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# STOCHASTIC DOMINANCE EFFICIENCY TESTS UNDER DIVERSIFICATION

Timo Kuosmanen
Department of Economics and Management Science
Helsinki School of Economics and Business Administration
P.O. Box 1210, FIN-00101 Helsinki, **FINLAND**
TEL. +358 9 4313 8537
FAX. +358 9 4313 8535
E-MAIL: kuosmane@hkkk.fi

**Abstract**
This paper focuses on *Stochastic Dominance* (SD) efficiency in a finite empirical panel data. We analytically characterize the sets of unsorted time series that dominate a given evaluated distribution by the First, Second, and Third order SD. Using these insights, we develop simple Linear Programming and 0-1 Mixed Integer Linear Programming tests of SD efficiency. The advantage to the earlier efficiency tests is that the proposed approach explicitly accounts for *diversification*. Allowing for diversification can both improve the power of the empirical SD tests, and enable SD based portfolio optimization. A simple numerical example illustrates the SD efficiency tests. Discussion on the application potential and the future research directions concludes.

**Key Words**
Stochastic Dominance, Portfolio Choice, Efficiency, Diversification, Mathematical Programming

**JEL Classification**: D81, G11, C61, C14

## 1. Introduction

*Stochastic Dominance* (SD) criteria for the choice under uncertainty date back at least to Quirk and Saposnik (1962), and Fishburn (1964).[1] The fully blown economic theory of SD is due to the work of Hadar and Russell (1969), Hanoch and Levy (1969), Rotschild and Stiglitz (1970), and Whitmore (1970). Today, the SD criteria have established as valuable analytical tools for studying theoretical probability distributions of random variables as well as empirical cumulative frequency distributions in virtually all areas of Economics. The SD approach can be viewed 'nonparametric' in the sense that the SD

[1] Some research in parallel with SD had been undertaken prior to 1960's. In a special case of equal mean distributions, Karamata (1932) proved a theorem akin to the second order stochastic dominance. Moreover, such approaches as *Lorenz dominance* in income and poverty studies, and *majorization theory* in Statistics have substantial similarity with SD.

criteria do not impose any explicit specification of the agent's utility function, or restrictions on the function form of the probability/frequency distribution. Rather, the SD approach resorts to some very general conditions of non-satiation and risk preferences, and takes into account the entire distribution. Consequently, the SD criteria are compatible with the traditional Von Neumann – Morgenstern Expected Utility theory, as well as a wide class of alternative non-expected utility theories. Furthermore, the existing empirical evidence from choice experiments seems to support generality of the SD criteria as rational choice rules (see e.g. Starmer, 2000, for a recent survey). In addition to serving as a convenient research instrument in both the theory and applications, the SD criteria have had a profound impact on the perception and the definition of "risk" in Economics.

In the empirical portfolio analysis, the two-moment *Mean-Variance* (MV) model[2] and its extensions have remained the dominating empirical research instrument, despite the theoretical superiority of the SD criteria which take into account the entire distribution. The appeal of the MV model lies to a substantial degree in its ability to test and build efficient *diversification* strategies (see e.g. the sharp note by Frankfurter and Phillips, 1975). By contrast, finding the SD efficient subset under diversification is highly complicated. Since Porter, Wart, and Ferguson (1973), simple enumerative algorithms for finding the SD efficient subset in a given finite set of empirical data have been developed, but the number of distributions becomes infinitely large when diversification is allowed. Although a few algorithms for some special cases exist (see e.g. Markowitz, 1977; Gavish, 1977, Ziemba, 1978), to the best of our knowledge, the long-awaited general computationally tractable algorithm for finding the efficient SD set under diversification has not been presented thus far. This computational difficulty with the SD approach under diversification seems the most natural explanation for the relatively marginal position of the SD approach in the financial and other such application areas where the diversification plays an important role.[3]

This paper proposes a novel solution to this well-recognized problem. Instead of developing a new algorithm for finding the SD efficient frontier, we analytically characterize the sets of unsorted time series vectors that dominate a given empirical distribution to be evaluated by the First, Second, and Third order SD. Interestingly, these *dominating sets* exhibit a relatively simple polyhedral structure. Utilizing these insights, we develop general SD efficiency tests, which compare the given distribution against an optimally diversified portfolio that can consist of multiple securities. The test statistics are formulated in terms of standard Linear Programming and 0-1 Mixed Integer Linear Programming, and can hence be computed using the standard, generally available algorithms. Accounting for the diversification can substantially enhance the power of the SD efficiency criteria as a screening device e.g. in ex post evaluation of mutual funds, pension funds, and other financial institutions. In addition, the tests provide information

[2] Popularized by Markowitz (1952, 1959)

[3] To quote Levy (1992, pp. 580): "It is well known that one of the disadvantages of SD analysis in comparison to the MV analysis is that in SD framework we do not have yet an algorithm to find the SD efficient *diversification* strategies."

of the efficient diversification strategies, which can improve the applicability of the SD approach to new research areas.

The plan for the rest of the paper is the following: Section 2 introduces the notation, defines the SD concepts, reviews the existing SD tests, and illustrates the difficulties involved with diversification. In Section 3 we analytically derive the dominating sets in the cases of the First, Second, and Third order SD. In Section 4 we take advantage of these results to derive SD efficiency tests that only involve solving a single LP or 0-1 MILP problem. To gain additional intuition, Section 5 presents a simple numerical illustration. We conclude by discussing the application potential and directions for future research in Section 6.

## 2. Preliminaries

In general, we can think of the SD concepts as properties of the probability distributions, putting aside the economic interpretation of the underlying random variable. Speaking of diversification, however, it contributes to our understanding to assume the rate of return of an investment portfolio as our random variable.[4] Throughout the text, the focus is on risky portfolios, i.e. there is some variation in the returns over time.

Consider two risky portfolios $j$ and $k$ with the return distributed according to the cumulative distribution functions (cdfs) $G_j$ and $G_k$, $G_j \neq G_k$, respectively.

**Definition 1:** Portfolio $j$ <u>dominates</u> portfolio $k$ by FSD, SSD, and TSD, denoted by $jD_1k$, $jD_2k$, and $jD_3k$ respectively, if and only if

FSD: $G_k(x) - G_j(x) \geq 0 \;\; \forall x \in \Re$, and $G_k(x) - G_j(x) > 0$ for some $x \in \Re$

SSD: $\int_{-\infty}^{x} [G_k(t) - G_j(t)]dt \geq 0 \;\; \forall x \in \Re$, and $\int_{-\infty}^{x} [G_k(t) - G_j(t)]dt > 0$ for some $x \in \Re$

TSD: $\int_{-\infty}^{x}\int_{-\infty}^{v} [G_k(t) - G_j(t)]dtdv \geq 0 \;\; \forall x \in \Re$, $\int_{-\infty}^{x}\int_{-\infty}^{v} [G_k(t) - G_j(t)]dtdv > 0$ for some $x \in \Re$.

The SD criteria have the following well-known economic interpretation in terms of the Expected Utility Theory: Consider a continuously differentiable Bernoullian utility function $U : \Re \to \Re$. If the investor is non-satiated, i.e. $U' \geq 0$, then $jD_1k$ implies the investor prefers portfolio $j$ over $k$. If the investor is risk-aversive in addition to non-satiation, i.e. $U'' \leq 0$, then $jD_2k$ implies preference of portfolio $j$ over $k$, and conversely. Furthermore, if the investor exhibits decreasing absolute risk-aversion (Pratt, 1964), i.e. $U''' \geq 0$, then $jD_3k$ implies portfolio $j$ is preferred over $k$. Converse relationships also hold: If the investor prefers portfolio $j$ over $k$ whenever $jD_1k$, then the investor is non-

[4] It should be stressed that this financial terminology is merely for illustrative purposes. For example, we could equally well phrase in terms of cost distributions of alternative production plants, or error distributions of alternative estimators.

satiated. If the investor prefers portfolio $j$ over $k$ whenever $jD_2k$, then the investor is risk aversive. Finally, if the investor prefers portfolio $j$ over $k$ whenever $jD_3k$, then the investor exhibits decreasing absolute risk aversion.

The SD concepts extend to the $n$th order SD in the analogous manner to TSD, except that we need $n$ – 1 definite integrals instead of two. Since the first three of the SD criteria are the most frequently applied ones, and they also have the most natural economic interpretation, as discussed above, in the following we will abstract from the higher order SD criteria. Extending the presented results to the higher order criteria is straightforward.

At least since Porter *et al.* (1973), there has been considerable interest in testing the SD efficiency conditions, analogous to the Mean-Variance efficiency á la Markowitz. The notion of *efficiency* requires some scarcity for the investment opportunities. Therefore, we assume there are a finite number of $n$ alternative securities indexed as $I \equiv \{1,2,...,n\}$ available for the investor, and we denote the set of alternative portfolios that can be composed of these securities by $J$. We present the following formal definition for the SD efficiency:

**Definition 2:** Portfolio $k \in J$ is FSD (SSD, TSD) efficient in $J$, if and only if, $jD_1k$ ($jD_2k$, $jD_3k$) $\Rightarrow j \notin J$ . Otherwise $k$ is FSD (SSD, TSD) inefficient.

It is standard knowledge that the SD efficiency criteria are transitive in the following sense: TSD efficiency implies SSD efficiency, which in turn implies FSD efficiency. Conversely stated, FSD inefficiency implies SSD inefficiency, which in turn implies TSD inefficiency. However, FSD, SSD, and TSD criteria are generally not equivalent. The FSD efficiency criterion is generally the weakest one in terms of discriminatory power, involving the largest efficient subset of $J$. Proceeding towards the higher order efficiency criteria can generally improve the discriminatory power of the SD test since the SD efficient subsets become smaller. However, the greater power of the efficiency tests should be balanced against the additional restrictions concerning the risk-preferences of the investor (discussed above).

In this paper, we focus on *empirical* tests of portfolios, and to this end it is natural to consider a finite (and therefore discrete) sample of return observations of the $n$ securities from $m$ time periods indexed as $T \equiv \{1,2,...,m\}$. This gives a panel data represented by the matrix $Y \equiv (y_1 ... y_n)$ with $y_j \equiv (y_{j1} ... y_{jm})^T$. We can sort each column vector $y_i$ in ascending order, and denote the resulting ranked return vector by $x_i$, i.e. $x_{i1} \le x_{i2} \le ... \le x_{im}$ for each $i \in I$. Let $X \equiv (x_1 ... x_n)$ with $x_j \equiv (x_{j1} ... x_{jm})^T$ denote the matrix of the sorted returns of all $n$ assets. (We will henceforth reserve $y, Y$ for the time series, and $x$, $X$ for the ranked data.) Based on $X$, we further construct the cumulative sum matrix $X' \equiv [x'_{it}]_{m \times n}$ where each element of the matrix is defined as $x'_{it} \equiv \sum_{l=1}^{t} X_{il}$ . In the similar fashion, we define $x''_{it} \equiv \sum_{l=1}^{t} x'_{il} = \sum_{l=1}^{t} \sum_{v=1}^{l} x_{iv}$ and $X'' \equiv [x''_{it}]_{m \times n}$.

Given a sufficiently large sample, the observed empirical return distribution can be viewed as an approximation of the true underlying probability distribution. For simplicity, however, in the following we restrict to SD sample statistics, and abstract away from statistical inference regarding the true underlying probability distributions.[5] After all, we need to walk before we can run. Testing SD efficiency in a sample should be interesting as such, e.g. for ex post performance evaluation of mutual funds, pension funds, and suchlike financial institutions. In addition, the proposed tests may turn out a valuable starting point for statistical tests that account for the sampling and/or data errors.

We record the basic SD tests in the following theorem, which forms an ample starting point for the generalizations in the subsequent sections:[6]

**Theorem 1**: The following equivalence results hold for empirical distributions of all portfolios *j* and *k*:

FSD: $j\mathrm{D}_1 k \Leftrightarrow x_{jt} \geq x_{kt} \quad \forall t \in T$, and $x_{jt} > x_{kt}$ for some $t \in T$.

SSD: $j\mathrm{D}_2 k \Leftrightarrow x'_{jt} \geq x'_{kt} \quad \forall t \in T$, and $x'_{jt} > x'_{kt}$ for some $t \in T$.

TSD: $j\mathrm{D}_3 k \Leftrightarrow x''_{jt} \geq x''_{jt} \quad \forall t \in T$, and $x''_{jt} > x''_{jt}$ for some $t \in T$.

**Proof**: FSD: On the basis of the vector $x_i$, we can construct the empirical cdf as $G_i(x) = \underset{t \in T}{Max}\{t/m \mid x \geq x_{it}\}$. Hence, for portfolios *j* and *k*, the FSD condition $G_k(x) - G_j(x) \geq 0 \ \forall x \in \Re$ can be written as $\underset{t \in T}{Max}\{t/m \mid x \geq x_{jt}\} \geq \underset{t \in T}{Max}\{t/m \mid x \geq x_{kt}\}$ $\forall x \in \Re$, which is equivalent to $x_{jt} \geq x_{kt} \ \forall t \in T$. The strict inequality is obvious from the definition of SD.

SSD and TSD: Using $x'_i$ and $x''_i$ we obtain $\int_{-\infty}^{x} G_i(t)dt = \underset{t \in I}{Max}\{t/m \mid x' \geq x'_{it}\}$ and $\int_{-\infty}^{x}\int_{-\infty}^{v} G_i(t)dtdv = \underset{t \in I}{Max}\{t/m \mid x'' \geq x''_{it}\}$ respectively. The equivalence result follows directly analogous to the FSD case. *Q.E.D.*

The right-hand side inequalities of Theorem 1 can be easily checked by enumeration. Hence, Theorem 1 outlines a simple but effective algorithm for testing SD efficiency of a finite number of portfolios by a pair-wise comparison against each other. Today, any basic spreadsheet software run on a desktop PC can handle the computation, even in relatively large data sets.

[5] Statistical inference is possible e.g. along the lines of Porter and Pfeffenberger (1975) and McFadden (1989), provided that the underlying probability distributions are relatively stationary over time, or the nonstationarities can be adjusted in some satisfactory way.

[6] We here summarize and slightly extend the results of Levy (1992; Appendix A). Since this theorem forms the key to the extensions of the subsequent sections, we chose to include a compact proof.

However, if diversification is possible, then there exist an infinite number of alternative portfolios that can be composed of the given securities. Consequently, any finite number of pair-wise comparisons does not suffice to confirm SD efficiency, because an infinite number of possible comparisons remain to be checked. The finite pair-wise comparison algorithm can already reject SD efficiency hypotheses, i.e. a SD inefficient security cannot become efficient if more alternative portfolios are introduced. Therefore, the pair-wise tests give necessary but not sufficient conditions. The necessary conditions can already be useful for portfolio *screening*. By accounting for diversification we also obtain the sufficient condition, which can both improve the power of the SD criteria as screening devises, and guide us in portfolio *building*.

It seems not exaggerated to claim that our inability to deal with diversification is the single most serious barrier in the empirical application of the SD criteria, given that the appeal of the widely applied two-moment MV approach essentially lies in its capability to build efficient portfolios and spread the portfolio risk by diversification. Indeed, the SD analysis of Hadar and Russel (1971) already aptly showed how a risk-aversive investor typically favors diversified (rather than pooled) portfolios. Consequently, the SD criteria have attracted substantial application in such areas where diversification is either impossible or unimportant: the prime examples include comparing income distributions in poverty studies and crop yield distributions in agricultural economics. The search for a general algorithm for identifying the SD efficient securities under diversification attracted a lot of research three decades ago (see e.g. the survey by Levy, 1992), but the results turned out quite modest, pertaining to some special cases (see e.g. Markowitz, 1977; Gavish, 1977, Ziemba, 1978). As far as we can see, the current rapid development of both the hardware and the algorithms has not changed the matters.

The computational complexity associated with the diversified portfolios essentially arises from the fact that the SD tests build on the "ranked data" (i.e. matrix *X*), see Theorem 1. However, the rankings of the returns of the 'benchmark portfolio' (i.e. the one we compare the evaluated portfolio to) depend on the weights of securities in a highly complex fashion. We can easily model a diversified 'benchmark portfolio' as $Yl,\ l \in \Lambda \equiv \left\{ l \in \Re^n_{(+)} \middle| \vec{1} l = 1 \right\}$,[7] obtained as the convex combination of the given set of $n$ securities. However, the convex combinations need not preserve the original ranking of the observations. Therefore, resorting to the conventional approach (i.e. Theorem 1) would necessitate a procedure for sorting the data of the benchmark portfolio in the ascending order, which is the ultimate complication. Alternatively stated, we cannot use $Xl$ directly (unless the observed security returns are perfectly correlated), since the time series structure is the key to spreading the risk by diversification.

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[7] We use $\vec{1} \equiv (1 \cdots 1)$ for a unit row vector with dimensions conforming with the rules of matrix algebra, and $\Lambda$ for the set of $n$x1 weighting vectors $l$ that sum up to unity. The domain of the weighting vectors may optionally consist of the entire $n$ dimensional Euclidean space $\Re^n$ if shortselling is allowed, or its positive orthant $\Re^n_+$ if the shortselling is prohibited. This is expressed by the symbol (+).

For illustration, consider the simplest thinkable case of 2 securities and 3 time periods, with return data given on Table 1. The rank correlation of the returns of the two securities equals –1, since the returns of portfolio A ranked in the ascending fashion follow the chronological order [1.; 2.; 3.], while returns of portfolio B follow the order [3.; 2.; 1.]. Since both A and B are the limiting special cases of the portfolios that can be composed of these securities, these are two examples of period orderings that can be obtained by alternative values of weighting vector $l = (l_1, l_2)$, e.g. $l = (1,0)$ and $l = (0,1)$. However, alternative orderings are obtained at different values of $l$. Consider e.g. a diversified portfolio with the equal (.50-.50) weights for both securities. The ranking of the portfolio returns is [1.; 3.; 2.], which differs from those of both A and B. (Note how diversification reduces the portfolio risk due to the negative correlation between the returns of A and B!) For another .35-.65 portfolio, the ordering is still different [2.; 3.; 1.]. This simple example illustrates how large numbers of alternative orderings may become possible by diversification already in very small data sets.

**Table 1: Example of the complex rankings under diversification**

| Period | Security A return | rank | Security B return | rank | 50-50 Portfolio return | rank | 35-65 Portfolio return | rank |
|---|---|---|---|---|---|---|---|---|
| 1 | **1** | 1. | **7** | 3. | **4** | 1. | **4.9** | 2. |
| 2 | **5** | 2. | **5** | 2. | **5** | 3. | **5** | 3. |
| 3 | **8** | 3. | **1** | 1. | **4.5** | 2. | **3.45** | 1. |

Of course, from the purely mathematical point of view the problem is trivial. Since the number of alternative rankings is finite, one could simply adopt to the brute force strategy, and test SD efficiency separately for all possible return rankings. Specifically, suppose we evaluate SD efficiency of the evaluated portfolio $y_0$, and denote an arbitrary permutation of the return vector by $\tilde{y}_0$. In theory, we could solve the FSD test problem

$$\text{(1)} \qquad \underset{l \in \Lambda}{Max} \left\{ \vec{1}\left(Yl - \tilde{y}_0\right) \middle| Yl - \tilde{y}_0 \geq 0 \right\}$$

for all possible permutations $\tilde{y}_0$. If there exists such a permutation for which the optimal solution is strictly positive, then the inequality $Yl - \tilde{y}_0 \geq 0$ must hold by a strict inequality for some dimensions, and by Theorem 1 the portfolio 0 must be FSD inefficient. Otherwise, the portfolio 0 is FSD inefficient. Moreover, the test extends to the higher order SD criteria in a straightforward fashion.

The only problem of this approach is that the total number of possible orderings is generally as high as *m*!. This becomes an astronomically large number in almost any non-trivial application (e.g. $10! \cong 3.63 \cdot 10^6$; $100! \cong 9.33 \cdot 10^{157}$). Solving the test problem (1) is simple when the ranking (i.e. permutation $\tilde{y}_0$) is fixed, and can be handled by standard Linear Programming (LP) algorithms. Still, solving millions of LP problems (or a single LP problem with millions of variables) is not operationally tractable even with the modern computation technology. Of course, if short-selling is not allowed, and if the asset returns are highly positively correlated, then one might be able to eliminate a large number of permutations, and hence preserve this strategy tractable. Nevertheless, in those situations we do not have that much to gain relative to the existing simple pair-wise tests. These observations motivate us to develop practically tractable tests of SD efficiency, which do account for all possible diversified portfolios, but do not depend on the ranking of the data.

## 3. SD Dominating Sets

Since ranking the the portfolio returns seems to present insurmountable complexity under diversification, our strategy is to attack the problem in terms of the unsorted time series vectors. This route, which apparently has not been pursued before, will prove a pivotal insight for building the SD efficiency tests. In this section, we analytically characterize the sets of such hypothetical (time series) return vectors $y \equiv (y_{.1} ... y_{.m})^T$ (with the corresponding ranked vector $x \equiv (x_{.1} ... x_{.m})^T$) which dominate a given portfolio $y_0 \equiv (y_{01} ... y_{0m})^T$ by FSD, SSD, and TSD, respectively.

### *3.1 FSD*

Define first the *permutation matrix* $P \equiv \lfloor P_{ij} \rfloor_{mxm}, P_{ij} \in \{0,1\}, \ \vec{1}P = \left(P\vec{1}^T\right)^T = \vec{1}$, which allows us to sort the elements of a return vector in any arbitrary order, i.e. the set of all permutations of vector $y_0$ is expressed as $\{y_0 P\}$. Note that permuting the elements of vector $y_0$ does not influence the cdf. That is, all permutations in $\{y_0 P\}$ have identical cdfs. We are now equipped to present our first dominating set.

**Definition 3**: The set $\Delta_1(0) \equiv \{y \in \Re^m | y \geq y_0 P\} \backslash \{y_0 P\}$ is called the *FSD dominating set* of the evaluated portfolio 0.

The following theorem relates this set intimately to the FSD condition:

**Theorem 2**: $yD_1 y_0 \Leftrightarrow y \in \Delta_1(0)$

**Proof**: Consider the set $\Delta_1(0)$: If $y \geq y_0 P$ for some permutation matrix *P*, we can safely sort the elements of both *y* and $y_0$ in the ascending order, i.e. use vectors *x* and $x_0$ instead, and clearly $x \geq x_0$. The strict inequality must hold in some dimension, otherwise $y \in \{y_0 P\}$. The equivalence follows directly from Theorem 1. *Q.E.D.*

**Corollary**: The necessary and sufficient condition of FSD efficiency of portfolio 0 is that $\{Yl \,|\, l \in \Lambda\} \cap \Delta_1(0) = \varnothing$, i.e. the FSD dominating set does not contain any feasible portfolio.

Conveniently, the set $\Delta_1(0)$ is closed, monotonous, and symmetric with respect to the diagonal of $\Re^m$ (see e.g. Figure 1 below). Unfortunately, the set is non-convex for all risky portfolios (i.e. $y_{0i} \neq y_{0j}$ for some $i, j \in T$), which is a source of some computational inconvenience, as discussed in more detail in the subsequent sections. In any event, this result forms a good starting point for extensions towards the higher order SD criteria.

*3.2 SSD.*

To proceed towards the SSD criterion, it is illustrative to consider what happens if we relax the binary integer constraint from the permutation matrix $P$, and hence take a liberty to form convex combinations of the elements of vector $y_0$.[8] For sake of transparency, denote the "relaxed permutation matrix" by $W \equiv \lfloor W_{ij} \rfloor_{mxm} : 0 \leq W_{ij} \leq 1,\ \vec{1}W = \left(W\vec{1}^T\right)^T = \vec{1}$. It is worth to observe that multiplying the null-portfolio from the right by matrix $W$, i.e. $y_0W$, results as an option that is generally "less risky" than the original portfolio $y_0$. For example, in the limiting case of $W_{ij} = 1/m\ \forall i,j$ we obtain a risk-free return, which is equal to the mean return of the original portfolio. Clearly, such risk-free option dominates any risky $y_k$ by SSD. Below we take this line of reasoning a bit further, but first we present our next dominating set:

**Definition 4**: The set $\Delta_2(0) \equiv \{y \in \Re^m | y \geq y_0W\} \setminus \{y_0P\}$ is called the *SSD dominating set* of the evaluated portfolio 0.

Analogous to the FSD case, we can relate the set $\Delta_2(0)$ to the SSD condition by the following theorem:

**Theorem 3**: $yD_2y_0 \Leftrightarrow y \in \Delta_2(0)$

To gain additional insight, we formulate the proof in the constructive fashion.

**Proof:** Observe first that the condition $yD_2y_0$ can be equivalently expressed in terms of the ranked vectors $x$ and $x_0$ by the following $m$ inequalities:

$$x_{.1} \geq x_{01}; x_{.1} + x_{.2} \geq x_{01} + x_{02};...; \sum_{i=1}^{m} x_{.i} \geq \sum_{i=1}^{m} x_{0i}$$

Hence, the set of vectors in $\Re^m$ which either dominate $y_0$ by SSD or have the identical cdf can be written as

$$D_2(0) \equiv \left\{ y \in \Re^m \middle| x_{.1} \geq x_{01}; x_{.1} + x_{.2} \geq x_{01} + x_{02};...;\sum_{i=1}^{m} x_{.i} \geq \sum_{i=1}^{m} x_{0i} \right\}.$$

[8] We here wish to acknowledge the influence from Koopmans and Beckmann (1957), who similarly relaxed the integer constraint from the permutation matrix to give a seemingly complicated combinatorial optimization problem (the linear *assignment problem*) an equivalent Linear Programming formulation.

This set can be equivalently written as the intercept of $m$ halfspaces corresponding to the previous inequalities as:

$$D_2(0)=\{y|x_{.1}\geq x_{01}\}\cap\{y|x_{.1}+x_{.2}\geq x_{01}+x_{02}\}\cap\ldots\cap\left\{y\left|\sum_{i=1}^{m}x_{.i}\geq\sum_{i=1}^{m}x_{0i}\right.\right\},$$

where each halfspace is bounded by a linear hyperplane

$$x_{.1}=x_{01};x_{.1}+x_{.2}=x_{01}+x_{02};\ldots;\ \sum_{i=1}^{m}x_{.i}=\sum_{i=1}^{m}x_{0i}\ .$$

(Note: the sums on the right-hand side take constant values.) Each of these hyperplanes of the ranked vector space ($x$) can be equivalently expressed in terms of the time series vectors ($y$) by the following hyperplane segments

$$y_{.j}=x_{01}\text{ if }y_{.j}=x_{.1};$$
$$y_{.j}+y_{.k}=x_{01}+x_{02}\text{ if }\left(y_{.j}=x_{.1},y_{.k}=x_{.2}\right)\text{or}\left(y_{.k}=x_{.1},y_{.j}=x_{.2}\right);$$
$$\vdots$$
$$\sum_{i=1}^{m}y_{.i}=\sum_{i=1}^{m}x_{0i}$$

Since the dimensionality of $y$ is $m$, the first equality gives $m$ hyperplane segments (corresponding to the alternative rankings), the second one $m(m-1)/2$ segments, and so forth. Thus, the total number of hyperplane segments becomes $p\equiv\sum_{i=1}^{m}\binom{m}{i}$. The linear structure implies the set $D_2(0)$ is a convex monotone polyhedron supported by these $p$ hyperplane segments. A vector $y\in D_2(0)$ is an *extreme point* of the polyhedron $D_2(0)$ if it satisfies at least $m$ of the above $p$ equalities. Obviously, $y_0$ is one of the extreme points, and so are all permutations of $y_0$. It is straightforward to verify that no other extreme points exist, and thus $\{y_0P\}$ is the set of the extreme points of the polyhedron $D_2(0)$. Index the $m!$ different permutation matrices as $P_1$, $P_2$, …, $P_{m!}$. Using duality theory, we can write $D_2(0)$ equivalently as the convex monotone hull of its extreme points, i.e.

$$D_2(0)=\left\{y\in\Re^m\left|y\geq y_0P_1w_1+y_0P_2w_2+\ldots+y_0P_{m!}w_{m!};\sum_{i=1}^{m!}w_i=1;w_i\geq 0,i=1,\ldots,m!\right.\right\}$$
$$=\left\{y\in\Re^m\left|y\geq y_0\left[\sum_{i=1}^{m!}P_iw_i\right];\sum_{i=1}^{m!}w_i=1;\ w_i\geq 0,i=1,\ldots,m!\right.\right\}.$$

Finally, observe that the weighted sum of permutation matrices $\sum_{i=1}^{m!}P_iw_i$ is a matrix which can be equivalently (and more simply) expressed by the $mxm$ weight matrix $W$: $0\leq W_{ij}\leq 1$, $\vec{1}W=\left(W\vec{1}^T\right)^T=\vec{1}$, i.e. $D_2(0)=\left\{y\in\Re^m\left|y\geq y_0W\right.\right\}$, and hence $D_2(0)\backslash\{y_0P\}=\Delta_2(0)$. *Q.E.D.*

**Corollary 1**: If $y_0$ and $y_0W:0\leq W_{ij}\leq 1,\ \vec{1}W=\left(W\vec{1}^T\right)^T=\vec{1}$ have non-identical cdfs, then $y_0W$ dominates $y_0$ by SSD. The convex combination $y_0W$ can be viewed as a mean preserving "anti-spread" of the null-portfolio, i.e. $y_0W$ and $y_0$ have the equal mean return, but the variance of the former one can be smaller.

**Corollary 2**: The necessary and sufficient SSD efficiency condition for portfolio 0 is that $\{Yl \mid l \in \Lambda\} \cap \Delta_2(0) = \varnothing$, i.e. the SSD dominating set does not contain any feasible portfolio.

Like the FSD dominating set, the set $\Delta_2(0)$ is closed, monotonous, and symmetric with respect to the diagonal of $\Re^m$. In addition, it is a convex set, which is very convenient from the operational point of view, as we will see in the subsequent section.

*3.3 TSD*

To further generalize towards the higher order SD criterion, we first introduce the following auxiliary vectors

(2) $$z^{kl} \equiv \left(z_1^{kl} \cdots z_m^{kl}\right); k < l; k, l \in T,$$

where

(3) $$z_i^{kl} \equiv \begin{cases} x_{0i}, & i = 1,\dots,k-1 \\ r_{kl}, & i = k,\dots,l \\ \mathrm{r}_{kl}, & i = l+1 \\ x_{0i}, & i = l+2,\dots,m \end{cases};$$

(4) $$r_{kl} \equiv \frac{\sum_{i=k}^{l}(l-i+1)x_{0i}}{\sum_{i=k}^{l}(l-i+1)};$$

and

(5) $$\mathrm{r}_{kl} \equiv \frac{\sum_{i=1}^{l+1}(l-i+2)x_{0i}}{\sum_{i=1}^{k-1}(l-i+2)x_{0i} + (k-l)r_{kl}}.$$

These auxiliary vectors have identical elements to $x_0$, i.e. the ranked return vector of the null portfolio, except for the elements from $k$ through $l+1$. For the elements from $k$ to $l$ these vectors contain a constant (risk-free) value of $r_{kl}$, and the value of element $k+1$ is $\mathrm{r}_{kl}$. Note that in the special case of $k = 1$, $l = m$ we have

(6) $$z^{1m} = \left(r_{1m} \cdots r_{1m}\right),\ r_{1m} = \frac{\sum_{i=1}^{m}(m-i+1)x_{0i}}{\sum_{i=1}^{m}(m-i+1)},$$

which turns out the smallest risk-free return that dominates the portfolio 0 by TSD.

In addition, analogous to matrix $W$, define the following additional $mxm$ weight matrices $W^0, W^{ij}, i < j; i, j \in T$ where $W^0_{kl}, W^{ij}_{kl} \in [0,1] \forall k, l \in T$; $\vec{1}\left(W^0 + \sum_{j=i}^{m}\sum_{i=1}^{m} W^{ij}\right) = \left(\left(W_0 + \sum_{j=i}^{m}\sum_{i=1}^{m} W^{ij}\right)\vec{1}^T\right)^T = \vec{1}$.

**Definition 5**: The set $\Delta_3(0) \equiv \left\{ y \in \Re^m \middle| y \geq y_0 W^0 + \sum_{j=i}^{m}\sum_{i=1}^{m} z^{ij} W^{ij} \right\} \backslash \{y_0 P\}$, is called the *TSD dominating set* of the evaluated portfolio 0.

**Theorem 4**: $yD_3 y_0 \Leftrightarrow y \in \Delta_3(0)$

**Proof:** Directly analogous to the SSD case:
The set of vectors $y \in \Re^m : yD_3 y_0$ or $x = x_0$, can be written as

$$D_3(0) \equiv \left\{ y \middle| x_{.1} \geq x_{01}; 2x_{.1} + x_{.2} \geq 2x_{01} + x_{02}; ...; \sum_{i=1}^{m}(m-i+1)x_{.i} \geq \sum_{i=1}^{m}(m-i+1)x_{0i} \right\},$$

which can be equivalently written as the convex monotone polyhedron supported by the hyperplane segments:

$$y_{.j} = x_{01} \text{ if } y_{.j} = x_{.1};$$
$$2y_{.j} + y_{.k} = 2x_{01} + x_{02} \text{ if } y_{.j} = x_{.1}, y_{.k} = x_{.2};$$
$$3y_{.j} + 2y_{.k} + y_{.l} = 3x_{01} + 2x_{02} + x_{01}, \text{ if } y_{.j} = x_{.1}, y_{.k} = x_{.2}, y_{.l} = x_{.3}$$
$$\vdots$$
$$\sum_{i=1}^{m}(m-i+1)y_{.i} = \sum_{i=1}^{m}(m-i+1)x_{0i} \text{ if } y_{.1} \leq y_{.2} \leq \ldots \leq y_{.m}, \text{ etc.}$$

Note that the rankings play a more essential role than in the SSD case: The first equality gives $m$ hyperplane segments, the second one $m(m$-1) segments, the third one $m(m$-1)($m$-2) and so forth, so in the last case we have as many as $m!$ equalities. The total number of hyperplanes becomes as huge as $q \equiv \sum_{i=1}^{m} \frac{m!}{(i-1)!}$. Nonetheless, similar to the SSD case, the linear hyperplane structure implies the set $D_3(0)$ is the convex monotone polyhedron supported by these $q$ hyperplane segments. Moreover, a vector $y$ is an extreme point of the polyhedron $D_3(0)$ if it satisfies at least $m$ of the above $p$ equalities. Obviously, $y_0$ is one of the extreme points, and so are all permutations of $y_0$. But in addition to these obvious ones, we have a series of additional extreme points in the edges of the hyperplane segments where the ordering of the dimensions changes, i.e. $y_{.k} = y_{.l}$ for some $k,l \in T$. The vectors $z^{kl}$, $k,l = 1,\ldots,m$ capture all those cases: these vectors contain $l$ - $k$ elements with the equal value, and consequently, the only way to satisfy the additional $m$ - ($l$ - $k$) equalities is to set $z_i^{kl} = x_{0i}$ for $i = 1,...,k-1$, $z_i^{kl} = r_{kl}$, $i = l+1$, and $z_i^{kl} = x_{0i}$, $i = l+2,...,m$. This proves that the permutations of $y_0$ and the vectors $z^{kl}$ constitute the exhaustive set of extreme points of the polyhedron $D_3(0)$. Like in the SSD case, we can write $D_3(0)$ equivalently as the convex monotone hull of its extreme points, i.e.

$$D_3(0) \equiv \left\{ y \in \Re^m \middle| y \geq y_0 W^0 + \sum_{j=i}^{m} \sum_{i=1}^{m} z^{ij} W^{ij} \right\},$$

where $W_{kl}^0, W_{kl}^{ij} \in [0,1] \forall k,l \in T$; $\vec{1}\left( W^0 + \sum_{j=i}^{m} \sum_{i=1}^{m} W^{ij} \right) = \left( \left( W_0 + \sum_{j=i}^{m} \sum_{i=1}^{m} W^{ij} \right) \vec{1}^T \right)^T = \vec{1}$. This gives $D_3(0) \setminus \{ y_0 P \} = \Delta_3(0)$. *Q.E.D.*

**Corollary 1**: The necessary and sufficient SSD efficiency condition for portfolio 0 is that $\{ Yl \mid l \in \Lambda \} \cap \Delta_3(0) = \varnothing$, i.e. the TSD dominating set does not contain any feasible portfolio.

**Corollary 2**: The special case of $r_{1m} = \dfrac{\sum_{i=1}^{m} (m-i+1) x_{0i}}{\sum_{i=1}^{m} (m-i+1)}$ is indeed the smallest risk-free rate of return that dominates the portfolio 0 by TSD.

Like the SSD dominating set, the set $\Delta_3(0)$ is a closed, monotonous, and convex set, which is very convenient from the operational point of view, as we will see in the next section.

## 4. SD Efficiency Tests

This section presents convenient LP/MILP tests of SD efficiency, building on the SD dominating sets introduced in the previous section.

First, add vector $y_0$ as the column "0" of matrix *Y*, and denote the resulting matrix by the resulting matrix by $Y^{+0}$. For the weighting vector $l$ we use the domain $\Lambda^{+0} \equiv \left\{ l \in \Re_{(+)}^{n+1} \middle| \vec{1}\, l = 1 \right\}$. To test FSD efficiency of portfolio 0, consider the following test statistic:[9]

(7) $$q_1(0) = \underset{l \in \Lambda^{+0}, P}{Sup} \left\{ \vec{1}(Y^{+0} l - y_0 P) \middle| Y^{+0} l \geq y_0 P \right\}$$

**Theorem 5**: The following conditions are equivalent: 1) Portfolio 0 is FSD efficient relative to *all* portfolios $Yl$, $l \in \Lambda$. 2) $q_1(0) = 0$.

**Proof**: Observe that $q_1(0) = 0$ is always a feasible solution, obtainable by setting $l_0 = 1$, $l_j = 0, j \neq 0$, and $P = I$. Therefore, if the supremum differs from zero, then $Y^{+0} l \geq y_0 P$ holds as a strict inequality for some dimension. The equivalence follows directly from Theorem 1. *Q.E.D.*

The test statistic (7) can be solved by standard 0-1 Mixed Integer Linear Programming algorithms. The most standard ones include the numerous variations of the branch-and-bound approach (see e.g. Bazaraa, Sherali, and Shetty, 1993; Nemhauser, Rinnooy Kan, and Todd, 1994; Wolsey, 1998)). Unfortunately, the number of binary integers equals $m^2$, which may introduce a barrier for practical application. We leave development of more efficient computational strategies for future research.

Before we introduce the SSD test statistic, we need an additional technical pre-requisite:

**Assumption**: $y_{0j} \neq y_{0i}$ for all $j, i \in T$, i.e. there are no ties in vector $y_0$.

Assuming away the ties might appear both brute and overly restrictive from empirical point of view. Fortunately, we can always transform the data matrix $Y$ to eliminate the ties, while still preserve the original ranking of the vector $y_0$. For example, if $y_{0j} = y_{0i}$, we can add a small perturbation constant $e > 0$ to both the evaluated portfolio $y_0$ and to all elements of the row $i$ in matrix $Y^{+0}$. Note that the SD efficiency criteria only depend on the relative magnitudes of the observations, not on their absolute values. Hence, this innocent data transformation does not change the relative SD efficiency status of the portfolio. Therefore, this technical assumption should be interpreted as a pre-requisite for data pre-processing rather than a limitation of the theory.

After eliminating the ties, we can test SSD efficiency in terms of the following test statistic:

[9] We use supremum rather than maximum because infinitely large returns may be feasible (in some dimension / time period) when short-selling is allowed.

(8) $$q_2(0) = \underset{\substack{l \in \Lambda^{+0}, W \\ s^+, s^-}}{Sup} \left\{ \vec{1}\left(Y^{+0} l - y_0 W\right) - \left(\vec{1} s^+ \vec{1}^T + \vec{1} s^- \vec{1}^T\right) \middle| Y^{+0} l \geq y_0 W; W - \tfrac{1}{2}\vec{1}\vec{1}^T = s^+ - s^- \right\}.$$

This test statistic is identical to (7), except for two modifications. First, we relaxed the binary integer constraint of the permutation matrix, i.e. we substituted $P$ by $W$. Second, we introduced the *mxm* matrices $s^+$ and $s^-$ so as to distinguish between integer and non-integer valued *W*.

**Theorem 6**: The following conditions are equivalent: 1) Security 0 is SSD efficient relative to *all* portfolios $Yl$, $l \in \Lambda$. 2) $q_2(0) = -\frac{m^2}{2}$.

**Proof**: Note first that $\frac{m^2}{2}$ is the maximum value for the the sum of $s^+, s^-$ variables within the feasible domain. Therefore, if the maximum sum of $s^+, s^-$ variables equals $\frac{m^2}{2}$, then $W$ is a permutation matrix. To minimize the sum of $s^+, s^-$ variables in case of permutation matrix, it is optimal to match $W_{ij} = 1$ with $s_{ij}^+ = \frac{1}{2}, s_{ij}^- = 0$, and $W_{ij} = 0$ with $s_{ij}^+ = 0, s_{ij}^- = \frac{1}{2}$ $\forall i, j = 1,...,m$. Since the total number of $s^+, s^-$ variables is $2m^2$, of which one half are set equal to 0 and the other half equal to $\frac{1}{2}$, the optimal solution becomes $-\frac{m^2}{2}$. Hence, analogous to the FSD case, $q_{SSD}(k) = -\frac{m^2}{2}$ is always a feasible solution, obtained e.g. by setting $l_0 = 1$, $l_j = 0, j \neq 0$, and $W = I$.

Now, if the optimal solution is greater than $-\frac{m^2}{2}$, then either the inequality $Yl \geq y_0 W$ holds as a strict inequality in some dimension, or the sum of $s^+, s^-$ variables is less than $\frac{m^2}{2}$ implying $Yl \notin \{y_0 P\}$. Since we assumed that $y_{kj} \neq y_{ki}$ for all $j, i \in T$, the condition $Yl \notin \{y_0 P\}$ implies $Yl$ and $y_0$ have non-identical cdfs. The equivalence result follows from Theorem 3. *Q.E.D.*

From computational point of view, the SSD test statistic (8) is more convenient than the FSD test statistic (7): The objective function as well as all the constraints are expressed in linear form, and hence the problem can be immediately solved by standard Linear Programming codes. Very effective simplex and interior point methods are generally available (see e.g. Martin, 1998, for further details), and therefore, the SSD test is tractable even for large-scale applications.

In the TSD case we consider the following test statistic:

(9)
$$q_3(0) = \underset{\substack{f\in\Re^n_+, W^0,\\ W^{ij}, s^+, s^-}}{Min} \left\{ \vec{1}f + \left( \vec{1}s^+\vec{1}^T + \vec{1}s^-\vec{1}^T \right) \middle| \; Yf \geq y_0 W^0 + \sum_{j=i}^{m}\sum_{i=1}^{m} z^{ij}W^{ij}; W^0 - \tfrac{1}{2}\vec{1}\vec{1}^T = s^+ - s^- \right\}$$

The TSD test (9) differs from the SSD test (8) in two important respects. The first one is obvious: We use the convex monotone hull of $y_0$ and the auxiliary vectors $z^{ij}$ to make use of Theorem 4. The second difference is that we minimize the sum of the weights $f$ rather than maximize the sum (or the average) of the difference between the reference portfolio and the evaluated security returns. This is because a reference portfolio with a lower mean return than the evaluated security can still dominate by TSD (see e.g. the example of Figure 3 below). Observe that minimizing $\vec{1}f$ is equivalent to minimizing a distribution shift parameter $q$ >0, i.e. we could use $q(\vec{1}\,l)$ in the objective function, and still maintain the usual constraint $l \in \Lambda$. However, introducing such a shift parameter would make the problem a nonlinear one, so we prefer to introduce the shift parameter implicitly in the weighting vector $f$.

**Theorem 7**: The following conditions are equivalent: 1) Portfolio 0 is TSD efficient relative to *all* portfolios $Yl$, $l \in \Lambda$. 2) $q_3(0) = 1 + \frac{m^2}{2}$.

The proof is directly analogous to the SSD case, and is hence omitted.

Like the SSD test statistic (3), the TSD test statistic (5) is expressed in the linear form, and hence the problem can be readily solved by standard Linear Programming. The total number of the $z^{kl}$ vectors is as large as $\frac{m^2}{2}$ if $m$ is an odd number and $\frac{m^2+1}{2}$ if $m$ is an even number, so the number of the weighting parameters $W_{kl}^{ij}$ increases at the power of four as $m$ increases. In other words, the TSD case involves solving a large-scale LP problem. Still, this number of parameters is significantly less than $m!$ associated with the brute force strategy discussed in Section 2 above.

## 6. Illustrative Example

To gain intuition for the SD tests derived above, we consider the simplest thinkable example with two return observations, to facilitate graphical illustration. Suppose the return data for the evaluated portfolio 0 take the following values: $y_0 = (1,4)$. Before turning attention to the benchmark portfolio, we characterize the FSD, SSD, and TSD dominating sets.

Consider first the FSD case, as illustrated by Figure 1. Obviously, any return vector $(y_1, y_2) \geq (1,4)$, $(y_1, y_2) \neq (1,4)$ dominates portfolio 0 by FSD. The key insight of the efficiency tests developed above is to also consider alternative orderings, i.e. the mirror image $(y_1, y_2) \geq (4,1)$. In the figure, the broken line distinguishes the alternative orderings, and represents the risk-free options. In this two-dimensional case the smallest risk-free return that dominates (1,4) by FSD equals 4. Thus, Figure 1 aptly illustrates the non-convex nature of the FSD dominating set, which is the ultimate source of the need to resort to Mixed Integer Linear Programming in the computation of the FSD test statistic.

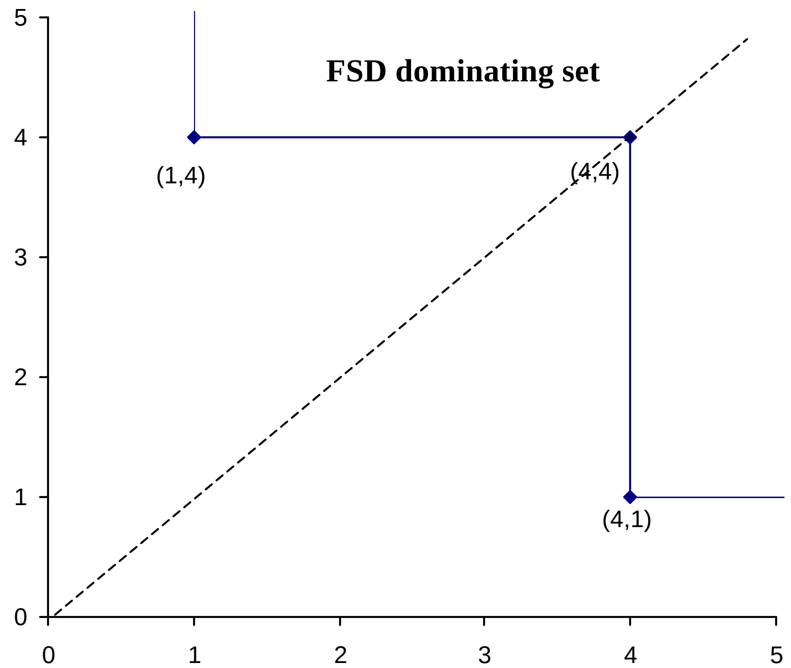


**Figure 1: The shaded area represents the FSD dominating set of the vector (1,4).**

Consider next the SSD case, as illustrated by Figure 2. The FSD dominating set is naturally a subset of the SSD dominating set. In addition, all return vectors that dominate a convex combination of the permuted return vector of portfolio 0 (= its mean-preserving anti-spread) are contained in the SSD dominating set by Theorem 3. Therefore, the triangular [(1,4), (4,4), (4,1)] is contained in the SSD dominating set. Note that the smallest risk-free return that dominates portfolio 0 by SSD equals 2.5, i.e. 2.5 > 1 and 2.5 + 2.5 = 1 + 4, which equals the mean return of the portfolio 0. This is a general property of SSD: An option with the smaller mean cannot dominate by SSD (Hadar and Russel, 1969). Conveniently, the SSD efficient set is convex, and therefore, the SSD test can be formulated in terms of Linear Programming.

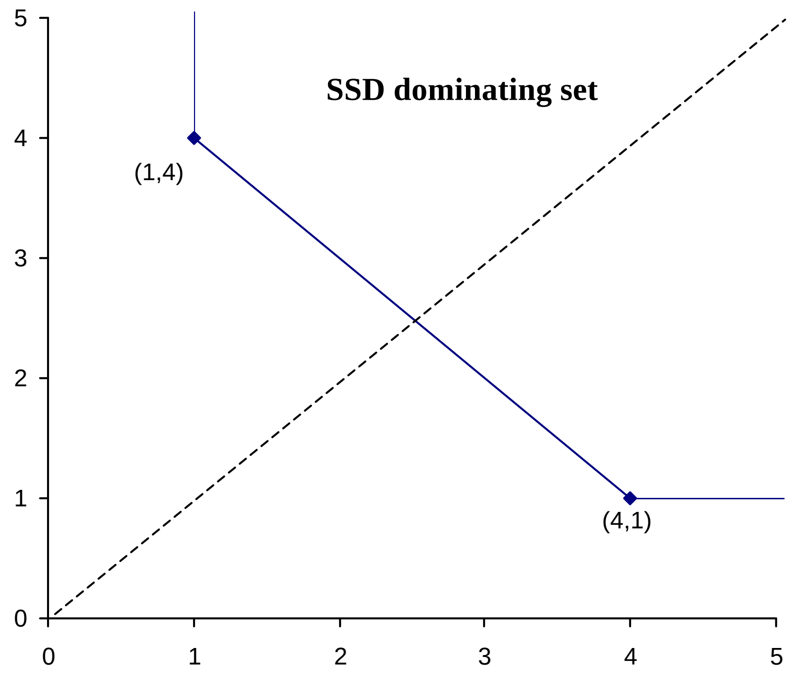


**Figure 2: The shaded area represents the SSD dominating set of the vector (1,4).**

Next, consider the TSD case, as illustrated by Figure 3. Again, the SSD dominating set is naturally a subset of the TSD dominating set. The risk-free return $r_{1m} = 2$ gives one of the extreme points of the TSD dominating set: 2 > 1 and 2 + (2+2) = 1 + (1 + 4). Interestingly, (2,2) dominates (1,4) by TSD, even though its mean return (2) falls below the mean return of portfolio 0 (2.5). In addition, Theorem 4 implies all return vectors that dominate a convex combination of the risk-free rate 2 and a permuted return vector $y_0$ are included in the TSD dominating set. Therefore, the triangular [(1,4), (2,2), (4,1)] belongs to the TSD dominating set. Like the SSD efficient set, the TSD efficient set is convex, and therefore, the TSD test can be formulated in terms of Linear Programming.

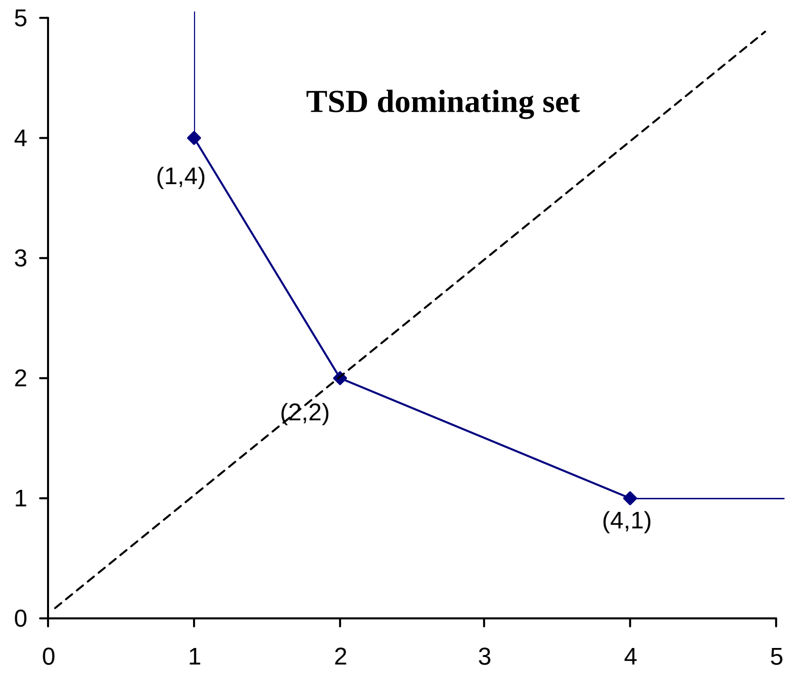


**Figure 3: The shaded area represents the return vectors that dominate (1,4) by TSD.**

Now, how do we test for SD efficiency under diversification? Suppose our sample includes three securities A, B, C: $y_A = (0.5, 4.5)$; $y_B = (2.5, 1.5)$; and $y_C = (3, 0.5)$. It is easy to verify that the basic pair-wise comparison test without diversification (Theorem 1) diagnoses portfolio 0 FSD, SSD, and TSD efficient. However, it is not so obvious whether portfolio 0 is FSD, SSD, or TSD efficient relative to all portfolios that can be constructed from these four portfolios.

Figures 4 and 5 illustrate the set of feasible portfolios obtainable as a convex combination of these given three securities and the evaluated portfolio 0, when short-selling is disallowed vs. allowed. It is worth to stress one of the key insights behind the SD efficiency tests developed in the previous sections: In sharp contrast to the evaluated security, we form the convex combinations from the original time series, and totally ignore the combinatorial possibilities for the reference portfolio. Therefore, characterizing and optimizing the reference portfolio is a trivially simple undertaking. The fundamental idea behind this solution is that when we try to match two sets of return observations, i.e. the reference portfolio returns and the evaluated security returns, there is no need to rank both sets. It clearly suffices to sort one of the sets, while the other set can remain in its original order. We choose to avoid sorting the reference portfolio to preserve the linear structure of the test problem. Since the return data of the evaluated security is given, we can safely permute the given evaluated vector without compromising the linear structure.

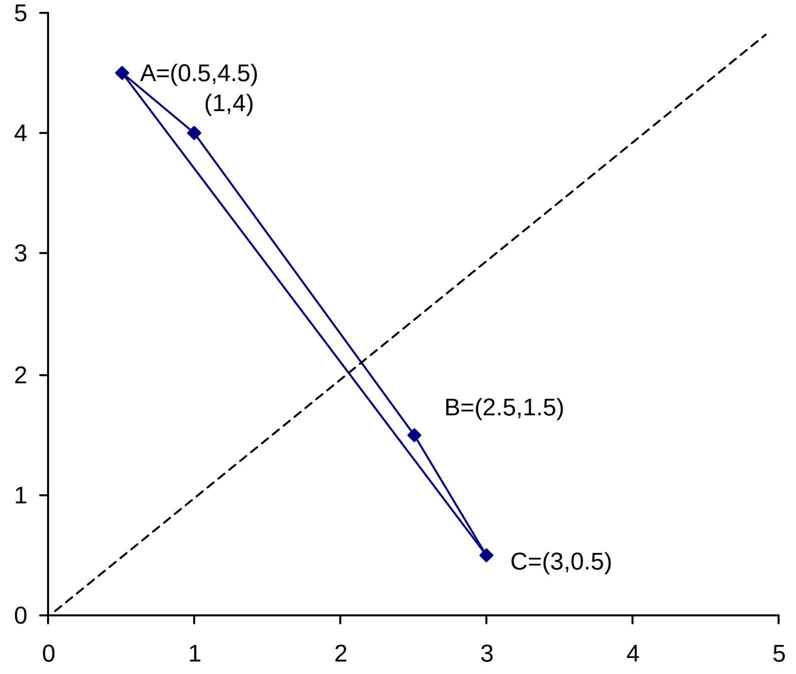


**Figure 4: The shaded area represents the feasible portfolios composed from securities A, B, and C, and vector (1,4), short-selling disallowed.**

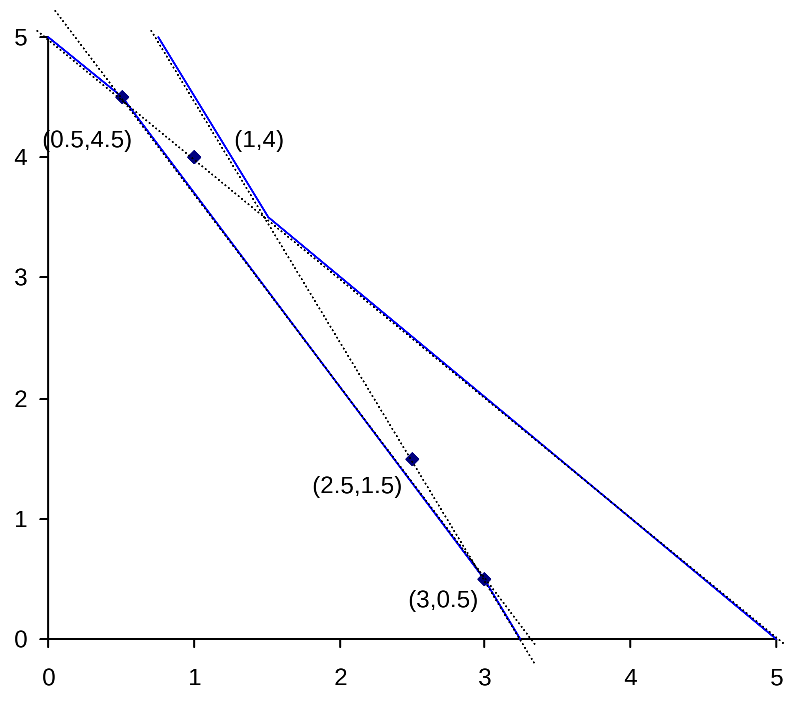


**Figure 5: The shaded area represents the feasible portfolios composed from securities A, B, and C, and vector (1,4), short-selling allowed.**

Our test statistics can be given the following graphical interpretations: In case of FSD, we can plot the FSD dominating set and the set of reference portfolios in the same figure, as illustrated by Figure 6 in the no short-selling case. The test problem selects a vector from the set of reference portfolios and another vector from the FSD dominating set to maximize the difference of the mean return. The broken line illustrates the isoquant of the mean return. Since no reference portfolio lies in the interior of the FSD dominating set. The maximum difference (equal to zero) is obtained by selecting portfolio 0 both for the reference portfolio and for the evaluated permutation. This proves portfolio 0 FSD efficient.

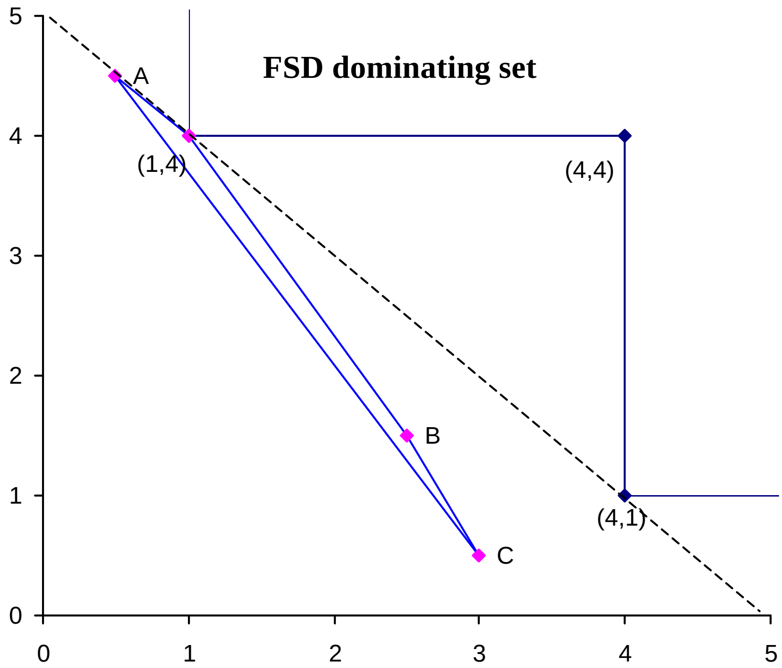


**Figure 6: Illustration of the FSD test, sort-selling disallowed. Vector (1,4) is FSD efficient because no reference portfolio lies in the interior of the FSD dominating set.**

Interestingly, the portfolio 0 turns out FSD inefficient under short-selling. This is illustrated by Figure 7, where the FSD dominating set and the set of feasible portfolios overlap. Specifically, by taking the short position with -3 units of security C and 4 units of security B we obtain the return distribution (1, 4.5), which obviously dominates (1,4) by FSD. The test problem (7) selects (1, 4.5) as the reference distribution because it yields the highest mean return among the FSD dominating feasible portfolios. Since FSD inefficiency implies SSD and TSD inefficiency, the short-selling option is suppressed from the further examples.

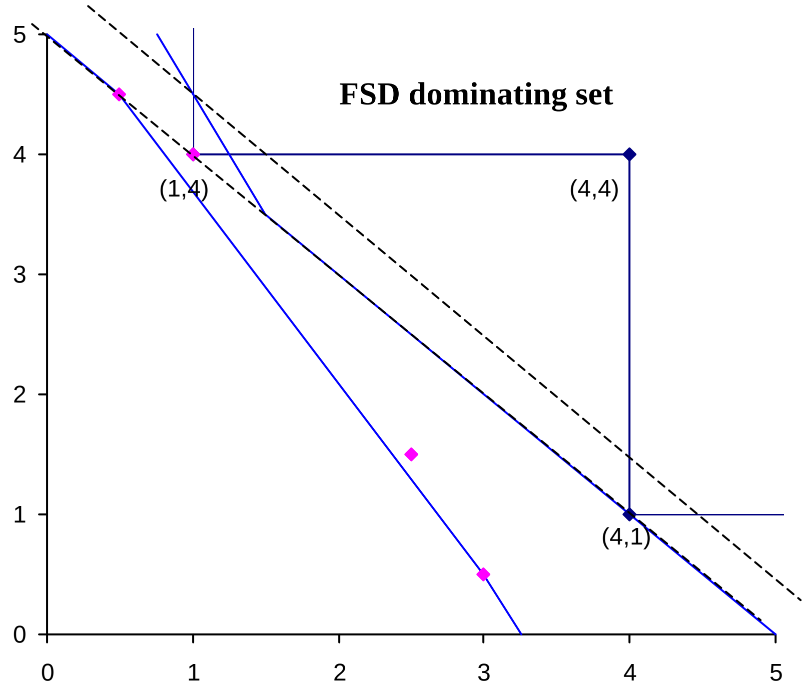


**Figure 7: Illustration of the FSD test, sort-selling allowed. Vector (1,4) is FSD inefficient because some reference portfolios lie in the interior of the FSD dominating set.**

In the SSD case we follow a similar approach and can plot the SSD dominating set and the set of reference portfolios in the same figure as illustrated in Figure 8. We test by selecting vectors from the set of reference portfolios and from the SSD dominating set respectively, so as to maximize the difference of the mean return illustrated by the broken line. Analogous to the FSD case, since no reference portfolio lies in the interior of the SSD dominating set, portfolio 0 is diagnosed SSD efficient.

By contrast, in the TSD case the set of reference portfolios and the TSD dominating sets overlap, as shown by Figure 9. It turns out a TSD inefficient strategy to hold portfolio 0, since we can compose a reference portfolio consisting of security B and securities in portfolio 0, e.g. a risk-free benchmark portfolio (2 1/8, 2 1/8) if obtained by assigning 25 per cent weight to portfolio 0 and 75 per cent weight to security B. This risk-free portfolio clearly dominates the original portfolio by TSD. Interestingly, the basic pair-wise comparison of securities diagnosed portfolio 0 TSD efficient, so this very simple example already demonstrates the greater power of the SD tests which account for the diversification strategies.

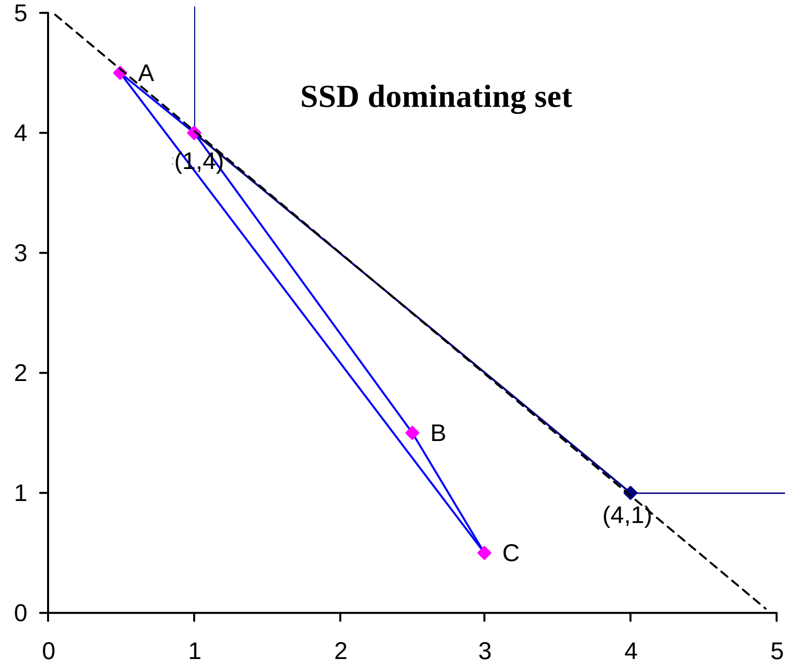


**Figure 8: Illustration of the SSD test, no short-selling. Security *k* is SSD efficient because no reference portfolio lies in the interior of the SSD dominating set.**

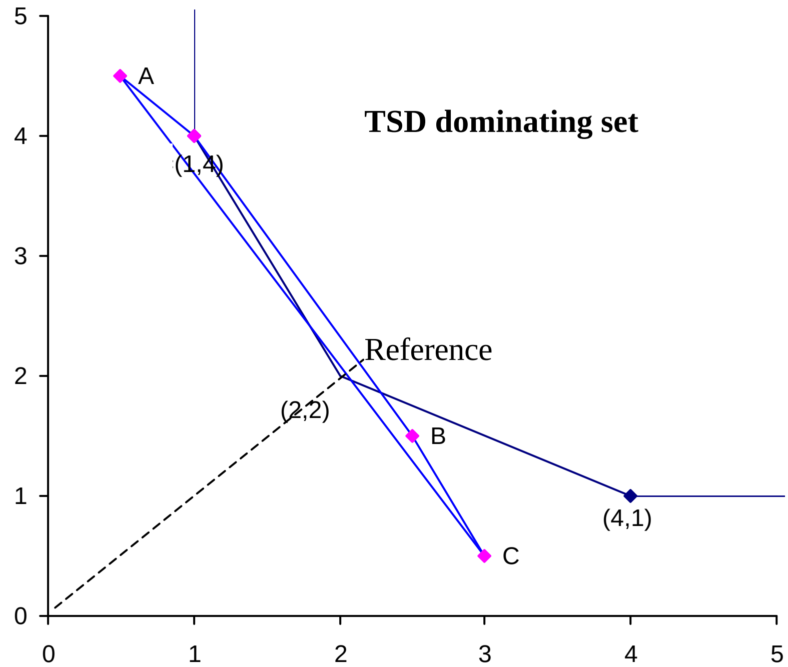


**Figure 9: Illustration of the TSD test, no short-selling. Security *k* is TSD inefficient because there exist a reference portfolio that lies in the interior of the TSD dominating set.**

In the TSD test problem (5), we seek to scale down this reference portfolio to the minimal return vector that is still contained in the TSD dominating set. This is illustrated by the broken line - the ray from the origin. We find that the greatest "down-sizing" potential of 5.88 per cent is associated with the risk-free portfolio (2 1/8, 2 1/8). This gives the optimal solution to the TSD test problem (9). Since the boundary of the TSD dominating set generally has the kink in the diagonal where the ranking of the dimensions changes, the TSD test tends to favor the least risky portfolios.

As the conclusion, we find that this very simple example gives a lot of insight to the basic principles behind the test statistics, and aptly illustrates the value added of taking the diversification into account.

## 7. Concluding remarks

We have analytically characterized the sets of time series vectors that dominate a given evaluated portfolio by FSD, SSD, and TSD, respectively. Interestingly, these sets have a relatively simple polyhedral structure. Based on these insights, we proposed tests of SD efficiency. The major innovation in contrast to the earlier tests is that our tests account for diversification. We formulated the FSD efficiency test as a 0-1 Mixed Integer Linear Programming Problem, while the SSD and the TSD tests took the form of the standard Linear Programming problems. The generalizations to the higher order SD criteria follow in a straightforward manner. We expect that computationally tractable SD efficiency tests that do allow for diversified portfolios will significantly enhance the power of the SD criteria, as well as extend their empirical applicability to areas where diversification plays an important role, such as financial applications.

We find the presented SD tests also fruitful as for the new research directions. The present paper confined attention to testing SD efficiency in the finite historical return data. As such, the tests may be applied e.g. for *ex post* performance analysis of mutual funds, pension funds, and other suchlike financial institutions. In addition, the past performance may be a useful guideline for the portfolio selection / building applications. Interestingly, the convenient LP/MILP structure of our SD tests is well in line with some recent work on portfolio optimization, e.g. the Mean - Absolute Deviation model by Yamakazi and Konno (1991) and the minimax portfolio selection rule suggested by Young (1998). As noted by these authors, framing the portfolio selection process as a LP problem makes it feasible to constrain certain decision variables to be integer valued, which facilitates the use of more complex decision models, e.g. accommodating fixed transaction costs.

It would naturally be advantageous to be able to draw statistical inference of SD efficiency in the underlying unobservable probability distribution, based on the sample SD tests in the empirical distribution. Of course, if our data set consists of a series of historical observations, then the probability distribution should exhibits sufficient stationarity over time. As aptly discussed by Porter and Pfeffenberger (1975), we face a trade-off between the sampling error in the short time series versus the risk of specification error due to the nonstationarity in the long time series. These observations motivate developing more elaborate statistical tests that use the sampling theory for the cdfs to account for sampling error as well as error components that could accommodate minor deviations from the stationarity. In the Mean-Variance framework, one can build on the elementary sampling theory for the estimators of the mean and the variance parameters. The situation is more complicated in the SD framework, but -as discussed by Porter and Pfeffenberger (1975)- not entirely hopeless. These authors already suggest utilizing the distribution-free Kolmogorov-Smirnov test. More recently, McFadden (1989) has investigated this route in the greater detail and elaboration in the cases of FSD and SSD. Alternative routes have also been considered: For example, Chow (1989) and Zheng *et al.* (2000) derive statistical tests taking advantage of the relation between the SD ordinates and the lower partial moments, while Anderson (1996) proposed a test based on the goodness-of-fit approach. We consider developing statistical tests of SD efficiency under diversification - building on these earlier works - a fascinating topic for future research. The SD dominating sets introduced above could be a good starting point for this undertaking.

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