Using the definition of human wealth, we may rewrite the consumption rule given in (19) and (15) as follows:

$$\bar{c}(w, y) = r\left(w + h\right) + \frac{\beta - r}{\gamma r} - \frac{\gamma \sigma^2}{2r}\,. \tag{21}$$

The first term in (21) is the annuity value of the sum of financial wealth w and human wealth h. If this were the only term in the consumption rule, then the consumption rule would correspond to Friedman's seminal permanent-income hypothesis and the implied consumption is a martingale (Hall (1978)). This is the core of consumption smoothing if the agent does not have any precautionary motive and if his subjective discount rate is equal to the riskless rate. The second term in (21) incorporates the agent's preference for intertemporal consumption arising solely from the differential between his subjective discount rate and the interest rate.

Most importantly, the third term in (21) captures the precautionary saving motive, which is induced by the cash flow risk after investment is made. It is increasing in

[11]This consumption rule is obtained in discrete time by Caballero (1991) and extended to more general income processes allowing for conditional heteroskedasticity of income by Wang (2004).

risk aversion γ and volatility σ of the cash flow.[12] The precautionary saving lowers the consumption after investment, and hence lowers the project value f.

Turn to the consumption rule before investment given in (17). It admits a similar interpretation. However, we do not have a closed form solution for g because of the presence of the last nonlinear term in (11). Intuitively, this term reflects the precautionary saving effect. It also lowers the option value g. If the entrepreneur obtained a cash stock at the instant of investment and did not obtain any cash flows in the future, then the decreased option value g would speed up investment. This is actually the main reason leading to the result in Hugonnier and Morellec (2004).

Observe that the entrepreneur's consumption is influenced by the cash flow y, even though he does not actually receive any cash flows before investment. This is because the entrepreneur is a forward-looking agent. Although he does not receive any income from y before investment, he rationally anticipates that the evolution of the future cash flow attainable upon his investing is relevant for his consumption decision making even before investment. This idea is at the core of permanent-income hypothesis. Alternatively, we may view that the agent uses saving to partially hedge against changes in his investment opportunity set, the "future" cash flow process in our setting. This interpretation leads us to link to the enormous portfolio choice literature pioneered by Merton (1969).

Finally, notice that the investment threshold $\bar{y}$ is independent of the discount rate β. This is because it has no impact on the project value f and the option value g

[12] More precisely, it is the third derivative of utility function that matters for our analysis. That is, the convex marginal utility gives rise to precautionary saving motive (Kimball (1990)) and thus affects investment timing decisions. For CARA utility, the coefficient of absolute prudence $-u'''/u''$, which measures the precautionary motive, is equal to the coefficient of absolute risk aversion $-u''/u' = \gamma$. For CRRA utility, precautionary saving is also positively related to the constant risk aversion coefficient. The classic example that differentiates risk aversion from precautionary saving is quadratic utility. An agent with quadratic utility is risk averse, but has no precautionary motive (the marginal utility is linear, not strictly convex.) The investment timing decision for an entrepreneur with quadratic utility will thus not be affected by his risk aversion. Quadratic utility is viewed by economists as an implausible utility specification because it implies increasing absolute risk aversion, inconsistent with empirical evidence.

given our CARA specification. Consequently, in our simulations below, we always set $\beta = r$.

Before delving into the details on the effect of risk aversion on investment timing, we first sketch out a simple case in which there is no cash flow risk. We define the value maximizing investment policy as the solution to the following problem

$$\max_{\tau} \int_{\tau}^{\infty} e^{-rt} Y_t dt - e^{-r\tau} I. \tag{22}$$

The following proposition summarizes the relation between the utility-maximizing investment policy and the value-maximizing investment policy when cash flow is deterministic.

Proposition 2 *Suppose that the investment cash flow process* $(Y_t)_{t \geq 0}$ *is deterministic and that the entrepreneur can only trade a riskless asset, then the entrepreneur invests when the cash flow reaches a trigger value* $\bar{y}_0 = rI$*, which is the same as the value-maximizing policy. Furthermore, this result holds true for any strictly increasing utility function* U*.*

This result is intuitive since risk aversion should not matter in the absence of uncertainty. The entrepreneur's utility maximization can be decomposed into two stages: (i) choose the investment policy to maximize the net present value of investment (22); and then (ii) finance consumption out of the maximum attainable total wealth, the sum of initial wealth and the net present value of investment.

When there is cash flow risk, the above result does not hold true generally. Under incomplete markets, the value maximizing policy is not well defined because there are multiple stochastic discount factors (state prices) and the computation of the market value of cash flows depend on a particular state price. When markets are complete, there is a unique state price. In this case, Proposition 4 below provides a similar result to Proposition 2. The intuition is also similar.

2.3 Risk Aversion and Investment Timing

When there is cash flow risk and markets are incomplete, risk aversion plays an important role in determining investment timing. Because there is no analytical solution to the free boundary problem (11)-(14), we resort to numerical simulations. To this end, baseline parameter values must be assigned. We set the risk-free rate $r = 2\%$ and the discount rate $\beta = r = 2\%$. We consider a project with $I = 10$, $Y_0 = 0$, $\alpha = 0.1$, and $\sigma = 0.1$. We leave the coefficient of absolute risk aversion γ as a free parameter since its consensus estimate is not available in the literature.

Figure 2 plots the investment threshold as a function of the volatility σ and risk aversion parameter γ. As is well known in the real options models of investment, the investment threshold increases with the cash flow volatility. However, here the mechanism is different from the standard one, which states that the increased volatility raises the option value of waiting. Within the present model, there is an important consumption smoothing (precautionary saving) effect. Specifically, an increase in the cash flow volatility raises the precautionary saving motive, thereby reducing the value of the project f (as seen from the last term in (15)). Moreover, it lowers the option value of waiting g (as seen from the last term in (11)), thereby mitigating the positive option effect. Simulations reveal that the former effect dominates. This is illustrated in Figure 3, which plots the changes of the functions f and g when volatility σ is increased from 5% to 30%. This figure also reveals that the negative precautionary saving effect dominates the option effect so that g shifts down.

Figure 2 also shows that the impact of volatility becomes larger for higher values of the risk aversion parameter. For example, for $\gamma = 0.1$, when σ is increased from 5% to 30%, the investment threshold increases from 0.2125 to 0.6472, which implies that investment is delayed by 4.3 years on average.[13] This also implies that the investment probability within 5 years is lowered by 39%. By contrast, for $\gamma = 1$,

[13] The average hitting time for the process $(Y_t)_{t\geq 0}$ between two points y and z is given by $|y-z|/\alpha$.

when σ is increased from 5% to 30%, the investment threshold increases from 0.2128 to 1.1655, which implies that investment is delayed by 9.5 years on average and the investment probability within 5 years is lowered by 74%.

[Insert Figures 2-3]

Turn to the impact of changes in the degree of risk aversion. Importantly, Figure 2 reveals that the investment threshold increases with the degree of risk aversion. That is, risk aversion delays investment. The intuition behind the impact of risk aversion is related to the discussion following Proposition 1. Recall that we interpret f as the value of the project and g as the option value to invest. Figure 4 plots the changes of the functions f and g when risk aversion γ is increased from 0.1 to 1. When γ is increased, the precautionary saving rises. This lowers consumption and hence the project value f. In the mean time, due to precautionary saving, consumption before investment also decreases and hence the option value g falls. Simulation results reveal that the former effect dominates the latter so that the entrepreneur delays investment. This dominance is intuitive since the entrepreneur does not bear directly cash flow risk before investment is actually taken place. Therefore, the precautionary saving effect before investment is not as strong as that after investment.

[Insert Figure 4]

However, for low volatilities, the response of the investment threshold is quite small. This is intuitive since when risk is low, risk aversion should not play a significant role. By contrast, when volatility is high, the investment threshold varies significantly with the degree of risk aversion. For example, for $\sigma = 10\%$, when γ is

The investment probability within T years is given by

$$P\left(\max_{0\leq t\leq T} Y_t \geq \bar{y}\right) = \Phi\left(\frac{-\bar{y}+\alpha T}{\sigma\sqrt{T}}\right) + e^{2\alpha\bar{y}/\sigma^2}\Phi\left(\frac{-\bar{y}-\alpha T}{\sigma\sqrt{T}}\right),$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function.

increased from 0.1 to 1, the investment threshold increases from 0.2500 to 0.2548, which implies that investment is delayed by only 18 days (0.05 years) and the investment probability within 5 years is lowered by 0.4%. By contrast, for $\sigma = 30\%$, when γ is increased from 0.1 to 1, the investment threshold increases from 0.6472 to 1.1655, which implies that investment is delayed by 5.2 years and the investment probability within 5 years is lowered by 35%.

3 Hedging and Investment

So far, we have assumed that the entrepreneur can trade a riskless asset only to smooth consumption and diversify cash flow risk. This is clearly unrealistic since in reality entrepreneurs can trade financial assets to hedge against cash flow risk. In this section, we study the implications of hedging.

3.1 Setup

Assume that the entrepreneur can trade a risky asset to hedge against the cash flow risk, in addition to the risk-free asset. One can think of this asset as a futures contract or a market portfolio. Let P_t denote the risky asset's price at date t. Let its returns be governed by the process

$$dP_t/P_t = \mu_e dt + \sigma_e dB_t, \tag{23}$$

where μ_e and σ_e are positive constants, and B is a standard Brownian motion correlated with the Brownian motion Z and defined on the probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P}\right)$. Let $\rho \in [-1, 1]$ be the correlation coefficient. Here the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motions Z and B.

One can alternatively rewrite (2), the cash flow generated from investment as

$$dY_t = \alpha dt + \rho dB_t + \sqrt{1-\rho^2} dB_t^1, \tag{24}$$

where B and B^1 are two independent standard Brownian motions. One can think of B as the Brownian motion describing the market risk and B^1 as the Brownian motion describing the idiosyncratic project risk. The market risk can be diversified away, while the idiosyncratic risk may be not. The correlation ρ describes the extent to which the riskiness of the project is correlated with the market.

Let π_t be the dollar amount invested in the risky asset at time t. A trading strategy $(\pi_t)_{t\geq 0}$ is admissible if it is progressively measurable and satisfies $E\left[\int_0^T \pi_t^2 dt\right] < \infty$ for any $T > 0$. Denote by $\mathcal{A}$ the set of all admissible trading strategies.

The entrepreneur's problem is to choose consumption, portfolio and investment timing $(C, \pi, \tau) \in \mathcal{C} \times \mathcal{A} \times \mathcal{T}$ so as to maximize (1) subject to the wealth dynamics:

$$dW_t = \left[rW_t + \pi_t\left(\mu_e - r\right) - C_t\right]dt + \pi_t \sigma_e dB_t,\ 0 \leq t < \tau,\ W_0 \text{ given}, \tag{25}$$

$$dW_t = \left[rW_t + \pi_t\left(\mu_e - r\right) + Y_t - C_t\right]dt + \pi_t \sigma_e dB_t,\ \tau \leq t,\ W_\tau = W_{\tau-} - I. \tag{26}$$

The wealth dynamics (25)-(26) admit an interpretation similar to that for (3)-(4). The difference is that here the entrepreneur can invest π_t dollars in the risky hedging asset, and thus affects the drift and volatility of wealth accordingly.

3.2 Model Solution

Similar to our solution methodology in Section 2.2, we solve the entrepreneur's problem backward by dynamic programming. The following proposition characterizes the solution.

Proposition 3 *Define the Sharpe ratio $\eta = (\mu_e - r)/\sigma_e$. Let (g, y^*) be the solution to the free boundary problem*

$$rg(y) = \left(\alpha - \rho\eta\sigma\right)g'(y) + \frac{\sigma^2}{2}g''(y) - \frac{\gamma r \sigma^2}{2}g'(y)^2\left(1 - \rho^2\right), \tag{27}$$

subject to the boundary conditions

$$\lim_{y \to -\infty} g(y) < \infty, \tag{28}$$

$$g(y^*) = f(y^*) - I, \tag{29}$$

$$g'(y^*) = \frac{1}{r}, \tag{30}$$

where $f(y)$ is given by

$$f(y) = \left(\frac{1}{r}y + \frac{\alpha}{r^2} - \frac{\eta\sigma\rho}{r^2}\right) - \frac{\gamma\sigma^2}{2r^2}\left(1 - \rho^2\right). \tag{31}$$

If $g(y) > f(y) - I$ for $y < \bar{y}$, then the threshold value y^* partitions the state space into an investment region $\{(w, y) \in \mathbb{R}^2 : y \geq y^*\}$ and a waiting region $\{(w, y) \in \mathbb{R}^2 : y < y^*\}$. In the waiting region, the optimal consumption and portfolio rules are given by

$$c^*(w, y) = r\left(w + \frac{\beta - r + \eta^2/2}{\gamma r^2} + g(y)\right), \tag{32}$$

$$\pi^*(w, y) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r} - \frac{\sigma\rho}{\sigma_e}g'(y). \tag{33}$$

In the investment region, the optimal consumption and portfolio rules are given by

$$c^*(w, y) = r\left(w + \frac{\beta - r + \eta^2/2}{\gamma r^2} + f(y)\right), \tag{34}$$

$$\pi^*(w, y) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r} - \frac{\sigma\rho}{\sigma_e r}. \tag{35}$$

Finally, the entrepreneur invests in the project the first time the process $(Y_t)_{t\geq 0}$ hits the threshold y^*.

Much intuition behind this proposition is similar to that described in Section 2. Specifically, one can think of the investment problem as an option exercise problem where the underlying project value is given by $f(y)$ and the option value is given by $g(y)$. Both f and g depend on model parameters related to asset value and preferences. Unlike the model in Section 2, f and g also depend on the hedging asset's Sharpe ratio η and the correlation coefficient ρ. Note that the investment threshold y^* is independent of the discount rate β, same as in Section 2. In addition, comparing the free boundary problem (27)-(30) with (11)-(14), one can see that the investment threshold y^* when $\rho = 0$ is the same as $\overline{y}$. The intuition is as follows. While the new risky asset allows the agent to take advantage of the expected excess returns, it does not offer any hedging benefits. While the entrepreneur enjoys the same gains in expected excess returns before and after investment, his cash flow risk remains the same with or without the risky asset (whose correlation is zero with cash flow Y. As a result, the investment timing strategy remains the same as the one studied in Section 2.

The key new element of the model in this section is that the entrepreneur can also invest in a risky hedging asset to diversify cash flow risk. The demand for this asset is given in (33) and (35). The first term in these expressions represents the standard mean-variance efficient rule (Merton (1969)). The second term represents the hedging demand. In order to minimize the variation of consumption, the entrepreneur holds a short position on the risky asset if $\rho > 0$, and a long position if $\rho < 0$.

The conventional wisdom is that hedging can reduce investment risk. In our model, this effect is manifested in the consumption rules before and after investment. Consider first the consumption rule after investment, particularly the last term in (31). After investment, cash flows bring income fluctuations. This induces a precautionary saving term $\gamma\sigma^2 \left(1-\rho^2\right) / \left(2r^2\right)$. Compared to the model without hedging, the precautionary saving term is lowered by $\rho^2\gamma\sigma^2 / \left(2r^2\right)$. When markets are complete ($\rho = 1$), investment risk can be perfectly diversified and hence the precautionary saving term disappears.

On the other hand, there is another important *return effect* of hedging. Since the entrepreneur holds a short position on the risky asset when $\rho > 0$, the entrepreneur loses returns from the hedging asset and hence wealth. This causes consumption to decrease by an amount of $\eta\sigma\rho/r^2$ in (31). Depending on parameter values, hedging may increase or decrease consumption and utility after investment. By contrast, when $\rho < 0$, the entrepreneur holds a long position on the risky asset. The return effect is always positive, thereby enhancing wealth and consumption.

Hedging has a similar effect on the consumption rule before investment. In particular, one can interpret the last term in (27) as a consumption smoothing (precautionary saving) effect. One can also interpret the term $-\rho\eta\sigma g'(y)$ as the return effect.

Because of the presence of hedging opportunities, the extent of hedging measured by ρ and the risk characteristic of the hedging asset measured by η are important determinants of investment timing. Before turning to the detailed analysis of hedging effect on investment timing, we first briefly sketch out the investment timing decision under complete markets.

When markets are complete ($\rho = \pm 1$), we can derive an explicit solution to the free boundary problem (27)-(30). Here, we present the solution for $\rho = 1$ only.[14] We shall compare it with the value-maximizing policy defined as the solution to the following problem

$$F(y) = \max_{\tau} E\left[\int_{\tau}^{\infty} \xi_t Y_t dt - \xi_\tau I \,\middle|\, Y_0 = y\right], \tag{36}$$

where $(\xi_t)_{t\geq 0}$ is the unique state price density process $(\xi_t)_{t\geq 0}$ satisfying $-d\xi_t/\xi_t = r dt + \eta dZ_t$, $\xi_0 = 1$. We summarize the relation in the following proposition.

Proposition 4 *Let $\rho = 1$. Then the investment threshold y^*, the option value to invest g, and the hedging demand before investment $\Pi(w, y)$ are respectively given by*

$$y^* = rI - \frac{\alpha - \eta\sigma}{r} + \frac{1}{\lambda}, \tag{37}$$

$$g(y) = F(y) = \frac{1}{r\lambda} e^{\lambda(y - y^*)}, \tag{38}$$

[14]When the project cash flow is perfectly negatively correlated with the hedging asset ($\rho = -1$), a similar analysis applies.

where $\lambda = -(\alpha - \eta\sigma)/\sigma^2 + \sqrt{(\alpha - \eta\sigma)^2 + 2\sigma^2 r}/\sigma^2 > 0$. *This utility maximizing policy is the same as the value-maximizing policy for any strictly increasing utility* U.

This proposition demonstrates that when markets are complete, the subjective option value to invest $g(y)$ is identical to the market option value $F(y)$. In addition, the investment threshold given in (37) is independent of preference parameters. This is consistent with the general principle that the option value and exercise trigger are independent of preferences if markets are complete (e.g., Dixit and Pindyck (1994)). Indeed, since markets are complete, we can apply the martingale method to rewrite the dynamic budget constraint as a static one, using the unique state price density.[15] The entrepreneur's decision problem can then be formulated as a two-stage problem as in the deterministic case described in Section 2.2: (*i*) choose an investment policy to maximize the option value (36) so that total wealth is maximized; (*ii*) choose optimal consumption given this total wealth.

3.3 Implications for Investment Timing

We now turn to the general case where markets are incomplete. We analyze the important question of how investment timing is affected by uncertainty, risk aversion and hedging opportunities. We use parameter values in Section 2.3 as baseline values. In addition, we set the Sharpe ratio $\eta = 0.3$. For example, this corresponds to a risk premium of 6% and a volatility of 20%. Finally, we treat the risk aversion parameter γ and correlation ρ as free parameters.

Cash Flow Risk Consider first the impact of changes in the cash flow volatility σ. Figures 5b, d, f plot the investment threshold as a function of σ for the case of positive correlation $\rho > 0$ and for various values of the risk aversion parameter γ,correlation

[15] See Cox and Huang (1989), and Karatzas, Lehoczky, and Shreve (1987) on martingale methods. Duffie (2001) provides a textbook treatment.

ρ, and Sharpe ratio η. Figures 6b, d, f plot the same function for the case of negative correlation $\rho < 0$. These figures reveals that the investment threshold increases with volatility σ, as in standard models. Importantly, the impact of volatility is sensitive to the values of γ, ρ, and η. As in the model in Section 2, it is intuitive that, under incomplete markets, the impact of volatility should be larger for more risk averse entrepreneurs. Surprisingly, we also find that the impact of volatility is quite different for $\rho > 0$ than for $\rho < 0$. Specifically, Figure 6b reveals that when the project cash flows are negatively correlated to the hedging asset, the impact of the project risk is smaller if the extent of hedging is higher (i.e., $|\rho|$ is bigger). By contrast, Figure 5b indicates an opposite result, implying that hedging destabilizes investment timing when the project cash flows are positively correlated with the hedging asset.

The above sensitivity is in sharp contrast to the standard result under complete markets or risk neutrality, which is explicitly stated in Dixit and Pindyck (1994, p.153):

> "Investment is highly sensitive to volatility in project values, irrespective of entrepreneurs' or managers' risk preferences, and irrespective of the extent to which the riskiness of V [the project value] is correlated with the market."

The intuition behind this difference is similar to that described in Section 2.3. Specifically, under incomplete markets, an increase in σ has the precautionary saving and option effects. In addition, there is an extra return effect. The return effect is positive for $\rho < 0$ and negative for $\rho > 0$. These effects influence both the project value f and the option value to invest g. Furthermore, the magnitude of changes of f and g depends on the values of parameters γ, ρ, and η.

[Insert Figures 5-6]

Risk Aversion Consider next the impact of changes in the coefficient of absolute risk aversion γ. Figures 5c-e and Figures 6c-e plot the investment threshold as a function of risk aversion γ for various levels of volatility σ, correlation ρ, and Sharpe ratio η. These figures reveal that, when markets are incomplete, the investment threshold increases with the degree of risk aversion. The intuition behind this result is similar to that in Section 2. When the entrepreneur is more averse to the cash flow risk, to smooth consumption over time, he saves more for precautionary motive. This lowers consumption, thereby reducing the project value f and the option value g. Simulation results reveal that the former effect dominates. Consequently, the entrepreneur prefers to delay investment. The preceding figures also reveal that the impact of risk aversion is quite large for high values of σ and low values of $|\rho|$. This is intuitive since risk aversion should not matter much if risk is low or if the financial markets are close to be complete.

Correlation Turn to the impact of changes in the correlation coefficient ρ. The correlation between the hedging asset and the investment cash flow provides a measure of the extent of hedging, or the extent to which the project risk is correlated with the market. When $\rho = \pm 1$, the cash flow risk is hedged perfectly. This corresponds to the case of complete markets. When $\rho = 0$, the cash flow risk cannot be hedged. This corresponds to the model in Section 2.

The impact of changes in correlation depends on whether the correlation takes positive or negative values. Figures 5a-c plot the investment threshold as a function of $\rho \geq 0$ for various values of volatility σ, risk aversion γ, and Sharpe ratio η. These figures reveal that the investment threshold increases with ρ for low risk aversion γ, low volatility σ, and high Sharpe ratio η. This result is in contrast to the conventional view that hedging should speed up investment because of reduced risk exposure. Surprisingly, these figures also indicate that the investment threshold first increases and then decreases with $\rho > 0$ for high risk aversion γ, high volatility σ and low Sharpe ratio η. In particular, investment timing is not monotonic with the incompleteness of markets. The top panel of Figure 7 illustrates the impact of the increase of ρ from 0.3 to 1. It reveals that the negative return effect dominates so that investment is delayed.

[Insert Figure 7 Here]

The intuition behind the above result is related to the discussion following Proposition 3. Specifically, on the one hand, an increase in $\rho > 0$ reduces the entrepreneur's exposure to the cash flow risk. The reduced risk exposure lowers the precautionary saving and raises consumption, thereby raising the project value f and the option value to invest g. On the other hand, an increase in $\rho > 0$ raises the short position on the hedging asset (see (35) and (33)). The increased short position results in losses of returns from the hedging asset and reduces wealth, thereby lowering the project value f and the option value to invest g. The overall effect of the impact of an increase in $\rho > 0$ depends on these two opposite effects and the magnitude of changes in f and g. Consequently, investment timing is not monotonic with the degree of hedging when the hedging asset and the project cash flows are positively correlated.

We now turn to the case where the hedging asset and the cash flows are negatively correlated, $\rho < 0$. Figures 6a-c plot the impact of correlation for this case. These figures reveal that, in contrast to the positive correlation case, the investment threshold decreases with the extent of hedging (i.e., $|\rho|$). This is because the entrepreneur holds a long position on the hedging asset and hence hedging has a positive return effect. Consequently, as $|\rho|$ increases, the entrepreneur benefits more from investment, thereby preferring to invest earlier. This is illustrated in the bottom panel of Figure 7.

Sharpe Ratio We finally analyze the impact of changes in the Sharpe ratio η, which measures the market price of risk of the hedging asset. Panels a, e and f in

Figures 5-6 plot the investment threshold for various values of σ, γ, and ρ. These figures reveal that the impact of Sharpe ratio depends crucially on the sign of ρ. In particular, when $\rho > 0$, the investment threshold increases with the Sharpe ratio η, implying that using hedging assets with a high market price of risk delays investment. By contrast, when $\rho < 0$, an opposite result follows.

Again, the intuition behind the above result is related to the discussion following Proposition 3. When $\rho > 0$, hedging has a negative return effect. In particular, an increase in η results in losses of returns from the hedging asset and reduces wealth, thereby reducing the project value f and the option value to invest g. Simulation results show that f decreases more than g, and hence investment is delayed. By contrast, when $\rho < 0$, hedging has a positive return effect and hence leads to the opposite result.

3.4 Welfare Implications of Hedging

It is clear that with an additional hedging asset available for trade, the entrepreneur is always better off compared with the case where the risk-free asset is the only financial investment opportunity. Consequently, to examine the welfare implications of hedging, we assume that the entrepreneur always has the opportunity to invest in a risk-free asset and in a risky asset as well. A risky asset is characterized by its Sharpe ratio and the extent to which it is correlated with the cash flow risk. We ask the following question: What kind of risky asset should the entrepreneur choose to hedge against the cash flow risk? The common intuition is that the entrepreneur should invest in a risky asset which is highly correlated with the cash flow risk. We will show below that this intuition is not the whole story.

In order to address this issue, we compute the additional amount of wealth the entrepreneur can be gained when he invests in a risky asset correlated with the cash flow risk, compared with the case where he invests in a risky asset with the same

Sharpe ratio, but uncorrelated with the cash flow risk. Specifically, let $V(w, y; \rho)$ be the value function before investment for the model with hedging when the correlation coefficient is ρ. Then the welfare gain x is the solution to equation: $V(w, y; \rho) = V(w + x, y; 0)$. By Propositions 3, one can show that $x = g(y; \rho) - g(y; 0)$. We assume initially $y = 0$, which implies that the investment project has not been undertaken in our simulations.

The welfare gains from hedging after investment can be defined similarly. It follows from Proposition 3 that these welfare gains are given by

$$x = f(y; \rho) - f(y; 0) = \frac{\gamma\sigma^2\rho^2}{2r^2} - \frac{\eta\sigma\rho}{r^2}. \tag{39}$$

This expression illustrates explicitly the two effects of hedging discussed earlier. On the one hand, hedging reduces cash flow risk, thereby increasing wealth by $\gamma\sigma^2\rho^2/(2r^2)$. On the other hand, hedging has a return effect. If the hedging asset is negatively correlated with the cash flows ($\rho < 0$), then hedging increases wealth by $\eta\sigma|\rho|/r^2$. Thus, hedging is always welfare improving if $\rho < 0$. Moreover, the welfare gains increase with the cash flow risk, Sharpe ratio and degree of risk aversion.

By contrast, if the hedging asset is positively correlated with the cash flows ($\rho > 0$), then hedging results in losses of returns from the hedging asset, thereby reducing wealth by $\eta\sigma\rho/r^2$. The overall effect depends on parameter values. In particular, if and only if $\rho > \eta/(\gamma\sigma)$, the welfare gains increases with the extent of hedging ρ. Moreover, the gains are larger for more risk averse entrepreneurs, riskier cash flows, and lower Sharpe ratios of hedging assets. Surprisingly, hedging may incur welfare losses if $\rho > 0$. This happens for less risk averse entrepreneurs, safer cash flows, and higher Sharpe ratios of hedging assets.

Hedging has similar implications for the welfare before investment. This is confirmed in Figure 8, which plots the welfare gains before investment for various parameter values.

[Insert Figure 8 Here]

4 Conclusion

The standard real options approach to investment under uncertainty typically adopts one of the two assumptions: complete markets and risk neutrality. Motivated by many real-world problems such as entrepreneurial investment decisions, we relax these two assumptions and consider how market incompleteness and risk aversion affect a risk averse entrepreneur's real investment decision. We show that risk aversion delays investment. Furthermore, the impact of risk aversion is quite large if the cash flow risk is high or if the extent of hedging is small. We also show that the impact of the cash flow risk on investment is sensitive to the degree of risk aversion, Sharpe ratios of the hedging assets, and the extent of hedging. Finally, we show that investment timing and welfare may be not monotonic with the extent of hedging opportunity.

These results are in sharp contrast to the standard real options models under complete markets. They have a number of empirical implications.[16] For example, when conducting empirical analysis using cross sectional data, entrepreneurial risk aversion should be an important factor to consider. One of the most tested predictions of real options theory is the investment-uncertainty relationship. Our analysis suggests that entrepreneurial risk aversion, the extent of hedging, and Sharpe ratios of hedging assets are important factors influencing this relationship. Moreover, the characteristics of hedging assets such as Sharpe ratios and the correlation with the cash flow risk are important determinants of investment decisions. Finally, our analysis suggests that to maximize welfare, entrepreneurs should use a hedging strategy to long assets negatively correlated with the project cash flow risk. Shorting assets positively correlated with the project cash flow risk may lower welfare.

[16]See Quigg (1993), Berger et al. (1996), and Moel and Tufano (1998) for empirical testing of real options theory.

Appendices

A Proofs

Proof of Proposition 1: The value function after investment is defined as

$$J(w,y) = \max_{C \in \mathcal{C}} E\left[\int_t^\infty e^{-\beta t} U(C_t)\, dt \,\middle|\, (W_t, Y_t) = (w,y) \right] \tag{A.1}$$

subject to $dW_s = (rW_s - C_s + Y_s)\, ds,\ s \geq t$. We conjecture that J takes the form given in (18), where $f(y)$ is a function to be determined. To solve for this function, we use the first-order condition $U'(c) = J_w(w,y)$ to derive the optimal consumption rule given in (19). Substitute it back into the HJB equation (5) to derive the ODE

$$0 = (y - rf(y)) + \alpha f'(y) + \frac{\sigma^2}{2}\left[f''(y) - \gamma r f'(y)^2 \right]. \tag{A.2}$$

It can be verified that its solution is given by (15). Moreover, it is such that the value function satisfies the transversality condition.

We now consider the case before investment. By the principle of optimality, the value function $V(w,y)$ satisfies

$$V(w,y) = \max_{(\tau, C) \in \mathcal{T} \times \mathcal{C}} E\left[\int_0^\tau e^{-\beta t} U(C_t)\, dt + J(W_\tau - I, Y_\tau) \,\middle|\, (W_0, Y_0) = (w,y) \right] \tag{A.3}$$

subject to $dW_t = (rW_t - C_t)\, dt$, $t \geq 0$. We conjecture that V takes the form in (16), where $g(y)$ is a function to be determined. From the first-order condition $U'(c) = V_w(w,y)$, we can derive the consumption policy before investment given in (17). Substituting it into the HJB equation (6), we can show that $g(y)$ satisfies the ODE (11). Given the functional forms of the value functions, one can show that the no-bubble condition, the value matching and smooth pasting conditions become (12)-(14). Finally, by (18), (16) and assumption, the set

$$\left\{ (w,y) \in \mathbb{R}^2 : V(w,y) > J(w,y) \right\} = \left\{ (w,y) \in \mathbb{R}^2 : y < \bar{y} \right\}.$$

is the waiting region. Q.E.D.

Proof of Proposition 2: When $\sigma = 0$, the free-boundary problem can be easily solved. One can derive that the solution is

$$g(y) = \frac{\alpha}{r^2} e^{\frac{r}{\alpha}(y - \bar{y}_0)}, \text{ and } \bar{y}_0 = rI. \tag{A.4}$$

It is easy to show that the value maximizing investment threshold is also given by rI. Finally, we can equivalently rewrite the wealth dynamics (3)-(4) as

$$\int_0^\infty e^{-rt} C_t dt = W_0 + \int_\tau^\infty e^{-rt} Y_t dt - e^{-r\tau} I, \tag{A.5}$$

where we have imposed a no-Ponzi game assumption, $\lim_{T\to\infty} e^{-rT} W_T = 0$. Now it is clear that for any strictly increasing utility function U, the utility maximizing investment policy must maximize the net present value (22). Q.E.D.

Proof of Proposition 3: The value function after investment is defined as

$$J(w, y) = \max_{(C,\pi)\in\mathcal{C}\times\mathcal{A}} E\left[\int_t^\infty e^{-\beta t} U(C_t)\, dt \middle| (W_t, Y_t) = (w, y)\right] \tag{A.6}$$

subject to

$$dW_s = [rW_s + \pi_s(\mu_e - r) + Y_s - C_s]\, ds + \pi_s \sigma_e dB_s, \ s \geq t. \tag{A.7}$$

By a standard argument, $J(w, y)$ satisfies the HJB equation

$$\begin{aligned} \beta J(w, y) &= \max_{(c,\pi)\in\mathbb{R}^2} U(c) + [rw + \pi(\mu_e - r) + y - c]\, J_w(w, y) \\ &\quad + \alpha J_y(w, y) + \frac{\sigma^2}{2} J_{yy}(w, y) + \frac{(\pi\sigma_e)^2}{2} J_{ww}(w, y) + \pi\sigma_e\sigma\rho J_{wy}(w, y). \end{aligned} \tag{A.8}$$

The transversality condition $\lim_{T\to\infty} E\left[e^{-rT} J(W_T, Y_T)\right] = 0$ must also be satisfied. We conjecture that $J(w, y)$ takes the form

$$J(w, y) = -\frac{1}{\gamma r} \exp\left[-\gamma r\left(w + \frac{\beta - r + \eta^2/2}{\gamma r^2} + f(y)\right)\right], \tag{A.9}$$

where the function f is to be determined. By the first order conditions

$$U'(c) = J_w(w,y)\,,\ \pi = \frac{-J_w(w,y)}{J_{ww}(w,y)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-J_{wy}(w,y)}{J_{ww}(w,y)}\frac{\sigma_e\sigma}{\sigma_e^2}, \tag{A.10}$$

one can derive the optimal consumption and portfolio policies after investment given in (32)-(33). Substituting them back into the HJB equation (A.8), one can derive

$$0 = (y - rf(y)) + \alpha f'(y) + \frac{\sigma^2}{2}\left[f''(y) - \gamma r f'(y)^2\right] + \frac{1}{2\gamma r}\left(\eta - \sigma\gamma r\rho f'(y)\right)^2 \tag{A.11}$$

Solving yields (31). It can be verified that this solution satisfies the transversality condition.

We now turn to the case before investment. By the principle of optimality, the value function before investment $V(w,y)$ satisfies

$$V(w,y) = \max_{(\tau,C,\pi)\in\mathcal{T}\times\mathcal{C}\times\mathcal{A}} E\left[\int_0^\tau e^{-\beta t}U(C_t)\,dt + J(W_\tau - I, Y_\tau)\,\middle|\,(W_0,Y_0) = (w,y)\right] \tag{A.12}$$

subject to

$$dW_t = \left[rW_t + \pi_t(\mu_e - r) - C_t\right]dt + \pi_t\sigma_e dB_t,\ t \geq 0. \tag{A.13}$$

Then $V(w,y)$ satisfies the following HJB equation

$$\begin{aligned}\beta V(w,y) &= \max_{(c,\pi)\in\mathbb{R}^2} U(c) + \left[rw + \pi(\mu_e - r) - c\right]V_w(w,y) + \alpha V_y(w,y) \qquad \text{(A.14)}\\ &\quad + \frac{\sigma^2}{2}V_{yy}(w,y) + \frac{(\pi\sigma_e)^2}{2}V_{ww}(w,y) + \pi\sigma_e\sigma\rho V_{wy}(w,y)\,.\end{aligned}$$

We conjecture that the value function V takes the form

$$V(w,y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + \frac{\beta - r + \eta^2/2}{\gamma r^2} + g(y)\right)\right], \tag{A.15}$$

where $g(y)$ is a function to be determined. Using the first-order conditions,

$$U'(c) = V_w(w,y)\,,\ \pi = \frac{-V_w(w,y)}{V_{ww}(w,y)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-V_{wy}(w,y)}{V_{ww}(w,y)}\frac{\sigma_e\sigma}{\sigma_e^2}, \tag{A.16}$$

one can derive the optimal consumption and portfolio policies before investment given in (34)-(35). Plugging these expressions into the HJB equation gives a differential equation for $g(\cdot)$.

$$rg(y) = \alpha g'(y) + \frac{\sigma^2}{2}\left[g''(y) - \gamma r g'(y)^2\right] + \frac{1}{2\gamma r}\left(\eta - \sigma\gamma r\rho g'(y)\right)^2. \qquad \text{(A.17)}$$

Re-arranging and simplifying gives (11). As in Section 2, the boundary conditions are given by the no-bubble, value-matching, and smooth-pasting conditions similar to (7)-(10). Using these boundary conditions, one can derive (28)-(30). Finally, by (A.9), (A.15) and assumption, the set

$$\left\{(w,y) \in \mathbb{R}^2 : V(w,y) > J(w,y)\right\} = \left\{(w,y) \in \mathbb{R}^2 : y < \bar{y}\right\}.$$

is the waiting region. Q.E.D.

Proof of Proposition 4: When $\rho = 1$, the solution to ODE (11) is given by

$$g(y) = A_1 e^{\lambda y} + A_2 e^{\widetilde{\lambda} y}, \qquad \text{(A.18)}$$

where A_1 and A_2 are constants to be determined, λ is given in the proposition, and $\widetilde{\lambda} = -(\alpha - \eta\sigma)/\sigma^2 - \sqrt{(\alpha - \eta\sigma)^2 + 2\sigma^2 r}/\sigma^2 < 0$. Use the no-bubble condition (28) to deduce $A_2 = 0$. Use the value-matching and smooth-pasting conditions to solve for A_1 and y^*. Simple algebra delivers (37)-(38).

By (36), $F(y)$ satisfies ODE

$$rF(y) = (\alpha - \eta\sigma) F'(y) + \frac{1}{2}\sigma^2 F''(y). \qquad \text{(A.19)}$$

The general solution is given by $F(y) = Ae^{\lambda y}$, where A is a constant to be determined. Notice that we have used the no-bubble condition $\lim_{y\to-\infty} F(y) < \infty$ to rule out the exponential associated with the negative root. The constant A and the value-maximizing investment threshold y^{**} are determined by the value-matching and smooth-pasting conditions. To derive these conditions, observe that the market value of cash flows is given by

$$E\left[\int_0^\infty \xi_t Y_t dt \middle| Y_0 = y\right] = \frac{y}{r} + \frac{\alpha - \eta\sigma}{r^2}. \tag{A.20}$$

Thus, the value-matching and smooth-pasting conditions are

$$F\left(y^{**}\right) = \frac{y^{**}}{r} + \frac{\alpha - \eta\sigma}{r^2} - I, \text{ and } F'\left(y^{**}\right) = \frac{1}{r}. \tag{A.21}$$

Simple algebra implies that $y^{**} = y^*$ and $F(y) = g(y)$. Thus, under complete markets the utility-maximizing investment policy is the same as the value-maximizing policy.

To show that this result holds for any strictly increasing utility function U, it suffices to note that we can apply the martingale method to rewrite the wealth dynamics (25)-(26) as the static budget constraint

$$E\left[\int_0^\infty \xi_t c_t dt\right] \leq W_0 + E\left[\int_\tau^\infty \xi_t Y_t dt - \xi_\tau I\right], \tag{A.22}$$

where we have imposed the no Ponzi game assumption $\lim_{T\to\infty} E\left[\xi_T W_T\right] = 0$. Q.E.D.

B Computation Method

We describe the solution method to the free boundary problem described in proposition 1. The problem described in Proposition 3 can be solved similarly. We use the projection method implemented with collocation (Judd (1999)). We do not use the traditional shooting method or finite difference method because these methods are inefficient for our nonlinear problem and extensive simulations.

We first rewrite the second order ODE (11) as a system of first-order ODEs. Let $h(y) = g'(y)$. Then (11) can be rewritten as

$$\begin{aligned} g'(y) &= h(y), \\ h'(y) &= \frac{2}{\sigma^2}\left(rg(y) - \alpha h(y)\right) + \gamma r h(y)^2. \end{aligned} \tag{B.1}$$

The boundary conditions are

$$\lim_{y\to-\infty} g(y) = 0, \tag{B.2}$$

$$g(\overline{y}) = f(\overline{y}) - I, \tag{B.3}$$

$$h(\overline{y}) = 1/r. \tag{B.4}$$

where (B2) is derived by the fact that when $y \to -\infty$, the entrepreneur never undertakes the investment project and hence his subjective option value equals zero.

The idea of the algorithm is to first ignore the smooth-pasting condition (B4) and then solve for a two point boundary value problem with a guessed threshold value $\overline{y}_0$. The true value of the threshold is found by adjusting $\overline{y}_0$ so that the smooth pasting condition (B4) is satisfied. Since the boundary condition (B2) is open end, we pick a very small negative number $\underline{y}$ and rewrite it as $g\left(\underline{y}\right) = 0$. We then adjust $\underline{y}$ so that the solution is not sensitive to this value. The algorithm is outlined as follows.

Step 1. Start with a guess $\overline{y}_0$ and a preset order n.

Step 2. Use Chebyshev polynomial to approximate g and h :

$$g(y;a) = \sum_{i=0}^{n} a_i T_i(y), \; h(y;b) = \sum_{i=0}^{n} b_i T_i(y), \tag{B.5}$$

where $T_i(y)$ is the Chebyshev polynomial of order i, and $a = (a_0, a_1, ..., a_n)$ and $b = (b_0, b_1, ..., b_n)$ are $2n+2$ constants to be determined. Substitute the above expressions into the preceding system of ODEs and evaluate it at n roots of $T_n(y)$. Together with the two boundary conditions, we then have $2n+2$ equations for $2n+2$ unknowns $a = (a_0, a_1, ..., a_n)$ and $b = (b_0, b_1, ..., b_n)$. Let the solution be $\widehat{a}$ and $\widehat{b}$.

Step 3. Search for $\overline{y}_0$ such that the smooth-pasting condition, $h\left(\overline{y}_0; \widehat{b}\right) = 1/r$, is approximately satisfied.

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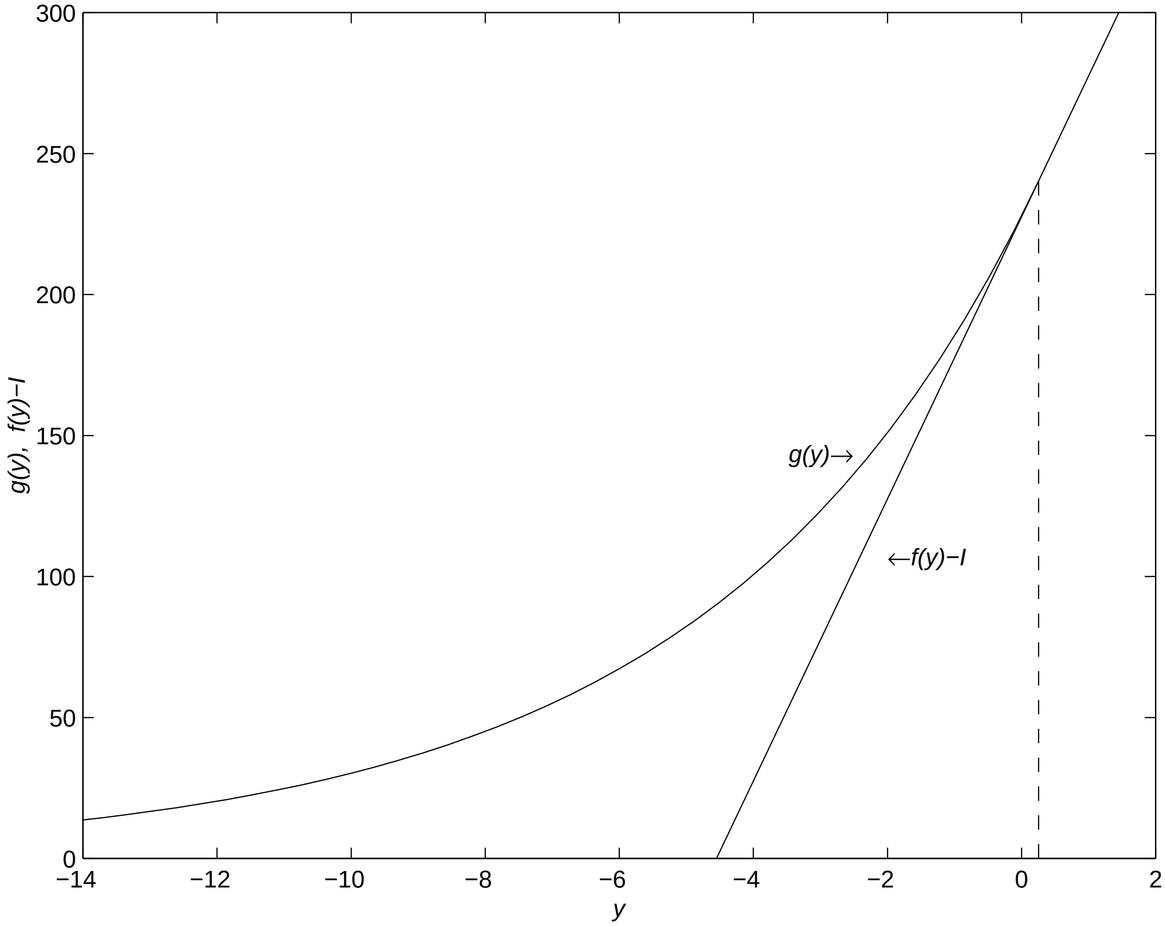


Figure 1: **The project value f and the option value g.** This figure plots the functions f and g for the model in section 2. The parameter values are set as follows: $\beta = r = 2\%$, $\alpha = 0.1$, $\sigma = 10\%$, $\gamma = 1$, and $I = 10$.

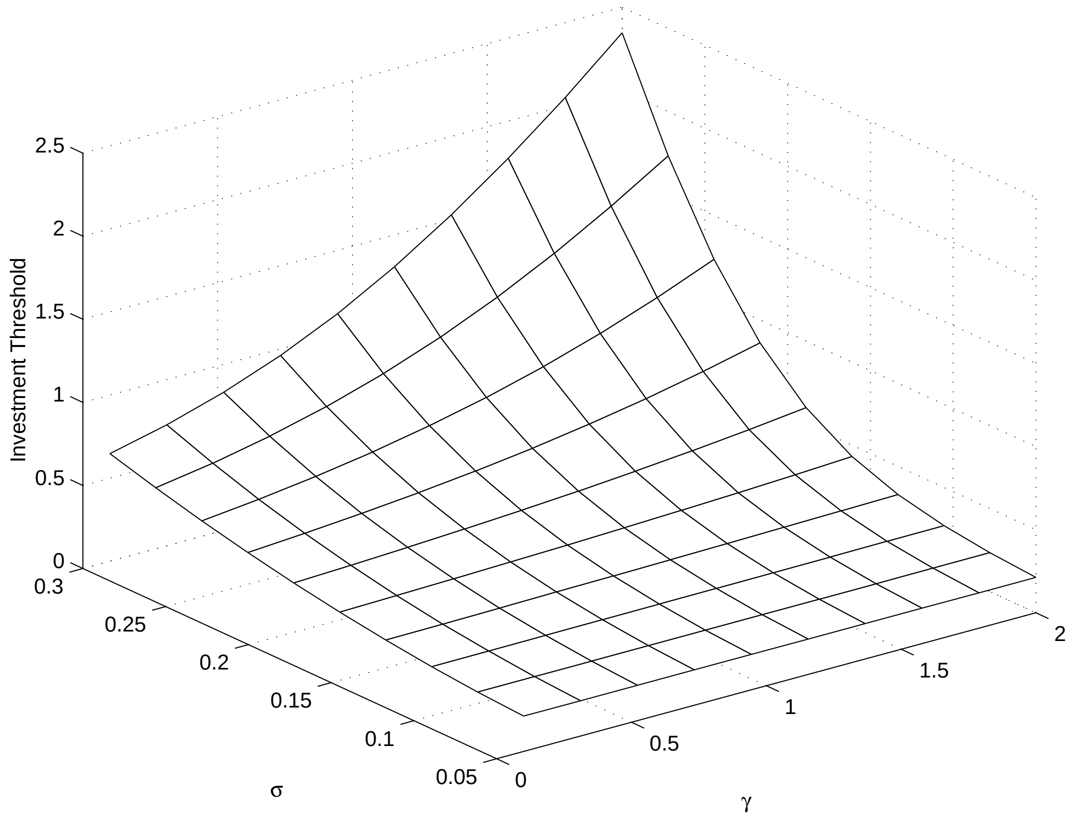


Figure 2: **Investment threshold, risk aversion, and project volatility.** This figure plots the investment threshold at varying levels of risk aversion and project volatility for the model in section 2. Other parameter values are set as $\beta = r = 2\%$, $\alpha = 0.1$, and $I = 10$

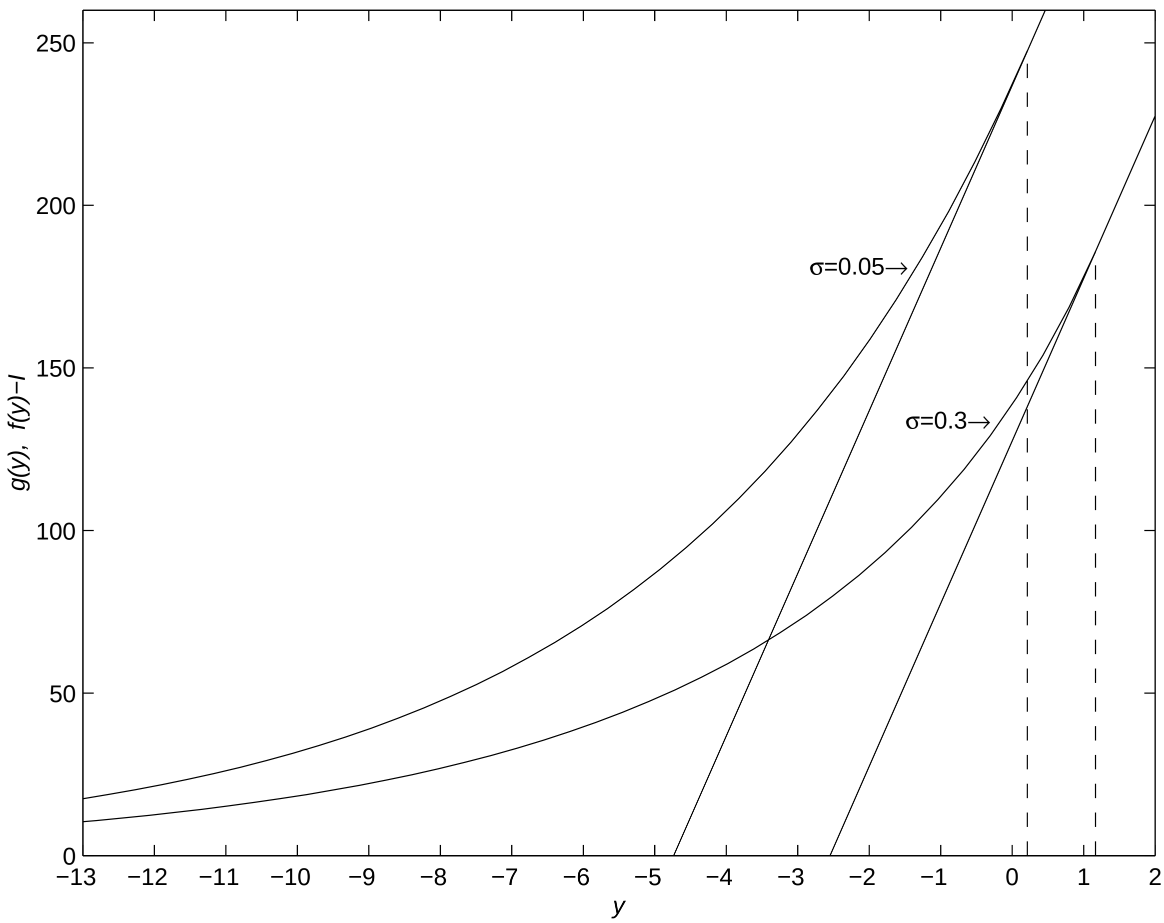


Figure 3: **Impact of changes in volatility.** This figure plots the impact on the investment threshold and the functions f and g for the model in section 2 when the risk aversion parameter σ is increased from 5% to 30%. Other parameter values are set as $\beta = r = 2\%$, $\gamma = 1$, $\alpha = 0.1$, and $I = 10$.

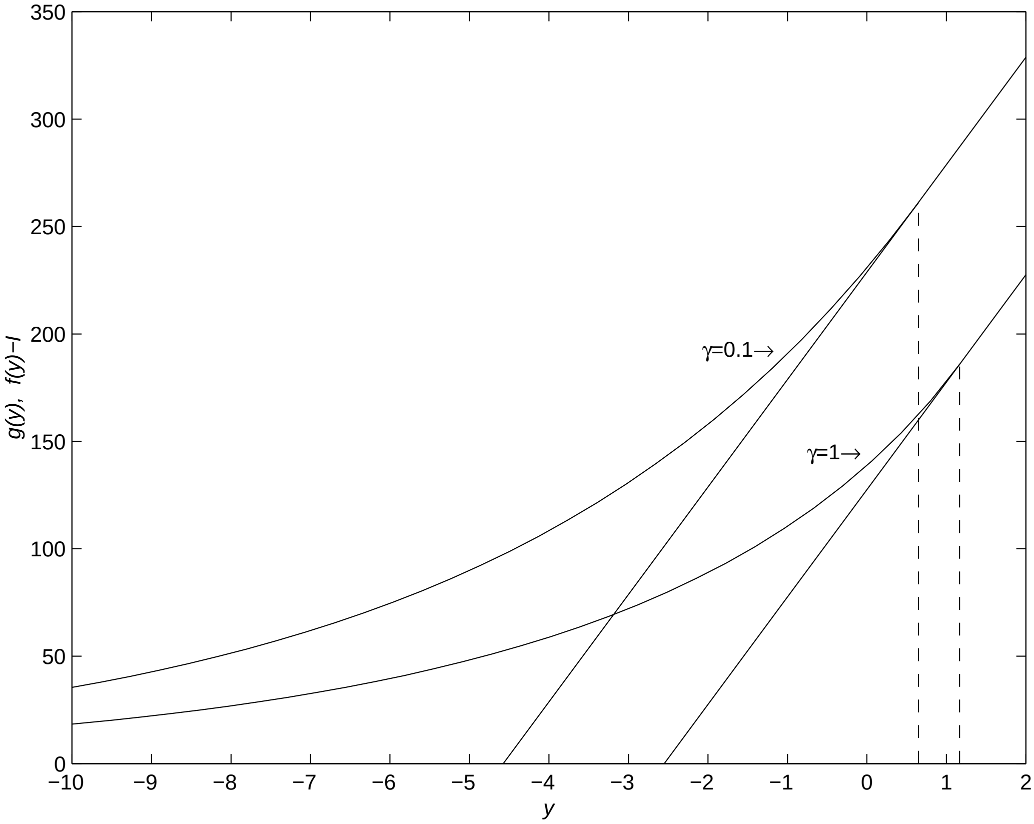


Figure 4: **Impact of changes in risk aversion.** This figure plots the impact on the value function and investment threshold for the model in section 2 when the risk aversion parameter γ increases from 1 to 2. Other parameter values are set as $\beta = r = 2\%$, $\alpha = 0.1$, $\sigma = 30\%$, and $I = 10$.

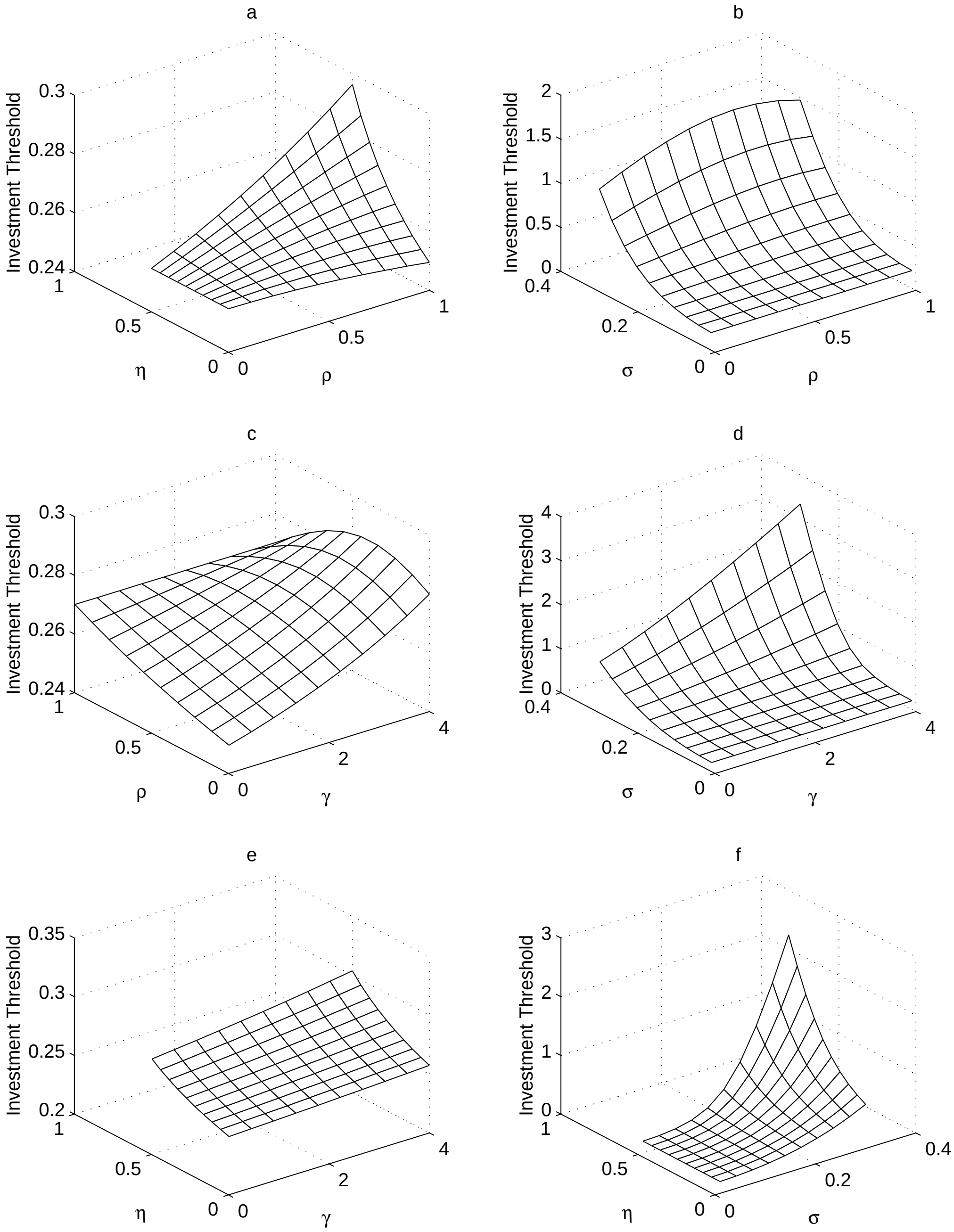


Figure 5: **Impact of parameters on investment threshold.** These figures plot the investment threshold as functions of various parameters for the model in section 3 when the correlation is positive. Baseline parameter values are set as $\beta = r = 2\%$, $\gamma = 1$, $\alpha = 0.1$, $\sigma = 10\%$, $\rho = 0.8$, $\eta = 0.3$, and $I = 10$.

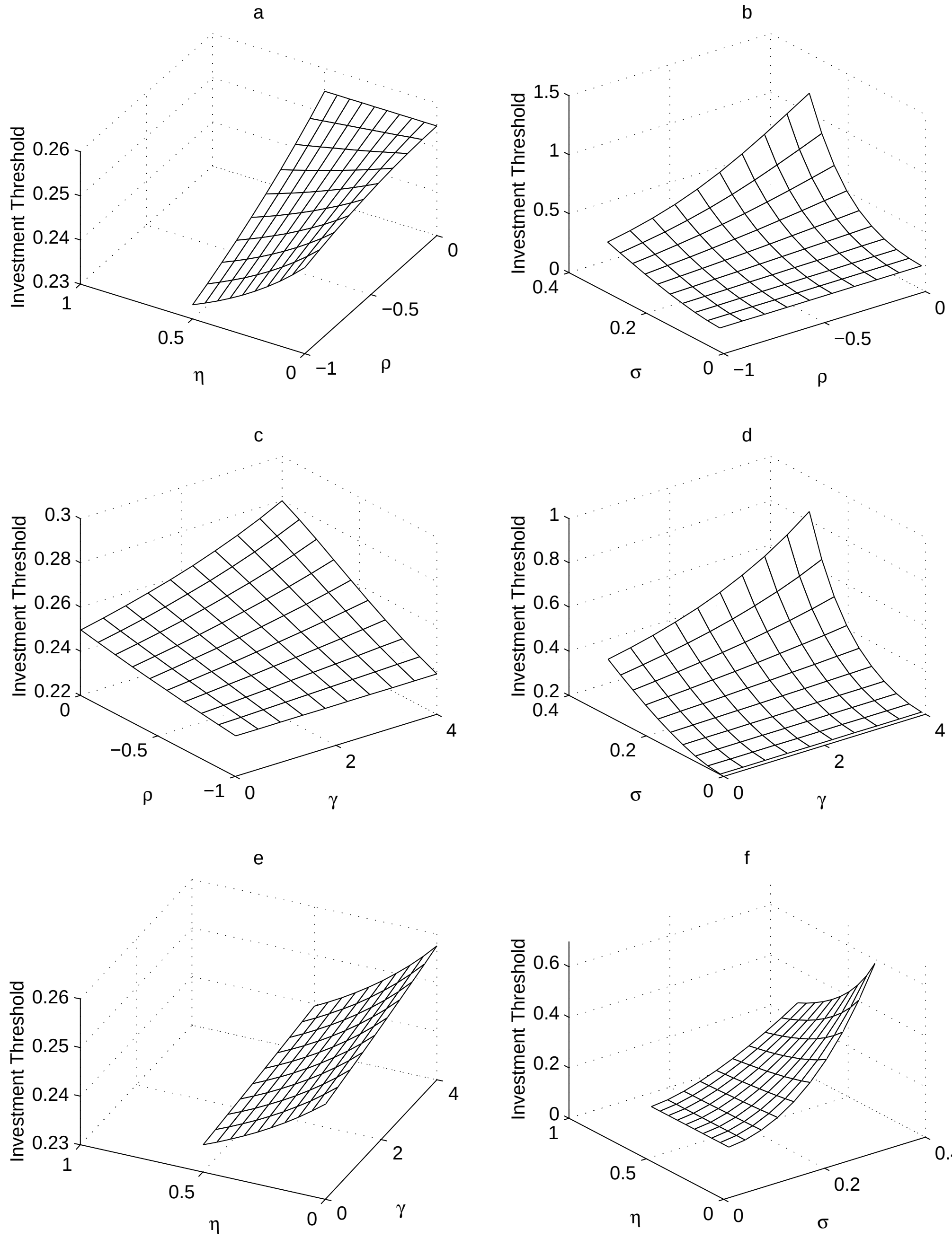
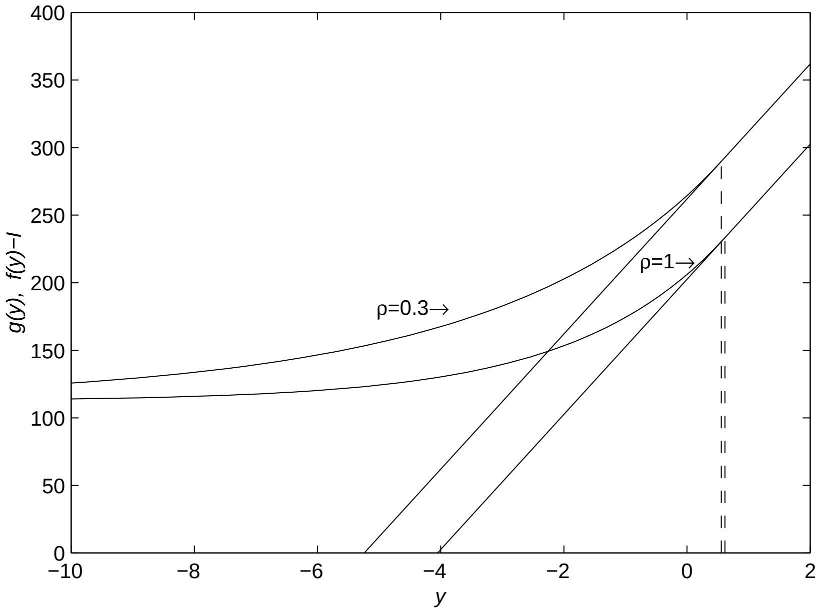


Figure 6: **Impact of parameters on investment threshold.** These figures plot the investment threshold as functions of various parameters for the model in section 3 when the correlation is negative. Baseline parameter values are set as $\beta = r = 2\%$, $\gamma = 1$, $\alpha = 0.1$, $\sigma = 10\%$, $\eta = 0.3$, $\rho = -0.8$, and $I = 10$.



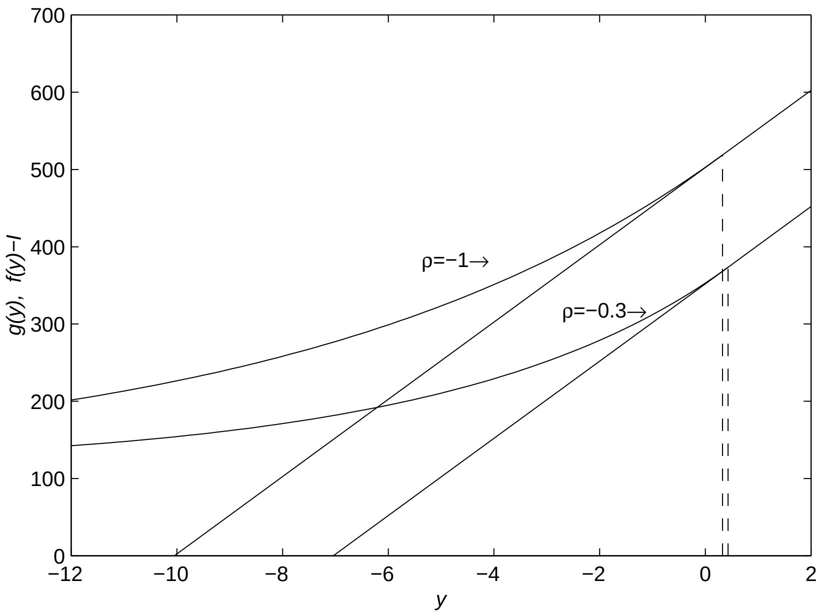


Figure 7: **Impact of changes in the correlation.** These figures plot the changes of functions $f(y) - I$ and $g(y)$ as ρ changes. Parameter values are set as $\beta = r = 2\%$, $\gamma = 1$, $\alpha = 0.1$, $\sigma = 20\%$, $\eta = 0.3$, and $I = 10$.

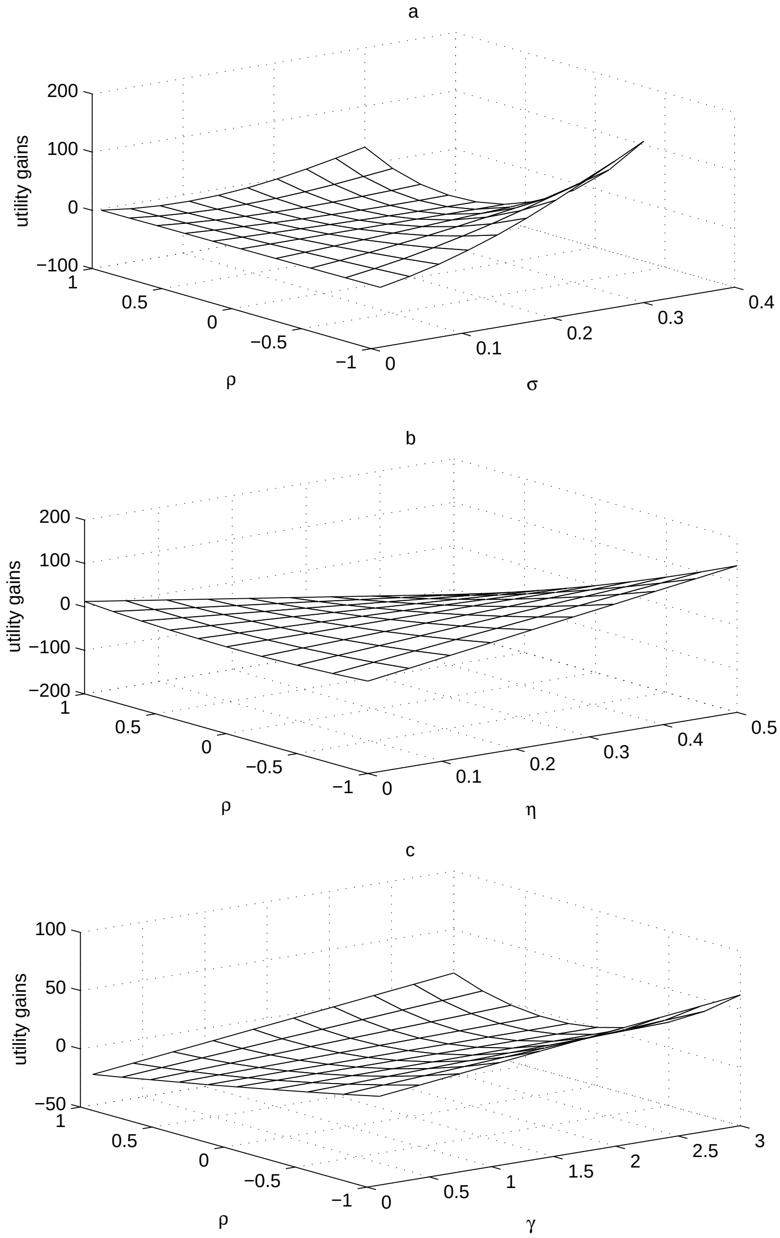


Figure 8: **Utility gains of hedging.** These figures plot the utility gains of hedging at varying levels of parameter values. The baseline parameter values are set as $\beta = r = 2\%$, $\gamma = 1$, $\alpha = 0.1$, $\sigma = 10\%$, $\eta = 0.1$, and $I = 10$.