UTILITY BASED PRICING OF CONTINGENT CLAIMS IN INCOMPLETE MARKETS

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Abstract. In a discrete setting, we develop a model for pricing a contingent claim in incomplete markets. Since hedging opportunities influence the price of a contingent claim, we first introduce the optimal hedging strategy assuming that a contingent claim has been issued: a strategy implemented by investing initial wealth plus the selling price is optimal if it maximizes the expected utility of the agent's net payoff, which is the difference between the outcome of the hedging portfolio and the payoff of the claim.
Next, we introduce the 'reservation price' as a subjective valuation of a contingent claim. This is defined as the minimum price that makes the issue of the claim preferable to stay put given that, once the claim has been written, the writer hedges it according to the expected utility criterion. We define the reservation price both for a short position (reservation selling price) and for a long position (reservation buying price) in the claim. When the contingent claim is redundant, both the selling and the buying price collapse in the usual Arrow-Debreu (or Black-Scholes) price. If the claim is non-redundant, then the reservation prices are interior points of the bid-ask interval. We provide also two numerical examples with different utility functions and contingent claims. Some qualitative properties of the reservation price are shown.

Keywords. Contingent claims, incomplete markets, reservation price, expected utility, optimization.

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1. Introduction

In this paper we develop a model for hedging and pricing a non-redundant contingent claim when the financial market is incomplete.

Hedging and pricing are two sides of the same problem. In a complete financial market, according to the 'replication approach', the price of a contingent claim is the cost of the hedging portfolio. On the other hand, by the 'martingale approach', the price is the present value of the random payoff with respect to the unique state (or Arrow-Debreu) price vector. This duality between the 'replication' and the 'martingale' approaches permits to solve the pricing problem easily under market completeness. It is less obvious that the duality can be exploited also when the market is incomplete and the claim to be priced is redundant.

Essentially, given a contingent claim hedging and pricing are the same problem solved in two different spaces: hedging works in the space of portfolios; pricing in the space of payoffs. When the contingent claim is non-redundant, the duality relation cannot be usefully exploited, because it does not provide a unique price. Actually, on portfolios side, there is no replicating portfolio and the hedging strategy could involve a risky position; on the payoffs side, there is an infinite number of martingale measures and each of them provides a different price for the contingent claim. This leads to an interval where the minimum is the 'ask price' and maximum is the 'bid price'. In correspondence to each price of the bid-ask interval there is a hedging strategy, once the hedging criterion has been chosen.

Assume that a new (non-redundant) contingent claim is issued by, say, an investment banker so that a proper price for this claim should be found. The *bid price* is the minimum cost of a super-replicating portfolio for a short position in the claim and the *ask price* is the maximum cost of a sub-replicating portfolio for a long position in the claim. If the potential counterparts of the writer are risk averse, no one would buy a claim offered at the bid price (and a risk averse writer would not sell the claim at the ask price). A risk averse agent will buy the claim at a price lower than the bid price and will sell at a price higher than the ask price. If the claim is issued for a price lower than the bid price, the writer must commit his wealth in the deal because the hedging strategy becomes risky.

In the financial literature several models have been proposed to partially hedge the claim if the writer accepts a risky hedging strategy. The first was the risk-minimizing strategy proposed by Föllmer and Sondermann in their 1986 seminal article [11]. According to this model, given a European non-redundant contingent claim, the writer's goal is the minimization of the quadratic additional cost of revision of the hedging portfolio at each trading date. The most important features of Föllmer and Sondermann's model are that the hedging portfolio can be obtained by backward recursion and that the strategy is mean self-financing. Other models based on the same idea have been proposed by Schweizer [17] and Schäl [16]. All these models are limited mainly by the fact that, according to a quadratic criterion, both positive and negative net payoffs resulting from the hedging strategy are assumed to be equally disliked by the agent. One of the consequences of this drawback is that, in a general setting, pricing a non-redundant contingent claim by taking the cost of the hedging portfolio can lead to a negative price even if the claim has a non-negative payoff. Obviously, this is not consistent with the absence of arbitrage opportunities.

Since risk is involved, a hedging criterion based on the maximization of the expected utility for the net payoff of the writer comes up as the most natural solution. A model based on a utility gradient approach is due to Davis [7], exploiting the idea first offered in Lucas [14] (see also Duffie and Skiadas [10]). Davis' model proposes a (fair) price for the contingent claim based on the assumption that the agent, an expected utility maximizer, is willing to divert only an *infinitesimal* amount of his initial wealth to sell (or to purchase, as for the buyer) the claim. Once the fair price has been determined, Davis finds out the optimal hedging strategy for a *given* (not necessarily infinitesimal) number of units of the claim by maximizing the expected utility of the net payoff.

The problem of risky hedges has been addressed also in the literature on pricing and hedging in presence of transaction costs. Actually, if there are transaction costs, the perfect replication of a contingent claim could not be an efficient strategy. It has been proved by Bensaid, Lesne, Pagés and Scheinkman [3] that transaction costs can make it cheaper to dominate than to replicate a claim. This amounts to say that transaction costs create market incompleteness since not all risks can be hedged; moreover, the issue of a derivative security when the market presents frictions involves unavoidable risks. Since the hedging of the contingent claim involves risk, then its price (i.e. the cost of the hedging portfolio) depends on the risk attitude of the agent. The concept of *reservation price* for a contingent claim (Andersen and Damgaard [1], Clewlow and Hodges [5], Davis, Panas and Zariphopoulou [8], Hodges and Neuberger [13]) takes into account both the agent's preferences and the financial market structure.

More specifically, the *reservation selling price* of a claim is defined as the price that makes the following two situations equivalent from the writer's standpoint: (a) writing a prespecified number of the claim and hedging the liability deriving from the deal with a portfolio of existing assets financed with the revenue of the sale of the claim, and (b) leaving his wealth optimally invested in the existing assets. The *reservation buying price* can be defined in the same way from a buyer viewpoint. It should be clear that the price is, as a rule, dependent on the number of claims issued and on the wealth, utility and beliefs of the writer.

The reservation price above defined is different from Davis' *fair price.* The fair price of the claim does not depend on the number of units written, as should be conceivably expected, and the hedging strategy is consistent with the price only if the number of units sold is "infinitesimal". We will show that the reservation price is consistent with Davis' fair price when the quantity issued is infinitesimal.

We frame the model in a discrete-time discrete-state space setting. We examine the problem of hedging and pricing a non redundant contingent claim mainly from the viewpoint of the writer. Hence, throughout the paper we will be talking of "hedging a liability" because (once issued) the contingent claim gives the writer the obligation to pay a state-contingent payoff. The description of the buyer's point of view is straightforward and can be easily derived from the results we present below.

The paper is organized as follows: after introducing the notation (Section 2), we first study the agent's hedging problem assuming that he has already issued a given amount of contingent claims (Section 3). We provide also some sufficient conditions on the agent's preferences and on the financial market to make the problem meaningful (Section 4). Next, we introduce the reservation price and study its properties. In particular we show that the reservation price is the Arrow-Debreu price if the claim is redundant, that the reservation price is always greater than the ask price and lower than the bid price and that it is consistent with Davis' *fair price* if the quantity traded is infinitesimal (Section 5). In the same section we provide also two numerical examples regarding option pricing in incomplete markets. We finally give some concluding remarks in Section 6. Proof of propositions are relegated in the Appendix A. In Appendix B, we describe the non-trivial procedure used to calculate the reservation price in the examples.

2. Notation

The notation we introduce is standard in financial economics.[1] Let there be given $T+1$ dates $\{t = 0, 1, \dots, T\}$. We assume that an agent can correctly anticipate the states of nature: let there be given S states of nature $\Omega = \{\omega_1, \dots, \omega_S\}$. Uncertainty concerns the prevailing state at T. We assume that the agent's information unfolds gradually as time proceeds: $\{\mathcal{F}_t, t = 0, \dots, T\}$ is the sequence of partitions of Ω.

This can be described by means of an event-tree. The couple (t, A_t) represents a node at time t with A_t in $\mathcal{F}_t$. At $t = 0$ there is one node; at each $0 < t \leq T$ there are $\mathcal{N}_t$ nodes ($\mathcal{N}_T = S$ is the cardinality of Ω). Let $\mathcal{N} = \sum_{t=1}^{T} \mathcal{N}_t$ be the number of non-initial nodes of the event-tree (the total number of nodes being $\mathcal{N} + 1$). We will number the nodes of the tree from 0 to $\mathcal{N}$, where 0 is the initial node, the subsequent $\mathcal{N}_1$ are the nodes at time 1, the next $\mathcal{N}_2$ are the time-2 nodes and so on.

Given a node $\xi = (t, A_t)$, $t < T$,

$$\xi^{\rightarrow} = \left\{ \xi' \mid \xi' = (t+1, A_{t+1}), A_{t+1} \subset A_t \right\}$$

denotes the *set of immediate successors* of node ξ. The cardinality of $\xi^{\rightarrow}$ is the *branching number* of node ξ. The *sub-tree* of a node ξ is the set of nodes of the tree following ξ: formally, the sub-tree of node $\xi = (t, A_t)$ is the set of nodes (τ, A_τ) such that $\tau > t$ and $A_\tau \subset A_t$.

Let there be a frictionless financial market where K financial contracts are traded. A financial security is characterized by the sequence of its prices at the nodes of the sub-tree of the node of issue. *We assume that the securities pay no dividend* and that the prices of securities are exogenously given.

[1] For a reference, see Duffie and Shafer [9] or Magill and Quinzii [15].

Let[2] $\{P_k(\xi), \xi = 0, \ldots, \mathcal{N}\}$ be the prices of security k at the nodes of the event-tree, $k = 1, \ldots, K$. We can consider, with no loss of generality, the number of securities available for trading as a constant throughout the event-tree.

A long term position in security k can be split into a sequence of two-period strategies. At each node $\xi = (t, A_t)$, with $t = 0, \ldots, T-1$, (at the terminal date T the securities cannot be traded) the security k is bought for $-P_k(\xi)$ and, in each subsequent node ξ', the security is sold for $P_k(\xi')$. If there is no transaction cost, assuming that the portfolio is rebalanced at each trading date does not reduce the generality of the model.

Let $\mathbf{P}(\xi) = (P_1(\xi), \ldots, P_K(\xi))$ be the (row) vector of prices of securities in node ξ at time $t = 0, \ldots, T-1$ and let

$$\mathbf{P}(\xi^{\rightarrow}) = \left(\mathbf{P}(\xi')\right)_{\xi' \in \xi^{\rightarrow}}$$

be the matrix of prices in the successors of node ξ. The rank of $\mathbf{P}(\xi^{\rightarrow})$ is called the *spanning number* of node ξ. Obviously, the spanning number can not be strictly greater than the branching number.

Let $\mathcal{N}^- = \sum_{t=0}^{T-1} \mathcal{N}_t$ be the number of non-final nodes of the event-tree; i.e. $\mathcal{N}^-$ is the number of nodes where the securities are traded. Let $\mathcal{M} = K\mathcal{N}^-$. We represent the financial market by the following $(\mathcal{N}+1) \times \mathcal{M}$ matrix

$$\mathbf{W} = \begin{pmatrix} -\mathbf{P}(0, A_0) & \cdots & \mathbf{0} & \cdots \\ \mathbf{P}\left(1, A_1^1\right) & \cdots & \mathbf{0} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \mathbf{P}\left(1, A_1^{\mathcal{N}_1}\right) & \cdots & \mathbf{0} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \mathbf{0} & \cdots & -\mathbf{P}(\xi) & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \mathbf{0} & \cdots & \mathbf{P}(\xi^{\rightarrow}) & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

[2]With a slight abuse of notation we identify the node with its number.

Consider the first K columns of this matrix: the first row represents the investment in one unit of each of the K securities at the initial node of the event-tree. The following $\mathcal{N}_1$ rows are the payoff of the subsequent sale in each node at $t = 1$. The same meaning can be given to the K columns of node ξ.

Sometimes it is easier to refer to the following representation of $\mathbf{W}$:

$$\mathbf{W} = \begin{pmatrix} -\mathbf{W}_0 \\ \mathbf{W}_1 \end{pmatrix}$$

where $\mathbf{W}_0$ is the (row) vector of initial prices augmented by a vector of zeroes and $\mathbf{W}_1$ is the matrix of prices at non-initial nodes. Throughout the paper we assume that $\mathbf{W}_1$ has *full rank*, that is at any date-state couple the prices of the securities are linearly independent. This is not restrictive: if it were not the case, we could simply skip the redundant assets.

A portfolio $\mathbf{x}$ will be a (column) vector in $\mathbb{R}^{\mathcal{M}}$, whose components are the holdings of the K securities at the nodes of the event-tree. The vector $\mathbf{x}$ can be decomposed as

$$\mathbf{x} = \big(\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(\mathcal{N}^-)\big)^\top$$

where the K-vector $\mathbf{x}(\xi)$ is the portfolio at the node ξ.

In what follows it will be assumed that the agent can take both long or short positions in all securities at any node. Given a portfolio $\mathbf{x}$, $\mathbf{W}_0\mathbf{x}$ is the initial cost and $\mathbf{W}_1\mathbf{x}$ is the payoff produced in the subsequent dates by $\mathbf{x}$. At an intermediate date, the agent changes his positions in the assets selling the portfolio he has been holding for one period and buying a new portfolio. At the final date the agent must sell the portfolio and consume his wealth. Hence, the payoff in node ξ can be described as follows:

$$(2.1) \qquad (\mathbf{W}_1\mathbf{x})_\xi = \begin{cases} \mathbf{P}(\xi)\mathbf{x}(\xi^-) - \mathbf{P}(\xi)\mathbf{x}(\xi) & \text{if } \xi \in \mathcal{F}_t, t = 1, \dots, T-1 \\ \mathbf{P}(\xi)\mathbf{x}(\xi^-) & \text{if } \xi \in \mathcal{F}_T \end{cases}$$

where ξ^- is the (unique) predecessor of node ξ and $(\cdot)_\xi$ denotes the ξ-th component of a vector. Equation (2.1) holds because of the particular "block" structure of matrix $\mathbf{W}$.[3]

3. Hedging a non-redundant contingent claim

Assume that there is an agent with time-separable and Von Neumann-Morgestern preferences, with a smooth utility function $u(\cdot)$ such that $u' > 0$ (strict non-satiation) and $u'' < 0$ (strict risk-aversion) and with subjective probability

$$\mathbf{p} = (\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^T)$$

at the various dates. $\mathbf{p}^t = (p^t(1), \dots, p^t(\mathcal{N}_t))$ is the vector of unconditional probabilities of nodes (t, A_t), $A_t \in \mathcal{F}_t$ at time t.

The agent has an initial wealth v_0 and, in order to maximize the expected utility for his wealth in the futures dates, he invests v_0 in the financial market. The cash flow produced by this strategy is $\mathbf{v}_1$, and must be non-negative.[4] The expected utility of $\mathbf{v}_1$ at $t = 0$ is

$$U(\mathbf{v}_1) = d(1) \sum_{\xi=1}^{\mathcal{N}_1} p^1(\xi) u(v^1(\xi)) + \ldots + d(T) \sum_{\xi=1}^{\mathcal{N}_T} p^T(\xi) u(v^T(\xi)) \tag{3.1}$$

where $v^t(\xi)$ is the wealth at time t and node $\xi \in \mathcal{F}_t$ and $0 < d(t) \leq 1$ is the discount factor.

The investment problem at time $t = 0$ is:[5]

$$\max_{\mathbf{z}} \{ U(\mathbf{v}_1) \mid \mathbf{v}_1 = \mathbf{W}_1 \mathbf{z},\ \mathbf{W}_0 \mathbf{z} = v_0,\ \mathbf{v}_1 \geq \mathbf{0} \}. \tag{3.2}$$

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[3] See Example 1 in Section 5.

[4] We can model different strategies: for instance, if we constrain the payoff to be equal to zero for all intermediate dates and non negative at the last date, we would have a self-financing portfolio strategy.

[5] The investment problem could have been written as $\max_{\mathbf{z}} \{ U(\mathbf{v}_1) \mid \mathbf{v}_1 = \mathbf{W}_1 \mathbf{z},\ \mathbf{W}_0 \mathbf{z} \leq v_0,\ \mathbf{v}_1 \geq \mathbf{0} \}$. The assumption on non satiation of the agent makes the two forms equivalent.

Let $\mathbf{z}^* \in \mathbb{R}^{\mathcal{M}}$ be the portfolio strategy[6] which produces the optimal payoff: $\mathbf{v}_1^* = \mathbf{W}_1\mathbf{z}^*$.

Now, assume that the agent is willing to write a contingent claim. In doing so, the agent will incur the obligation to pay an amount $l(t, A_t) \geq 0$ for all $t = 1, \ldots, T$: this time-and-state dependent liability is represented by a (column) vector $\mathbf{l}_1 \in \mathbb{R}^{\mathcal{N}}$. Assume that the agent sells *a given number* $\delta \neq 0$ of units of the contingent claim for l dollars a piece. The agent's goal is to hedge $\delta\mathbf{l}_1$ with a portfolio financed with the revenue obtained by writing the claims: δl. If the claim in non-redundant, the deal involve some risk for the writer.

The risk of the writer can be reduced in several ways. The most obvious of these is a super-replication strategy (perfect hedging): the selling price should be high enough to buy a self-financing portfolio whose payoff is at any date and state not lower then the liability. This is an insurance strategy: the agent writes the claim but wants to keep his future wealth unaffected by the deal. It is easy to see that the price that permits the writer to insure its wealth at any node is too high to be paid by any risk-averse buyer.

On the contrary, if the selling price, l, of the claim is too low, the deal turns to be attractive for some risk-averse counterpart but the writer can possibly commit his wealth in the deal in some unfavourable state of nature.

There are two problems facing the writer. The first one is the valuation of the claim in order to establish the minimum price that makes issuing a good trade. Let's call this *reservation selling price*. Obviously, any price above the reservation price would make selling the claim more and more desirable. The second problem is the choice of a portfolio financed with δl to partially hedge the liability. The agent will honor the liability at each date and state with the payoff of the hedging portfolio and, if needed, with his own wealth.

[6] In Section 4 we will give conditions such that Problem (3.2) is meaningful and the solution is unique.

The solution of the hedging problem influences the valuation problem: actually, the reservation price depends on the hedging opportunities offered in the financial market. If the claim is redundant, the reservation price must be equal to the cost of the hedging (replicating) portfolio if arbitrage opportunities are ruled out. In case of non-redundancy, the agent may find it optimal to partially hedge the claim and then to pay the liability also with his own wealth. In this case, the reservation price depends on the risk attitude and on the wealth of the agent.

For the sake of exposition, the two steps of the deal will be considered in reverse order: first we will be concerned with the selection of a hedging portfolio assuming that the claim has already been issued. This provides the optimal solution as a function of the selling price, considered as a parameter. Next, we will provide the reservation price of the claim as the value of the parameter that leaves unaffected the optimal expected utility of the payoff.

As for the hedging strategy, the agent uses the (whole) revenue δl to buy a portfolio $\mathbf{x}$ to hedge the liability $\mathbf{l}_1$. The net payoff of the deal, denoted $\mathbf{w}_1 = \mathbf{W}_1\mathbf{x} - \delta\mathbf{l}_1$, is added to the optimal payoff $\mathbf{v}_1^*$ obtained by the investment of initial wealth v_0. Hence, the agent does not re-discuss the optimal allocation of the initial wealth as a consequence of the deal, but can resort to his state-and-time contingent wealth to finance a loss in some unfavorable nodes and can benefit of any possible gain. A natural constraint is $\mathbf{v}_1^* + \mathbf{w}_1 \geq \mathbf{0}$; that is, the agent can resort only to his wealth to pay the liability.

Summing up, the hedging problem is

$$\max_{\mathbf{x}} \{U(\mathbf{v}_1^* + \mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x} - \delta\mathbf{l}_1, \mathbf{W}_0\mathbf{x} = \delta l, \mathbf{v}_1^* + \mathbf{w}_1 \geq \mathbf{0}\}\,. \tag{3.3}$$

Remark 3.1. Problem (3.3) is equivalent to

$$(\mathcal{H}) \qquad \max_{\mathbf{y}} \{\, U(\mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{y} - \delta\mathbf{l}_1,\ \mathbf{W}_0\mathbf{y} = v_0 + \delta l,\ \mathbf{w}_1 \geq \mathbf{0}\,\}\,.$$

To see that this is true, let it be given $\mathbf{z}^*$, the optimal portfolio for problem (3.2). Since the optimal contingent wealth is traded, by plugging $\mathbf{v}_1^* = \mathbf{W}_1\mathbf{z}^*$ in (3.3) we obtain:

$$\max_{\mathbf{x}} \left\{ U(\mathbf{W}_1 (\mathbf{z}^* + \mathbf{x}) - \delta \mathbf{l}_1) \mid \mathbf{W}_0\mathbf{x} = \delta l, \mathbf{W}_1 (\mathbf{z}^* + \mathbf{x}) - \delta \mathbf{l}_1 \geq \mathbf{0} \right\}$$

and, letting $\mathbf{y} = \mathbf{z}^* + \mathbf{x}$ and observing that $\mathbf{W}_0\mathbf{z}^* = v_0$, problem $(\mathcal{H})$ follows immediately.

Problem $(\mathcal{H})$ and problem (3.2) have the same structure. Assuming that $\mathbf{z}^*$ is known, once the optimal portfolio $\mathbf{y}^*$ for problem $(\mathcal{H})$ has been found, the hedging portfolio $\mathbf{x}^*$ is given by $\mathbf{y}^* - \mathbf{z}^*$. We will find is easier to solve problem $(\mathcal{H})$ than problem (3.3).

Observe that problem $(\mathcal{H})$ can model also a partial commitment of the agent in the deal: assume that the agent's purpose, for the sake of prudence, is to put only a small fraction of his initial wealth in the deal of writing a contingent claim. Denote b the budget he is going to put in the deal, $b < v_0$. As in the case of total commitment ($b = v_0$), the budget can be used to finance a portfolio whose payoff maximizes the expected utility for state contingent wealth. If the agent writes a contingent claim, he will find out the optimal hedging portfolio by solving problem $(\mathcal{H})$ with b in place of v_0. Moreover, if the deal is marginal with respect to the agent's total wealth and the agent partially commits with a budget $b \ll v_0$, then we can assume that the agent will always be so well off to be able to pay any outcome from writing the claim. Formally, if the deal is marginal with respect to the agent previous investment, the net payoff always satisfies the budget constraint and the nonnegativity constraint in problem $(\mathcal{H})$ can be dropped (see also Hodges and Neuberger [13]).

With small changes we can model also the problem of the buyer of the claim. The investment problem from the buyer's viewpoint has the same structure of problem $(\mathcal{H})$ with $-\delta$ in place of δ. Given $\mathbf{v}_1^*$, the payoff of the optimal portfolio obtained by investing v_0 (see problem in (3.2)), the buyer problem's is

$$\max_{\mathbf{x}} \{U(\mathbf{v}_1^* + \mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x} + \delta\mathbf{l}_1, \mathbf{W}_0\mathbf{x} = -\delta l, \mathbf{v}_1^* + \mathbf{w}_1 \geq \mathbf{0}\} \tag{3.4}$$

and $\mathbf{x}$ is a portfolio which alters the previous positions in the existing assets. The equivalence between problem (3.4) and $(\mathcal{H})$ with $-\delta$ can be proved with the argument used above.

4. Existence of a solution and no-arbitrage

In this section we will provide conditions for existence of an optimal solution to Problem $(\mathcal{H})$.[7] These conditions are mainly related to the absence of arbitrage opportunity in the financial market.[8] This is related also to the existence of strictly positive price (row) vectors $\boldsymbol{\pi} = (1, \boldsymbol{\pi}_1) \in \mathbb{R}_{++}^{\mathcal{N}+1}$ called *state-price vectors* ($\boldsymbol{\pi}_1$ is the part of $\boldsymbol{\pi}$ which refers to non-initial nodes) such that $\boldsymbol{\pi}\mathbf{W} = \mathbf{0}$.

A financial market is *complete* if any claim can be replicated by a portfolio of traded securities. Hence, a market is incomplete if there is at least one claim that cannot be replicated by trading in the existing securities. It is known that the financial market $\mathbf{W}$ is complete if at each node the spanning number is equal to the branching number [15, Proposition 22.4]. This type of completeness is referred to as *dynamic completeness*; that is, although there are less financial securities (K) than states of nature (S), the market can be completed (in each node) by trading the existing securities if the number of securities is equal to the number of contingencies at each node. A financial market is *incomplete* if at some node ξ the spanning number is strictly smaller that the branching number.

[7] Since Problem $(\mathcal{H})$ and problem (3.3) have the same structure, the conditions apply also to the latter one.

[8] An *arbitrage* is a portfolio $\mathbf{y}$ such that $\mathbf{W}\mathbf{y} \geq \mathbf{0}$ with at least a positive component.

In an arbitrage-free financial market, completeness can be characterized also by the existence of a state-prices vector $\boldsymbol{\pi}$. If the market is incomplete, there is an infinite number of state-price vectors such that $\boldsymbol{\pi}\mathbf{W} = \mathbf{0}$ whereas, if it is complete, there is a unique state price that satisfies that condition.

By relation $\boldsymbol{\pi}\mathbf{W} = \mathbf{0}$, the price at $t < T$ of the k-th security is the present value of its payoff at $t+1$: in node $\xi = (t, A_t)$, the price is

$$P_k(\xi) = \frac{1}{\pi_\xi} \sum_{\xi' \in \xi^{\rightarrow}} \pi_{\xi'} P_k(\xi'). \tag{4.1}$$

Hence $\pi_{\xi'}/\pi_\xi$ is called *node-price*: $\pi_{\xi'}/\pi_\xi$ represents the cost in node ξ of an additional unit of payoff available in node $\xi' \in \xi^{\rightarrow}$.

Remark 4.1. If $\mathbf{l}_1$ is redundant, then there is *only one price*, denoted $q(\mathbf{l}_1)$, compatible with the absence of arbitrage opportunities. We call $q(\mathbf{l}_1)$ the Arrow-Debreu price of the claim.

If there is an arbitrage opportunity, Problem $(\mathcal{H})$ has no solution because a non satiated agent could increase his utility by increasing the position in the arbitrage portfolio. The next proposition states that also the converse is true: if there is no arbitrage, then Problem $(\mathcal{H})$ has a solution.

Proposition 4.2. *Assume that the utility function $u(\cdot)$ is continuous and increasing. Then, Problem $(\mathcal{H})$ has a solution if and only if there is no arbitrage opportunity in the financial market.*

To make problem $(\mathcal{H})$ meaningful, the solution must be unique.

Remark 4.3. If the utility function $u(\cdot)$ is smooth and $u' > 0$ and $u'' < 0$, the solution for Problem $(\mathcal{H})$ is unique.

Throughout the following section we will assume that u satisfies the hypotheses which ensure that the solution for the hedging problem is unique.

5. Reservation price

In this section we introduce the notion of *reservation price* of a contingent claim. Loosely speaking, the reservation price from the point of view of the writer of the claim is the minimum price that makes the deal better than staying put.

Given an agent with utility function u, subjective probability $\mathbf{p}$ and initial wealth v_0, first assume that the agent sells a number $\delta \neq 0$ of contingent claims for l a piece. The time-and-state contingent payoff of the claim is $\mathbf{l}_1$. The related hedging problem is $(\mathcal{H})$: the agent maximizes the expected utility for the net payoff $\mathbf{w}_1 = \mathbf{W}_1\mathbf{x} - \delta\mathbf{l}_1$ of a portfolio strategy $\mathbf{x}$ financed with $v_0 + \delta l$. We denote by $\mathcal{U}(\cdot)$ the (indirect) utility for initial wealth plus the revenue:

$$\mathcal{U}(v_0+\delta l) = \max_{\mathbf{x}}\{\, U(\mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x}-\delta\mathbf{l}_1, \mathbf{W}_0\mathbf{x} = v_0+\delta l,\ \mathbf{w}_1 \geq \mathbf{0}\,\}. \tag{5.1}$$

On the other hand, if he does not write the claim, his initial wealth remains optimally invested according to problem (3.3). In this case, the indirect utility for initial wealth, denoted $\mathcal{V}(\cdot)$, is

$$\mathcal{V}(v_0) = \max_{\mathbf{x}}\{\, U(\mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x}, \mathbf{W}_0\mathbf{x} = v_0,\ \mathbf{w}_1 \geq \mathbf{0}\,\}. \tag{5.2}$$

Obviously, $\mathcal{V}(v_0) = \mathcal{U}(v_0 + \delta l)$ when no claim is written ($\delta = 0$). The following proposition is important in order to define the reservation price.

Proposition 5.1. *The indirect utility function $\mathcal{U}(\cdot)$ is strictly increasing.*

We define the *reservation selling price* of the claim with payoff $\mathbf{l}_1$ and denote it by l^s as the minimum price that makes writing the claim better than doing nothing; that is

$$l^s = \min\{\, l \mid \mathcal{U}(v_0 + \delta l) \geq \mathcal{V}(v_0)\,\}.$$

Needless to say, l^s depends, among the others, both on the initial wealth, v_0, and on the units sold, δ. Since $\mathcal{U}(\cdot)$ is strictly increasing, δl^s is the only increment of the initial wealth due to selling $\delta \mathbf{l}_1$, that leaves the optimal expected utility unchanged. Hence l^s satisfies equation

$$(\mathcal{P}) \qquad \mathcal{U}(v_0 + \delta l^s) = \mathcal{V}(v_0)$$

and is called the *reservation selling price* of a claim with payoff $\mathbf{l}_1$. We observe that l^s does not depend on the representation of the preferences but only on preferences themselves: that is, l^s is unchanged if we take an affine increasing transformation of the utility function u.

The *reservation buying price*, i.e. the reservation price for the buyer, (denoted l^b) can be defined in the same way by putting $-\delta$ in place of δ in condition $(\mathcal{P})$.

The reservation price can be thought of as the certainty equivalent (c.e.) of the random payoff $\mathbf{l}_1$ under the assumption that the agent is optimally investing his wealth v_0 (see Bellini and Frittelli [2]).

The next proposition states that, when the claim with payoff $\mathbf{l}_1$ is redundant, that is, there exists a portfolio $\bar{\mathbf{x}}$ such that $\mathbf{l}_1 = \mathbf{W}_1\bar{\mathbf{x}}$, the reservation selling (and buying) price collapses into the Arrow-Debreu price $q(\mathbf{l}_1)$.

Proposition 5.2. *If the contingent claim is redundant then:*

(1) $l^s = l^b = q(\mathbf{l}_1)$;

(2) *the optimal portfolio strategy for problem in* (5.1), *denoted* $\mathbf{x}^*$, *is equal to* $\mathbf{x}^* = \mathbf{y}^* + \delta\bar{\mathbf{x}}$, *where* $\mathbf{y}^*$ *is the optimal investment portfolio for problem in* (5.2).

According to this proposition, when the claim is redundant the agent's strategy is given by the sum of two portfolios: the hedging (replicating) portfolio financed by δl^s and the optimal investment portfolio financed by v_0. This means that, in case of redundancy, the deal in the contingent claim does not affect the optimal state-contingent wealth $\mathbf{v}_1^*$ from problem (3.2).

On the other hand, according to the very definition (problem (3.3)), when the claim is non-redundant the net losses from the deal will be paid also with $\mathbf{v}_1^*$. The next proposition states that the reservation prices always lie between the bid and the ask price: if the claim is not redundant, the selling price is strictly lower than the bid price and the buying price is strictly higher than the ask price.

To this aim, define the *bid* (respectively, ask) *price* as the optimal value of the problem

$$
\begin{aligned}
l^{\text{bid}}(l^{\text{ask}}) = \sup_{\boldsymbol{\pi}} (\inf_{\boldsymbol{\pi}}) \quad & \boldsymbol{\pi}\mathbf{l}_1 \\
\text{subject to} \quad & \boldsymbol{\pi}\mathbf{W}_1 = \mathbf{W}_0, \\
& \boldsymbol{\pi} > \mathbf{0}.
\end{aligned}
\tag{5.3}
$$

The bid (ask) price is the maximum (minimum) price compatible with no arbitrage. If we take the dual of the linear program (5.3) we can define the bid (ask) price also as

$$
\begin{aligned}
l^{\text{bid}}(l^{\text{ask}}) = \inf_{\mathbf{y}} (\sup_{\mathbf{y}}) \quad & \mathbf{W}_0\mathbf{y} \\
\text{subject to} \quad & \mathbf{W}_1\mathbf{y} \geq \mathbf{l}_1 \\
& (\mathbf{W}_1\mathbf{y} \leq \mathbf{l}_1)
\end{aligned}
\tag{5.4}
$$

so that the bid price is the minimum cost of a perfect hedge (the ask price is the maximum cost of a dominated portfolio).

Remark 5.3. If there is no arbitrage in the financial market, then the solution for problem (5.4) is unique.

Proposition 5.4. *If the claim is not redundant, then $l^{ask} < l^b$ and $l^s < l^{bid}$.*

To add more intuition to the concepts described above, we provide two examples regarding option pricing problems. The details of the numerical procedure can be found in Appendix B.

Example 1. The first example concerns a european call option in a three-date setting in incomplete financial market. We assume that there is a bond (interest rate equal to 0.1 per period) and a non-dividend paying stock. The event-tree is as in Figure 1. Assume also that a single european call option on the stock with strike price $X = 110$ is sold ($\delta = 1$) (respectively, bought, $\delta = -1$) by an agent with utility function $u(w)$. We consider exponential and Hyperbolic Absolute Risk Aversion (HARA) utility functions

$$u(w) = -\exp(-\alpha w), \qquad u(w) = \frac{1-\gamma}{\gamma}\left(\frac{\eta w}{1-\gamma} + \beta\right)^{\gamma}.$$

Let $\rho(v_0) = -u''(v_0)/u'(v_0)$ be the De Finetti-Arrow-Pratt absolute risk aversion coefficient at v_0 and assume the agent has uniform subjective probability

$$\mathbf{p} = (1/3, 1/3, 1/3, 1/9, \ldots, 1/9),$$

and let $v(1) = v(2) = 1$.

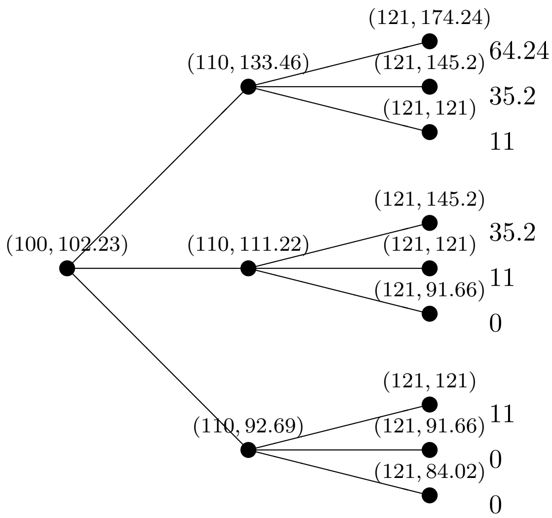

FIGURE 1. Example 1: in the nodes of the event-tree there are the security prices (bond and stock, respectively) and in the leaves the payoff of a european call option ($X = 110$) written on the stock.

For illustrative purposes, the matrix $\mathbf{W}$ for this example is

$$\begin{pmatrix}
-100 & -102.23 & 0 & 0 & 0 & 0 & 0 & 0 \\
110 & 133.46 & -110 & -133.46 & 0 & 0 & 0 & 0 \\
110 & 111.22 & 0 & 0 & -110 & -111.22 & 0 & 0 \\
110 & 92.69 & 0 & 0 & 0 & 0 & -110 & -92.69 \\
0 & 0 & 121 & 174.24 & 0 & 0 & 0 & 0 \\
0 & 0 & 121 & 145.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 121 & 121 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 121 & 145.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 121 & 121 & 0 & 0 \\
0 & 0 & 0 & 0 & 121 & 91.66 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 121 & 121 \\
0 & 0 & 0 & 0 & 0 & 0 & 121 & 91.66 \\
0 & 0 & 0 & 0 & 0 & 0 & 121 & 84.02
\end{pmatrix}$$

In Table 1 we show the reservation selling and buying prices of the european call option for different risk aversions and wealth. In order to have somehow comparable results across various utility functions, we adjust the parameters of the functions to get the same $\rho(v_0)$.

TABLE 1. Example 1: reservation selling and buying prices for different initial wealth v_0 and absolute risk aversion ρ's. The prices are obtained with $\alpha = \rho$ for exponential and $\eta = 0.1, \gamma = -1$ for HARA utility function. We adjust β in the HARA case in order to get the desired absolute risk aversion at v_0. All figures are rounded to the third decimal place.

	EXP		HARA	
$\rho(v_0)$	$v_0 = 10$	$v_0 = 20$	$v_0 = 10$	$v_0 = 20$
0.05	16.533	16.507	16.527	16.535
	10	16.379	10	16.394
0.1	16.562	16.560	16.590	16.631
	10	16.304	10	16.228
0.15	16.609	16.609	16.658	16.779
	10	16.228	10	15.782

Observe that an agent with little initial wealth $v_0 = 10$ is willing to spend his whole endowment to buy the security and that the spread between the buying and the selling price is larger the higher the risk aversion. This is also more clearly shown in Figure 2 that depicts the reservation buying and selling prices against the absolute risk aversion coefficient for an exponential agent with $v_0 = 20$.

Figure 3 shows reservation selling prices for different initial wealth. Note that the selling price is generally not increasing in the risk aversion coefficient and that, given the risk aversion, the lower the initial wealth the higher is the selling price. The three selling prices collapse to a unique price for ρ around 0.1: nonnegativity constraints are redundant for such risk averse agents and it then applies the result stating that the price is independent on wealth for exponential agents.

The reservation prices of two agents having different subjective probabilities are shown in Figure 4 as a function of the absolute risk aversion coefficient. The probabilities are respectively $\mathbf{p}$ and

$$\mathbf{q} = \left(\frac{1}{10}, \frac{1}{10}, \frac{8}{10}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{1}{4}\right).$$

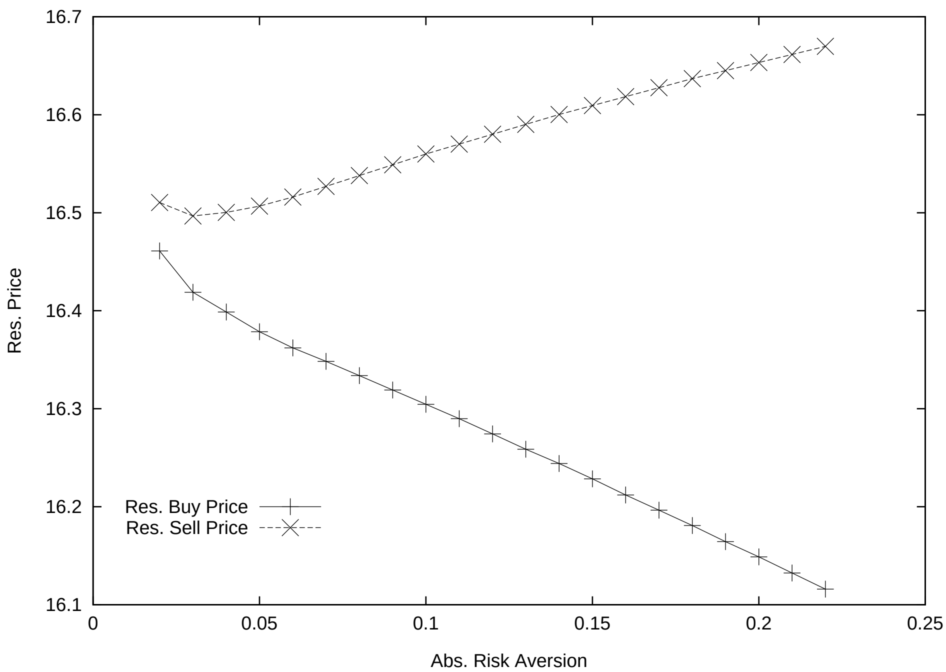

FIGURE 2. Reservation selling and buying prices against the absolute risk aversion coefficient in the case of exponential utility ($v_0 = 20$).

Example 2. This example is intended to mimic the pricing of a *spread option* (which has final payoff $\max(0, S_2 - S_1 - X)$ where S_1 and S_2 are the prices of two risky assets). No analytical formula is known to price such derivative and simulations or other numerical approximations must be used. We compute

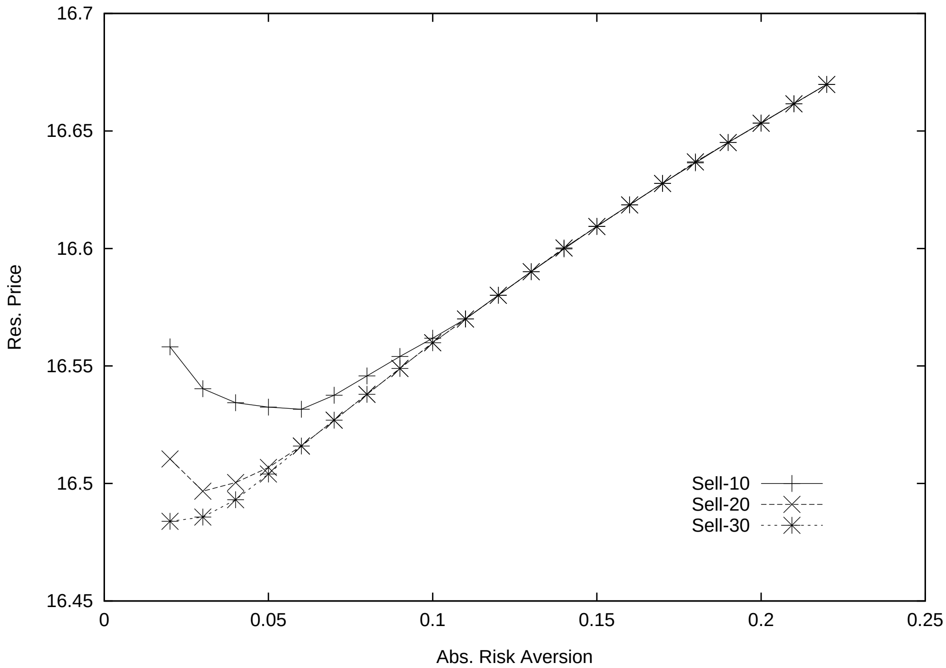

FIGURE 3. Reservation selling prices against the absolute risk aversion coefficient in the case of exponential utility for different initial endowments ($v_0 = 10, 20, 30$).

the selling ($\delta = 1$) (respectively, the buying) price of the spread option with $X = 5$, in an incomplete financial market where just the securities S_1 and S_2 (and no risk-less bond) are available. The event-tree is described in Figure 5. Note that we are implicitly assuming a positive correlation among assets, which need not to be the case (see [4]). The subjective probabilities and discounting of the agent are the same as in Example 1.

The agent's preferences are represented by exponential, HARA (as defined above) and logarithmic utility function, $u(w) = \log(w + M)$, where M is suitably chosen. The reservation prices obtained for some wealth and absolute risk aversion coefficients are shown in Table 2. Again, we adjust parameters β and M to have same $\rho(v_0)$. In two cases, however, this normalization give raise to unfeasible problems when logarithmic utility is used.

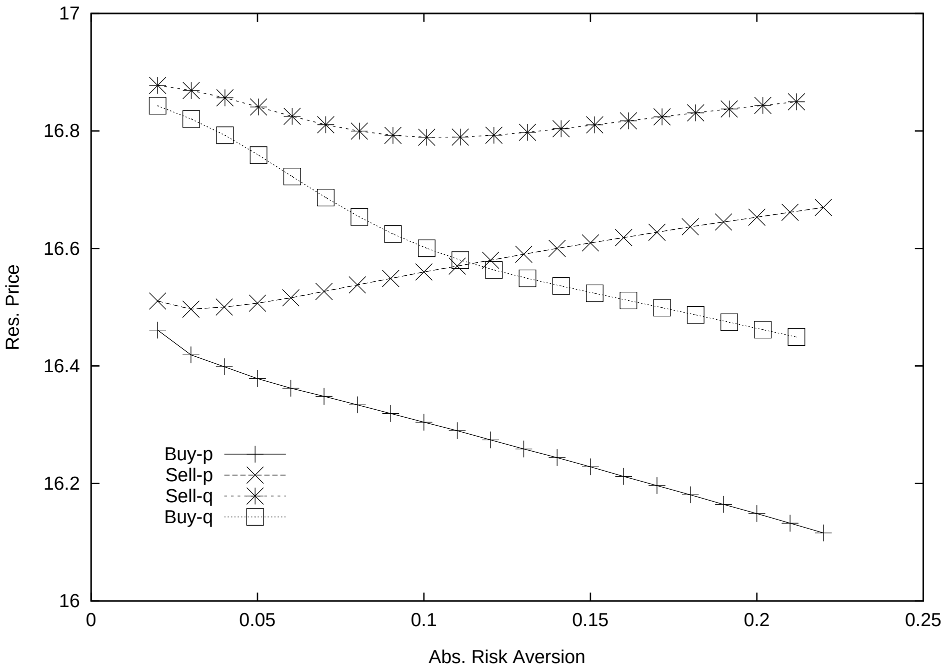

FIGURE 4. Reservation selling and buying prices against the absolute risk aversion for exponential agents with different subjective probabilities, respectively **p** and **q**.

TABLE 2. Example 2: reservation selling and buying prices for different budget and absolute risk aversions. The figures for exponential utility are obtained setting $\alpha = \rho$ for exponential and $\eta = 0.1, \gamma = -1$ for HARA utility function. We adjust M and β in order to get absolute risk aversion ρ at v_0 with logarithmic and HARA utilities. All figures are rounded to the third decimal place.

	EXP		HARA		LOG	
$\rho(v_0)$	$v_0 = 10$	$v_0 = 20$	$v_0 = 10$	$v_0 = 20$	$v_0 = 10$	$v_0 = 20$
0.05	3.371	3.375	3.375	3.387	3.381	3.411
	3.309	3.314	3.305	3.306	3.230	3.290
0.1	3.405	3.413	3.425	3.481	3.462	-
	3.282	3.291	3.263	3.242	3.222	-
0.15	3.437	3.449	3.489	3.693	3.642	-
	3.254	3.268	3.201	3.049	2.972	-

Figure 6 shows the reservation prices obtained with logarithmic utility against v_0. The interval spanned by reservation selling and buying prices is wider the smaller the wealth. Note, however, that being absolute risk

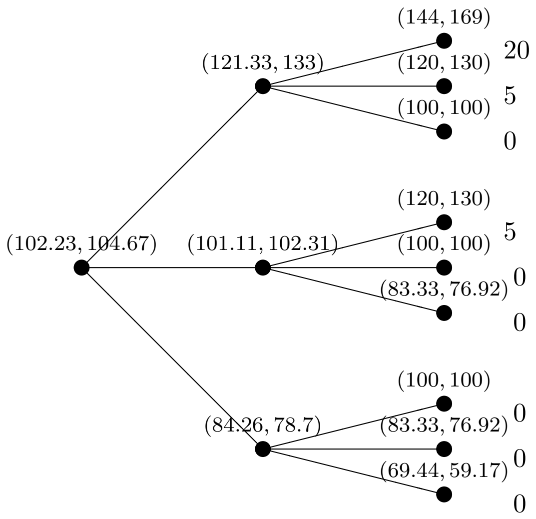

FIGURE 5. Example 2: security prices (in the nodes) and payoff of a spread option ($X = 5$) (on the leaves).

aversion dependent on v_0, a bigger endowment comes together with smaller risk aversion. Finally it is interesting to contrast reservation, bid and ask prices. The latter are respectively 4.359 and 1.328 (not plotted for graphical convenience), and span a much wider interval than utility based prices.

6. CONCLUDING REMARKS

In this article we introduced the reservation price as a valuation criterion for a newly-issued non-redundant contingent claim in an incomplete financial market. We described also some properties of the price both from the seller's and the buyer's viewpoints.

There are two issues that need to be addressed at the end of this work.

The first is that we assumed that the equilibrium in the financial market is unaffected by the introduction of the new claim. This is unlikely to last for a long time. Essentially, when a non-redundant contingent claim is issued, the prices of the existing financial securities change because the opportunities offered by the financial market to hedge risky positions are different. This influences the equilibrium consumption and portfolio policies of the agents

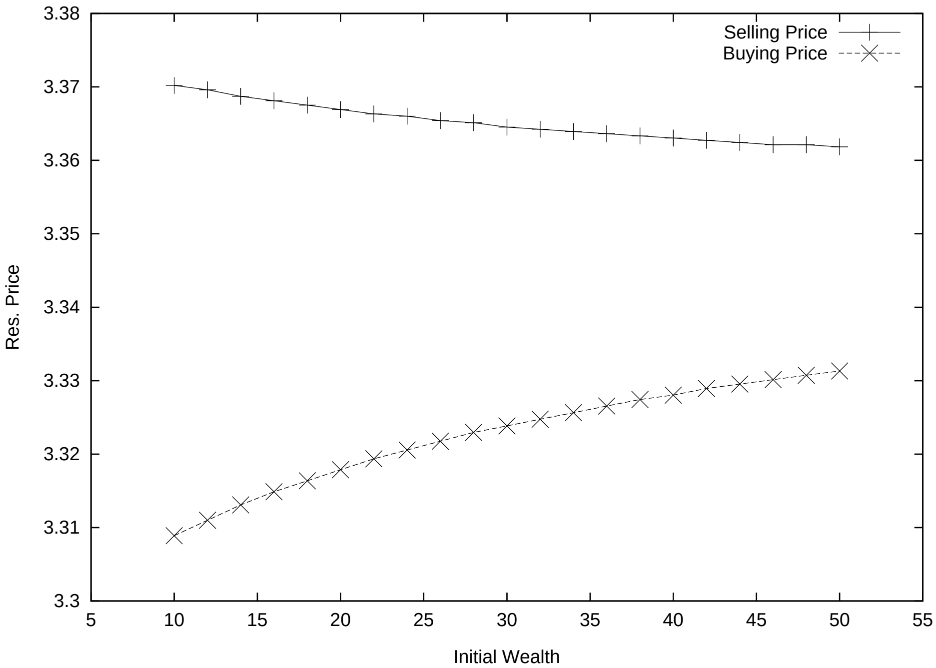

FIGURE 6. Reservation selling and buying prices against initial wealth v_0 in the case of logarithmic utility ($M = 15$).

and, as a consequence, the prices of the existing assets. The same can be said for other pricing criteria in incomplete markets offered in the literature [1, 5, 7, 8, 13]. With respect to this, our pricing criterion is best suited to determine the initial (offered or requested) price of the claim.

The second issue is that our model describes the behavior of a single part: either the writer or the buyer. It is quite natural to extend the model in order to describe the bargaining process between the parts. Assuming that both parts select the hedging portfolio by maximizing the expected utility of the net payoff, and if the reservation price for the writer is lower than the reservation price for the buyer, then there is room for bargaining. The existence and the properties of the bargaining solution will be the subject of future research.

Appendix A. Proof of propositions

Let $\mathbb{B}$ be the set of feasible net payoffs:

$$\mathbb{B} = \left\{ \mathbf{w}_1 \in \mathbb{R}^{\mathcal{N}} \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x} - \delta\mathbf{l}_1, \mathbf{W}_0\mathbf{x} = v_0 + \delta l,\ \mathbf{w}_1 \geq \mathbf{0} \right\}.$$

We will denote the set of arbitrage-free state-price vectors by

$$\Pi = \left\{ \boldsymbol{\pi} = (1, \boldsymbol{\pi}_1) \in \mathbb{R}^{\mathcal{N}+1}_{++} \mid \boldsymbol{\pi}\mathbf{W} = \mathbf{0} \right\}$$

and by $\mathcal{cl}\Pi$ its closure. Let $\mathcal{B}_{\boldsymbol{\pi}}$ denote the *budget-feasible set* according to a given state-price vector $\boldsymbol{\pi} \in \mathcal{cl}\Pi$:

$$\mathcal{B}_{\boldsymbol{\pi}} = \left\{ \mathbf{w}_1 \in \mathbb{R}^{\mathcal{N}} \mid \boldsymbol{\pi}_1(\mathbf{w}_1 + \delta\mathbf{l}_1) = v_0 + \delta l,\ \boldsymbol{\pi} = (1, \boldsymbol{\pi}_1) \in \Pi,\ \mathbf{w}_1 \geq \mathbf{0} \right\}.$$

The following preliminary results can be easily proved:

Lemma A.1. *For all $\boldsymbol{\pi} \in \mathcal{cl}\Pi$, $\mathbb{B} \subset \mathcal{B}_{\boldsymbol{\pi}}$.*

Proof of Proposition 4.2. This proposition is drawn on a well known theorem stating the equivalence between the existence of a solution for a consumption - investment problem and the absence of arbitrage opportunities in a financial market [15, Th. 9.3].

To prove necessity, if Problem $(\mathcal{H})$ has a solution, then there is no arbitrage. Assume that $\mathbf{x}$ is an optimal portfolio such that $\mathbf{w} = \mathbf{W}\mathbf{x} - \delta\mathbf{l}$, with $\mathbf{w} = (-v_0, \mathbf{w}_1)$ and $\mathbf{l} = (-l, \mathbf{l}_1)$. If there was an arbitrage $\mathbf{y}$ such that $\mathbf{W}\mathbf{y} \geq \mathbf{0}$ with at least a positive component, then we would have

$$\mathbf{w} = \mathbf{W}\mathbf{x} - \delta\mathbf{l} \leq \mathbf{W}\mathbf{x} + \mathbf{W}\mathbf{y} - \delta\mathbf{l} = \mathbf{w}',$$

where $\mathbf{w}' = (-v_0, \mathbf{w}'_1)$, that is, we would have $\mathbf{w} \leq \mathbf{w}'$ with a strict inequality for at least a component, against the assumption that $\mathbf{w}$ is optimal. This ends the necessity part.

To prove sufficiency, absence of arbitrage opportunities is equivalent to the existence of a state-price vector $\boldsymbol{\pi} \in \mathbb{R}^{\mathcal{N}+1}_{++}$. According to [15, Proposition 7.3], this implies that the budget-feasible set, $\mathcal{B}_{\boldsymbol{\pi}}$, is compact. Since $\mathbb{B}$ is closed, also $\mathbb{B} \subset \mathcal{B}_{\boldsymbol{\pi}}$, from Lemma A.1, is compact. Hence, Problem $(\mathcal{H})$ has a solution because $U(\cdot)$ is continuous. $\square$

Proof of Proposition 5.1. Let $\theta = v_0 + \delta l$. First we prove that $\theta' > \theta$ implies $\mathcal{U}(\theta') \geq \mathcal{U}(\theta)$. To see that this is true, define the set of feasible portfolios

$$\mathcal{D} = \{\mathbf{x} \mid \mathbf{W}_0\mathbf{x} \leq \theta, \mathbf{W}_1\mathbf{x} \geq \delta\mathbf{l}_1\}$$

It is easy to see that $\mathcal{D}(\theta) \subset \mathcal{D}(\theta')$. Hence, the optimal expected utility of $\mathbf{w}_1 = \mathbf{W}_1 - \delta\mathbf{l}_1$ is not lower in $\mathcal{D}(\theta')$ than in $\mathcal{D}(\theta)$.

We have to prove that, if $\theta' > \theta$, we cannot have $\mathcal{U}(\theta) = \mathcal{U}(\theta')$. Actually, if the last equality held, then denoting with $\mathbf{w}_1(\theta)$ the optimal net payoff as a function of θ, this would give $\mathbf{w}_1(\theta) = \mathbf{w}_1(\theta')$ for the uniqueness of the solution. Since $\mathbf{W}_1$ has full rank, then this would imply that $\mathbf{x}(\theta) = \mathbf{x}(\theta')$, where $\mathbf{x}(\theta)$ is the optimal portfolio supporting $\mathbf{w}_1(\theta)$, that is $\mathbf{w}_1(\theta) = \mathbf{W}_1\mathbf{x}(\theta) - \delta\mathbf{l}_1$. The last equality gives $\theta = \mathbf{W}_0\mathbf{x}(\theta) = \mathbf{W}_0\mathbf{x}(\theta') = \theta'$. And this contradicts the assumption. $\square$

Proof of Proposition 5.2.

(1) Since there is a $\bar{\mathbf{x}}$ such that $\mathbf{W}_1\bar{\mathbf{x}} = \mathbf{l}_1$, let $q(\mathbf{l}_1) = \mathbf{W}_0\bar{\mathbf{x}}$ be the Arrow-Debreu price of $\mathbf{l}_1$. By letting $\mathbf{x} = \mathbf{y} + \delta\bar{\mathbf{x}}$, the hedging problem $(\mathcal{H})$ becomes

$$\max\{\, U(\mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x} - \delta\mathbf{l}_1, \mathbf{W}_0\mathbf{x} = v_0 + \delta l, \mathbf{w}_1 \geq \mathbf{0}\,\} =$$
$$\max\{\, U(\mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{y}, \mathbf{W}_0\mathbf{y} = v_0 - \delta q(\mathbf{l}_1) + \delta l, \mathbf{w}_1 \geq \mathbf{0}\,\}.$$

Since

$$\mathcal{V}(v_0) = \max\{\, U(\mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x}, \mathbf{W}_0\mathbf{x} = v_0, \mathbf{w}_1 \geq \mathbf{0}\,\},$$

clearly $\mathcal{V}(v_0) = \mathcal{V}(v_0 - \delta q(\mathbf{l}_1) + \delta l)$. Since $\mathcal{V}(\cdot)$ is a strictly increasing function (Proposition 5.1), then $l^s = q(\mathbf{l}_1)$. The proof for the buying price is the same, by taking $\mathbf{x} = \mathbf{y} + \delta\bar{\mathbf{x}}$ in place of $\mathbf{x} = \mathbf{y} - \delta\bar{\mathbf{x}}$.

(2) Let $\mathbf{x}^*$ be optimal for problem (5.1) and let $\mathbf{x}^* = \mathbf{y}^* + \delta\bar{\mathbf{x}}$. Since $\mathbf{l}_1 = \mathbf{W}_1\bar{\mathbf{x}}$ and using part (1) of Proposition 5.2, we have that $\mathbf{y}^*$ is optimal for problem (5.2).

On the other hand, let $\mathbf{y}^*$ be the optimal strategy for problem (5.2). Then by taking $\mathbf{y}^* = \mathbf{x}^* - \delta\bar{\mathbf{x}}$ we have that $\mathbf{x}^*$ is optimal for problem (5.1).

□

Proof of Proposition 5.4. We prove that $l^s \leq l^{\text{bid}}$ for any claim. Let $\mathbf{w}_1$ be the optimal net payoff according to problem (3.3) with $\delta = 1$ for convenience. Moreover, let define the net payoff $\widehat{\mathbf{w}}_1 = \mathbf{W}_1\widehat{\mathbf{x}} - \mathbf{l}_1$, where $\widehat{\mathbf{x}}$ is the perfect hedging portfolio such that $\mathbf{W}_0\widehat{\mathbf{x}} = l^{\text{bid}}$ (see problem in (5.4)). Since $\widehat{\mathbf{w}}_1 \geq \mathbf{0}$, then $U(\mathbf{v}_1 + \widehat{\mathbf{w}}_1) \geq U(\mathbf{v}_1)$ because $u(\cdot)$ is strictly increasing, where $\mathbf{v}_1$ is optimal for problem (3.2). Moreover, $\mathbf{v}_1 + \widehat{\mathbf{w}}_1$ is feasible. Since

$$\max_{\mathbf{x}} \left\{ U(\mathbf{v}_1 + \mathbf{w}_1) \mid \mathbf{w}_1 = \mathbf{W}_1\mathbf{x} - \mathbf{l}_1, \mathbf{W}_0\mathbf{x} = l^{\text{bid}}, \mathbf{v}_1 + \mathbf{w}_1 \geq \mathbf{0} \right\} \geq$$
$$\geq U(\mathbf{v}_1 + \widehat{\mathbf{w}}_1),$$

then we have, by definition of reservation selling price,

$$\mathcal{U}(v_0 + \delta l^{\text{bid}}) \geq \mathcal{V}(v_0) = \mathcal{U}(v_0 + l^s),$$

and since $\mathcal{U}(\cdot)$ is monotone (Proposition 5.1), $l^s \leq l^{\text{bid}}$. Strict inequality $l^s < l^{\text{bid}}$ in case of non-redundancy of the claim follows immediately from condition $\widehat{\mathbf{w}}_1 \geq \mathbf{0}$ and $\widehat{\mathbf{w}}_1 \neq \mathbf{0}$.

To prove that $l^{\text{ask}} < l^b$, let it be given $\widehat{\mathbf{w}}_1 = \mathbf{W}_1\widehat{\mathbf{x}} - \mathbf{l}_1$, where $\widehat{\mathbf{x}}$ is the perfect hedging portfolio such that $\mathbf{W}_0\widehat{\mathbf{x}} = l^{\text{ask}}$ (from problem (5.4)). Condition $\widehat{\mathbf{w}}_1 \leq \mathbf{0}$ and $\widehat{\mathbf{w}}_1 \neq \mathbf{0}$ implies that $U(\mathbf{v}_1 + \widehat{\mathbf{w}}_1) \leq U(\mathbf{v}_1)$. Since

$$\max_{\mathbf{x}} \left\{ U(\mathbf{v}_1 + \mathbf{w}_1) \mid \mathbf{w}_1 = -\mathbf{W}_1\mathbf{x} + \mathbf{l}_1, \mathbf{W}_0\mathbf{x} = l^{\text{ask}}, \mathbf{v}_1 + \mathbf{w}_1 \geq \mathbf{0} \right\} \leq$$
$$\leq U(\mathbf{v}_1 + \widehat{\mathbf{w}}_1),$$

then the inequality can be proved with the same argument used before. □

Appendix B. The numerical algorithm

The reservation price, as seen in section 5, is defined as the unique solution of equation $(\mathcal{P})$. Once $v_0, \mathbf{W}_0, \mathbf{W}_1$ and δ are fixed the left hand side of $(\mathcal{P})$ is a function of the price l^s while the right hand side $\mathcal{V}(v_0)$ is a fixed constant, namely the optimal utility that can be attained if no security is written. Equation $(\mathcal{P})$ can be solved by bisection, after l^s is bracketed in an interval $[l_*, l^*]$ such that

$$\mathcal{U}(v_0 + \delta l_*) \leq \mathcal{V}(v_0) \leq \mathcal{U}(v_0 + \delta l^*).$$

Clearly, each evaluation of function $\mathcal{U}$ need the solution of the constrained optimization problem $(\mathcal{H})$. To this aim, we used the "NEOS Server for Optimization" hosted at Argonne National Laboratories.[9] Actually, NEOS is an acronym for Network-Enabled Optimization System. This site offers a variety of solvers for general optimization problems. The user is asked to precisely define the problem, to submit it to the server in one of several possible ways and to specify a suitable solver. In particular, we adopt the AMPL language (Fourer et al. [12]) to submit the problem $(\mathcal{H})$ and use the LANCELOT solver to get a solution (Conn et al. [6] or see `www.cse.clrc.ac.uk/Activity/LANCELOT` for details). Each result was checked using the MINOS solver, which is available at NEOS server too: the two programs always produced the same solution, up to required precision. Further details on MINOS can be found in Fourer's book or at the NEOS server.

Note that it is not trivial to execute a bisection algorithm in a pure optimization environment. In fact, although it is quite easy to solve an optimization problem using the AMPL language, it is not straightforward to program an iterative procedure like bisection. However, taking profit of

[9] URL: `http://www-neos.mcs.anl.gov`

the AMPL statements `let` and `if`, it is possible to iterate the solution of problem $(\mathcal{H})$ to compute l^s repeating the following AMPL code

```
let l_down := if optimal_U < VV_0 then l else l_down;
let l_up := if optimal_U > VV_0 then l else l_up;
let l := if optimal_U < VV_0 then (l+l_up)/2 else (l+l_down)/2;
solve;
```

that performs a bisection step. The notation of the code is suggestive of that in the paper, namely `l_down` $= l_*$, `l_up` $= l^*$, `VV_0` $= \mathcal{V}(v_0)$ and `optimal_U` is the optimal value of problem $(\mathcal{H})$, given `l` $= l^s$. Observe that each step of the bisection algorithm is followed by the solution of problem $(\mathcal{H})$ (`solve`) which allows to perform another step of the bisection procedure.

In summary, we numerically compute the reservation price in two steps: solution of a constrained multidimensional optimization problem by submission to the LANCELOT solver and solution of the equation $(\mathcal{P})$ by bisection method, iterating the previous point until convergence is reached. In particular, we stop iterating when two successive values of the price l_s differ by less than 10^{-4}.

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