

Exponentially Damped Lévy Flights

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Abstract

Since real processes seem to depart from standard Lévy distributions, modifications to the latter have been suggested in literature. These include (abruptly) truncated [3], smoothly truncated [4, 5] and gradually truncated Lévy flights [6, 7]. We put forward what we call an exponentially damped Lévy flight which encompasses the previous cases. In the presence of increasing and positive feedbacks, our distribution is assumed to deviate from the Lévy in both a smooth and gradual fashion. We estimate the truncation parameters by nonlinear least squares to optimally fit the distribution tails. That is a novel approach for estimating parameters α and γ of the Lévy. The method is illustrated with daily data on exchange rates for 15 countries against the US dollar. Our results show that the exponentially damped Lévy flight fits the data well when increasing and positive deviations are present.

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1. Introduction

Since returns of financial series are usually larger than those implied by a Gaussian distribution [1], research interest has revisited the hypothesis of a stable Pareto-Lévy distribution [2]. Ordinary stable Lévy distributions have fat power-law tails that decay more slowly than an exponential decay. Such a property can capture extreme events, and that is plausible for financial data. But it also generates an infinite variance, which is implausible. Real world data points are necessarily finite, and so it is the variance.

Truncated Lévy flights (TLFs) are an attempt to overturn such a drawback [3]. The standard Lévy distribution is thus abruptly cut to zero at a cutoff point. The TLF is not stable though, but has finite variance and slowly converges to a Gaussian process as implied by the central limit theorem.

Owing to the sharp truncation, the characteristic function of a TLF is no longer infinitely divisible as well. However, it is still possible to define a TLF with a smooth cutoff that yields an infinitely divisible characteristic function [4]. In a smoothly truncated Lévy flight (STLF), the cutoff is done by asymptotic approximation of a stable distribution valid for large values [5].

The cutoff can be further combined with a statistical distribution factor to generate a gradually truncated Lévy flight (GTLF) [6, 7]. It has been pointed out [6, 7] that the STLF breaks down in the presence of a positive feedback. This also implies fatter tails for the Lévy. But the GTLF itself also breaks down if the positive feedback

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is strong enough. This seems to happen because the truncation function is decreasing exponentially.

This paper puts forward what we call an exponentially damped Lévy flight (EDLF), in which the gradually truncated Lévy is modified and then combined with the smoothly truncated one. In the presence of an increasing and positive feedback, our distribution is assumed to smoothly and gradually deviate from the Lévy. The truncation parameters are estimated by nonlinear least squares to provide an optimized fit for the tails. We then show that our EDLF fairly fits data on daily exchange rates for 15 countries against the US dollar.

The rest of the paper is organized as follows. Section 2 presents the ordinary Lévy distribution and its extensions, namely the TLF, the STLF, and the GTLF. Section 3 presents our suggested EDLF. Section 4 shows our novel approach for estimating parameters α and γ . Section 5 applies our methodology to the exchange rate data. Section 6 concludes.

2. The ordinary Lévy distribution and its extensions

Let S_n be the sum of n independent and identically distributed random variables X_i ,

$$S_n = X_1 + X_2 + X_3 \dots + X_n, \quad (1)$$

with $E(X_i) = 0$. And consider

$$Z_{\Delta t}(t) = S_t - S_{t-\Delta t} = X_t + X_{t-1} + \dots + X_{t-\Delta t+1}, \quad (2)$$

where Δt is a time lag. The ordinary Lévy distribution of index α and scale factor γ is symmetric and given by

$$L(Z_{\Delta t}) \equiv \frac{1}{\pi} \int_0^{\infty} \exp(-\gamma \Delta t q^\alpha) \cos(qz_{\Delta t}) dq, \quad (3)$$

where $0 < \alpha < 2$ and $\gamma > 0$.

The natural logarithm of the characteristic function $\ln[\varphi(K)] \equiv \ln E[e^{iKZ_{\Delta t}}]$ is given by $\ln[\varphi(K)] = -\gamma \Delta t |K|^\alpha$. Such a characteristic function satisfies $\Delta t \ln[\varphi(K)] = \ln[\varphi(\Delta t^{1/\alpha} K)]$, which means that the corresponding probability distribution is

$$L(Z_{\Delta t}) = \Delta t^{-1/\alpha} L(\Delta t^{-1/\alpha} Z_{\Delta t}) = \Delta t^{-1/\alpha} L(Z_s), \quad (4)$$

where $Z_s = \Delta t^{-1/\alpha} Z_{\Delta t}$ is a scaled variable at Δt .

The Lévy distribution above can be modified to

$$P(z_{\Delta t}) = \eta L(z_{\Delta t}) f(z_{\Delta t}), \quad (5)$$

where η is a normalizing constant, $L(z_{\Delta t})$ is given by Eq. (3), and function $f(z_{\Delta t})$ characterizes the change carried out on the distribution. Eq. (5) can be dubbed as a modified Lévy flight (MLF).

The TLF is an extension in which

$$f(z_{\Delta t}) = f_{abrupt}(z_{\Delta t}) = \begin{cases} 0, & (|z_{\Delta t}| > l_c) \\ 1, & (|z_{\Delta t}| \leq l_c) \end{cases}, \quad (6)$$

where l_c is the step size at which the distribution begins to departure from the ordinary Lévy.

The TLF is not stable. With a finite variance, it converges slowly to a Gaussian. Due to its sharp cutoff, the characteristic function of the TLF is no longer infinitely divisible. As a result, scaling turns out to be approximate and valid for a finite time interval only. For longer time intervals, scaling must break down.

Now consider the smooth case, i.e. the STLFL. Here the cutoff parameter $\lambda > 0$ is introduced into Eq. (5) as

$$f(z_{\Delta t}) = f_{smooth}(z_{\Delta t}) = \begin{cases} Ca |z_{\Delta t}|^{-1-\alpha} e^{-\lambda|z_{\Delta t}|}, & (z_{\Delta t} < 0) \\ Cbz_{\Delta t}^{-1-\alpha} e^{-\lambda z_{\Delta t}}, & (z_{\Delta t} > 0) \end{cases} \quad (7)$$

Function $f_{smooth}(z_{\Delta t})$ is based on the asymptotic approximation of a stable distribution of index α valid for large values of $|Z|$ when $\gamma = 1$, and exhibits a power law behavior. For $0 < \alpha < 1$, the first term of the expansion of $L(Z_{\Delta t})$ can be approximated by

$$L(Z_{\Delta t}) \approx \frac{\gamma \Delta t \Gamma(1+\alpha) \sin(\pi\alpha/2) |z_{\Delta t}|^{-(1+\alpha)}}{\pi}. \quad (8)$$

By taking into account the particular case of the Lévy where $a = b$, we obtain

$$\ln[\varphi_{STLF}(K, \lambda)] = \begin{cases} \gamma((\lambda^2 + K^2)^{\alpha/2} \cos(\alpha\theta) - \lambda^\alpha) & (0 < \alpha < 1) \\ \gamma\lambda^\alpha ((1 + iK/\lambda)^\alpha - 1), & (1 < \alpha < 2) \end{cases} \quad (9)$$

where $\theta = \arctan(K/\lambda)$ and $\gamma = C\Gamma(-\alpha)$. Now the characteristic function ends up infinitely divisible. The convolutions of its corresponding distribution can be collapsed onto $\Delta t = 1$ by scaling $Z_{\Delta t}$ and λ . By considering (5), (7), and (8), the approximate variance of the STLFL obtains, i.e.

$$\sigma_{smooth}^2 = \Delta t^{2/\alpha} 2\eta\gamma\pi^{-1}\Gamma(1+\alpha)\sin(\pi\alpha/2)\lambda_s^{\alpha-2}\Gamma(2-\alpha), \quad (10)$$

where $\lambda_s = \Delta t^{1/\alpha}\lambda$ is the scaled value of λ .

Finally, the GTLF is defined as

$$f(z_{\Delta t}) = f_{gradual}(z_{\Delta t}) = \begin{cases} 1, & (|z_{\Delta t}| \leq l_c) \\ \exp\left\{-\left(\frac{|z_{\Delta t}| - l_c}{k}\right)^\beta\right\}, & (|z_{\Delta t}| > l_c) \end{cases} \quad (11)$$

where l_c is the step size at which the distribution starts to deviate from the Lévy. Here k and β are the constants related to the truncation. By taking (5), (8), and (11) into account, the approximate variance is now given by

$$\sigma_{gradual}^2 = \frac{\Delta t^{2/\alpha} 2\eta\gamma\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi(2-\alpha)} l_{efs}, \quad (12)$$

where

$$l_{efs} = l_s^{2-\alpha} + (2-\alpha) \left(l_s + \frac{k_s \Gamma(2/\beta)}{\Gamma(1/\beta)} \right)^{1-\alpha} \frac{k_s}{\beta} \Gamma(1/\beta) \quad (13)$$

with $\beta \neq 1$, $l_s = \Delta t^{-1/\alpha} l_c$, and $k_s = \Delta t^{-1/\alpha} k$. Our factor $2 - \alpha$ in Eq. (13) is not originally considered when the GTLF is first proposed [6, 7]. So our Eq. (13) corrects the original one.

We now turn to our suggested distribution.

3. Exponentially damped Lévy flight

Elsewhere [8] some of the authors have dealt with the currency data also presented here only to realize that their distributions deviate from the Lévy in a smooth and gradual fashion after $|z_{\Delta t}| > l_c$. The deviations were also caught increasing.

Such class of deviations was found to be positive [6, 7], which means fatter tails. It has been argued [6, 7] that, since the physical capacity of the system is limited, the

feedback begins to decrease exponentially (and not abruptly) after a certain critical step size. In contrast, in the presence of our previously found increasing deviations, we argue that an abrupt truncation is still necessary. In such cases, using the truncation approaches as in Eqs. (6), (7), and (11) might not be appropriate.

For this very reason, here we put forward a broader formulation for $f(z_{\Delta t})$.

We suggest an exponentially damped Lévy flight that encompasses the previous abrupt, gradual, and smooth truncation of the Lévy. We thus define

$$f(z_{\Delta t}) = f_{damped}(z_{\Delta t}) = \begin{cases} 1, & |z_{\Delta t}| < l_c \\ (\Delta t^{-1/\alpha} |z_{\Delta t}| + \mathfrak{G})^{\beta_1} \exp\{H(z_{\Delta t})\}, & (l_c < |z_{\Delta t}| < l_{max}), \\ 0, & |z_{\Delta t}| > l_{max} \end{cases}, \quad (14)$$

where

$$H(z_{\Delta t}) = \lambda_1 + \lambda_2 [1 - |z_{\Delta t}| / l_{max}]^{\beta_2} + \lambda_3 (|z_{\Delta t}| - l_c)^{\beta_3}, \quad (15)$$

and \mathfrak{G} , λ_1 , $\lambda_2 \leq 0$, $\lambda_3 \leq 0$, β_1 , β_2 , and β_3 are parameters describing the deviations from the Lévy, l_c is (as before) the step size at which the distribution begins to deviate from the Lévy, and l_{max} is the step size at which an abrupt truncation is done.

Note that when $l_{max} \rightarrow \infty$, $H(z_{\Delta t}) = \lambda_1 + \lambda_2 + \lambda_3 (|z_{\Delta t}| - l_c)^{\beta_3}$. By setting $\mathfrak{G} = 0$, $\beta_1 = -1 - \alpha$, $l_c = 0$, and $\beta_3 = 1$, the resulting function is thus equivalent to the smooth case given by Eq. (7). When $l_{max} \rightarrow \infty$, the similar function for the gradual case can be found by setting $\mathfrak{G} = \lambda_1 = \lambda_2 = \beta_1 = 0$. The abrupt case is given by setting $l_c = 0$ and choosing the appropriate parameters such that $H(z_{\Delta t}) \rightarrow -\infty$. By using (5), (8), and (14), and considering the case with l_{max} finite and $\lambda_3 = 0$, the approximate variance is given by

$$\sigma_{damped}^2 = \frac{\Delta t^{2/\alpha} 2\eta\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi} R, \quad (16)$$

where

$$R = \frac{l_{cs}^{(2-\alpha)/\alpha}}{2-\alpha} + l_{ms}^{2-\alpha+\beta_1} e^{\lambda_1} \sum_{k \geq 0} \frac{\lambda_2^k}{k!} B_d(k\beta_2 + 1, 2 - \alpha + \beta_1), \quad (17)$$

$l_{ms} = \Delta t^{-1/\alpha} l_{max}$, $l_{cs} = \Delta t^{-1/\alpha} l_c$, and $B_d(k\beta_2 + 1, 2 - \alpha + \beta_1)$ is the incomplete Beta function with $k\beta_2 + 1 > 0$, $2 - \alpha + \beta_1 > 0$, and $d = 1 - l_c/l_{max}$.

It is worth noting that expressions (10), (12), and (16) all have the power law form $\sigma_{\Delta t}^2 = \nu \Delta t^{2/\alpha}$, where ν is a constant describing the ‘‘quasi-stable processes’’ that emerge from the truncation parameters for some interval $\Delta t_1 \leq \Delta t \leq \Delta t_2$ [8].

4. Estimation

Our parameter estimation departs from the previous approaches in literature. Parameters α and γ have been so far estimated [3-11] by plotting $L(0)$ versus Δt , where $L(0)$ is the probability of return to the origin for a given Δt .

Here a hybrid estimation process is advanced. We take a maximum likelihood approach for α and γ , and nonlinear least squares for the other parameters.

From Eq. (5), the log likelihood function is given by

$$\sum_z \ln(P(z_{\Delta t})) = \sum_z \ln(L(z_{\Delta t})) + \sum_z \ln(f(z_{\Delta t})) + \text{constant}. \quad (18)$$

The maximum likelihood estimates are obtained by minimizing Eq. (18) as a function of the distribution parameters. Note that (18) has three parts. For $\Delta t = 1$, the

first part depends only on α and γ , and the second one depends on the other parameters. As a result, the estimation process can be done separately for $\Delta t = 1$. The maximum likelihood estimates of α and γ are the same as those obtained if the process were a Lévy. Such a practice is discussed elsewhere [12, 13], and a computer program for implementing it (called STABLE.EXE) is available online at <http://academic2.american.edu/~jpnolan/>.

Now let $\hat{\alpha}$ and $\hat{\gamma}$ be the maximum likelihood estimates of α and γ . And let $\hat{P}(z_s)$ be the sample probabilities of the scaled variable at Δt , i.e. $z_s = \Delta t^{-1/\hat{\alpha}} z_{\Delta t} = \Delta t^{-1/\hat{\alpha}} Z_{\Delta t}$. Also let $\hat{L}(z_1)$ be the Lévy distribution using $\hat{\alpha}$, $\hat{\gamma}$ for $\Delta t = 1$.

The difference $\ln \hat{P}(z_s) - \ln \hat{L}(z_1)$ shows how data deviate from the original Lévy process. Assuming that $\ln \hat{P}(z_s) - \ln \hat{L}(z_1) = \ln f_{damped}(z_{\Delta t})$, which equals

$$\beta_1 \ln(|z_s| + \mathcal{G}) + H(z_s), \quad (\Delta t^{-1/\alpha} l_c < |z_s| < \Delta t^{-1/\alpha} l_{\max}), \quad (19)$$

then the parameters describing the deviations from the Lévy can be estimated by a nonlinear least squares procedure. Next section illustrates our approach with real world financial data.

5. Data analysis

The data sets employed were taken from the Federal Reserve website at <http://www.federalreserve.gov/releases/H10/hist/>. They refer to currency values in US dollar terms. These foreign exchange rates were collected by the Federal Reserve Bank of New York from a sample of market participants, and are noon buying rates in New York from cable transfers payable in foreign currencies. As standard, “holes” from weekends and holidays were ignored, and analysis focuses on trading days. We choose to take the historic values of the 15 currencies in Table 1. Table 1 shows a currency together with its historical time period and number of data points.

Our analysis takes returns Z rather than raw data (as usual), i.e. $Z_{\Delta t}(t) = S_{t+\Delta t} - S_t$, where S_t is a rate at day t . Fig. 1 displays the logarithm of the probability density functions (PDFs) of currency returns of the countries in Table 1 for $\Delta t = 1, 2$, and 5 trading days (a week) until 240 trading days (a year).

For $\Delta t = 1$, parameters α and γ were estimated by the maximum likelihood method. Results are in Table 2. The sample probabilities of a scaled variable at Δt , $\hat{P}(z_s)$ together with a Lévy distribution using $\hat{\alpha}$ and $\hat{\gamma}$ were then calculated. The differences $\ln \hat{P}(z_s) - \ln \hat{L}(z_1)$ are shown in Fig. 2. For all countries, these differences were fitted by employing Eq. (16). The resulting curves are shown as the continuous lines in Fig. 2. Their parameter estimates obtained by the nonlinear least squares method are also presented in Table 2. In all cases, parameters λ_3 and β_3 were dropped from the model. Here zero estimates for λ_3 and β_3 mean that the scaled PDFs exhibit heavy tails with increasing and positive feedbacks.

The examples in Fig. 2 show that our EDLF fits the exchange rate data reasonable well. It should also be noted that the abrupt truncation makes sense whenever increasing deviations are present.

It is worth emphasizing that the data in Fig. 2 show log differences $\ln \hat{P}(z_s) - \ln \hat{L}(z_1)$ that are increasing. This is suggestive that the earlier truncation procedures described in the previous sections might not be appropriate. To meet a

convergence condition, the exponential argument is set at $1-|Z_{\Delta t}|/l_{max}$, where l_{max} is an abrupt cutoff. Also, parameters λ_3 and β_3 are dropped from the model for us to get feasible estimates (the same goes as far as restriction $\lambda_3 \leq 0$ is concerned). A zero estimate for λ_3 means that parameter λ_2 explains the function exponential behavior in $1-|Z_{\Delta t}|/l_{max}$ rather than in $|Z_{\Delta t}|-l_c$. And that is why fatter tails with increasing (instead of decreasing) and positive feedback emerge.

6. Conclusion

The ordinary Lévy distribution should be truncated because variances are finite in real-world financial data. The suggested modifications in literature include (abruptly) truncated [3], smoothly truncated [4, 5] and gradually truncated Lévy flights [6, 7]. This paper advances an exponentially damped Lévy flight which encompasses all previous cases. Incidentally we correct the expression originally presented [6, 7] for the gradual case. Interestingly the expressions for the extensions of the Lévy presented here all have the power law form $\sigma_{\Delta t}^2 = v\Delta t^{2/\alpha}$, where v is a constant describing the “quasi-stable processes” [8] that emerge from the truncation parameters for some interval $\Delta t_1 \leq \Delta t \leq \Delta t_2$.

The need for a novel methodology of truncation comes from the previous experience of some of the authors [8] in analyzing currency data. Their distributions were found to deviate from the Lévy in a smooth and gradual fashion after $|z| > l_c$. What is more, the deviations were caught increasing. And this contrasts with earlier suggestions [6, 7] that the deviations should decrease. It has been argued [6, 7] that, due to the limited physical capacity of the system, the feedback begins to decrease exponentially (and not abruptly) after a certain critical step size. However here we argue that an abrupt truncation is still necessary in the presence of increasing deviations.

Indeed data for selected exchange rates against the US dollar (Fig. 2) showed increasing log differences $\ln \hat{P}(z_s) - \ln \hat{L}(z_1)$. This is suggestive that the earlier truncation procedures might not be appropriate.

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Table 1
Description of data sets

Country	Currency	Time Period	Data Points
Australia	Australian Dollar	4Jan71 – 10Jan03	8025
Brazil	Real	2Jan95 – 10Jan03	2014
Britain	Pound	4Jan71 – 10Jan03	8032
Canada	Canadian Dollar	4Jan71 – 10Jan03	8038
Denmark	Krone	4Jan71 – 10Jan03	8031
Hong Kong	Hong Kong Dollar	2Jan81 – 10Jan03	5532
Japan	Yen	4Jan71 – 10Jan03	8026
Mexico	Peso	8Nov93 – 10Jan03	2300
New Zealand	New Zealand Dollar	4Jan71 – 10Jan03	8016
Norway	Krone	4Jan71 – 10Jan03	8031
Singapore	Singapore Dollar	2Jan81 – 10Jan03	5531
South Korea	Won	13Apr81 – 10Jan03	5416
Sweden	Krona	4Jan71 – 10Jan03	8031
Switzerland	Swiss Franc	4Jan71 – 10Jan03	8032
Taiwan	Taiwan Dollar	30Oct83 – 10Jan03	4548

Table 2

Parameter estimates for the currencies in Table 1 using Eq. (19). Estimates $\hat{\alpha}$ and $\hat{\gamma}$ are obtained by the maximum likelihood method for $\Delta t = 1$ from the STABLE computer program available at <http://academic2.american.edu/~jpnolan/>. Estimates $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$, and \hat{g} are nonlinear least square estimates for the truncation parameters of the function using the SAS system (<http://www.sas.com>); and l_c and l_{max} are empirically found from our data.

Country	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	\hat{g}	l_c	l_{max}
Australia	1.4753	0.00180229	8.3372	0.0000	0.00000	2.2366	0.0000	0.00000	0.00428	0.0000	0.27
Brazil	0.5960	0.00157227	3.3317	0.0000	0.00000	1.0683	0.0000	0.00000	0.00277	0.0000	0.27
Britain	1.5513	0.00520366	1.0086	-3.7788	0.00000	-0.0276	2.4653	0.00000	0.00000	0.0050	0.09
Canada	1.5599	0.00181284	2.8675	-2.4219	0.00000	0.8203	3.3156	0.00000	0.00000	0.0035	0.04
Denmark	1.5117	0.02321810	4.7664	-3.4188	0.00000	0.8453	2.4294	0.00000	0.00000	0.0350	1.10
Hong Kong	0.7600	0.00085047	4.0242	-1.5663	0.00000	0.8316	4.6757	0.00000	0.00000	0.0000	0.70
Japan	1.4341	0.49351300	1.3605	0.0000	0.00000	1.9274	0.0000	0.00000	0.33850	0.7500	30.00
Mexico	1.4017	0.02356360	5.0413	-1.5195	0.00000	1.4748	2.7723	0.00000	0.00000	0.0350	1.30
New Zealand	1.4061	0.00154540	5.0986	-3.3148	0.00000	0.7782	4.4101	0.00000	0.00000	0.0000	0.22
Norway	1.5180	0.02001750	4.1884	-2.0023	0.00000	1.0801	1.3332	0.00000	0.00000	0.0350	0.50
Singapore	1.4696	0.00270447	5.4180	-0.7819	0.00000	1.4317	1.4658	0.00000	0.00000	0.0055	0.08
South Korea	0.6394	0.33273200	2.6583	0.0000	0.00000	1.1408	0.0000	0.00000	0.44520	0.0000	400.00
Sweden	1.4062	0.01853350	4.7843	-2.9571	0.00000	1.0297	2.3466	0.00000	0.00000	0.0250	1.10
Switzerland	1.5984	0.00717332	4.3446	-1.8323	0.00000	1.1556	3.2261	0.00000	0.00000	0.0125	0.17
Taiwan	1.0052	0.01627530	3.6706	0.0000	0.00000	1.6571	0.00000	0.00000	0.05930	0.0000	1.70

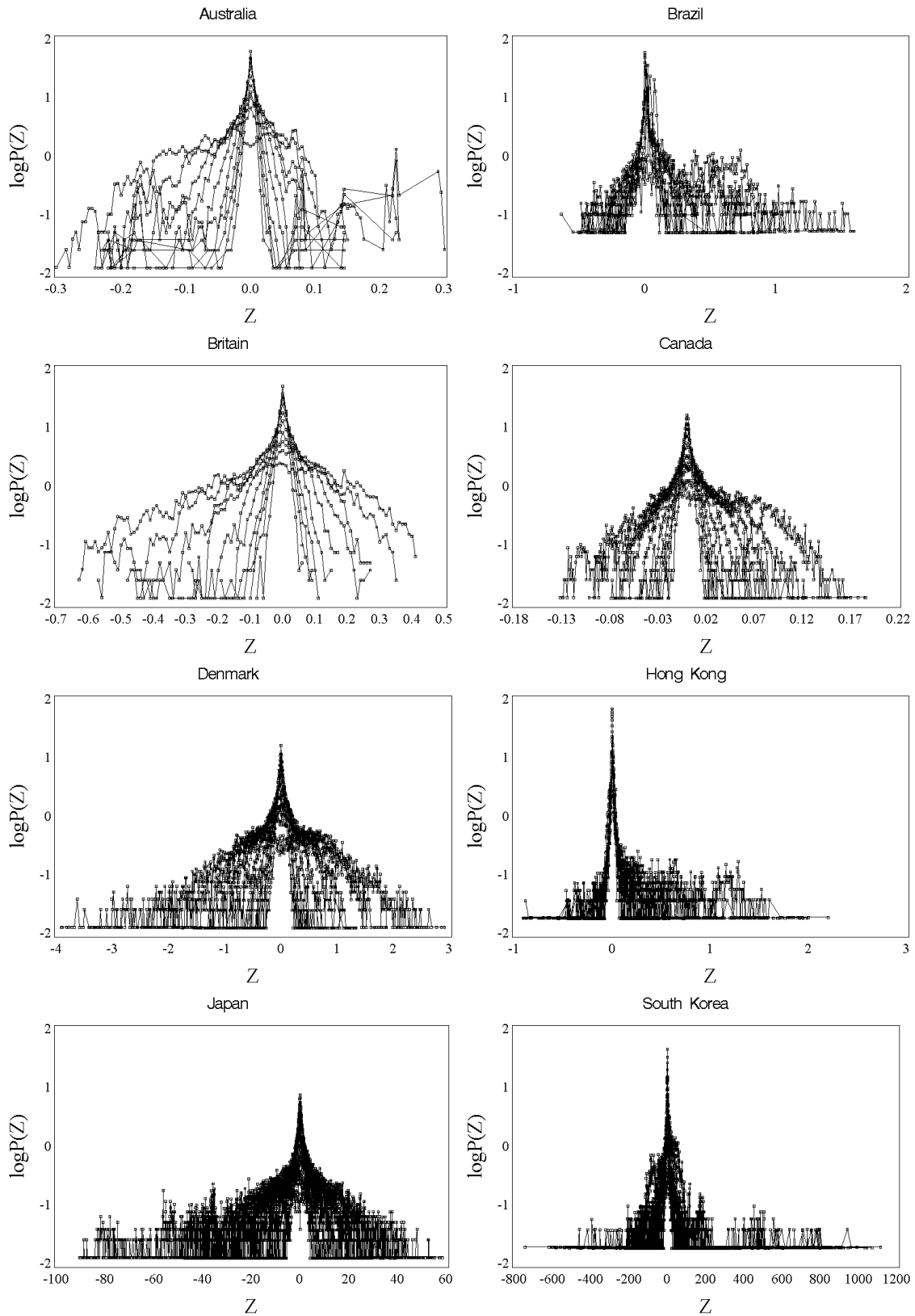


Fig. 1. Probability density functions of currency returns of selected countries in Table 1 observed at time intervals Δt , which range from 1 to 240 trading days. As Δt is increased, a spreading of the probability distribution characteristic of any random walk is observed.

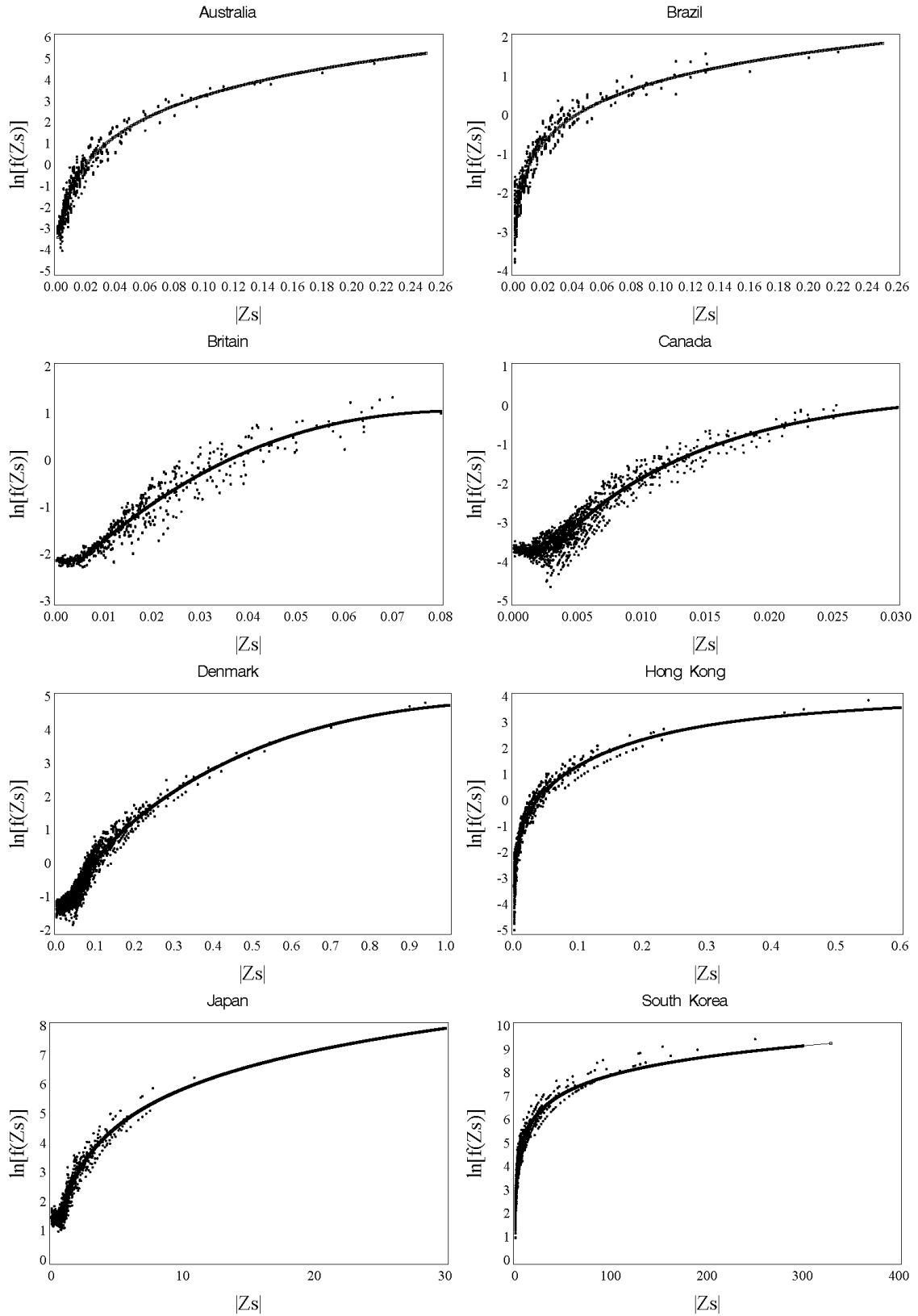


Fig. 2. Log of the differences in Eq. (16) showing how the observed log PDFs of currency returns in Fig. 1 deviate from the original log Lévy process. The continuous lines represent the fittings using Eq. (16) and $z_s = \Delta t^{-1/\hat{\alpha}} z_{\Delta t}$.

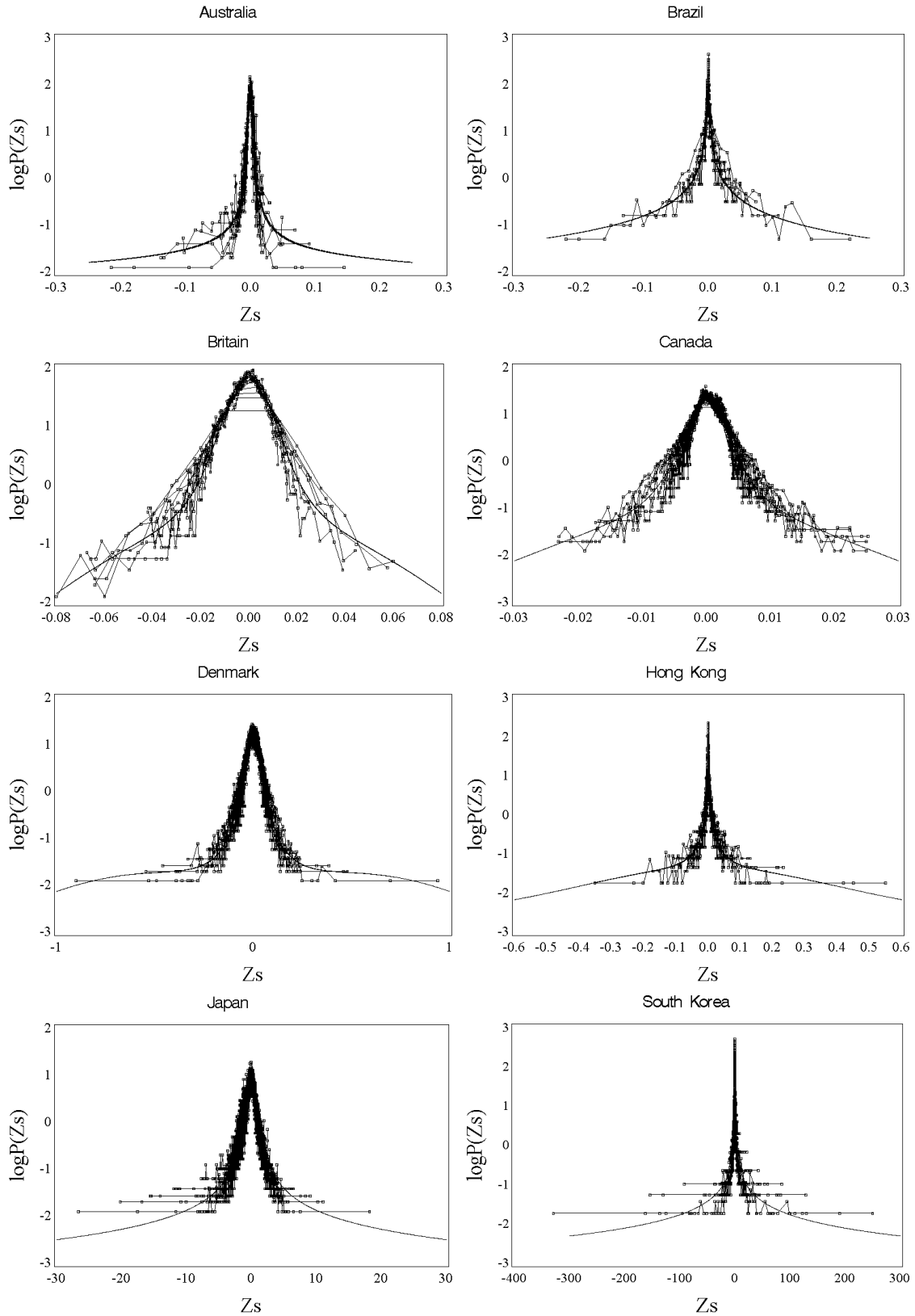


Fig. 3. The same PDFs as in Fig. 1 but now plotted in scaled units $P(Z_s)$, where $z_s = \Delta t^{-1/\hat{\alpha}} z_{\Delta t}$. Given the scaling index α for a currency, the data are made to collapse onto a $\Delta t = 1$ distribution. The curves are our suggested exponentially damped Lévy flights estimated from the data.

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