Long Memory Options: Valuation

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Abstract

This paper graphically demonstrates the significant impact of the observed financial market persistence, i.e., long term memory or dependence, on European option valuation. Many empirical researchers have observed non-Fickian degrees of persistence or long memory in the financial markets different from the Fickian neutral independence (i.i.d.) of the returns innovations assumption of Black-Scholes' geometric Brownian motion assumption. Moreover, Elliott and van der Hoek (2003) have now also provided a theoretical framework for incorporating these findings in the Black-Scholes risk-neutral valuation framework. This paper provides the first graphical demonstration why and how such long term memory phenomena change European option values and provides thereby a basis for informed long term memory arbitrage. Risk-neutral valuation is equivalent to valuation by real world probabilities. By using a mono-fractional Brownian motion, it is easy to incorporate the various degrees of persistence into the binomial and Black-Scholes pricing formulas. Long memory options are of considerable importance in Corporate remuneration packages, since warrants are written on a company's own shares for long expiration periods. Therefore, we recommend that for a proper valuation of such warrants, the degrees of persistence of the companies' share markets are measured and properly incorporated in the warrant valuation.

1 Introduction

In the past decade interest in the phenomenon of "Long Memory" in financial market time series has drastically increased due to the availability of better measurement methodologies and their more precise empirical measurements (Taqqu, 1986; Robinson, 1994; and Baillie, 1996). Unfortunately, the studies of Fractal Brownian motion (FBM) with the LM property had faded out from financial literature after the FBM was proven to be an inappropriate process for financial asset prices due to its possible arbitrage opportunities. Nevertheless, Calvet and

Fisher (2002) has recently shown that the compound process between the FBM and trading time can be used to construct Multi-Fractal process that seems to be superior to the Geometric Brownian Motion (GBM) and GARCH processes. Since the properties of the FBM are transferred to the Multi-Fractal process via the compounding procedure, the investigation of the LM effects on financial asset prices following the FBM is essential for further development of multi-fractal modeling in finance.

Long Memory (LM) is sometimes also referred to as "global dependence," "strong dependence," or "persistence." In LM financial series, autocovariances are not summable and spectral densities are unbounded.[1] These mathematical properties lead to distinctive asymptotic behavior of statistics of financial interest, like volatility or "risk."

The non-Fickian scaling of the volatility of the financial market rates of return is now frequently corroborated, thereby falsifying the Fickian volatility scaling assumption of i.i.d. log price innovations of the Black-Scholes (1973) option pricing formula. The empirical results corroborate the prevalent existence of LM in various financial markets, both in its persistent and anti-persistent versions. Therefore, this article proposes the incorporation of these non-Fickian volatility scaling results in the option valuation literature. Black and Scholes (1972) were aware of the problem of the degree of market efficiency or persistence, but had no proper test for it and therefore maintained their assumption of independent (martingale) innovations. Black and Scholes (1973) were also aware of the fact that their second important maintained assumption of constant instantaneous return volatility was invalid for the valuation of options, but wanted to keep their valuation approach simple.

This paper presents the implications of LM, non-neutral or non-Fickian persistence as a second-order (autocovariance or spectral) property of financial market pricing series for European option valuation.[2] This LM correction of the existing option pricing literature is of considerable importance for the valuation of corporative remuneration packages. These packages contain often options written on the companies own stock, These options are long term, *i.e.*, have long expiration times and their values are impacted by the degree of LM or persistence inherent in the market of the company's shares or, in case such shares

[1] Cf. Robinson, 2003, Chapter 1, pp. 4 - 32 "Long-Memory Time Series" for a detailed definition of the long memory phenomenon and a comprehensive overview of the statistical time series literature.

[2] These second-order property does not completely describe non-Gaussian stable processes, which need higher order moments, like skewness and kurtosis or "shape stability," for a complete description. In these non-Gaussian stable processes the long term "memory" also occurs in these higher order moments. It has been noted, for example, that asset returns $x(t)$ frequently exhibit little autocorrelation (= linear dependence), as is consistent with the efficient market hypothesis, whereas $x^2(t)$ are noticeably correlated, indicating nonlinear dependence of the asset returns $x(t)$. This phenomenon has been modeled by Engle's (1982) ARCH(p) models and Bollerslev's GARCH(p, q) models. However, these models imply that the autocorrelations of $x^2(t)$ either eventually cut off completely or decay exponentially. But empirical evidence shows slower decay consistent with LM (Ding, Granger, Engle, 1993; Ding and Granger, 1996). There is also not yet a rigorous asymptotic distribution theory for the more general ARCH(p) and GARCH(p, q) models for $p > 1$ and $q > 1$.

are not traded, of the market of a comparable company, of which the shares are traded. Thus proper valuation of these options requires the measurement of the degree of persistence of the underlying price formation.

Options can be binomially priced by using (a) real world probabilities or (b) risk-neutral probabilities. Both methods are equivalent, as can be easily demonstrated. Consequently, the exact shape or form of the real world distribution is immaterial, as long as independence and stationarity of the instantaneous return shocks are assumed, and, consequently, the expiration-time-adjusted variance of the stock returns remains constant, as Black and Scholes (1973) assumed.

But the option valuation turns out quite differently when the Black-Scholes Fickian i.i.d. assumption for the instantaneous return innovations is not true and the financial markets show non-neutral, non-Fickian persistence. However, even then the risk-neutral option valuation method can still be used, since the only variable of any significance is the, now non-Fickian, time-scaling volatility. This empirically observed non-Fickian time-scaling volatility can now be incorporated in the risk-neutral probabilities following the theoretical analysis of fractal white noise rate of return innovations in the financial markets by Elliott and van der Hoek (2003) and by using various LM simulation approaches.[3]

The scaling binomial distribution with a non-Fickian mono-fractal memory, persistence, or Hurst exponent may no longer converge to a scaling Gaussian distribution, but converges to a stable, scaling, non-Gaussian distribution. Both scaling distributions are stable distributions in the sense that their shape is immutable, although their distribution sizes scales over time. A more troubling aspect is that for some small values of the memory exponent, these accumulated distributions are no longer probability distributions, since their right tails veer outside the $[0, 1]$ range, leading to the possibility of intermittence and turbulence in the log pricing sequence.

The following three Sections set the stage for LM option valuation. Section 2 of this paper surveys the basic concepts of fractional log price diffusion and non-Frickian volatility scaling, the statistical measurement and testing theory, as well as the most salient empirical values in stock, bond and foreign exchange markets for the long memory exponent. This section demonstrates that the proven existence of LM in those markets is becoming a very relevant topic for option valuation. Then, following McDonald (2003), Section 3 discusses the binomial pricing of options by using both real world probabilities and risk-neutral probabilities, to show that both approaches are equivalent and that the measurement of the volatility scaling is relevant for proper option valuation. In Section 4 we incorporate the LM assumption in the Black-Scholes pricing model and compare it with the, now classical, neutral memory Black-Scholes pricing

[3]Strictly, Elliott and Van der Hoek (2003) go beyond the simple mono-fractional Brownian motions discussed in this paper and deliver their proof for the even more general multi-fractional Brownian motions. In this paper we discuss only mono-fractional Brownian motions for reasons of pedagogical exposition. This paper presents the implications of LM or non-neutral persistence as a second-order (autocovariance or spectral) property of financial market pricing series for European option valuation. Moreover, a multifractal white noise spectrum would logically lead to a spectrum of option prices and contradict the fundamental economic Law of One Price. Such a fundamental contradiction is the subject of another paper.

model.

Section 5 forms the bread and butter of this paper by graphically showing the impact of various degrees of LM or persistence on out-of-the-money (OTM) options, at-the-money (ATM) options, and in-the-money options, both on call and put values. Most corporate warrants are priced as ATM options. In Section 6 we summarize the discussions, formulate some conclusions and recommendations for empirical option valuation and point out some pitfalls of LM for such valuation.

In a companion paper, we discuss the consequences of LM for secondary issues, like for delta hedging, market-making and risk arbitrage.

2 Empirical Measurement of Long Memory

In the past two decades the mainstream econometric time series literature has demonstrated considerable interest in LM in its focus on unit root series. Unit root models assume a known degree of memory in the integer (unit) order of differencing, which reduces a series to short memory stationarity and invertability. But more recently a greater willingness as been shown by empirical financial modelers to consider the more flexible, fractional differencing models, which arose from considerations of self-similarity over time and frequencies (For extensive surveys, *cf.* Los, 2003, and Robinson, 2003).

A continuous time stochastic return process $\{x(t); -\infty < t < \infty\}$ is self-similar with a Hurst or memory exponent $H \in (0, 1)$ introduced by Hurst (1951) if for any $a > 0, \{x(at); -\infty < t < \infty\}$ has the same time and frequency distribution as the process $\{a^H x(t); -\infty < t < \infty\}$.

If the first differences $\Delta x(t) = x(t) - x(t-1) = \varepsilon(t)$, for integer time t, are covariance stationary, their autocorrelation function (ACF) is

$$\gamma(\tau) = \frac{\gamma(0)}{2}[|\tau + 1|^{2H} - 2|\tau|^{2H} + |\tau - 1|^{2H} \tag{1}$$

It is easy to show that as the time horizon $\tau \to \infty$, this ACF decays over time like

$$\gamma(\tau) \sim \sigma_{\varepsilon}^2 \tau^{2H-2} \tag{2}$$

The formula for the corresponding spectral density, i.e. the Fourier transform of this ACF, can, for example, be found in Sinai (1976) and Los (2003) and satisfies also a power law, with radian frequency $\omega = \frac{2\pi}{\tau}$:

$$P(\omega) = \sigma_{\varepsilon}^2 \omega^{-(2H-1)} \tag{3}$$

An example of such price diffusion is the fractional Brownian Motion for stock market prices $S(t)$. Such "fractional" log price diffusion has been extensively studied by Mandelbrot and Van Ness (1968), Granger and Joyeux (1980), Hosking (1981) and Sowell (1990).

Definition 1 ***Fractional Brownian Motion (FBM)*** *is defined by the fractionally differenced time series*

$$(1-L)^d x(t) = \varepsilon(t),\ d \in (-\frac{1}{2}, \frac{1}{2}),\ \textit{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{4}$$

where $x(t) = \Delta \ln S(t) = (1-L)\ln S(t)$*, so that* $x(t)$ *are the rates of return, L is the lag operator, and* $d = H - \frac{1}{2}$ *is the fractional differencing exponent.*

A completely equivalent definition of the FBM is that $x(t)$ is fractionally integrated white noise, since, by inversion

$$x(t) = (1-L)^{-d}\varepsilon(t),\ d \in (-\frac{1}{2}, \frac{1}{2}),\ \text{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{5}$$

Alternative forms of the FBM can be found in Marinucci and Robinson (1999). This self-similar generalization of the geometric Brownian motion has also been called an α_Z-stable Lévy Motion, and has been the subject of two monographs by Samorodnitsky and Taqqu (1994) and by Janicki and Weron (1994). Such a fractional motion is "almost certain" (a.c.) dense with discontinuities ("singularities"), in contrast to the geometric Brownian motion, which is a.c. everywhere continuous. The discontinuities in an $\alpha-$stable Levy Motion imply that there may be occasionally a drawdown in the stock market with $0.5 < H < 1$ faster than trades can be executed, as occurred in October 1929 and, more recently, in October 1987 (McCulloch, 1996, p. 397), so that program hedging breaks down, which should be of considerable concern for banks and insurance companies.

The corresponding spectral density is obtained by the Fourier Transform of this integration FBM (Adenstedt, 1974):

$$F_{FBM}(\omega) = (1 - e^{-j\omega})^{-d}\mathcal{F}[\varepsilon(t)] \tag{6}$$

Next, by apply the two exponential series expansions for $e^{j\omega}$ and $e^{-j\omega}$, with $j = \sqrt{-1}$, the imaginary number and ω is the radian frequency, and take the limit for $\omega \to 0$ (or $\tau \to \infty$), we obtain the aforementioned power law for the spectral density from which the differencing exponent d can be identified by measuring the slope of $logP(\omega)$ versus the frequency ω.

Thus these fractional differencing models allow for the memory Hurst exponent, which is $H = d + \frac{1}{2}$,to be fractional, unknown, and identifiable from the noisy financial data. It has been argued that statistical inferences and financial modeling based on an incorrect order of differencing are liable to be invalid and may lead to misleading pricing conclusions. For such a discussion of the (asymptotic) statistical behavior of the various parametric statistics to identify H, see Robinson (2003, pp. 4 - 25). However, the differencing rule should be simple: differentiate by integer numbers, until the residual series has a differencing exponent $d \in (-\frac{1}{2}, \frac{1}{2})$.

Thus the current approach is to compute the H exponent from the negative slope coefficient $(2H-1)$ of the log periodogram, *i.e.*, the data set based spectral density. The crucial test for financial analysts is the test of the existence of LM. Or, more precisely, the test of the hypothesis of neutral short memory ($H_0 : H = 0.5$, or $d = 0$) against the two alternative hypotheses of LM, persistence ($H_1 : 0.5 < H < 1.0$, or $0 < d < 0.5$) and anti-persistence ($H_2 : 0 < H < 0.5$, or $-0.5 < d < 0$).

The connection of this fractional time series literature with the stable frequency distribution literature is direct (McCulloch, 1996): for $0 < H < 1$, the characteristic or shape (Zolotarev) exponent of a stable distribution $\alpha_Z = \frac{1}{H}$. Thus for $H = 0.5$, $\alpha_Z = 2$ and the stable distribution is Gaussian. For $0.5 < H < 1$, we have $2 > \alpha_Z > 1$, and the variance of the distribution is non-existing or "infinite," due to a long upper "Paretian" tail. This is, for example, the case for the empirical S&P Index since its measured $H = 2/3$, so that its shape exponent $\alpha_Z = 3/2$. This strongly suggests that the S&P500 Index is statistically unsuitable as an "underlying" index for option valuation.

For the Geometric Brownian (GBM) motion $x(t)$ of Black and Scholes (1973), the log prices $\ln S(t)$ themselves have a memory exponent of $1+H = 1.5$, so that their $\alpha_Z = 2/3$, and, indeed, their mean does not exist, *i.e.*, is nonconvergent.[4] One differentiation of the GBM provides the log errors , which have a memory exponent of $H = 0.5$, or $\alpha_Z = 2$, so that their mean exists $E\{\varepsilon(t)\} = 0$ as well as their variance and their distribution is equivalent to a Gaussian $N(0, \sigma_\varepsilon^2)$ in their first two moments.

When $0 < H < 0.5$ and $\infty > \alpha_Z > 2$, the resulting "distribution" is not a proper probability distribution, since one or both tails will then lie outside $[0, 1]$. This is the case, for example, in some foreign exchange markets, where it is found that $H = 0.25$, so that $\alpha_Z = 4$. Thus also options written in these anti-persistent markets do not make much pricing sense when the usual risk-neutral valuation is applied.

Statisticians have concentrated on finding a test statistic for $H = d + \frac{1}{2}$ with a limiting distribution that can be easily computed and that has good power for hypothesis testing. Based on some "slightly defective heuristics" (Robinson, 2003, p. 14) Geweke and Porter-Hudak (1983) argued that, asymptotically, the distribution of the log-periodogram regression estimate of d satisfies

$$\tau^{0.5}(\widehat{d} - d) \rightarrow^d N(0, \frac{\pi^2}{24}) = N(0, 0.41123) \tag{7}$$

giving rise to extremely simple inferential procedures. By employing a linear process for $x(t)$ based on martingale difference innovations, Robinson (1995a & b) rigorously and correctly established a more precise result based on a slightly different estimator for the whole range $-0.5 < d < 0.5$ or $0 < H < 1$:

$$\tau^{0.5}(\widehat{d} - d) \rightarrow^d N(0, \frac{1}{4}) \tag{8}$$

This result provides simple asymptotic interval estimates as well as a simple test of neutrality, $d = 0$ or $H = 0.5$. Robinson's treatment, based on the

[4] While $\alpha_Z = 2/3$ is a value for brown noise, a lower value than the black noise value of $\alpha_Z = 2/5$ has not been empirically found.

theoretical assumption of Gaussianness,[5] actually covered multiple time series, possibly differing memory parameters (= multifractality), and more efficient tests for equality of these parameters.

Additional asymptotic statistical theory for the log periodogram computation of d or H has also been provided by Velasco (2000), extending it to linear processes, and by Hurvich, Deo and Brodsky (1998), who proposed which bandwidths τ to select for inclusion in the d statistic based on the data characteristics. Such a bandwidth selection is required for periodograms based on the Fourier transform, which uses sinus and cosinus bases with infinite support, causing problems of overlapping ("double-counting" or "smearing"). However, periodograms based on the wavelet transform, which uses non-overlapping orthogonal wavelet bases with finite support don't require any "selection," since no "double-counting" occurs and the analysis is complete: all empirically observed bandwidths are properly included. Scalegrams (= periodograms based on a wavelet transform) provide a complete analysis complete.

Hurvich and Ray (1995) showed that the log-periodogram procedure also works for nonstationary or noninvertible fractional differencing processes. Moulines and Soulier (1999, 2000), Hurvich and Deo (1999) and Hurvich and Brodsky (2001), who used explicitly the fractional differencing process, extended the statistical measurement results over all frequencies. Thus there are now sufficient asymptotic distributional results for statistical testing of the LM hypothesis in both its persistent and anti-persistent ranges, against the null hypothesis of memory neutrality of the Geometric Brownian Motion, which was assumed by Black-Scholes (1973) for their option valuation procedures.

To emphasize: the relevance of these theoretical statistical discussions about asymptotic results is questionable, since in practice the distribution of H is very local. Moreover, the completeness of the wavelet multiresolution analysis (MRA) or scalogram drastically reduces the scientific relevance of the concept of statistical "sampling" on which these discussions are based. Flandrin and Gonçalves (1996) studied the theoretical time-frequency distributions of affine processes using wavelet MRA.

Notwithstanding this controversy about the scientific relevance of asymptotic statistical sampling theory for empirical (= finite data set based) analysis, there are already sufficient empirical measurement results to prove the existence of LM and of non-Fickian volatility scaling in the financial markets for a wide range of $H = d + \frac{1}{2}$ values.

Harvey and Whaley (1991, 1992) and Dumas, Fleming and Whaley (1998) are looking for parametric functions in the implied volatility of the S&P500 Index returns. But their approach is spanning the cart before the horse, since their implied volatilities are derived from the Black-Scholes option valuation formula based on the now falsified i.i.d. assumption. A similar mistake was made by Xu and Taylor (1994) in the foreign exchange markets, although their analysis confirmed that the implied *and* historical return volatility was definitely

[5] This assumption of Gausianness begs the scientific question - why? - since we know that the underlying is not Gaussian.

not constant.

Peters (1994) and Cizeau *et al.* (1997) correctly observe and measure that there exists non-Fickian volatility scaling in the S&P500 stock index returns with a persistent $H = 0.67$, which should be incorporated in the Black-Scholes formula.[6] The S&P500 Index underlies important futures and options valued and traded in Chicago. Cont, Potters, and Bouchaud (1997), Gopikrishnan *et al.* (1998) and Lo and MacKinlay (1999) also observe non-Fickian volatility scaling in stock markets return. Ramsey, Usikov and Zaslavsky (1995) were the first to perform the analysis of stock market volatility using wavelets MRA. Batten, Ellis and Mellor (1999), Batten and Ellis (1999) and Batten, Ellis and Hogan (1999) observed non-Fickian volatility scaling in Australian Dollar Eurobond and in foreign exchange market returns using more traditional scaled variance measurements.

A seminal comprehensive study of volatility scaling in several financial markets using high frequency data is Müller *et al.* (1995). Müller *et al.* (1990) were early observers of non-Fickian persistent volatility scaling in the smaller pre-Euro foreign exchange markets. But Karuppiah and Los (2004) found non-Fickian anti-persistent volatility scaling of $H \approx \frac{1}{3}$ in the ultra-liquid anchor currencies of the Japanese Yen/US dollar and (formerly) German Deutschemark/US dollar foreign exchange markets before and after the Asian Financial Crisis in 1997. This surprising anti-persistence value suggests that turbulence is, indeed, possible in foreign exchange markets, as already suggested by Ghasgaie *et al.* (1996) and Mantegna and Stanley (1996), and it inspired the theoretical study by Elliott and Van der Hoek (2003; originally presented in 2000). Similar volatility scaling behavior in foreign exchange and stock markets has also found by Gençay, Selçuk and Whitcher (2001) , Kyaw, Los, and Zong (2003), and Lipka and Los (2003) using wavelet MRA. In other words, replication research by complete MRA has now corroborated those initial statistical findings.

More LM financial market results can be found in the compilations of articles by Kondor and Kertesz (1999) and Robinson (2003), and in the monographs by Peters (1994), Mantegna and Stanley (2000) and Los (2003). The existence of these empirical results now warrants a serious correction of literature on the option valuation.

However, one is forewarned that in the fast growing interdisciplinary literature on LM time series various notational systems are used, sometimes with the same symbols meaning different concepts. The best is to accept one notational system and to translate all other concepts into it, as is done in Table 4.3 in Los, 2003, p. 124 which provided the equivalence of various critical irregularity exponents, such as the dependence, difference, spectral, Hurst, Zolotarev stability and Lipschitz exponents. In this paper we use the Hurst exponent H, since that has become an accepted LM parameter in the finance literature, even though the Lipschitz exponent is the most universally accepted in the mathematics and

[6] Interestingly, the Dow Jones Industrial Average index is exceptionally market neutral and have a Fickian Hurst exponent of $H = 0.5$ (Li, 1991). This is like the neutral memory of the River Rhine in Europe, which represents the similar exception to the rule that long rivers are persistent (Mandelbrot and Wallis, 1969).

physics literatures.

These critical exponents have to be identified or computed from the available "noise" data in the financial markets to determine if these markets are anti-persistent, neutral or persistent. This was pursued, for example, by Beran (1989, 1992, 1994) by conventional covariance stationary time series analysis with constant integer correlations (= linear dependencies) and frequencies and by Flandrin (1989, 1992) and Kaplan and Kuo (1993) by the complete, and therefore superior, wavelet MRA, which can be applied to non-stationary time series with time-varying fractional nonlinear dependencies and frequencies.

3 Binomial Pricing of Options

3.1 Pricing Options by Using Real World Probabilities

In the following two sub-sections we closely follow McDonald, 2003, pp. 337 - 338 and 358 - 359). Pricing options using real world probabilities is the conventional financial discounting-of-expected-cash-flows method. In fact, for option pricing the only distributional moment of importance is the second moment. All higher order moments are not considered by the Black-Scholes valuation method.

Suppose we have a non-dividend paying stock S_0 with an expected rate of return α. Then if p is the real world probability of the stock going up, this p must be consistent with the uptick u, the downtick d, and the expected stock return α according to the following expectation:

$$\begin{aligned} E_0\{S_1\} &= puS_0 + (1-p)dS_0 \\ &= e^{\alpha h}S_0 \end{aligned} \tag{9}$$

Solving for the real world probability p gives

$$p = \frac{e^{\alpha h} - d}{u - d} \tag{10}$$

with $u > e^{\alpha h} > d$. Thus, the actual expected payoff to the option one period hence is

$$E_0\{C_1\} = pC_u + (1-p)C_d \tag{11}$$

$$= \frac{e^{\alpha h} - d}{u - d}C_u + \frac{u - e^{\alpha h}}{u - d}C_d \tag{12}$$

The option is a leveraged investment in the stock and is thus riskier than the stock. Consequently, it must be discounted at an expected rate $\gamma > \alpha$. Replication theory tells us that an option is equivalent to a portfolio consisting of Δ shares of stock S_0 and B_0 bonds:

$$C_0 = \Delta S_0 + B_0 \tag{13}$$

The expected return on such a replicating portfolio is the weighted average:

$$e^{\gamma h} = \frac{\Delta S_0}{\Delta S_0 + B_0}e^{\alpha h} + \frac{B_0}{\Delta S_0 + B_0}e^{rh} \tag{14}$$

The option price is then the properly discounted payoff

$$C_0 = e^{-\gamma h}\left[\frac{e^{\alpha h}-d}{u-d}C_u + \frac{u-e^{\alpha h}}{u-d}C_d\right] \tag{15}$$

This formula using real world probabilities gives the same option price as the risk-neutral probability valuation, as we'll now show in the next section.

3.2 Pricing Options by Using Risk-Neutral Probabilities

Risk-neutral valuation states that the risk-neutrally expected payoff to the option one period hence is:

$$\begin{aligned} E_0^*\{C_1\} &= p^*C_u + (1-p^*)C_d \\ &= \frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d \\ &= e^{rh}\left[\frac{C_u-C_d}{u-d} + e^{-rh}\frac{uC_d-dC_u}{u-d}\right] \\ &= e^{rh}[\Delta S_0 + B_0] \\ &= e^{rh}C_0 \end{aligned} \tag{16}$$

by the definition of $\Delta = \frac{C_u-C_d}{S_0(u-d)}$ and $B_0 = e^{-rh}\frac{uC_d-dC_u}{u-d}$. This can also be written as

$$C_0 = e^{-rh}E_0^*\{C_1\} \tag{17}$$

This equation defines risk-neutral valuation (Los, 2001, Chapters 8 and 9; Tavella, 2002, Chapter 3). The Δ and B_0 values are the solutions of the two successfully replicating portfolio equations:

$$\Delta uS_0 + e^{rh}B_0 = C_u \text{ and} \tag{18}$$

$$\Delta dS_0 + e^{rh}B_0 = C_d \tag{19}$$

To check that this approach comes out the same as with real world probabilities, we can now rewrite the option price equation as follows:

$$\begin{aligned} C_0 &= (S\Delta + B)\left(\frac{1}{\Delta S_0 e^{\alpha h} + B_0 e^{rh}}\right)\left[\frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d + \frac{e^{\alpha h}-e^{rh}}{u-d}(C_u-C_d)\right] \\ &= (\Delta S_0 + B_0)\left(\frac{1}{\Delta S_0 e^{\alpha h} + B_0 e^{rh}}\right)\left[e^{rh}[\Delta S_0 + B_0] + (e^{\alpha h}-e^{rh})\Delta S_0\right] \\ &= \Delta S_0 + B_0 \end{aligned} \tag{20}$$

Q.E.D.

4 Long Term Memory and Black-Scholes i.i.d. Assumptions

Since the value of the expected stock return α does not matter in risk-neutral pricing, *any* consistent pair of α and γ will give the same option price. Risk-neutral pricing is thus valuable because setting the expected stock return $\alpha = r$, the risk-free rate, results in the *simplest* pricing procedure.

Black and Scholes (1973) assumed that the instantaneous return innovations followed a simple neutral i.i.d. Wiener process. Now, matters turn out somewhat differently, when the Black-Scholes independence and stationarity (i.i.d.) assumptions are violated.

For the Black-Scholes assumption of a GBM, the rate of stock return is

$$d \ln S_t = \alpha dt + \sigma dz_t,\ dz_T \sim i.i.d.(0, T) \tag{21}$$

and the long term memory or Hurst exponent has the Fickian value of $H = 0.5$ so that

$$Var\left\{\ln(S_T/S_0)\right\} = \sigma^2 T^{2H} = \sigma^2 T \tag{22}$$

Thus, the expiration time-adjusted, instantaneous variance of the rate of return is, indeed, a constant, as assumed by Black and Scholes (1973):

$$\frac{Var\left\{\ln(S_T/S_0)\right\}}{T} = \sigma^2 \tag{23}$$

Consequently, under this i.i.d. assumption, the (Cox, Ross and Rubinstein, 1979) uptick $u = e^{\sigma T^{0.5}}$ and downtick $d = e^{-\sigma T^{0.5}}$ remain constant for a particular expiration time T and the risk-neutral valuation works very well.[7]

For the assumption of a mono-FBM, the rate of stock return is similarly:

$$d \ln S_t = \alpha dt + \sigma dz_t^*,\ dz_T^* \sim i.i.d.(0, T^{2H}) \tag{24}$$

and the long term memory, or Hurst exponent has the non-Fickian value of $0 < H < 1$, $H \neq 0.5$ so that

$$Var\left\{\ln(S_T/S_0)\right\} = \sigma^2 T^{2H} \tag{25}$$

With the existence of LM, the expiration time-adjusted variance of the stock rate of return is not constant, except when $H = 0.5$, since

$$\frac{Var\left\{\ln(S_T/S_0)\right\}}{T} = \sigma^2 T^{2H-1} \tag{26}$$

[7] Notice that the Black-Scholes assumptions do NOT include a specific assumption regarding the shape of the distribution of the stochastic return shocks: i.i.d.= independent, identically distributed (= strict-sense stationarity, although in their methodology wide-sense stationarity suffices). The Gausianness equivalence stems from the neutral memory assumption of $H = 0.5$, which is implied by their not so innocuous i.i.d., "white noise" or "flat spectrum" assumption.

Indeed Holton (1992) called time the second dimension of risk, the first being the instantaneous return variance σ^2. Still, the LM uptick $u = e^{\sigma T^H}$ and downtick $d = e^{-\sigma T^H}$ remain constant for a particular market value of the (scaling) exponent H and a particular expiration time T and thus the risk-neutral binomial valuation can still be used for option valuation. Since the u and d ticks prominently figure in the risk-neutral probabilities, the option prices will differ according to various values of H, indicating the degrees of persistence of trading and pricing in the various financial markets.[8]

The Black-Scholes European option value based on a mono-FBM is as follows. The call option value is:

$$C_0 = S_0 e^{-gT} SD(d_1) - K e^{-rT} SD(d_2) \quad (27)$$

$$d_1 = \frac{\ln(S_0/K) + (r-g)T + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \quad (28)$$

$$d_2 = d_1 - \sigma T^H \quad (29)$$

with $0 < H < 1$.[9]

Why the changed notation from $N(.)$ to $SD(.)$? Strictly speaking, the d_1 variable is no longer a standardized Gaussian variable, since it scales in a non-Gaussian way. Therefore the accumulations $SD(d_1)$ and $SD(d_2)$ do no longer represent cumulative standard Gaussian distributions but cumulative (non-Gaussian) stable distributions. With the standard parametrization, they represent cumulative standard stable (scaling) distributions, *e.g.,* a self-similar Lévy, Cauchy, Beta, Gamma, etc. distribution, something that Mandelbrot (1971) had already observed in early computer generation of LM time series by aggregation:

$$SD(d_i) = \int_{-\infty}^{d_i} f(z^*) dz^* \quad (30)$$

Unfortunately, this "closed form" representation does not hold for most stable distributions, although there exist explicit Zolotarev parametrizations for their characteristic functions (Los, 2003, Chapter 3). This means that most stable distributions can only numerically be integrated by simulation (McCulloch, 1996).

Other simulation approaches of LM time series are based on Cholesky decomposition (Hipel and McLeod, 1978), on fast Fourier transform (Davies and Harte, 1987) and on fast Wavelet transforms Flandrin (1992).[10]

[8] Various financial markets have different H values for $0 < H < 1$. *Cf.* Cornelis A. Los, Financial Market Risk: Measurement & Analysis, Routledge International Studies in Money and Banking, Taylor & Francis Books Ltd, London, UK, 2003, who measures these various H values using wavelet multiresolution analysis.

[9] This is a corrected version of the formula in Los (2003, p. 439).

[10] All four simulation approaches and the various methods to identify H are incorporated in the Benoit 1.3 software package: "Fractal System Analysis for Windows" from Trusoft International Inc., St. Petersburg, FL 33704.

5 LM Options

As demonstrated by our review in Section 2, it has now empirically been established that various financial markets have different persistence exponents or H values for $0 < H < 1$. Fig. 1 shows that when the expiration time, maturity, or investment horizon T increases, the volatility of anchor foreign exchange (FX) rates tends to increase *slower* than that of a GBM, while the volatility of stocks and bonds tends to increase faster than that of a GBM. In popular opinion, in the long run FX markets are considered more risky than stock markets in the long term, while the opposite is actually true.

(1) The FX appreciation rates are usually antipersistent with Hurst exponents of the order $0.2 < H < 0.5$ (Karuppiah and Los, 2004). At Hurst values of $H = \frac{1}{3}$ *financial turbulence* may occur.

(2) Stock and bonds are traded securities. Their rates of return are persistent, with Hurst exponents $0.5 < H < 0.8$, *e.g.,* the S&P500 index has $H = \frac{2}{3}$.

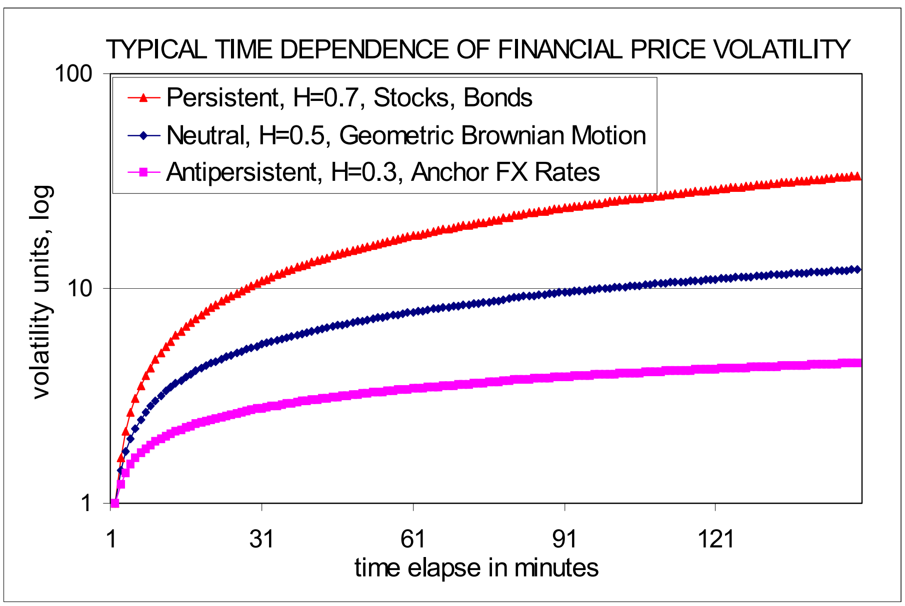

Figure 1: Typical time dependence of financial price volatility, $\log\sigma^2$. The volatility or second moment risk of the persistent stock and bond prices increases faster with the time horizon T than the volatility of the conventional Geometric Brownian Motion, while the volatility of FX rates increases slower

But does LM matter for option valuation? Let's look at the numerical impact of time decay within the whole range of degrees of persistence of a monofractional price diffusion process ($0 < H < 1$) on the values of out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) European call and put options. Thus, we model the price diffusion by a simple mono-FBM and use the corresponding memory (LM) Black-Scholes formula to derive some qualitative and quantitative statements regarding their correct LM pricing, relative to their theoretical neutral memory pricing

For the sake of these examples, at all times we assume a constant instantaneous risk-free rate of $r = 0.06$, an instantaneous volatility of $\sigma = 0.30$, and a strike price of $K = \$60$. For simplicity, we assume a non-dividend-paying stock S (dividend yield $g = 0$), but that assumption can, of course, be relaxed. Moreover, the underlying can be an asset, a commodity, a futures (resulting in a LM Black formula), etc., or a foreign exchange rate.

5.1 LM OTM Call Option

We begin this pictographic story with an long memory out-of-the-money (LM OTM) call option, with the underlying stock price at \$40, while we let the expiration time vary over $T = 1, 2, 5, 10$ and 20 years, for various degrees of persistence $0 < H < 1$, as in Fig. 2.

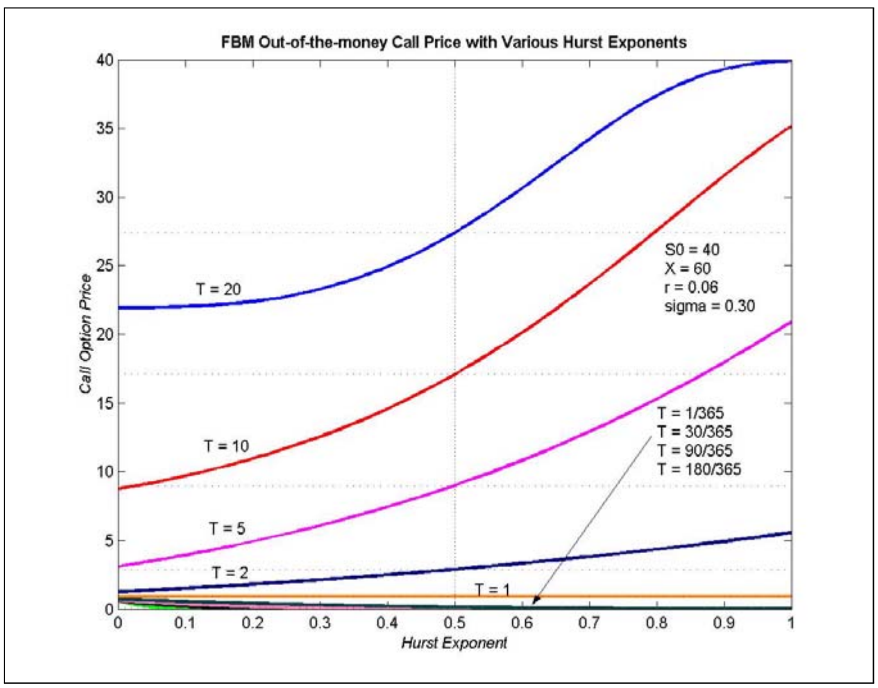

Figure 2:

There is, of course, no exponential effect when $T = 1$ year, when the call price is \$0.90, no matter what the degree of persistence of the market of the underlying stock. But notice that for a 2-year call option, the difference in value of a call option on a stock trading in an ultra-persistent market ($H \uparrow 0$) and in an ultra-anti-persistent (= ultra-fast mean-reversing or ultra-efficient) market would be already ca. \$3.00. That extreme LM difference in call value grows to ca. \$17.00 when the expiration time is $T = 5$ years and ca. \$26.00 when $T = 10$.

Such 10-year options are used in corporative remuneration packages. These options are written on the shares of the own corporation and are called warrants. The question whether these shares are trading in an anti-persistent, neutral or persistent market is thus a very relevant question for a manager, who finds a substantial part of his or her remuneration awarded in the form of warrants. The differences between option values based on the Black-Scholes GBM model and based on the mono-FBM are the largest in the persistent markets. Since most corporate shares are issued by medium-sized companies, these markets are always smaller and less liquid and traded by less traders and thus likely to be much more persistent than the perfect competition model of stock markets suggests. A small or medium sized stock trading on the NASDAQ is likely to show more persistent pricing than a blue chip technology stock trading on the New York Stock Exchange.[11] Notice that for 20-year options (are there any?) this extreme difference has again been reduced to ca. \$18.00, because of the theta (θ) effect.

But what happens with the intra-year option values? Is there still a LM effect? The answer is a resounding "yes," as demonstrated by Fig. 3. This Fig. 3 is the enlarged lower part of Fig. 2, for intra-year expiration times $T = 1/365$ (= 1 day), 30/365 (= 1 month), 90/365 (= three months), and 180/365 (=six months).

But now the LM effect works "in reverse." There is virtually no impact on the one-day options, except in the empirically unobserved area of the blue noise ultra-anti-persistence $H \downarrow 0$. The empirically observed range of the Hurst exponent (thus far) is about $0.2 < H < 0.8$. There is, for example, a \$0.20 90-day call value difference between a $H = 0.2$ anti-persistent (ultra-efficient, since ultra-fast reversing) market and a slightly persistent (= slightly inefficient) market. This grows to a \$0.30 difference for a 180-day call.

5.2 LM ATM Call Option

The LM effect is clearly less pronounced with the ATM options, as Fig. 4 shows

The call theta based on the first order Taylor expansion, using the stable distributions, is now:

$$\frac{\partial C_0}{\partial T} = -ge^{-gT}S_t SD(d_1) - rKe^{-rT}SD(d_2) - \frac{Ke^{-rT}SD'(d_2)\sigma}{2T^H} \quad (31)$$

$$= -rKe^{-rT}SD(d_2) - \frac{Ke^{-rT}SD'(d_2)\sigma}{2T^H} \text{ for } g = 0 \quad (32)$$

The following figures illustrate the LM effect on the theta values and they show that the theta values differ a lot between anti-persistent and persistent

[11] At Kent State University we are currently empirically researching that same issue.

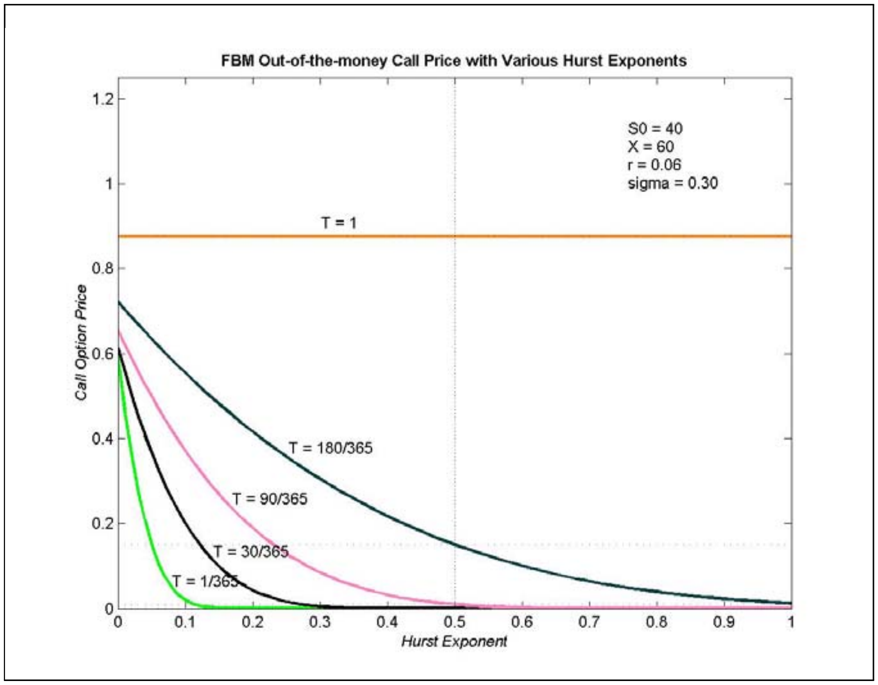

Figure 3:

markets for the one day options, but that those differences fade out when the expiration time or time-to-maturity T increases. Fig. 5 shows compares the theta values of a one-day option with the values of options with longer expiration times for the various degrees of persistence in the particular stock markets.

Fig. 6 shows that when the expiration time T increases, the difference in values of the theta between the stock markets of different degrees of persistence decreases. However, while anti-persistence is boosts theta values for expiration time $T < 90$ days, the reverse is true for $T > 90$ days. For expiration times of greater than five years, the differences between thetas are no longer important, no matter what the degree of persistence in the stock market. All options of various long term maturities have the same almost constant theta.

5.3 LM ITM Call Option

Interestingly, the differences in degrees of persistence are almost non-existent in the anti-persistent $0 < H < 0.5$ stock markets for LM ITM calls. In other words, for LM ITM call options the Black-Scholes GBM formula will provide the (almost) correct value, as is shown in Fig. 7. However, the different degrees of persistence do matter in the persistent $0.5 < H < 1$ stock markets for expiration times $T > 2$ years. For a 10-year LM ITM option at the extreme $H \uparrow 1$ value,

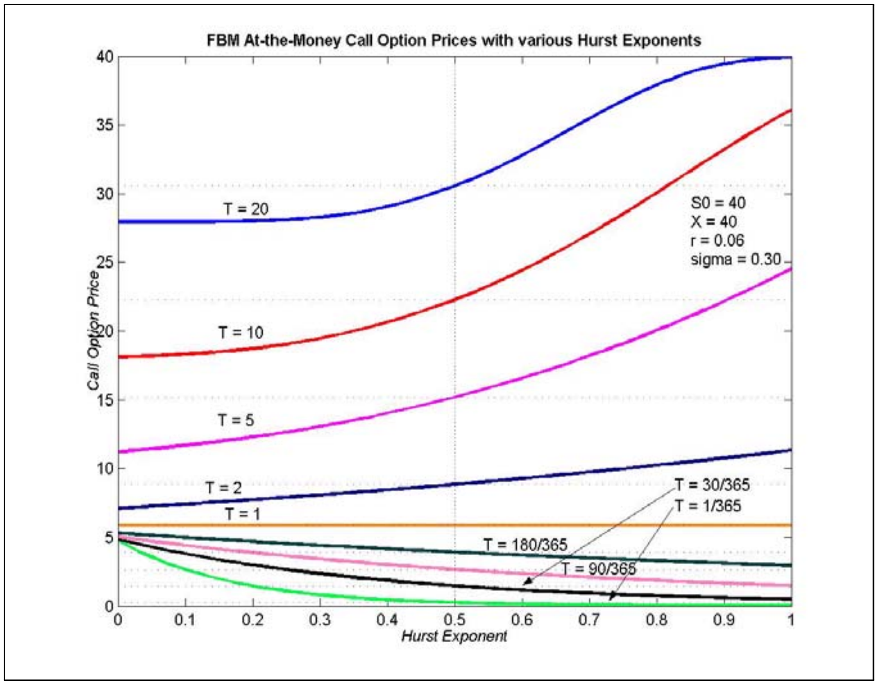

Figure 4:

the value difference compared with the GBM valuation is about \$8.50.

5.4 LM OTM Put Option

Let's first look at the supra-year options. As Fig. 8 shows, the LM effect is important almost exclusively in the persistent stock markets for the OTM puts and is almost unimportant in the anti-persistent markets. This is clearly different from the OTM calls, where the effect was noticeable in both anti-persistent and persistent markets.

Another interesting phenomenon is the difference between the 10-year and 20-year options in the extreme persistence markets where $H \uparrow 1$. This is caused by the deep discounting of the strike price (equivalent to a zero-coupon bond), which imposes a maximum value constraint on the present put option value P_0, since when $S_0 = 0$, the present put value is maximally

$$P_0 = Ke^{-rT} \tag{33}$$

Now, for $T = 10$, $Ke^{-rT} = 20 \times e^{-0.06 \times 10} = \$10.98$, but for $T = 10$, $Ke^{-rT} = 20 \times e^{-0.06 \times 20} = \$6.03$.

What is the intra-year effect on puts for the various degrees of persistence? This is dramatically illustrated in Fig. 9. It only applies to the ultra-anti-

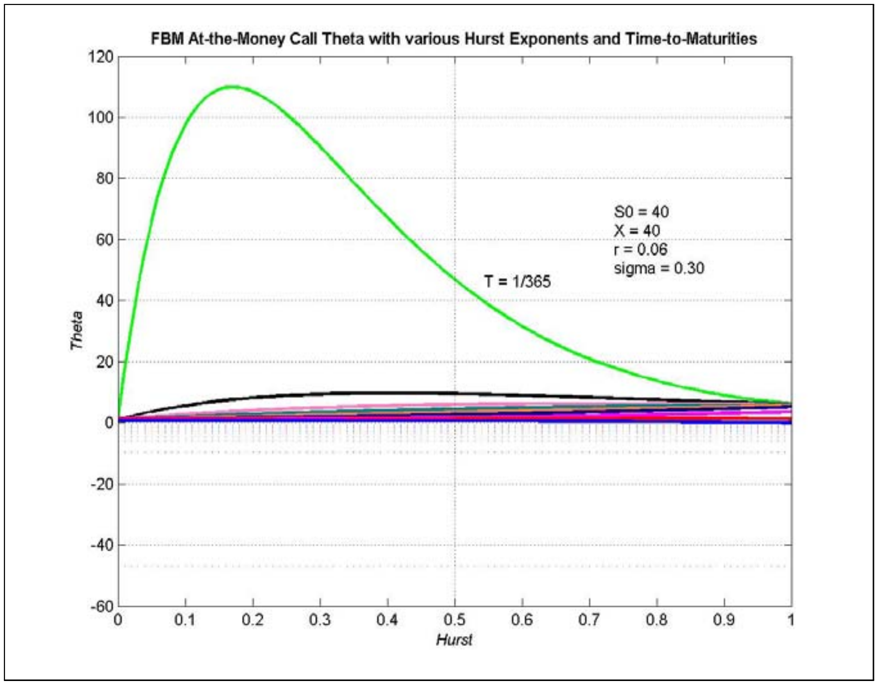

Figure 5:

persistent markets with $0 < H < 0.3$, and is therefore not relevant for empirically observed markets. An interesting dichotomy occurs: for the OTM put valuation the supra-year effects are very important for the empirically relevant LM range of $0.3 < H < 1$ while the intra-year effects are only relevant for the empirically irrelevant $0 < H < 0.3$ anti-persistence range.

5.5 LM ATM Put Option

For the LM ATM put options the degrees of persistence do matter over the whole range of $0 < H < 1$, as can be seen in Fig. 10. In other words, while this LM effect is important for the OTM put options in the persistent markets, but not in the anti-persistent markets, anti-persistence begins to matter for put valuation with the LM ATM put options.

5.6 LM ITM Put Option

For an ITM put option the LM effects are rather surprising, as observed in Fig. 11. First, all intra-year put option values lie, of course, above the one year put option values. Second, the supra-year ITM put option values are smaller than the one-year values for the relevant empirical range $0 < H < 0.7$. Thus

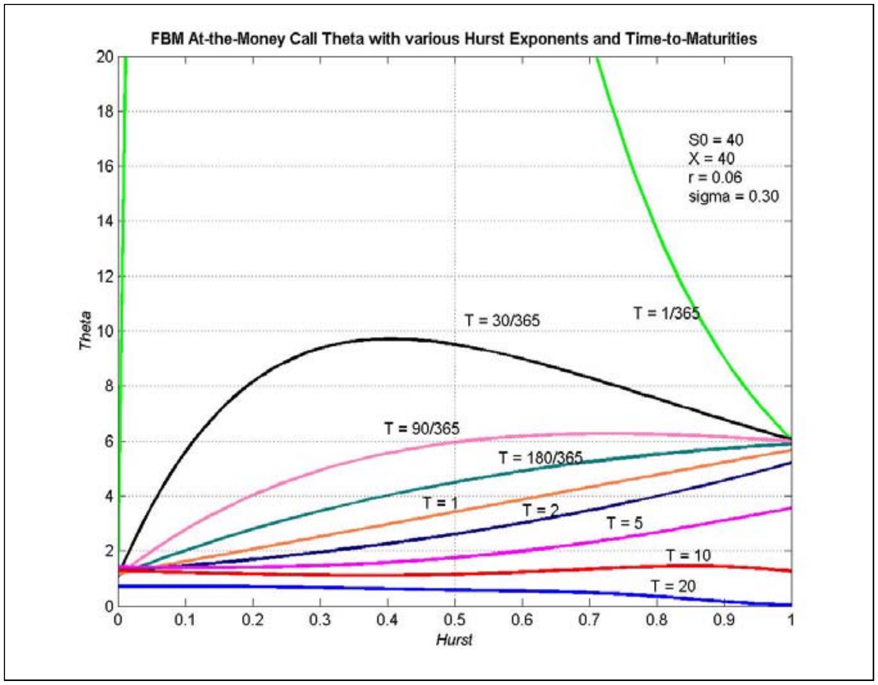

Figure 6:

for this persistence range, the put option becomes more valuable as the time to expiration decreases Third, the differences between ITM put option values for these empirically relevant degrees of persistence are striking: in an anti-persistent ($H = 0.3$) market a 10-year ITM put option is worth \$5.00, but in a persistent market the same put option is worth \$17.50, a \$12.50 difference. Fourth, the supra-year ITM put option values are larger than the one-year values in the empirically irrelevant range $0.7 < H < 1$.

6 Conclusion

This paper demonstrates the impact of observed financial market persistence, *i.e.*, long term memory on European option valuation. The degree of persistence or, equivalently, the type of long term memory of the market of the underlying does have a significant impact on the LM option values via their time-dependent volatility and thus via the risk-neutral probabilities used in their valuation. Some of these effects are rather counter-intuitive. Option traders should be aware of these LM phenomena and the arbitrage opportunities they entail between persistent and anti-persistent markets.

Proper long memory option valuation is of considerable importance in cor-

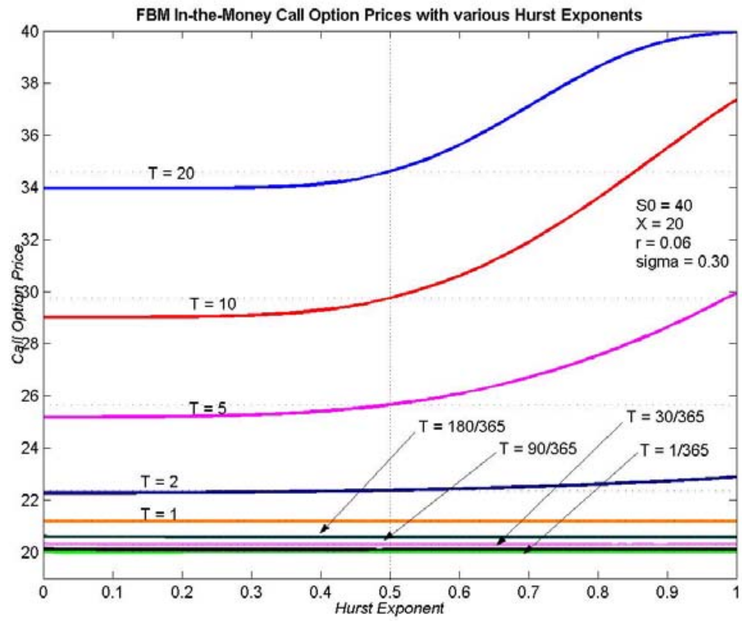

Figure 7:

porate remuneration packages, since such options are written on a company's own shares with long expiration periods. Therefore, we recommend that for a proper valuation of such warrants, the degrees of persistence of the companies' share markets are measured and properly incorporated in the warrant valuation.

As Elliott and van der Hoek (2003) show, this long memory financial market analysis can be extended to multi-fractional Brownian Motions (multi-FBMs), although such an extension raises another issue. Does this lead to a contradiction with the fundamental Law of One Price, since multiple intrinsic option prices may be coexistent within the same market, depending on the degree of persistence experienced by the various market participants. The empirical corroboration of the existence of a multi-FBM market points to a corroboration of the idea that a market for a particular financial instrument of a particular maturity or expiration time is actually segmented into buyers and sellers according to the differences in time horizon of the various market participants, as was suggested by Peters (1989 and 1994, p. 272).

Let us elaborate this point a bit. Spectrum analysis tells us how the energy (= risk or volatility) of a particular financial market price diffusion is allocated over various time scales or frequencies. What the multi-fractal spectrum of one particular financial market shows is that in that term T market there are trading participants who hold different investment time-horizons, and therefore

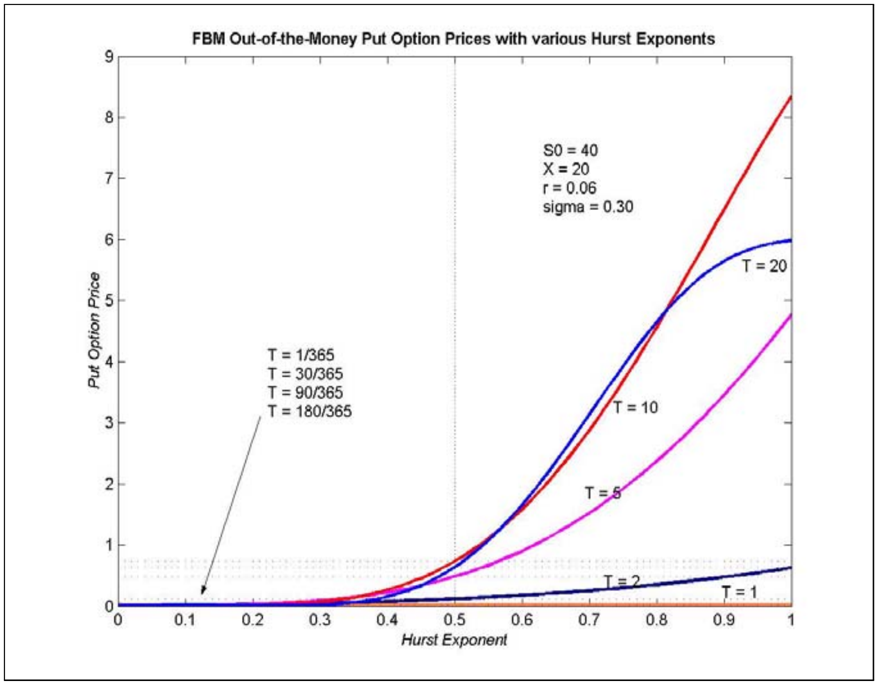

Figure 8:

trade slower or faster, *i.e.*, trade with more or less persistence. In other words, within the market, say, of $T = 2$ year options, there can be participants who have different $T = 0.5, 1, 1.5, 2, 5, 10, 20,$ etc. year investment horizons, like day traders, quarterly mutual fund managers, or very long term buy-and-hold pension fund managers.

The intrinsic option prices of a particular expiration time T market, which incorporates such different degrees of trading persistence, are different even for traders trading options with the same expiration time. This can explain why trading takes place at all within the same market: the persistent traders have high intrinsic option values relative to the prevailing market option price and tend to buy, while the anti-persistent traders have low intrinsic values relative to the prevailing option market price and tend to sell.

In this paper we did not analyze multi-fractal spectra of the multi-FBMs, but focused on the mono-fractal Hurst exponent. The mono-fractal Hurst exponent of such a particular internally segmented market, which is the most prevalent long memory exponent, tells us the what the most important dimension or prevailing constituency of traders is in that market. Stock markets tend to be persistent, because the traders in such markets tend to have longer investment horizons, while cash and anchor foreign exchange markets tend to be anti-persistent, because the traders in such markets tend to have ultra-short

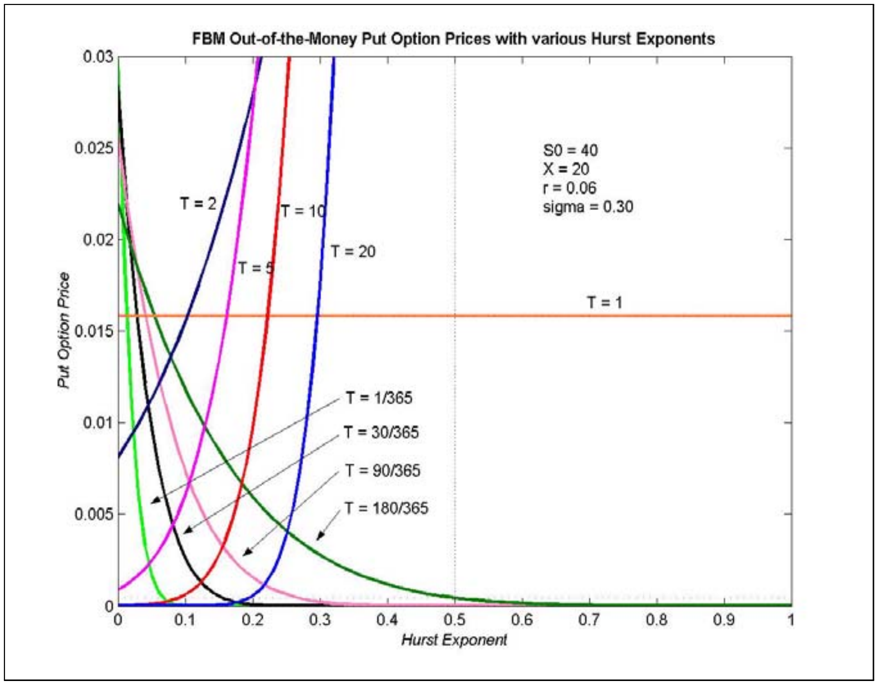

Figure 9:

(intra-day) investment horizons.

This also points to an indicator of potential emerging market malfunctioning. The wider the multi-fractal spectrum, the more varied the investment horizons of the market participants. The narrower the multi-fractal spectrum, the narrower the spectrum of market participants is. When the multi-fractal spectrum narrows, more and more market participants tend to have the same investment horizon or investment view and there are less buyers and sellers with different investment perspectives.[12]

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[12] This idea is explored by Rossitsa Yalmova in her successfully defended PhD thesis "Wavelet MRA of Index Patterns Around Stock Market Shocks," Kent State University, 2003.

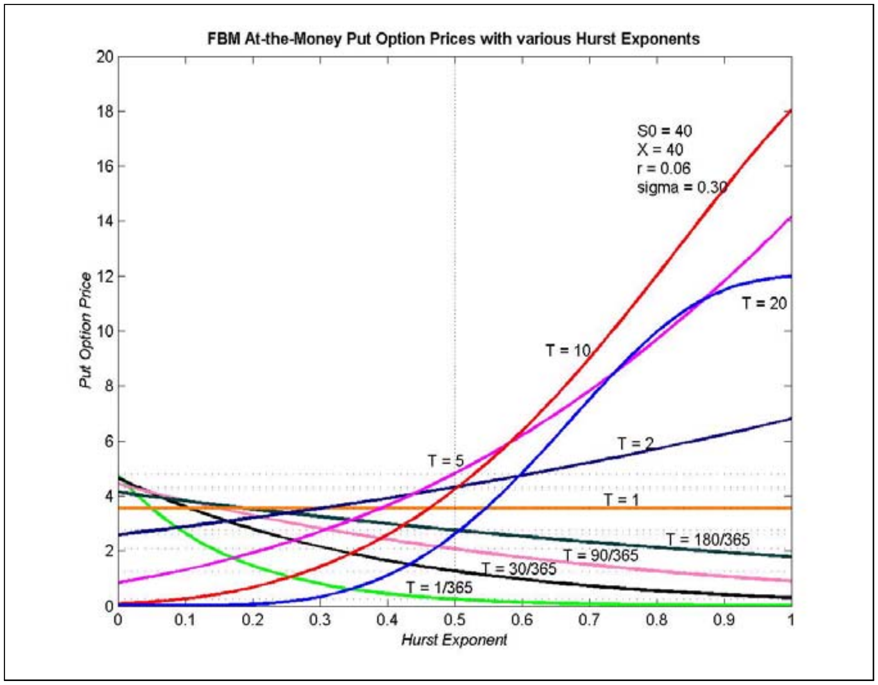

Figure 10:

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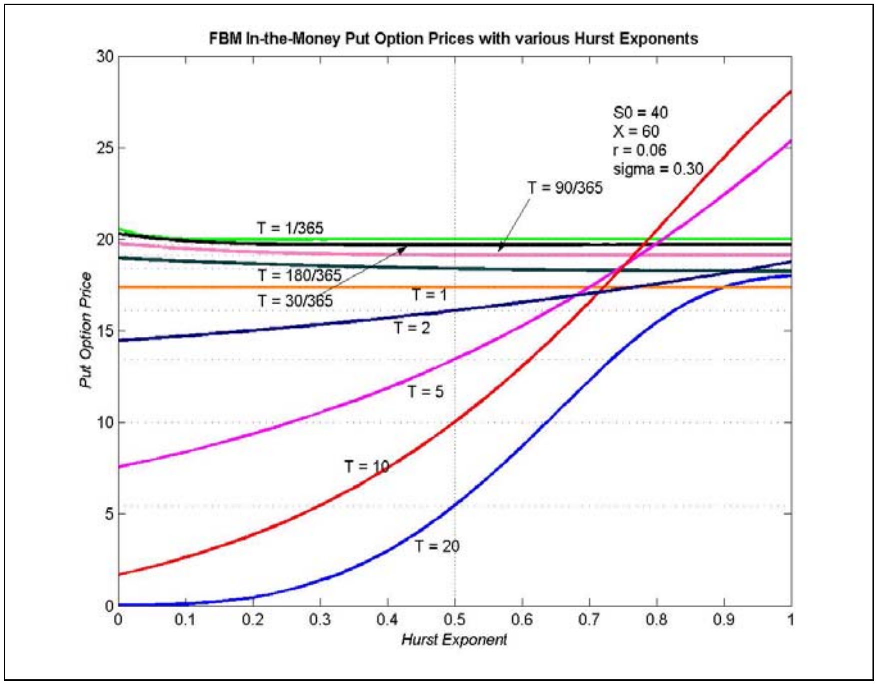

Figure 11:

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