LIBOR MARKET MODEL AND GAUSSIAN HJM EXPLICIT APPROACHES TO OPTION ON COMPOSITION

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Abstract. The twin brothers Libor Market and Gaussian HJM models are investigated. A simple exotic option, floor on composition, is studied. The same explicit approach is used for both models. Using an approximation the LLM price is obtained without Monte Carlo simulation. The results of the approximation are very good, with an error well below the uncertainty due to the simulation. The appendices proves the existence of the (modified) normal and shifted log-normal LLM used in the pricing. The link of the latter with the Ho and Lee continuous time model is described.

1. Introduction

Even if they are *twin brothers* (see [4]), the Libor Market Models (LLM) and Gaussian HJM models usually lead to very different numerical implementations techniques. Monte Carlo simulations is the main tool for the former while explicit formulas or trees are the standard for the later.

One simple case where the LLM are not used with Monte Carlo simulations is the pricing of vanilla caps and floors. There the explicit Black formula is used. This is probably the only case where the almost-always-working-but-heavy simulation approach is not used.

The instrument studied here is an option on the composition (described below). It is not a plain vanilla instrument but a relatively simple exotic one. It involves several Libor rates fixed at different dates but only one payment date.

In the simple one-factor Gaussian HJM, an explicit formula can be obtained for many instruments and in particular for that one (Section 3). Formulas for similar instruments, including overnight-indexed swaps, can be found in [6].

The LLM used is based on rates following an arithmetic Brownian motion (by opposition to a geometric Brownian motion in the BGM[1] model [1]) The reason for the choice is that the normal models on rates seems to represent better the dynamic of interest rate products, at least for the moment in USD. A comparison between models that leads to this conclusion in the case of swaptions is proposed in [8]. For the normal LLM an approximation of the drift *and volatility* terms are used to obtain an explicit formula in Section 4. The quality of the approximation is analysed through examples in Section 5. The volatility part is analysed through caplets. The drift an volatility approximations is applied to the options on composition.

The model used is not exactly based on a pure arithmetic Brownian motion. Appendix A proves that such equations for the forward Libor rates can not be embedded into a HJM framework. The equations are modified far away from any realistic rate to ensure the theoretical foundation of the model but with no impact on any practical computations.

Appendix C is dedicated to the theoretical analysis of the approximation used in Section 4 pricing formula. The approximated formula can also be considered as an exact formula for a LLM based on shifted log-normal dynamic for the forward rate. In its simplest version the same shifted

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[1]The acronym BGM is an anagram of GBM (Geometric Brownian Motion).

LLM is equivalent to the continuous time version of the Ho and Lee model [9]. The equivalence is explained in the same appendix. The shifted log-normal and normal LLM have been used in other places (in particular [13, Chapter 11] and [3]) but the conditions for their existence were not discussed.

The compounded instrument pays a floating rate (typically the Libor) compounded on several consecutive periods. The period dates are $0 \leq t_1 < t_2 < \cdots < t_n$. The rates are fixed at the dates $s_i \leq t_i$ ($0 = s_0 \leq s_1 < \cdots < s_{n-1}$). The accrual factors for the periods t_i–t_{i+1} are δ_i. The composition is

$$\prod_{i=1}^{n-1} 1 + \delta_i L(s_i, t_i) \tag{1}$$

The payment is subject to a floor (or a cap). Without the floor the value of the instrument would simply be 1 in t_1, like a floating rate note. What is special here is that the floor is on the total composition, not on each individual fixing. For a floor with an amount K, the payment at maturity is

$$\max\left(\prod_{i=1}^{n-1} 1 + \delta_i L(s_i, t_i), K\right).$$

2. Model and hypothesis

The two instances of the HJM framework used in this article are described in this section. They are a one-factor Gaussian version (deterministic volatility) and a multi-factors LLM version with normal Libor as base equations.

In general, the HJM framework describes the behavior of $P(t, u)$, the price in t of the zero-coupon bond paying 1 in u ($0 \leq t, u \leq T$). When the discount curve $P(t, .)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t, u)$ such that

$$P(t, u) = \exp\left(-\int_t^u f(t, s)ds\right). \tag{2}$$

The idea of Heath-Jarrow-Morton [5] was to exploit this property by modeling f with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u).dW_t$$

for some suitable (stochastic) μ and σ and deducing the behavior of P from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The volatility and the Brownian motion are m-dimensional while the drift and the rate are 1-dimensional. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of [10].

The probability space is $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a m-dimensional standard Brownian motion $(W_t)_{0 \leq t \leq T}$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t, u) = \int_t^u \sigma(t, s)ds.$$

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with $(r_s)_{0 \leq s \leq T}$ the short rate given by $r_t = f(t, t)$. The equations of the model in the numeraire measure associated to N_t are

$$df(t, u) = \sigma(t, u)\nu(t, u)dt + \sigma(t, u).dW_t$$

or

$$dP^N(t, u) = -P^N(t, u)\nu(t, u).dW_t$$

The notation $P^N(t, s)$ designates the numeraire rebased value of P, i.e. $P^N(t, s) = N_t^{-1} P(t, s)$.

The two following technical lemmas were presented in [7] for the Gaussian one-factor HJM. Similar formulas can be found in [2, (3.3),(3.4)] in the framework of coherent interest-rate models.

Lemma 1. *Let $0 \le t \le u \le v$. In HJM framework the price of the zero coupon bond is*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left(-\int_t^u \left(\nu(s,v) - \nu(s,u)\right).dW_s - \frac{1}{2}\int_t^u \left(|\nu(s,v)|^2 - |\nu(s,u)|^2\right) ds \right).$$

Lemma 2. *Let $0 \le u \le v$. In the HJM framework*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s ds \right) = P(u,v) \exp\left(-\int_u^v \nu(s,v) dW_s - \frac{1}{2}\int_u^v \nu^2(s,v) ds \right).$$

2.1. **Gaussian HJM.** The first version of the model used is the HJM model with $m = 1$ and a deterministic volatility function ($\sigma : [0,T]^2 \to \mathbb{R}^+$). The simplicity of the model allows explicit formulas for many products. Section 3 describes the formulas for the option on composition.

In one example of Section 5 caplet are used. Let $\theta \le t_i < t_{i+1}$ be the expiry, start and end dates of the caplet. The strike rate is K. The value of the caplet at 0 is

$$-(1+\delta_i K)P(0,t_{i+1})N(-\kappa - \alpha_{i+1}) + P(0,t_i)N(-\kappa - \alpha_i)$$

with

$$\alpha_i^2 = \int_0^\theta (\nu(s,t_i) - \nu(s,\theta))^2 ds$$

and

$$\kappa = \frac{1}{\alpha_{i+1} - \alpha_i} \ln\left(\frac{P(0,t_{i+1})(1+\delta_i K)}{P(0,t_i)} - \frac{1}{2}(\alpha_{i+1}^2 - \alpha_i^2) \right).$$

2.2. **Libor Market Model.** The idea behind the Libor Market model is to embed different Black-like equation for the forward (Libor) rate between standard dates ($0 \le t_1 < t_2 < \cdots < t_n$) into a unique HJM model. The Libor rates $L(t,t_j)$ are defined by

$$1 + \delta_i L(s,t_i) = \frac{P(s,t_i)}{P(s,t_{i+1})}.$$

The equations underlying the normal, Gaussian, or Bachelier Libor market model are

$$dL(t,t_j) = \gamma_j(L(t,t_j),t).dW_t^{j+1} \tag{3}$$

in the probability space with numeraire $P(t,t_{j+1})$. The γ_j ($0 \le j \le n-1$) are m-dimensional functions. To merit the full qualification of Bachelier model, γ_j should be purely deterministic (not involving L). For fundamental reasons explained in Appendix A such a model would be ill-defined. In this section γ is used with its most general form. Section 4 will consider it in its simple deterministic form. It can be considered also as an affine function leading to a displaced log-normal dynamic as described in Appendix C.

The Brownian motion change between the N_t and the $P(t,t_{j+1})$ numeraires is given by

$$dW_t^{j+1} = dW_t + \nu(t,t_{j+1})dt.$$

The difference $\nu(t,t_j) - \nu(t,t_{j+1})$ can be written as

$$\nu(t,t_j) - \nu(t,t_{j+1}) = \frac{1}{L(t,t_j) + \frac{1}{\delta_j}} \gamma_j(L(t,t_j),t)$$

Recursively the change of numeraire gives

$$dW_t^{j+1} = -\sum_{i=j+1}^{n-1} \frac{1}{L(t,t_i) + \frac{1}{\delta_i}} \gamma_i(L(t,t_j),t)dt + dW_t^n.$$

All the rates can be written with respect to the same (last) numeraire

$$dL(t,t_j) = -\left(\sum_{i=j+1}^{n-1} \frac{1}{L(t,t_i) + \frac{1}{\delta_i}} \gamma_i(L(t,t_j),t).\gamma_j(L(t,t_j),t) \right) dt + \gamma_j(L(t,t_j),t).dW_t^n.$$

When δ_i is small the rate dependency of drift almost disappear. Let $\delta_i \sim 1/n$ (one year final rate). If the rates are bounded by $L_- \leq |L_i| \leq L_+$ we obtain for the drift term

$$\gamma_j \sum_{i=j+1}^{n-1} \gamma_i \frac{1}{L_- + n} \leq \text{drift} \leq \gamma_j \sum_{i=j+1}^{n-1} \gamma_i \frac{1}{L_+ + n}$$

If all the γ_i are equal and constant, both bounds are converging to γ^2. In all cases the ratio between the lower bound and upper bound is

$$\frac{L_- + n}{L_+ + n}$$

which is rapidly close to 1 when n growths. So for most of the rates, the drift is close to its initial value. This property of little dependency of the drift will be used later.

In the pure Bachelier model on rate $L(., t_j)$ the caplet price is given by ([12, Section 3.3.1])

$$P(0, t_{i+1})\delta_i \left((L(0, t_i) - K)N(\kappa) + |\gamma_i|\sqrt{\theta} n(\kappa) \right)$$

with

$$\kappa = \frac{L(0, t_i) - K}{|\gamma_i|\sqrt{\theta}}.$$

3. Gaussian one-factor HJM approach to option on composition

Theorem 1. *Let $0 \leq t_1 < t_2 < \cdots < t_n$, $0 = s_0 \leq t \leq s_1 < s_2 < \cdots < s_{n-1}$ with $s_i \leq t_i$ and $K > 0$. In the a HJM one factor model, the price of an instrument paying in t_n the maximum of a fixed amount K and of a principal gross-up by the discrete compounding of interest rates over the periods $[t_i, t_{i+1}]$ fixed in s_i (i.e. $Q \prod_{i=1}^{n-1} P(s_i, t_i)/P(s_i, t_{i+1})$) is given in 0 by*

$$F_0 = P(0, t_1)N(\kappa + \sigma) + KP(0, t_n)N(-\kappa)$$

where

$$\sigma^2 = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \int_0^{\min(s_i, s_j)} (\nu(s, t_{i+1}) - \nu(s, t_i))(\nu(s, t_{j+1}) - \nu(s, t_j))ds.$$

and

$$\kappa = \frac{1}{\sigma} \left(\ln \left(\frac{P(0, t_1)}{KP(0, t_n)} \right) - \frac{1}{2}\sigma^2 \right).$$

The price of an instrument paying in t_n the minimum of a fixed amount K and of a principal gross-up by the discrete compounding of interest rates over the periods $[t_i, t_{i+1}]$ fixed in s_i is given in 0 by

$$C_0 = P(0, t_1)N(-\kappa - \sigma) + KP(0, t_n)N(\kappa)$$

Proof. The price of the instrument is

$$F_0 = N_0 \, \mathrm{E}_{\mathbb{N}} \left(\max \left\{ \prod_{i=1}^{n-1} \frac{P(s_i, t_i)}{P(s_i, t_{i+1})}, K \right\} N_{t_n}^{-1} \right).$$

Using Lemma 1, we have

$$\prod_{i=1}^{n-1} \frac{P(s_i, t_i)}{P(s_i, t_{i+1})} = \frac{P(0, t_1)}{P(0, t_n)} \exp \left(\frac{1}{2} \sum_{i=1}^{n-1} \int_0^{s_i} \nu^2(s, t_{i+1}) - \nu^2(s, t_i)ds + \sum_{i=1}^{n-1} \int_0^{s_i} \nu(s, t_{i+1}) - \nu(s, t_i)dW_s \right)$$

By Lemma 2,

$$N_{t_n}^{-1} = P(0, t_n) \exp \left(-\int_0^{t_n} \nu(s, t_n)dW_s - \frac{1}{2} \int_0^{t_n} \nu^2(s, t_n)ds \right).$$

We denote this last exponential by L_{t_n}. Let $W_s^{\#} = W_s + \int_0^s \nu(\tau, t_n)d\tau$. By the Girsanov's theorem ([11, Section 4.2.2, p. 72]), $W_t^{\#}$ is a standard Brownian motion with respect to the probability $\mathbb{P}^{\#}$ of density L_{t_n} with respect to $\mathbb{N}$. This is the t_n maturity bond numeraire. Note that the probability $\mathbb{P}\#$ is not the same as the probability $\mathbb{P}^n$ used in the LLM as the models are not the same, but the idea is the same.

The sum of the integrals in the exponential can be written as

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_0^{s_i}\nu(s,t_{i+1})-\nu(s,t_i)dW_s+\frac{1}{2}\sum_{i=1}^{n-1}\int_0^{s_i}\nu^2(s,t_{i+1})-\nu^2(s,t_i)ds\\
=\;&\sum_{i=1}^{n-1}\int_0^{s_i}\nu(s,t_{i+1})-\nu(s,t_i)dW_s^{\#}\\
&-\frac{1}{2}\sum_{i=1}^{n-1}\int_0^{s_i}(\nu(s,t_{i+1})-\nu(s,t_i))(2\nu(s,t_n)-\nu(s,t_{i+1})-\nu(s,t_i))ds
\end{aligned}$$

Using the identity $\nu(s,t_n)-\nu(s,t_i)=\sum_{j=i}^{n-1}(\nu(s,t_{j+1})-\nu(s,t_j))$ and rearranging the terms the last sum can be written as

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\int_0^{\min(s_i,s_j)}(\nu(s,t_{i+1})-\nu(s,t_i))(\nu(s,t_{j+1})-\nu(s,t_j))ds.$$

The value of the instrument can now be written as

$$F_0=\mathrm{E}^{\#}\left(P(0,t_n)\max\left(\frac{P(t,t_1)}{P(t,t_n)}\exp\left(-\frac{1}{2}\sigma^2-\sigma X^{\#}\right),K\right)\right)$$

where $X^{\#}$ is a random variable with a standard normal distribution with respect to $\mathbb{P}^{\#}$. The first term of the maximum operator is the actual maximum when $X^{\#}<\kappa$. So we obtain

$$F_0=P(0,t_1)\,\mathrm{E}^{\#}\left(\exp\left(-\sigma X^{\#}-\frac{1}{2}\sigma^2\right)\mathbb{1}(X^{\#}<\kappa)\right)+KP(0,t_n)\mathbb{P}^{\#}\left(X^{\#}\geq\kappa\right)$$

which by standard manipulation on the expectation and on the normal distribution lead to the result.

The price of the capped instrument can be obtained by put-call parity. □

4. Libor Market model approach to option on composition

Theorem 2. *Let $0\leq t_1<t_2<\cdots<t_n$, $0=s_0\leq t\leq s_1<s_2<\cdots<s_{n-1}$ with $s_i\leq t_i$ and $K>0$. In the LLM, the price in 0 of an instrument paying in t_n the maximum of a fixed amount K and of a principal gross-up by the discrete compounding of interest rates over the periods $[t_i,t_{i+1}]$ fixed in s_i (i.e. $\prod_{i=1}^{n-1}P(s_i,t_i)/P(s_i,t_{i+1})$) is (approximately)*

$$F_0=P(0,t_1)N(\kappa+\sigma)+KP(0,t_n)N(-\kappa)$$

where $T=(\tau_{i,j})$ $(1\leq i,j\leq n-1)$ with

$$\tau_{i,j}=\int_0^{\min(s_i,s_j)}\gamma_i(s).\gamma_j(s)ds,$$

$$\lambda_i=\frac{1}{L(0,t_i)+\frac{1}{\delta_i}}$$

$$\sigma^2=\lambda^T T\lambda$$

and

$$\kappa=\frac{1}{\sigma}\left(\ln\left(\frac{P(0,t_1)}{KP(0,t_n)}\right)-\frac{1}{2}\sigma^2\right).$$

The price in 0 of an instrument paying in t_n the minimum of a fixed amount K and of a principal gross-up by the discrete compounding of interest rates over the periods $[t_i,t_{i+1}]$ fixed in s_i is (approximately)

$$C_0=P(0,t_1)N(-\kappa-\sigma)+KP(0,t_n)N(\kappa)$$

Proof. The price of the instrument using $P(0,t_n)$ as numeraire is

$$F_0 = P(0,t_n)\,\mathrm{E}^n\left(\max\left\{\prod_{i=1}^{n-1}\frac{P(s_i,t_i)}{P(s_i,t_{i+1})}, K\right\}\right).$$

In a similar way to the previous theorem, the integrals appearing in the expected value can be written in the LLM model as

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_0^{s_i}(\nu(s,t_{i+1})-\nu(s,t_i)).dW_s + \frac{1}{2}\int_0^{s_i}(|\nu(s,t_{i+1})|^2-|\nu(s,t_i)|^2)ds\\
&= \sum_{i=1}^{n-1}\int_0^{s_i}(\nu(s,t_{i+1})-\nu(s,t_i)).dW_s^n\\
&= -\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\int_0^{\min(s_i,s_j)}(\nu(s,t_{i+1})-\nu(s,t_i))(\nu(s,t_{j+1})-\nu(s,t_j))ds
\end{aligned}$$

The stochastic integrals $\int_0^{s_i}\gamma_i(s)dW_s^n$ are normally distributed with mean 0 and covariance T.

The (stochastic) value of the Libor rate is approximated in the formula by $L(s,t_i) = L(0,t_i)$[2]. As noted in Section 2, the impact of the rate level is only through the ratio $1/L+\frac{1}{\delta_i}$ and is relatively limited. The initial ratio is λ_i. The integrals can be written as

$$-\sum_{i=1}^{n-1}\lambda_i X_i - \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\lambda_i\lambda_j\tau_{i,j}.$$

The composition becomes

$$\prod_{i=1}^{n-1}\frac{P(s_i,t_i)}{P(s_i,t_{i+1})} = \frac{P(0,t_1)}{P(0,t_n)}\exp\left(-\sum_{i=1}^{n-1}\lambda_i X_i - \frac{1}{2}\lambda^T T\lambda\right).$$

With the approximation the large sum in the exponential is only a constant, denoted α. The sum of the random variables $\lambda_i X_i$ is a normally distributed variable with mean 0 and variance matrix $\lambda^T T\lambda$.

The maximum is the composition when $X < \kappa$. The price can now be written as

$$F_t = P(0,t_1)\,\mathrm{E}^n\left(\exp(-\sigma X - \frac{1}{2}\sigma)\mathbb{1}(X<\kappa)\right) + KP(0,t_n)\mathbb{P}^n(X\geq\kappa).$$

The result follows easily.

The price of the capped instrument can be found by put-call parity. □

The price of a caplet can be deduced from the above formula with $n = 2$.

5. Examples

In the first example a vanilla caplet is priced using different models. The goal of this example is to assess the quality of the approximation used to obtain the explicit solution in the LLM.

The price is computed with the exact and approximated formulas for the normal LLM, the Hull-White model, and the log-normal Black model. All the results are represented in Figure 1. The numbers are in term of normal implied volatility.

The figures used for the example are a (simplified) curve (with spot equal to today) which schematically represents the USD curve at the time of writing (Table 1). The (normal) volatility is constant for all rates and is 1%.

The caplet studied has one year to expiry and the underlying rate is the three months rate.

The Hull-White and Black model are calibrated to the at-the-money caplet. The Hull-White mean reversion parameter is 2%.

[2]The relation between this approximation and the shifted log-normal model is described in Appendix C.

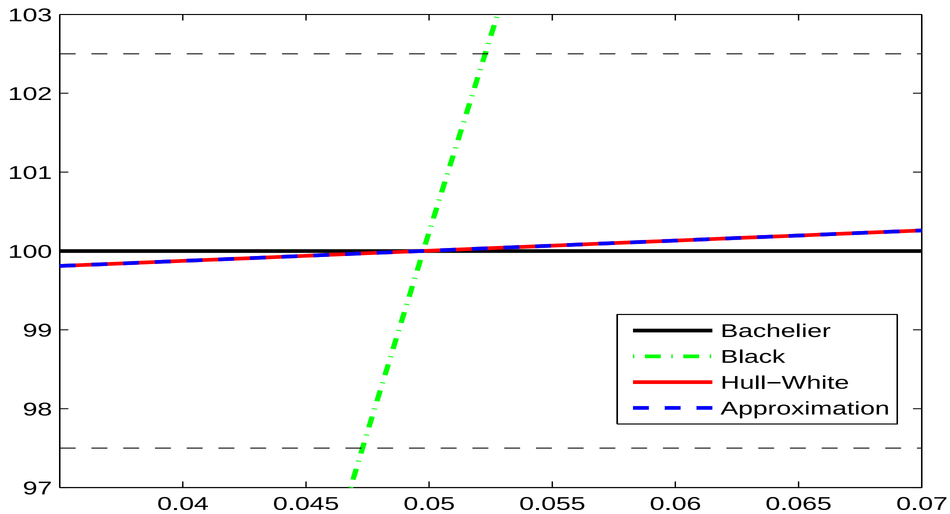


FIGURE 1. Implied volatility for the four models studied.

O/N	1m	3m	6m	1y	2y
4.00	4.25	4.25	4.50	4.75	4.75

TABLE 1. Yield curve.

The price of the caplet is computed for the four models and the implied volatility for the normal model is computed from them. In the price graph all the curves are undistinguishable; the implied volatility display the differences better. The two horizontal dotted lines represent a typical bid-offer spread.

The main part of the comparison is between the approximated and the exact solution for the normal LLM. The difference is small. At its maximum it is approximately one tenth of the bid-offer.

The Hull-White curve is almost equal to the approximated normal one. From Appendix C equivalence this is not surprising. The Ho and Lee model is equivalent to the Hull-White with a 0 mean reversion. For standard caplets, the correlation structure and multi-factor features of the normal LLM is not used.

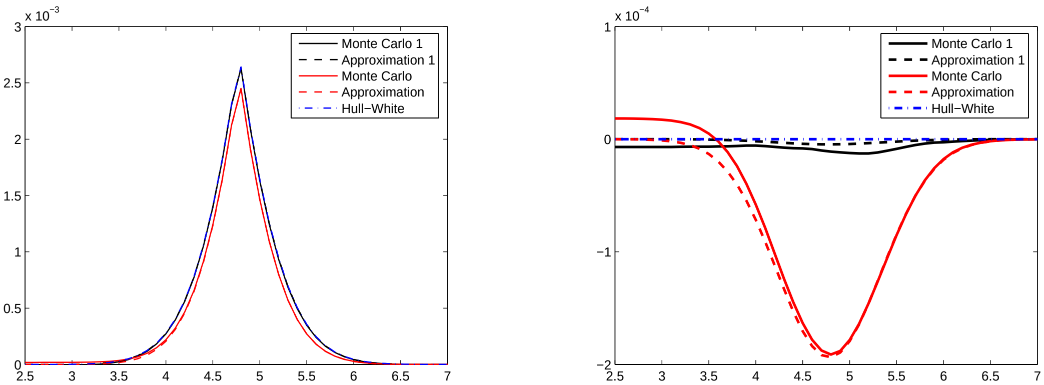
The Black curve, that was included only as a reference point, is very different. The similarity between the results for the three first model is not due to the fact that all possible models gives similar results but due to the fact that the three models studied are them-self very close for instruments for which no correlation is used.

The second example is related to the option on composition. The same curve and volatilities are used. For the option on composition the full structure of γ is used, not only its norm. Two different structures are used to see its impact. The first is a one-factor one for which no more details are required. The second one is a two factor model. The structure used is one suggested in [13, Section 7.3.1] with

$$\gamma_i = |\gamma_i|(\sin(\theta_i), \cos(\theta_i)).$$

The five θ_i are chosen equally spaced between 0 and $\pi/2$.

The the price for a five (three month) periods composition is presented in Figure 2. The first graph is the intrinsic value $(F_0 - \max(P(0,t_n)K, 1))$. In the second graph, to emphasize the differences, the figures are relative to the Hull-White price.



(a) Intrinsic value (b) Value relative to the HJM value

FIGURE 2. Price using the five different approaches

The prices are computed for a set of strikes rates between 2.5% and 7%. The prices are computed five times for each strike. Once with the Hull-White (Gaussian HJM) approach and four times with the LLM one. There are two instances of the LLM, one with one factor and the second with two factors. For both of them the price is computed with a Monte Carlo simulation approach and the explicit approximated formula. The Monte Carlo simulations are done using the *predictor-corrector* [13, Section 5.3.1] long-jump technique and 200,000 simulations. As the number of factors is not the same it was not possible to use the same seed for the two simulations. In term of speed the explicit approximated formula is obviously a lot more efficient.

The non-convergence to the same level for low strike is due to the imprecision inherent to the Monte Carlo simulation. The difference between the approximation and the exact figure is smaller than the error coming from the simulation.

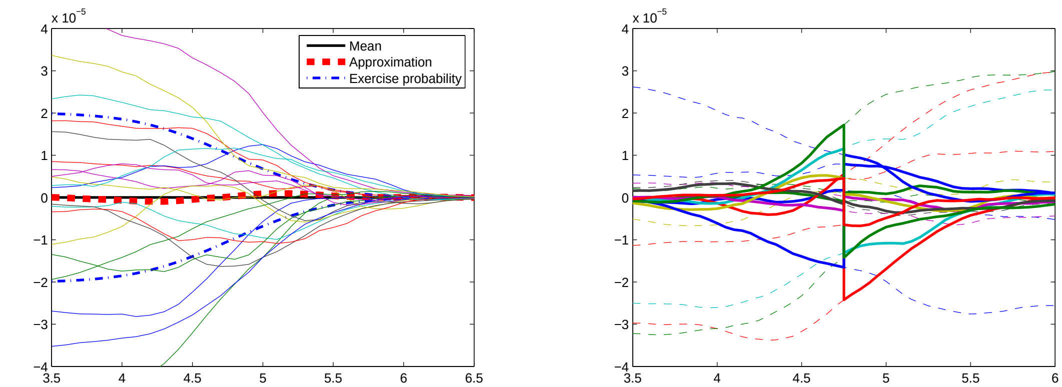
Figure 3(a) reports those errors. The 100,000 simulations were run 20 times. The graph reports the difference to the mean of the simulations. The approximated result is represented by the dotted line very close to 0. As can be seen the typical simulation error is well above the approximation error. The error is higher for low strikes. One could ask why this systematic error pattern appears[3]. For lower strikes more simulated rates and less strikes are used in the expected value; the rates are subject to simulation errors while the fixed strikes are not. The (rescaled positively and negatively) exercise probability is also indicated on the graph to corroborate the interpretation.

This suggest it is possible to use the put-call parity to improve the precision in Monte Carlo simulations. In our case the parity is

$$F_0 = P(0, t_n)K + P(0, t_1) - C_0.$$

The floor on composition are repriced by Monte Carlo simulations in Figure 3(b) using the initial approach and the put-call parity (only ten simulations are shown to simplify the graph). The graphs shows the floor and the cap approaches in dotted lines. The results are relative to the approximated formula. The errors are symmetrical between the two approaches, large on low strikes for the floor and large on high strikes for the cap approach. The a priori best approach is to take the cap approach for low strikes and floor approach for high strikes. The cut-off between the two approaches can be done at the forward rate. In the example it is around 4.75%. The combine approach is given in solid lines. The vertical lines at 4.75% indicate the two possible choices at the

[3]The question was actually asked to the author by Luis Bengoechea while discussing the paper first draft.



(a) Monte Carlo simulations reported to their average compared to the approximated formula.

(b) Put-call parity improvements

FIGURE 3. Monte Carlo simulations

cut-off strike. From the picture it is clear that most of the worst performing cases are removed. The standard deviation of error (10 simulations and 51 strikes) was 0.12 bps for the floors, 0.11 bps for the caps and 0.05 bps for the combined approach.

6. CONCLUSION

The normal version of the Libor market model is used to price simple exotic options. The (non-)existence of such a model and its link to the Ho and Lee model is described. The pricing is done through an approximated explicit formula. The result of the approximation on both the standard caplets and floor on composition is found to be very good. This approach combine the flexibility of the LLM in term of number of factors and correlation with the explicit results usually available only in the less flexible model like the Gaussian one-factor HJM.

APPENDIX A. NON-ARBITRAGE FREE NORMAL LLM

This appendix is devoted to analysing if the arithmetic Brownian equations for the forward rates (3) can be embedded in a HJM framework for γ deterministic. Unfortunately the answer is no and this can be seen easily.

In the HJM framework, the bond prices are given by Equation (2) and are always positive.

On the other side the link between forward rate and prices is

$$1 + \delta_i L(s, t_i) = \frac{P(s, t_i)}{P(s, t_{i+1})}.$$

If L is modelled by a pure arithmetic Brownian motion, it can become very negative (with a positive probability). When $L(s, t_i) < -1/\delta_i$, the ratio of the prices is negative. A contradiction with the previous assertion.

The dynamic of the forward rate has to be modified (artificially) to ensure that the model can be embedded in an well behaved HJM framework. Conditions sufficient to ensure the existence of such a framework are presented in Appendix B.

The impact of the function modification far away from the current rate level is very small. For a three months rate starting in one year, a current rate of 5% and a volatility of 1%, the probability to have a negative rate is $N(-L_0/\sigma\sqrt{\theta}) = N(-5) = 3.10^{-7}$.

Appendix B. Existence results

The first part is an existence result. Let $\gamma_i(L,t) = p_i(L)\gamma_i(t)$. The functions p_i are globally Lipschitz, $p_i(L(0,t_i)) > 0$[4] and p_i have zeros z_i with $-1/\delta_i \leq z_i < L(0,t_i)$. Like in With those conditions it is possible to prove the existence of a HJM model that contains the Equations 3. The argument is the same as in [10, Section 18.2.2] with. The first steps is to prove the existence of the solution of (3). This follows from the global Lipschitz condition with Itô's theorem [10, Theorem 6.27]. Note that because of the condition on the zero of p_i and the Lipschitz property, the solutions have a lower non-attainable barrier at $z_i \geq \delta_i$.

The second main step is to prove that the $P(.,t_n)$ numeraire rebased assets are martingales. For this it is useful to prove that the integrals

$$\int_0^t \frac{\delta_j p_j(L(s,t_j))}{1+\delta_j L(s,t_j)} \gamma_j(s).dW_s^j$$

are of bounded quadratic variation. Or by the identity between the quadratic variation of an Itô integral and a Lebesgue integral [10, Theorem 4.18] it is sufficient to prove that

$$\int_0^t \left(\frac{\delta_j p_j(L(s,t_j))}{1+\delta_j L(s,t_j)} \right)^2 |\gamma_j(s)|^2 ds$$

is bounded. The boundedness result comes from the global Lipschitz condition (in particular at z_i and at infinity), the lower barrier on the solution L in z_i and the fact that $z_i \geq -1/\delta_i$.

LMM with displaced log-normal or normal rates have been described in other places, in particular in [13, Chapter 11], but the question of existence of such a model was not discussed and the condition on the displacement not mentioned.

The first set of functions p_i to which the result is applied in this note is $p_i(L) = 1$ modified close to $L = -1/\delta_i$. The modification is done by keeping p continuous, setting $p_i(L) = 0$ for $L < -1/\delta_i$, $p_i(L)$ affine between $-1/\delta_i$ and $1/\delta_i + \epsilon$ and leaving $p_i(L) = 1$ unchanged for $L > -1\delta + \epsilon$.

Appendix C. Shifted log-normal Libor market model

The second set of functions is

$$p_i(L) = \frac{1+\delta_i L}{1+\delta_i L(0,t_i)}.$$

With that choice, the volatility differences simplify to

$$\nu(t,t_j) - \nu(t,t_{j+1}) = \frac{\delta_j}{1+\delta_j L(0,t_j)} \gamma_j(t).$$

This is exactly the value that was used as an approximation in Theorem 2.

The same framework can be linked to the continuous version of the Ho and Lee model [9]. For this take the simplified version of the shifted log-normal model were all volatilities are constant and such that $\gamma_j(t)(1+\delta_j L(0,t_j)) = \bar{\sigma}$ for a certain $\bar{\sigma} \in \mathbb{R}^+$.

Then the bond volatility function is given by

$$\nu(t,t_j) - \nu(t,t_{j+1}) = \delta_j \bar{\sigma} = \bar{\sigma}(t_{j+1} - t_j)$$

which is exactly the Ho and Lee volatility structure with short rate volatility $\bar{\sigma}$.

The (old fashioned) Ho and Lee model can now be renamed with the more fashionable name of one-factor shifted log-normal Libor market model. This emphasize once more the brotherhood between LLM and HJM models quoted in the first sentence of this note, a designation borrowed from Gątarek [4].

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

[4]The choice of p_i being positive at the current level of rate is arbitrary and without loss of generality. If $p_i(L(0,t_i)) < 0$, by changing the Brownian motion to $-W^j$, the condition is satisfied. The standardization to a positive volatility is more a tradition than a mathematical constraint.

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