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Portfolio Management for a Random Field of Bond Returns

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Abstract

This paper describes a new method of bond portfolio optimization based on stochastic string models of correlation structure in bond returns. The paper shows how to approximate the correlation function of bond returns with Padé approximations and compute the optimal portfolio allocation using Wiener-Hopf factorization. The technique is illustrated with an example of the Treasury bond portfolio.

Keywords: bond portfolio management, stochastic string, Toeplitz operators, Padé approximations, Wiener-Hopf factorization.

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1 Introduction

While a cross-section of monthly stock returns exhibits an irregular saw-tooth pattern, monthly bond returns plotted against maturity look like the sea wave on a breezy but calm day – the nearby points are close to each other and move in accord. This distinction explains why bond portfolio management is a long-standing financial problem: Bond returns are more structured but also more difficult to diversify. The present paper contends that the optimal bond portfolio can be found by approximating the return correlation structure with a rational function of the difference in maturities and reducing the optimization problem to the inversion of a Hilbert space operator.

The importance of bond portfolio management is difficult to overestimate. In 2000 the values of government and corporate debt outstanding were 7.7 and 5 trillions respectively, compared with 17 trillions of corporate equity outstanding. In addition, the relative importance of debt is rising: the equity value in 2000 decreased by 13 percent from its 1999 level, while the government and corporate debts were up 2 and 9 percent respectively.[1] Despite all the effort to balance its budget, the federal government is still spending more than it receives in revenues and therefore the likelihood that the Treasury securities market will be shriveling is scant. So, given the importance of debt markets how should investors optimize their bond portfolios?

In the early 1980s Heaney and Cheng applied to bonds the techniques for stock portfolio optimization. This approach, however, was later abandoned because bonds move together to a much greater extent than stocks and modeling these co-movements is harder.

Another idea appeared even earlier and got much greater application – the idea of immunization (Fisher and Weil (1971)). The immunization technique minimizes sensitivity of the portfolio with respect to small, parallel shifts in all interest rates. So, this approach directly takes into account the observation that interest rates are highly correlated. The modern development of this idea uses stochastic programming (Dembo (1993), Mulvey and Zenios (1994), Golub et al. (1995), Zenios et al. (1998), Consigli and Dempster (1998), Beltratti et al. (1999), Dupačová and Bertocchi (2001)), in which the investor formulates a set of scenarios for interest rate movements, prescribes them probabilities, and minimizes a certain loss function – for example, loss that can occur with 5% probability.

While practical, the immunization technique neglects many aspects of the interest rate dynamics. The scenario probabilities are usually extracted from a finite factor model, which does not capture all the information about the return correlations. In addition, the market models require continually re-calibrating parameters and are internally inconsistent. What is needed is a better method for modeling interest rate correlations.

The present paper explores a synthesis of the two methods of portfolio optimization: The correlations are directly estimated using a plausible assumption

[1] Source: Securities Industry Association Factbook 2001, page 22.

on their structure, and the estimated correlations are then used by a portfolio optimization model.

This new method has its provenance in modeling of bond returns as a random field (Kennedy (1994), Kennedy (1997), Goldstein (2000), Santa-Clara and Sornette (2001) and Baaquie et al. (2003)), an approach that provides more flexibility than finite-factor models for bond return correlations. According to the philosophy of these papers, correlations in bond returns should be approximated as a function of the difference in maturities. Once the function is estimated, the problem of optimization can be solved by inverting a special operator in the Hilbert space spanned by bond returns, a task that was already extensively studied in the communications engineering literature.

The rest of the paper is organized as follows. Section 2 explains assumptions and notation. Section 3 is about Padé approximations to the correlation function. Section 4 solves the portfolio optimization in terms of the Wiener-Hopf factorization. Section 5 applies the technique to a portfolio of the Treasury bonds. And Section 6 concludes.

2 Assumptions and Notation

It is assumed that the bond returns form a random field that has a two-dimensional correlation structure: across calendar time and across maturities. It is also assumed that the correlation of two contemporaneous bond returns is a function of the difference of bond maturities. Let us formulate this assumption in a more detailed form.

Let symbol t denote the time remaining to bond maturity and symbol s the calendar time. The market consists of infinite number of coupon-free bonds with times to maturity $t = n\delta$, $n = 0, 1, 2, 3, ...$, where δ is a fixed time period. The price of the bond with maturity $n\delta$ at time s is denoted $P_n(s)$. At maturity the price of the bond is 1: $P_0(s) = 1$. Let $R_n(s)$, $n = 1, 2, 3, ...$, denote the logarithmic return of investment in this bond after holding it for time δ :

$$R_n(s) = \log \frac{P_{n-1}(s+\delta)}{P_n(s)}, \tag{2.1}$$

Then it is assumed that

$$Cor(R_n(s), R_m(s)) = C(|n-m|). \tag{2.2}$$

The assumption may be motivated by analogy with the assumption of stationarity in time series where the autocovariance of the returns depends only on the difference between times of these returns. On a deeper level, the market percepts the bonds with close maturities as similar and the assumption says that the relevant measure of similarity is the difference between maturities. Since the question about appropriate measure of similarity is a question about market perceptions, it needs further empirical investigation. A priori, the assumption that is taken in this paper seems to be reasonable.

In the following the calendar time argument in $R_n(s)$ will be suppressed for the reader's convenience.

Let me introduce artificial securities S_n, $n = 0, 1, 2, ...$, that have unit variance and the following expected return

$$E_n = (ER_{n+2} - R_1)/\sqrt{V_{n+2}}, \tag{2.3}$$

where ER_n is the expected return of bond with maturity $n\delta$, and V_n is the variance of this return. The purpose of this normalization is two-fold: to restrict attention to bonds that have a stochastic return over the next period (R_1 is not stochastic), and also to focus on the correlation structure in bond returns, not on volatilities of individual bonds.

If holdings in the portfolio of bonds are denoted X_n for the bond with the maturity $n\delta$, let us also define a portfolio of the securities S_n that holds

$$Y_n = X_{n+2}\sqrt{V_{n+2}} \tag{2.4}$$

in S_n. Then any portfolio of bonds that have stochastic returns is mapped to a portfolio of securities S_n. Those two portfolios have the same variance and the expected return of the new portfolio is equal to the abnormal expected return of the original portfolio. The formulation it terms of securities S_n is useful as it emphasizes the importance of the correlation structure. In the same time, if the optimal portfolio of securities S_t is found, it is easy to translate it back to the optimal portfolio of bonds by inverting relationship (2.4).

We will use the following criterion for portfolio optimization:

$$U(\Pi) = E\Pi - \gamma Var(\Pi), \tag{2.5}$$

where Π is a portfolio, $E\Pi$ is its expected abnormal return, $Var(\Pi)$ is its variance, and γ is a parameter that captures investor's risk aversion. The task of the investor is to maximize $U(\Pi)$ with respect to all possible choices of Π. We are going to re-write this problem to show explicitly the dependence of the investment criterion on the correlation structure.

Let us introduce the generating functions for expected returns, correlations, and holdings of securities S_t :

$$\widehat{E}(z) \equiv \sum_{n=0}^{\infty} E_n z^n. \tag{2.6}$$

$$\widehat{C}(z) \equiv \sum_{n=0}^{\infty} C_n z^n, \tag{2.7}$$

$$\widehat{Y}(z) \equiv \sum_{n=0}^{\infty} Y_n z^n,$$

The second step is to introduce some machinery of Hilbert spaces. Let $\mathcal{H}$ be the linear space of the formal series in variable z with the bounded sum of squared coefficients:

$$a(z)=\sum_{-\infty}^{\infty} a_k z^k \text{ such that } \sum_{-\infty}^{\infty} |a_k|^2<\infty. \tag{2.8}$$

Scalar product $<a|b>=: \sum a_i \overline{b_i}$ turns $\mathcal{H}$ into a Hilbert space. When $a(z)$ and $b(z)$ are defined on $|z|=1$, this scalar product can also be written in the integral form:

$$<a|b>=\frac{1}{2\pi i}\int_{|z|=1} a(z)\overline{b}(z^{-1})\frac{dz}{z}. \tag{2.9}$$

To any function

$$F(z)=\sum_{-\infty}^{\infty} f_i z^i \text{ such that } \sup|f_i|<\infty,$$

corresponds an operator of multiplication by this function,

$$a(z)\rightarrow F(z)a(z),$$

that maps H to H. To distinguish operators from functions, the operators will have a multiplication sign in the subscript: $F_\times$. So, function F maps complex numbers to complex numbers, and operator $F_\times$ maps the Hilbert space $\mathcal{H}$ to itself.

Finally, let $\mathcal{H}_+$ be a subspace of series with non-negative powers, $\mathcal{H}_-$ its orthogonal complement, and P_+ the orthogonal projector on $\mathcal{H}_+$.

Using this notation it is easy to write the expectation and variance of portfolio Π that holds Y_n in each of the securities S_n :

$$E(\Pi) = <\widehat{E}|\widehat{Y}> \tag{2.10}$$

$$\mathrm{Var}(\Pi) = <\widehat{Y}|P_+A_\times\widehat{Y}>, \tag{2.11}$$

where

$$A(z)=\widehat{C}(z^{-1})+\widehat{C}(z)-1. \tag{2.12}$$

Therefore,

$$U(\Pi)=<\widehat{E}|\widehat{Y}> -\gamma <\widehat{Y}|P_+A_\times\widehat{Y}>. \tag{2.13}$$

If operator $P_+A_\times$ is invertible, the solution to this problem is:

$$\widehat{Y} = \frac{1}{2\gamma}\left[P_+A_\times\right]^{-1}\widehat{E}, \tag{2.14}$$

$$U = \frac{1}{4\gamma}<\widehat{E}|\left[P_+A_\times\right]^{-1}\widehat{E}>.$$

If the operator is non-invertible, its kernel is non-empty, and it may signal an arbitrage opportunity: Indeed if $\ker P_+A_\times$ is non-orthogonal to $\widehat{E}$, then the investor can achieve an arbitrary large level of utility.

Finding the solution requires methods for estimating the correlation function, $\widehat{C}(z)$ and for inverting the operator $P_+A_\times$. We will address these problems in two next sections.

3 Estimating Correlation

One method to estimate the correlation function is to compute empirical correlations, C_n, and fit them with a rational function. In more detail, let $P_M(z)$ and $Q_N(z)$ be a couple of polynomials of degrees M and N, respectively. Let their ratio has the following Tailor expansion:

$$\frac{P_M(z)}{Q_N(z)} = \sum_{i=0}^{\infty} f_i z^i. \tag{3.1}$$

This couple of polynomials is called a Padé approximation to the sequence $\{C_n\}$ if the first $N+M+1$ coefficients of the Tailor expansion coincide with the first terms in $\{C_n\}$:

$$f_i = C_i \text{ for } i = 0, 1, ..., M+N. \tag{3.2}$$

Padé approximations can be easily computed by solving a system of linear equations (see Baker and Graves-Morris (1996)).

A generalization of Padé approximations may be helpful when the data is noisy. The generalization requires that the coefficients of the Taylor expansion be only approximately equal to the terms of $\{C_n\}$:

$$f_i = C_i + \varepsilon_i \text{ for } i = 0, 1, ..., M+N+K. \tag{3.3}$$

By definition, the $[M, N, K]-$order generalized Padé approximation minimizes the sum of squared errors in (3.3).

4 Computing the Optimal Portfolio

In the engineering literature the operator $P_+A_\times$ is called a rational filter, and its inversion is a well-known problem. It can be solve by several efficient methods (Kailath et al. (2000)), from which one of the most elegant is given by the Wiener-Hopf factorization. Let $\ln A(z)$ be decomposed as follows:

$$\ln A(z) = A_+(z) + A_-(z), \text{ where } a_+(z) \in \mathcal{H}_+ \text{ and } a_-(z) \in \mathcal{H}_-. \tag{4.1}$$

Then the Wiener-Hopf factorization theorem claims that

$$[P_+A_\times]^{-1} = [\exp(-A_+)]_\times P_+ [\exp(-A_-)]_\times . \tag{4.2}$$

See Lax (2002) for the proof.

Example 4.1 *Let$A(z)$ be a ratio of polynomials:*

$$A(z) = a_0 \frac{\prod_{i=1}^{M}(z-\theta_i)}{\prod_{j=1}^{N}(z-\eta_i)}.$$

Let θ_i^+ and η_i^+ be zeros and poles outside the unit circle, and θ_i^- and η_i^- be zeros and poles inside the unit circle. Then

$$\begin{aligned} A_+(z) &= \ln\left[a_0 \frac{\prod(z-\theta_i^+)}{\prod(z-\eta_i^+)}\right], \\ A_-(z) &= \ln\left[\frac{\prod(z-\theta_i^-)}{\prod(z-\eta_i^-)}\right]. \end{aligned}$$

The technique of the Wiener-Hopf factorization allows writing analytic expressions for the optimal bond portfolio allocation and corresponding utility:

Theorem 4.1 *The optimal allocation is*

$$\widehat{Y} = \frac{1}{2\gamma}\left[\exp(-A_{+)}\right]_\times P_+ \left[\exp(-A_-)\right]_\times \widehat{E}$$

The corresponding utility function is

$$U = \frac{1}{4\gamma} < \widehat{E}| \left[\exp(-A_+)\right]_\times P_+ \left[\exp(-A_-)\right]_\times \widehat{E} > .$$

Proof: This theorem is a direct consequence of the Wiener-Hopf factorization theorem and expressions for optimal portfolio and utility (2.14).

Example 4.2 *AR(1) correlations and expectations*

Let correlations between bond returns be as they are in AR(1) time series model:

$$\widehat{C}(z) = 1 + \sum_{i=1}^{\infty} \alpha^i z^i = \frac{1}{1-\alpha z}. \tag{4.3}$$

Then

$$\begin{aligned} A(z) &= \widehat{C}(z^{-1}) + \widehat{C}(z) - 1 = \frac{1-\alpha^2}{(1-\alpha z)(1-\alpha z^{-1})}, \qquad (4.4) \\ A_+ &= \ln\frac{1-\alpha^2}{1-\alpha z}, \text{ and } A_- = \ln\frac{1}{1-\alpha z^{-1}}. \end{aligned}$$

Therefore,

$$\left[P_+ A_\times\right]^{-1} = \frac{1}{1-\alpha^2}(1-\alpha z)_\times P_+ (1-\alpha z^{-1})_\times. \tag{4.5}$$

Assume also for the purposes of this example that the normalized expectations for bonds with longer maturities are smaller – perhaps because of large variance of the returns on longer maturity bonds. More precisely, let the normalized expectations decline exponentially:

$$\widehat{E}(z) = E_0(1 + \sum_{i=1}^{\infty} \beta^i z^i) = \frac{E_0}{1-\beta z}, \tag{4.6}$$

where $\beta < 1$ is the rate of decline. Then, according to Theorem 4.1,

$$\widehat{Y} = \frac{E_0}{2\gamma}\frac{1-\alpha\beta}{1-\alpha^2}\frac{1-\alpha z}{1-\beta z}, \tag{4.7}$$

and

$$\begin{aligned} U &= \frac{E_0^2}{4\gamma}\frac{1-\alpha\beta}{1-\alpha^2} < \frac{1}{1-\beta z} | \frac{1-\alpha z}{1-\beta z} > \\ &= \frac{E_0^2}{4\gamma}\frac{1-\alpha\beta}{1-\alpha^2}\frac{1}{2\pi i}\int_{|z|=1}\frac{1}{1-\beta z^{-1}}\frac{1-\alpha z}{1-\beta z}\frac{dz}{z} \\ &= \frac{E_0^2}{4\gamma}\frac{(1-\alpha\beta)^2}{(1-\alpha^2)(1-\beta^2)}. \end{aligned} \tag{4.8}$$

Note the symmetry of the expression relative to parameters that govern expectations and correlations of bond returns. The symmetry shows neatly that the investor need to take into account both the expectations and correlations of future returns.

5 Application

We use Treasury interest rates data by J. Huston McCulloch that represent the 67 months from 8/1985 to 2/1991. These data give the zero-coupon yield curve implicit in coupon bond prices. The yields have been defined for each month using interpolation by cubic splines. From these data the returns on holding a particular bond for one month have been computed.

The correlations have been estimated according to the formula

$$C(\tau) = \frac{1}{N(s)N(t)}\sum_{s,t}(E_{s,t} - \overline{E_{s,t}})(E_{s,t+\tau} - \overline{E_{s,t+\tau}}), \tag{5.1}$$

where $E_{s,t}$ are returns normalized by their standard deviation, and $N(s)$ and $N(t)$ are number of dates and maturities available for estimation.

Figure 1 shows the actual estimates of the correlations and the correlations fitted by a Padé approximation. Figure 2 shows correlations fitted by a generalized Padé approximation. From the comparison of these figures, it is clear that the traditional approximation is good for small differences in maturities but severely underestimate the correlation between bonds with larger difference in maturities. The generalized Padé approximation is more balanced in the sense that it approximates equally well the correlations for all differences in maturities. On the other hand, the generalized approximation underestimate the correlations between bonds with small difference in maturities.

As usual, estimating expected returns is more tricky than estimating covariances. In particular, it depends on what theory the researcher holds about formation of interest rates. One possibility, assumed here for the purposes of illustration, is that expectations of the future interest rate curve coincide with the current interest rate curve. This allows to estimate expected return as follows:

$$ER(t,s) = \log \frac{P(t-1,s)}{P(t,s)}, \tag{5.2}$$

where $P(t,s)$ is the price of the bond with maturity t at time s. It should be emphasized that this is only a possible choice among many others. It is appropriate for illustrative purposes because of its simplicity.

Figure 3 compares a benchmark with results of investment in the portfolio optimized by using the generalized Pade approximation. The benchmark is the performance of the portfolio optimized using the assumption that the bond returns are uncorrelated. In this example, the risk-aversion parameter is chosen in such a way that the sum of investments is equal to 1.

From Figure 3 it is clear that the optimal portfolio performs much better than the benchmark portfolio. It has much lower variance and its monthly returns are always positive in a striking difference with the returns of benchmark portfolio. These results suggest that properly modelling the correlation structure pays off.

6 Conclusion

An investor entering the business of bond portfolio management faces the business that was regulated as early as in the time of Hammurabi when the law required putting to death as a thief any man who received a deposit from a minor or a slave without power of attorney, but also the business that is still shaken by unpredictable bubbles and demises of huge financial institutions, the business that attracts more bright mathematicians and physicists than all mathematics and physics departments in the country, that is operated by traders who play toy machine guns during their lunch, and that demands deeper insight than stock portfolio management will ever require, – but when he faces this fascinating business, the investor may perhaps be comforted by the thought that the business is the most scientific and precise in the whole area of financial speculation.

The present article is a contribution that shows how engineering techniques can be applied to calculating optimal bond portfolios. Pade approximations are used to estimate correlations between returns on bonds of different maturities and the Wiener-Hopf factorization technique is used for portfolio optimization. Preliminary empirical investigation show that the approach is practically useful.

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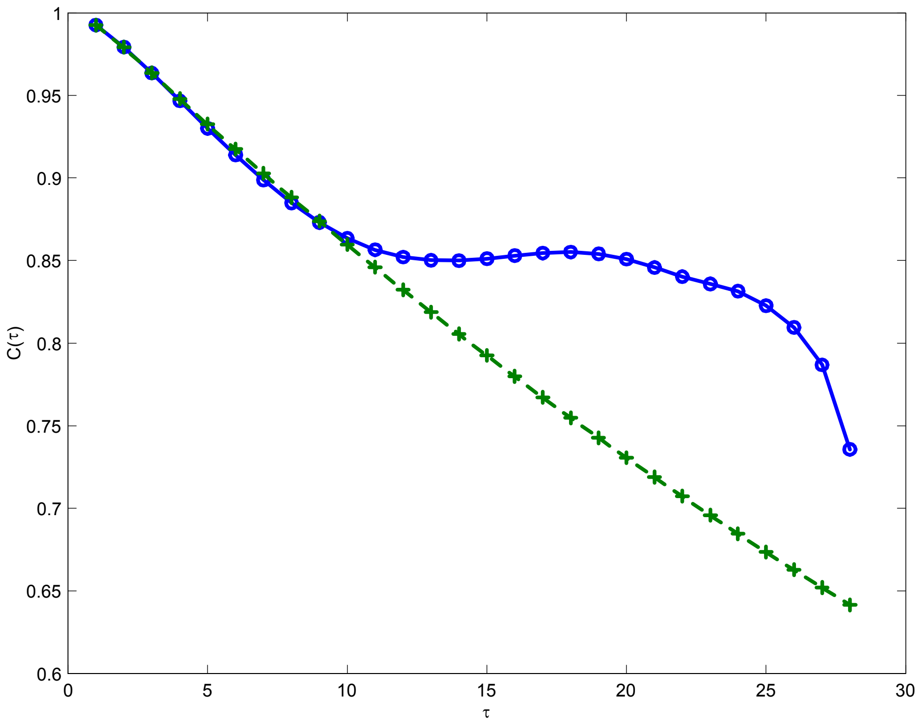
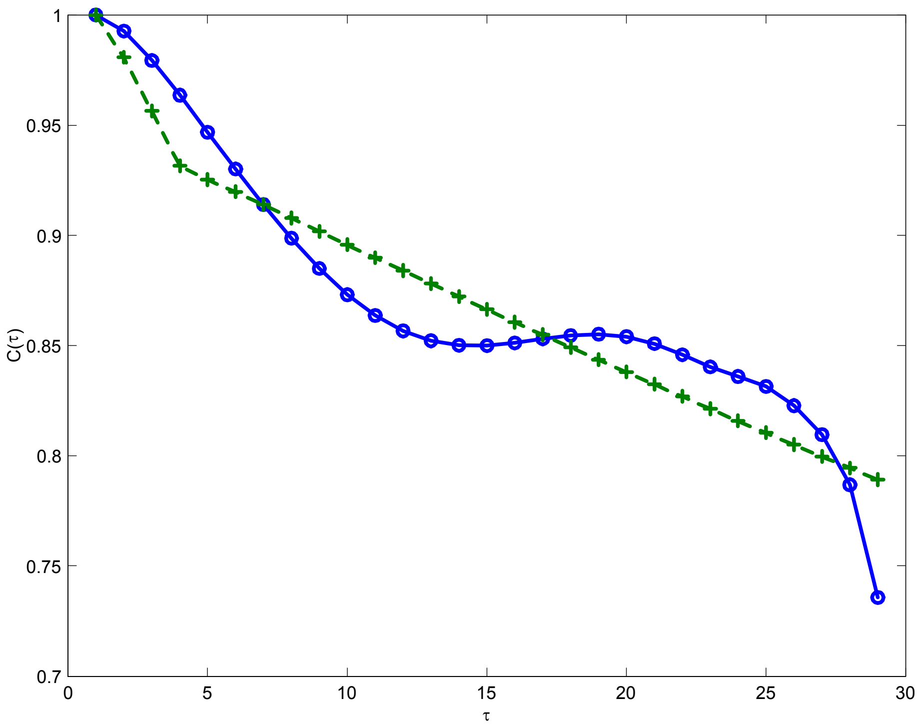
Figure 1: Actual and Fitted Correlation Functions – Pade Approximation



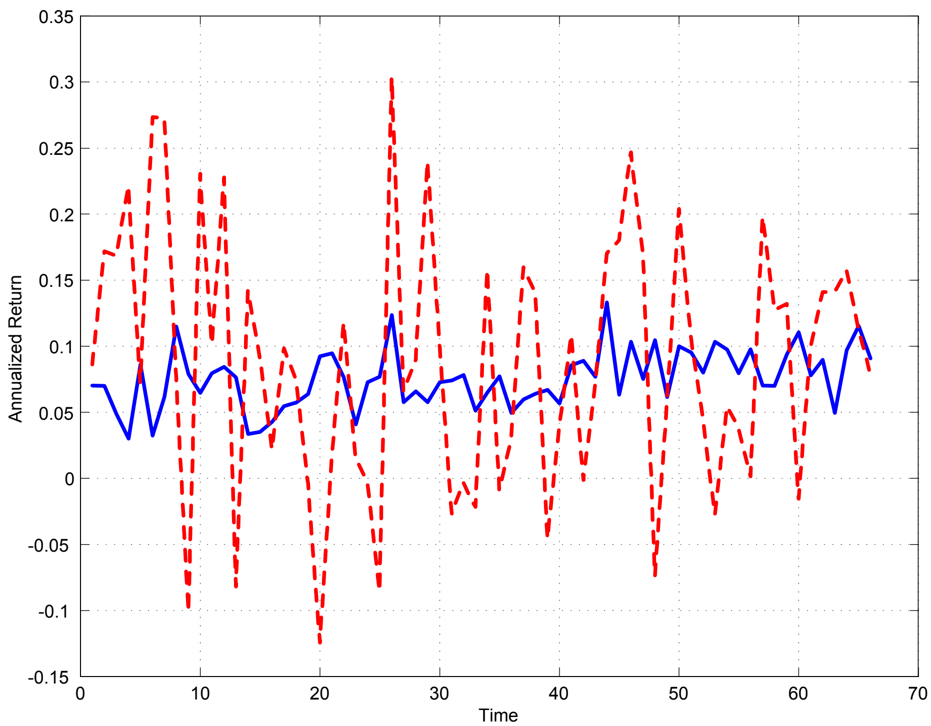
The solid line marked by circles is the actual correlation estimates. The dashed line marked by pluses is the fitted correlations from the Padé [1/2] approximation. The vertical axis shows size of correlations. The horizontal axis shows the differences between maturity times measured in years.

Figure 2: Actual and Fitted Correlation Functions – Generalized Pade Approximation



The solid line marked by circles is the actual correlation estimates. The dashed line marked by pluses is the fitted correlations from the generalized Padé [0/5/28] approximation. The vertical axis shows size of correlations. The horizontal axis shows the differences between maturity times measured in years.

Figure 3: Returns for Optimal and Benchmark Portfolios



The solid line shows the returns of the optimal portfolio computed using [0/5/28] generalized Padé approximation. The dashed line is the returns of the benchmark portfolio computed under assumption of zero correlations between bond returns. The horizontal axes shows time in months; the vertical axes shows annualized monthly returns.