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Measurement of Financial Risk Persistence

Abstract

This paper discusses various ways of measuring the persistence or Long Memory (LM) of financial market risk in both its time and frequency domains. For the measurement of the risk, irregularity or "randomness" of these series, we can compute a set of critical Lipschitz - Hölder exponents, in particular, the Hurst Exponent and the Lévy Stability Alpha, and relate them to the Mandelbrot-Hoskings' fractional difference operators, as occur in the Fractional Brownian Motion model (which is our benchmark). The main contribution of this paper is to provide a compaison table of the various critical exponents available in various scientific disciplines to measure the LM persistence of time seies. It also discusses why Markov- and (G)ARCH models cannot capture this LM, long term dependence or risk persistence, because these models have finite lag lengths, while the empirically observed long memory risk phenomenon is an infinite lag length phenomenon. Currently, there are three techniques of nonstationary time series analysis to measure time - varying financial risk: Range/Scale analysis, windowed Fourier analysis, and wavelet MRA. This paper relates these powerful analytic techniques to classical Box-Jenkins-type time series analysis and to Pearson's spectral frequency analysis, which both rely on the uncorroboated assumption of stationarity and ergodicity.

Key Words: persistence, long memory, dependence, time series, frequency, critical exponents, fractional Brownian motion, (G)ARCH, risk measurement

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1 Introduction

This paper focuses on several issues of the measurement of serial and global, or short - term and long - term, *temporal dependence* among asset returns. Speculative market returns (and other financial and economic time series) tend to be characterized by the presence of aperiodic cycles of all conceivable "periods" of uncertain length - short, medium, and long - where "long" means comparable up the length of the total available data set, and where the distinction between "long cycles" and "trends " is very fuzzy (Mandelbrot, 1972). Consider, for example, the business cycles in the USA which used to have, more or less defined, "periods" of somewhere between 3.5 and 10 years (Moore, 1980). In fact, the most recent business "cycle" in the USA had an expansion phase of about 12 years, from 1989 - 2001 and is one of the longest on record!

Although cyclical behavior of time series produced by economic models has been extensively studied, efforts to characterize the structure of actual empirical financial - economic time series have been minimal until recently. The historical exceptions are the elegant and heroic efforts by Granger and Morgenstern (1963) and Granger (1966), who tried to characterize time series of stock market prices by stationarity-based spectral analysis and who attempted to determine the "typical spectral shape of economic variables."[1] We'll have to understand the essence of these classical techniques to analyze stationary and semi - stationary financial time series first, before we can advance to the current technology of wavelet multiresolution analysis, also called multi - scale decomposition, to analyze nonstationary and unstable financial time series, and to analyze series of singularities. Such time series are not even convergent in their lower - order moments.

[1] The current unorthodox efforts to characterize nonstationary financial - economic time series using more advanced signal processing technology are comparable with these early out - of - the - mainstream technical efforts by Granger and Morgenstern. For example, econometrician J. B. Ramsey of New York University performed the first wavelet multiresolution analysis (MRA) of macroeconomic data series (Ramsey, 1997).

2 Serial Dependence

2.1 Mixing Random Processes

One way to describe serial, "weak," or short - term time dependence is that of *strong - mixing* processes. Informally, *mixing processes* are processes that gradually "mix" with new information and so also gradually "forget" their initial conditions over time. In particular, a process is strong - mixing if the maximal dependence between any two events at two different dates becomes trivially small as the time span between these two dates increases. By controlling the rate at which this dependence between past and future events declines, it is possible to extend the usual laws of large numbers and the central limit theorems from sequences of independent random variables to sequences of dependent random variables. A formal definition of a strong - mixing random process, using a non-linear distance measure, is the following.

Definition 1 *(**Strong - mixing process**) Let the random process* $\{X(t)\}$ *be defined on the probability space* $(\Omega, \mathcal{G}, P)$ *and define the distance measure:*

$$\gamma(\mathcal{A}, \mathcal{B}) \equiv \sup_{A \in \mathcal{A}, B \in \mathcal{B}} (|P(A \cap B) - P(A)P(B)|), \mathcal{A} \subset \mathcal{G}, \mathcal{B} \subset \mathcal{G} \tag{1}$$

The quantity $\gamma(\mathcal{A}, \mathcal{B})$ *is a measure of the dependence between the two* $\sigma-$*algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *in the measurable set* $\mathcal{G}$*. Denote by* $\mathcal{B}_s^t$ *the* $\sigma-$*algebra generated by the sequence* $\{X_s(\omega), ..., X_t(\omega)\}$*, i.e.,* $\mathcal{B}_s^t \equiv \sigma(X_s(\omega), ..., X_t(\omega)) \subset \mathcal{G}$*. Define the quantities*

$$\gamma(\tau) \equiv \sup \gamma(\mathcal{B}_{-\infty}^t, \mathcal{B}_{t+\tau}^\infty) \tag{2}$$

The random process $\{X(t)\}$ *is said to be **strong - mixing** if*

$$\lim_{\tau \to \infty} \gamma(\tau) = 0 \tag{3}$$

Such strong mixing conditions are satisfied by all finite - order stationary autoregressive - moving average (ARMA) models. These ARMA models can all be transformed into stable Markov processes.

2.2 Markov and Finite - Order ARMA Processes

The first efforts to characterize oscillatory behavior with exact periodicity was by postulating second - and higher - order affine Markov processes and their directly related cousins, the Box - Jenkins type ARMA models (Box and Jenkins, 1970; Anderson, 1994). Markov models provide only for short - term, or serial, time dependence. These models are identified by using autocovariance function analysis, or by using its cousin, spectral analysis.

Definition 2 *The **First - Order Markov Process** is defined by*

$$\begin{aligned} X(t) &= a_1 X(t-1) + \varepsilon(t) \\ &= a_1 L X(t) + \varepsilon(t), \;\; with\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \end{aligned} \tag{4}$$

which can also be written with the lag operator L as

$$(1 - a_1 L) X(t) = \varepsilon(t), \; with\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{5}$$

This first - order Markov process is stable when $0 < a_1 < 1$. The Random Walk is a first - order Markov process, which is marginally unstable (and has in the limit an infinite variance), since $a_1 = 1$. An unstable and geometrically exploding first - order Markov process has $1 < a_1$. This is easy to confirm, since this first - order autoregressive AR(1) Markov process $X(t)$ can also be viewed as an infinite - order moving average (MA) process with an infinite memory:

$$\begin{aligned} X(t) &= \frac{1}{(1 - a_1 L)} \varepsilon(t) \\ &= (1 + a_1 L + a_1^2 L^2 + a_1^3 L^3 + ...) \varepsilon(t) \\ &= (1 + \sum_j^{\infty} a_1^j L^j) \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \end{aligned} \tag{6}$$

When $0 < a_1 < 1$, the $\lim_{j \to \infty} \sum_j a_1^j L^j = q$ exists, where $0 < q < \infty$ is a real constant. Thus, in the limit, $\sigma_X^2 = (1+q)^2 \sigma_\varepsilon^2$ is a finite (equilibrium) variance and over time the financial market risk remains bounded and is stable. When $1 \leq a_1$, the limit diverges, $\lim_{j \to \infty} \sum a_1^j L^j \to \infty$, and, in the limit, the variance of $X(t)$ is unbounded, $\lim \sigma_X^2 \to \infty$. The financial market risk diverges: in the limit the financial risk of $X(t)$ becomes unbounded and infinite.

But first - order Markov processes tun out to be too simple processes to describe financial pricing processes. Financial pricing processes are characterized by uncertain "periodicity," *i.e.,* by oscillatory behavior of some sort, although without fixed periods, which can therefoe better be called "cyclicity." For such uncertain "periodicity" one needs at least two - to fourth - order Markov processes, or more likely, nonlinear processes, prefeably with a stochastic component.[2]

[2] Los (1999, 2000) provides some empirical measurement examples of such "periodicity" for Asian FX markets, using non - parametric methods, based on high frequency data for 1997.

Definition 3 *The **Second - Order Markov Process** is defined by*

$$(1 - a_1 L - a_2 L^2) X(t) = \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{7}$$

Remark 4 *From straightforward solution analysis of quadratic equations we know that this second - order Markov process is stable when $(a_1^2 - 4a_2) > 0$; it is oscillatory (= showing strict periodic behavior), when $(a_1^2 - 4a_2) < 0$; and is unstable when $(a_1^2 - 4a_2) = 0$.*

Such higher - order Markov processes are easier to represent in a generic fashion in vector - matrix notation, as follows.

Definition 5 *The **n - Order Markov Process** is defined by*

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1) + \boldsymbol{\varepsilon}(t), \text{ with } \boldsymbol{\varepsilon}(t) = \begin{bmatrix} \varepsilon_1(t) \\ \mathbf{0} \end{bmatrix} \text{ and } \varepsilon_1(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{8}$$

where $\mathbf{x}(t)$ is a $(n \times 1)$ vector and $\mathbf{A}$ a $(n \times n)$, which can also be written with the lag operator as

$$(\mathbf{I} - \mathbf{A}L)\mathbf{x}(t) = \boldsymbol{\varepsilon}(t), \text{ with } \boldsymbol{\varepsilon}(t) = \begin{bmatrix} \varepsilon_1(t) \\ \mathbf{0} \end{bmatrix} \text{ and } \varepsilon_1(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{9}$$

Example 6 *For $n = 3$, a $3-$ order autoregressive $AR(p, q) = AR(3, 0)$ process can be written in such vector matrix notation as*

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} x(t) \\ x(t-1) \\ x(t-2) \end{bmatrix} \\ &= \mathbf{A}\mathbf{x}(t-1) + \boldsymbol{\varepsilon}(t) \\ &= \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t-1) \\ x(t-2) \\ x(t-3) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1 x(t-1) + a_2 x(t-2) + a_3 x(t-3) + \varepsilon_1(t) \\ x(t-1) \\ x(t-2) \end{bmatrix} \\ &\text{with } \varepsilon_1(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \end{aligned} \tag{10}$$

Again, the behavior of this random process depends on the spectral analysis of the actual values of the $\mathbf{A}-$matrix, in particular, the parameters a_1, a_2 and a_2, which are to be determined from the A*cf.* If the determinant

$$|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i < 1 \tag{11}$$

then the process is stable or *implosive*; if $|\mathbf{A}| = 1$, it is *marginally stable*; and if $|\mathbf{A}| > 1$, the process is unstable or *explosive*.

Remark 7 *Even more general Markov processes can be described by this type of model when the innovations are covarying, e.g.,* $\boldsymbol{\varepsilon}(t) \sim i.i.d.(0, \boldsymbol{\Sigma})$, *with* $\boldsymbol{\Sigma} > \mathbf{0}$, *a positive definite* $(n \times n)$ *matrix. Such general Markov processes form the basic random system structure for the Kalman filter, which can track nonstationary processes* $\mathbf{x}(t)$ *(including unstable ones!) with time - varying covariance risk matrices symptomatic for the conditional heteroskedasticity of G(ARCH) processes to be discussed in Section 1.4.*[3]

3 Global Dependence

As mentioned, financial and economic time series do not exhibit exact periodicity, or even uncertain periodicity. They exhibit distinct *aperiodic cyclicity.* In the frequency domain such time series are said to have risk (= power) at low frequencies. Financial time series, in particular, exhibit such aperiodic cyclicity, or periods of relative stability, followed by periods of great turbulence. Such diverse behavior with uncertain periods of great intensity of movement followed by periods of low intensity of movement, is called *intermittency.* Intermittency is a property of nonlinear dynamic processes which are close to complete chaos. Chaos is the behavior of a deteministic dynamic system when it orbits through an infinite number of equilibrium states.

The occurrence of sharp discontinuities in otherwise trend - wise financial and economic time series is called the "Noah effect" by Mandelbrot (1965), an appropriate reference to the Old Testamental catastrophic Flood. Long - term aperiodic cyclicity is called the "Joseph Effect " by Mandelbrot and Wallis (1969). This is an appropriate biblical reference to the Old Testament prophet, who foretold of the seven years of plenty followed by the seven years of famine that Egypt was to experience. This uncertain cyclical phenomenon was explained by the long - term aperiodic, but somehow cyclical behavior of the water flows of the river Nile, which brought some time intervals of fertile sediment and thus rich harvests, followed by time intervals of drought, no sediments and consequently poor harvests in Egypt. This aperiodic cyclic behavior of the Nile's floodwaters has been carefully analyzed by Harold Edwin Hurst, the British hydrologist in the 1950s.

[3] *Cf. Los (1984) for theoretical discussions and Monte Carlo experiments with empirically estimated Kalman filters for econometric time - varying parameter models, including unstable ones!*

Hurst, who is known in Egypt as the "Father of the Nile," studied the behavior of the Nile's water level to determine the height and mass of the Aswan dam to be built by the Russians. In the process, he designed a new and powerful statistical measure, the "range - over - standard deviation," or R/S measure, to quantify such aperiodic cyclical persistence of floodwater levels. This R/S measure is related to various exponents measuring the irregularity (= "randomness") of financial - economic time series.

3.1 Long - Term Persistence of Speculative Prices

Optimal consumption, savings, portfolio and hedging decisions may become extremely sensitive to investment horizons τ_i, when the investment returns are long - term time dependent, *i.e.,* when they show Long Memory (LM) properties. Problems may also arise in the pricing of derivative securities (such as options and futures) with Fama's martingale methods, since the theoretical continuous - time random processes most commonly employed, *e.g.,* Geometric Brownian Motions (GBMs), are inconsistent with such empirical long - term memory effects. For example, persistent LM time series show unexpected discontinuities (extreme draw-downs and draw-ups) that are outside the range of historical experience and thus don't fit in the game-type martingale model (Sornette, 2003).

In such circumstances, traditional tests of the Capital Asset Pricing model (CAPM) and Arbitrage Pricing Theory (APT) are no longer valid, since the usual forms of statistical inference do not apply to time series exhibiting long - term persistence (Lo and MacKinlay, 1988, 1999). Mandelbrot (1971) was the first to consider the implications of such persistent statistical dependence in asset returns in terms of the limitations of Fama's martingale model. This particular line of research acquired a greater urgency in the 1990s, when the abnormal frequency of financial crises appeared to increase and financial analysts and traders became much more aware of aperiodic cyclicity and intermittency.

3.2 Fractionally Differenced (ARFIMA) Time Series

We will now introduce a theoretical model, which can represent such long - term time dependence and aperiodic cyclicity, which is inconsistenmt with martingale theory. Fractional Brownian Motion (FBM) is a nonstationary process with infinite time span of temporal dependence. *Fractional difference processes* were originally proposed by Mandelbrot and Van Ness (1968). But Hosking (1981) extended the range of these models in the form of *Autoregressive Fractionally Integrated Moving Average* or ARFIMA(p, d, q) models, with fractional $d \in \mathbb{R}$, where short - term, or serial, frequency effects are superimposed on the long - term, global, or long memory processes.[4] These fractionally differenced, respectively integrated, random processes are not strong-mixing. They are nonstationary, but have a risk spectrum with a *power law decay.* The autocorrelation functions (ACFs) of long memory or globally dependent processes decay at much slower rates than the better known and more intensely studied ACFs of serially dependent processes.[5]

Definition 8 *A* ***Fractionally Differenced Processes*** *is defined by*

$$(1 - L)^d X(t) = \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{12}$$

where L is the lag operator and $0 < d < 1$ is a fraction $\in \mathbb{R}$ and $\varepsilon(t)$ is some sort of shock or innovation.

Remark 9 *When the $-1 < d < 0$ is a fraction $\in \mathbb{R}$, we have a* ***fractionally integrated*** *process of order d.*

Since the expression $(1-L)^d$ can be expanded via the binomial theorem for fractional d powers, we have the general *autoregressive* (AR) process (Lo and MacKinley, 1999):

[4] Mandelbrot has questioned if Hosking's ARFIMA models were an improvement over his simpler fractionally differenced models, since such models with fractional exponents can trivially represent the integer exponent ARIMA models. But Hosking wanted to show the fractional and integer exponents separately within one modified framework, because they represent different phenomena: non - periodic and periodic cyclicity, respectively.

[5] *Cf.* Meerschaert (1999) for the continuous time form of these long memory dynamic processes.

$$
\begin{aligned}
(1-L)^d X(t) &= \sum_{\tau=0}^{\infty} \left[(-1)^\tau \begin{pmatrix} d \\ \\ \tau \end{pmatrix} \right] L^\tau X(t) \\
&= \sum_{\tau=0}^{\infty} a(\tau) X(t-\tau) \\
&= \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \qquad (13)
\end{aligned}
$$

where the AR coefficients

$$
a(\tau) = (-1)^\tau \begin{pmatrix} d \\ \\ \tau \end{pmatrix} \qquad (14)
$$

are often re - expressed in terms of the gamma function $\Gamma(u)$ as follows.

Definition 10 *The **gamma function** $\Gamma(u)$ is defined by*

$$
\Gamma(u) = \int_0^\infty x^{u-1} e^{-x} dx \qquad (15)
$$

Integration by parts and iterated substitution gives the following important result

$$
\begin{aligned}
\Gamma(u+1) &= u\Gamma(u) \\
&= u(u-1)\Gamma(u-1) \\
&= u(u-1)(u-2)\Gamma(u-2) \\
&= u(u-1)(u-2).....\Gamma(1) \\
&= u! \text{ for } u \text{ a positive integer} \qquad (16)
\end{aligned}
$$

since $\Gamma(1) = 1$.

Thus, we have for the AR coefficients:

$$
\begin{aligned}
a(\tau) &= (-1)^{\tau} \begin{pmatrix} d \\ \\ \tau \end{pmatrix} \\
&= (-1)^{\tau} \frac{d!}{\tau!(d-\tau)!} \\
&= (-1)^{\tau} \frac{d(d-1)....(d-\tau+1)}{\tau!} \\
&= \frac{(\tau-d-1)....(1-d)(-d)}{\tau!} \\
&= \frac{(\tau-d-1)!}{(-d-1)!\tau!} \\
&= \frac{\Gamma(\tau-d)}{\Gamma(-d)\Gamma(\tau+1)} \qquad (17)
\end{aligned}
$$

As the time horizon increases, $\tau \rightarrow \infty$, proportionally,

$$
a(\tau) \sim \frac{\tau^{-d-1}}{(-d-1)!} \qquad (18)
$$

Following Box and Jenkins (1970) and Anderson (1994), we can also view the AR process as an infinite - order *moving - average* (MA) process (= the so-called Wold's representation), since

$$
\begin{aligned}
X(t) &= (1-L)^{-d}\varepsilon(t) \\
&= \sum_{\tau=0}^{\infty} b(\tau)\varepsilon(t-\tau), \text{ with } \varepsilon(t) \sim i.i.d.(0, \sigma_{\varepsilon}^2) \qquad (19)
\end{aligned}
$$

where the MA coefficients $b(\tau)$ can also be expressed in terms of the gamma function

$$
\begin{aligned}
b(\tau) &= (-1)^{\tau} \begin{pmatrix} -d \\ \\ \tau \end{pmatrix} \\
&= \frac{(\tau+d-1)!}{(d-1)!\tau!} \\
&= \frac{\Gamma(\tau+d)}{\Gamma(d)\Gamma(\tau+1)} \qquad (20)
\end{aligned}
$$

as can be checked by following the preceding steps with $-d$ substituted for d.

As the time horizon increases, $\tau \rightarrow \infty$, proportionally,

$$
b(\tau) \sim \frac{\tau^{d-1}}{(d-1)!} \qquad (21)
$$

Viewed this MA way, any time series $X(t)$, even a fractionally integrated one, can thus be represented as a summation (integration) of white noise $\varepsilon(t)$.

We can characterize both such AR and MA processes by their *autocovariance function.*

Definition 11 *The **(non - normalized) Auto - Covariance Function (ACF)** of $x(t)$ is defined by the integral*

$$\begin{aligned}\gamma(\tau) &= \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \\ &= \int_{-\infty}^{\infty} x(t)L^{\tau}x(t)dt \end{aligned} \tag{22}$$

The ACFs of these long-term dependent random processes decay so slowly that for the case of *persistence,* when $d > 0$, the sum of the AR coefficients $a(\tau)$ diverges to infinity (= the financial market risk of investment returns increases) and for the case of *anti - persistence*, when $d < 0$, their sum collapses to zero (= the financial market risk of investment returns vanishes).[6] Of course, for the MA $b(\tau)$ coefficients the reverse is true. The main empirical research question is: how fast does financial risk divergence to infinity or financial risk convergence to zero occur?

In the next section, we'll discuss this persistence and anti - persistence of random (investment return) processes in terms of a variety of critical (Lipschitz) exponents. First, we need the definitions of regularly and slowly varying functions to be able to define the important concept of long - term time dependence, which we have used thus far in a rather loose fashion, but which now needs to be rigorously defined.

Definition 12 *A function $f(x)$ is said to be **regularly varying at infinity** with index λ if*

$$\lim_{\tau\to\infty} \frac{f(\tau x)}{f(x)} = x^{\lambda} \textit{ for all } x > 0 \tag{23}$$

*i.e., if it behaves asymptotically as a power function. When $\lambda = 0$, the function $f(x)$ is said to be **slowly varying at infinity,** since it behaves like a "constant" for a large horizon τ.*

We have finally arrived at the definition of a long - term time dependent random process. This random process figures now prominently in the financial literature concerned with the measurement of the efficiency and the microstructure of financial markets (*cf.* Lo and MacKinlay, 1999).

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[6] Such classical ACFs support the econometric measurements of Vector Auto - Regression (VARs) models. Classical VARs can represent higher order periodicities, but not the long term time dependent phenomenon of non - periodic cyclicities, because they are expressed in terms of integer Markov processes. Of course, one can also, unconventionally, model fractional VARs to properly represent globally dependent or long memory processes.

Definition 13 *A **long - term dependent** random process is a process with an ACF $\gamma(\tau)$, such that*

$$\gamma(\tau) = \left\{ \begin{array}{l} \tau^{\lambda} H(\tau) \text{ for } \lambda \in [-1, 0), \text{ or} \\ -\tau^{\lambda} H(\tau) \text{ for } \lambda \in (-2, -1] \end{array} \right\} \tag{24}$$

as the time interval lengthens, $\tau \to \infty$, where $H(\tau)$ is any slowly varying function at infinity.

The ACF of the afore - mentioned fractionally - differenced time series, when $\varepsilon(t) \sim i.i.d.(0, \sigma_{\varepsilon}^2)$ is given by:

$$\begin{aligned} \gamma(\tau) &= \frac{(-1)^{\tau}(-2d)!}{\tau!(-2d-\tau)!} \\ &\sim \sigma_{\varepsilon}^2 \tau^{2d-1} \text{ as } \tau \to \infty \end{aligned} \tag{25}$$

where $d \in (-\frac{1}{2}, \frac{1}{2})$. Thus, asymptotically, this ACF is *slowly decaying.*

We have now three important cases of noise processing in the financial markets:

(1) When $d \downarrow -\frac{1}{2}$, the market fractionally differentiates white noise $\varepsilon(t)$ and its ACF converges to $\gamma(\tau) \sim \sigma_{\varepsilon}^2 \tau^{-2}$, twice as fast as a hyperbolic decay. The market representing FBM produces an *antipersistent* financial time series.

(2) When $d = 0$, the market processes just white noise $\varepsilon(t)$, and its ACF converges to $\gamma(\tau) \sim \sigma_{\varepsilon}^2 \tau^{-1}$, a simple hyperbolic decay. The market representing FBM integrates the white noise once and produces thereby a neutrally persistent or *brown noise* financial time series.

(3) When $d \uparrow \frac{1}{2}$, the market fractionally integrate white noise $\varepsilon(t)$ and its ACF converges to $\gamma(\tau) \sim \sigma_{\varepsilon}^2$, a constant. The market representing FBM produces a *persistent* financial time series.

Remark 14 *One can measure these exponents by taking logarithms at both sides of the proportionality sign $\sim$:*

$$\ln \gamma(\tau) = (2d-1) \ln \tau + \ln \sigma_{\varepsilon}^2 + \ln C \tag{26}$$

for any constant C. The empirically measured slope $(2d-1)$ in this double - logarithmic picture provides us with the value of the differentiation exponent d.

We present here also the spectral density of the fractionally - differenced time series at frequencies close zero. The spectral density is the Fourier Transform of its ACF:

$$P(\omega) \cong \sigma_\varepsilon^2 (1 - e^{-j\omega})^{-d}(1 - e^{j\omega})^{-d}$$
$$\sim \sigma_\varepsilon^2 \omega^{-2d}$$
$$= \sigma_\varepsilon^2 \omega^{-\upsilon} \text{ as } \omega \to 0 \tag{27}$$

The spectral density $P(\omega)$ will be either infinite, as the frequencies approach zero, $\omega \to 0$, when $d > 0$: we differentiate the time series $X(t)$, *c.q.,* we integrate white noise $\varepsilon(t)$. Or, the opposite is true and the spectral density is zero, as the frequencies approach zero, $\omega \to 0$, when $d < 0$: we integrate the time series $X(t)$, *c.q.,* differentiate the white noise $\varepsilon(t)$. The exponent $\upsilon = 2d$ is called the *spectral exponent*.

Before we continue our discussion of the Fractional Brownian Motion (FBM) model and how to measure it, we'll discuss now first some strong, and popular, contenders of the FBM: the (G)ARCH processes, which are martinghale-consistent. We will demonstrate that the FBM dominates the GARCH model in representing long - term time dependence.

4 (G)ARCH Processes

There is strong empirical and theoretical evidence that the second moment, or variance, of the rates of return on financial assets are time - dependent random processes (Nelson, 1991). The ARCH (= Auto - Regressive Conditional Heteroskedastic) processes, introduced by Engle (1982) are the only plausible alternative to fractal distributions and fractionally differenced time - series. ARCH processes appear to fit the empirical data of stock returns, interest rates, inflation rates and foreign exchange rates, since they can have sharp modes and fat tails, *i.e.,* they can exhibit different degrees of leptokurtis for the same variances. Bollerslev (1986) generalizes the ARCH model further to GARCH (= Generalized ARCH) and IGARCH (= Integrated GARCH) models.

Although, by definition, ARCH models cannot explain correctly the measured long - term time dependence (LM) phenomena, the IGARCH models do a better, although still not perfect, job of explaining them, because of the incorporation of a unit root, *i.e.,* a marginally stable process. For a promotional overview of ARCH models in finance, *cf.* Bollerslev, Engle and Nelson (1994), the collection of articles by Engle (1995) and Bollerslev, Chou and Kroner (1998).

4.1 Statistical Properties of ARCH Processes

ARCH models describe random processes, which are *locally nonstationary*, but *asymptotically stationary*. This implies that the parameters of its conditional p.d.f. are time - varying. Still the random process has a well - defined asymptotic p.d.f.. ARCH processes are models for which the financial risk σ_t is conditioned on a finite series of past values of the square value of the process x_t itself, as follows.

Definition 15 *An **ARCH(τ)**, or **Autoregressive Conditional Heteroskedastic** random process x_t of order p is a random process defined by:*

$$\sigma_t^2 = a_0 + a_1 x_{t-1}^2 + ... + a_p x_{t-p}^2$$
$$\text{with } a_0, a_1, ..., a_\tau > 0,\ E\{x_t\} = 0 \text{ and } E\left\{x_t^2 | \mathcal{A}_{t-1}^{t-p}\right\} = \sigma_t^2 \tag{28}$$

where $E\left\{x_t^2 | \mathcal{A}_{t-1}^{t-p}\right\}$ is an expectation of a conditional p.d.f., conditioned on the information of a finite memory of x_t of a lagged horizon of p time periods from $t-1$ through $t-p$.

Remark 16 *An ARCH(p) process is completely determined when p and the shape of the p.d.f. are defined and parametrized by the coefficients $a_0, a_1, ..., a_p$. The conditional p.d.f. may be Gaussian or non - Gaussian.*

Example 17 *The, among currency traders popular, ARCH(1) process is*

$$\sigma_t^2 = a_0 + a_1 x_{t-1}^2 \tag{29}$$

with Gaussian conditional p.d.f., is characterized by the finite asymptotic or limit ("unconditional") variance (= the variance observed over an infinite horizon)

$$\sigma^2 = \lim_{t\to\infty} \sigma_t^2 = \frac{a_0}{1-a_1} \tag{30}$$

provided

$$1 - a_1 \neq 0, 0 \leq a_1 < 1 \tag{31}$$

The limiting normalized kurtosis of this ARCH(1) process is

$$\begin{aligned}\kappa &= \lim_{t\to\infty} \frac{E\left\{x_t^4\right\}}{E\left\{x_t^2\right\}^2} \\ &= \frac{m_4}{m_2^2} \quad (32) \\ &= \frac{c_4}{m_2^2} + 3 \\ &= \frac{c_4}{\sigma^4} + 3 \\ &= \frac{6a_1^2}{1-3a_1^2} + 3 \quad (33)\end{aligned}$$

which is finite if

$$0 \le a_1 < \frac{1}{\sqrt{3}} \quad (34)$$

Notice the potential excess kurtosis of this ARCH(1) process, since $\frac{6a_1^2}{1-3a_1^2}+3 \geq 3 =$ *the kurtosis of a Gaussian distribution. By varying* a_0 *and* a_1*, one can obtain random processes with the same limit variance* σ^2*, but with different values of limiting kurtosis. An example for an ARCH(1) process is given in the following Table 1. Successive increments of simulations of these three ARCH(1) processes are shown in Fig. 1 and their respective p.d.f.s in Fig. 2. Both figures are borrowed, with small modifications, from Mantegna and Stanley (2000, pp. 79 - 80).*

[TABLE 1 ABOUT HERE]

[FIGURE 1 ABOUT HERE]

[FIGURE 2 ABOUT HERE]

4.2 Statistical Properties of GARCH Processes

Bollerslev (1986, 1987) proposes a generalized ARCH random process, called GARCH(p, q) process, which can represent a greater degree of inertia in its conditional volatility or risk, as follows.

Definition 18 *A* ***GARCH(***p, q***)****, or* ***Generalized Autoregressive Conditional Heteroskedastic Random Process*** x_t *of orders* (p, q) *is a random process defined by:*

$$\sigma_t^2 = a_0 + a_1 x_{t-1}^2 + ... + a_p x_{t-p}^2 + b_1\sigma_{t-1}^2 + ... + b_q\sigma_{t-q}^2$$

$$\text{with } a_0, a_1, ..., a_p, b_1, ..., b_q > 0,\ E\left\{x_t\right\} = 0 \text{ and } E\left\{x_t^2 | \mathcal{A}_{t-1}^{t-p,t-q}\right\} = \sigma_t^2 \quad (35)$$

where $E\left\{x_t^2|\mathcal{A}_{t-1}^{t-p,t-q}\right\}$ *is an expectation of a conditional p.d.f., conditioned on the information of a finite memory of* x_t *of* p *or* q *time periods, whichever is longest.*

Example 19 *Baillie and Bollerslev (1992) show that the simplest GARCH(1,1) process, with a Gaussian p.d.f. has as the finite asymptotic or limit ("unconditional") variance*

$$\sigma^2 = \frac{a_0}{1 - a_1 - b_1} \quad (36)$$

The limit normalized kurtosis of this GARCH$(1,1)$ *process is given by*

$$\kappa = \lim_{t\to\infty} \frac{E\{x_t^4\}}{E\{x_t^2\}} = \frac{m_4}{m_2^2}$$
$$= \frac{6a_1^2}{1-3a_1^2-2a_1b_1-b_1^2} + 3 \quad (37)$$

which allows again excess kurtosis, depending on various configurations of the values of the parameters a_1 and b_1. When $a_1 = 0$, the process is Gaussian. When $b_1 > 0$ the variance feedback process of σ_t increases the kurtosis of the x_t process.

4.3 (G)ARCH Processes: Noncorroborated Time Scaling

(G)ARCH processes are empirically deficient models since they don't exhibit the observed empirical long - term dependence (LM) properties, in particular, the proper time-frequency scaling properties. For example, the empirical evidence shows that the variance of financial market returns is characterized by power law correlations. Since the correlation of the squared x_t of a GARCH$(1,1)$ process is exponential, a GARCH$(1,1)$ process cannot be used to properly describe this empirical phenomenon. In other words, (G)ARCH model processes can't represent the empirically observed long memories. They are investment - horizon $\tau-$specific and can represent only finite memories. They measure conditional variances for specific finite horizons of maximally $\tau = p$ or q length and not of infinite length. In contrast, fractionally differenced processes indiscriminately represent p.d.f.s for all possible investment horizons, finite and infinite and produce thus the proper scaling properties for the unconditional p.d.f.s.

Example 20 *Mantegna and Stanley (2000) compare empirical investigations of the S&P500 high frequency data with simulations of a GARCH*$(1,1)$ *process, characterized by the same limiting variance and kurtosis. Such equality is ensured by calibrating the three control parameters of the GARCH*$(1,1)$ *process,* a_0, a_1 *and* b_1 *subjectively and thus, non - scientifically. For example, Akgiray (1989) arbitrarily chooses* $b_1 = 0.9$*. From the empirical analysis of the S&P500 minute - by - minute data for the period January 1984 - December 1989 (*$493,545$ *minutes), Mantegna and Stanley find that the limit variance* $\sigma^2 = 0.00257$ *and the limit kurtosis* $\frac{m_4}{m_2^2} \approx 43$*. Using the preceding equations, with* $b_1 = 0.9$*, the parameter values* $a_0 = 2.30 \times 10^{-5}$ *and* $a_1 = 0.09105$ *are obtained. The resulting simulated p.d.f. fits the* $\Delta t = 1$ *minute p.d.f. data well. But, as Mantegna and Stanley (2000, p. 87) correctly conclude: "The fact that the GARCH*$(1,1)$ *process describes well the* $\Delta t = 1$ *minute p.d.f. does not ensure that the same process describes well the stochastic dynamics of the empirical data for any time horizon* Δt*." To describe the dynamics of the price changes in a complete way, in addition to the p.d.f. of the price changes at a given* *time horizon, the scaling properties of price change p.d.f.s must be also considered. Although there is no theoretical model for the scaling properties of the GARCH(*$1,1$*) process, one can perform numerical simulations of the GARCH(*$1,1$*) process, as reported in the double - logarithmic Fig. 3 (borrowed, with an important correction, from Mantegna and Stanley, 2000, p. 86). From Fig. 3 it is clear that although the GARCH(*$1,1$*) process can accurately describe the* $\Delta t = 10^0 = 1$ *minute empirical leptokurtic p.d.f. of price changes, it fails to describe the scaling properties of the empirical p.d.f.s of the high - frequency S&P500 data for all higher time horizons, using the same control parameters. The absolute value of the empirical slope of the GARCH (*$1,1$*) simulated price change data (black squares) is a Gaussian Hurst exponent* $H = 1/\alpha_Z = \frac{\ln 10^{1.5}}{\ln 10^3} = 0.5$, *while the slope of the high - frequency S&P500 data (white circles) has a Hurst exponent* $H = 1/\alpha_Z = \frac{\ln 10^2}{\ln 10^3} = 0.67$.[7]

[FIGURE 3 ABOUT HERE]

The *Integrated variance GARCH*, or IGARCH models of Bollerslev (1986), a further generalization of his GARCH model, are characterized by infinite unconditional variance, because they contain a unit root. In those models, current information remains important for the forecasts of conditional variance for all investment horizons. Although empirically clearly uncorroborated models, it is still an open theoretical research question if these models produce the proper dynamic scaling properties (*cf.* Alexander, 1998). Numerical simulations are easy to execute, but the derivation of the theoretical scaling properties of these models is quite a difficult matter and the possible topic for a doctoral dissertation.

5 Fractional Brownian Motion

Thus, we must conclude that one of the most useful generic research models for a random process currently in existence in the financial markets literature, the *Fractional Brownian Motion* (FBM). This random process model encompasses virtually all of the observed empirical phenomena in the time series of financial markets. A recent theoretical paper by Elliott and van den Hoek (2000) discusses the theoretical niceties of the FBM and shows how easy it is to replace the GBM by the FBM in all the familiar dynamic valuation and hedging models in the finance literature, to present models that are much closer to empirical observations in their scaling properties. In this paper,

[7] *And not the incorrect value of* $H = 0.53$ *provided by Mantegna and Stanley (2000, p. 86), who are proven wrong by their own Fig. 10.7, which we borrowed as our Fig. 3.*

we'll focus on the empirical measurement analysis of the FBM and the wide range of empirical phenomena it is able to represent.

Definition 21 ***Fractional Brownian Motion (FBM)*** *is defined by the fractionally differenced time series*

$$(1-L)^d x(t) = \varepsilon(t),\ d \in (-\frac{1}{2}, \frac{1}{2}),\ \textit{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{38}$$

where $x(t) = \ln X(t) - \ln X(t-1) = (1-L)\ln X(t)$*.*

A completely equivalent definition is that Fractionally Brownian Motion $x(t)$ is fractionally integrated white noise, since

$$x(t) = (1-L)^{-d}\varepsilon(t),\ d \in (-\frac{1}{2}, \frac{1}{2}),\ \text{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{39}$$

Remark 22 *The FBM can also be presented in terms of the original market price series* $X(t)$ *as*

$$\begin{aligned} &(1-L)^d (1-L)\ln X(t) \\ &= (1-L)^{d+1}\ln X(t) \\ &= \varepsilon(t),\ \textit{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \end{aligned} \tag{40}$$

Table 2 provides a comparison of the ACFs of two simulated fractionally differenced time series, $(1-L)^d x(t) = \varepsilon(t)$ for $d = -\frac{1}{3}$ and $\frac{1}{3}$, with long - term memory, with the ACF of a simulated AR(1) time series, $x(t) = \rho x(t-1) + \varepsilon(t)$ with $\rho = 0.5$ and short - term memory. The variance σ_ε^2 of the i.i.d. noise was chosen to yield a unit variance for $x(t)$ in all three cases. Notice the very gradual decline and infinite continuation of the ACF when $d = \frac{1}{3}$ or when $d = -\frac{1}{3}$ and the initial steep decline and virtual non - existence of the ACF of the AR(1) after only 10 lags.

[TABLE 2 ABOUT HERE]

The standard Geometric Brownian Motion (GBM) is the special case of a fractionally differenced time series, when $d = 1$, so that

$$\Delta x(t) = (1-L)x(t) = \varepsilon(t), \tag{41}$$

$$\text{or}\ x(t) = (1-L)^{-1}\varepsilon(t),\ \text{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{42}$$

with its ACF decaying hyperbolically:

$$\gamma(\tau) \sim \sigma_\varepsilon^2 \tau^{-1} \tag{43}$$

which is proportional to the variance of the *i.i.d.* innovations $\varepsilon(t)$: σ_ε^2. Thus, obviously, the GBM is *self - similarly scaling.* Brownian Motion is once integrated white noise, since its innovations are white noise, *i.e.,* they exhibit a flat, constant spectral density: $P_\varepsilon(\omega) = \sigma_\varepsilon^2$.

Example 23 *Fig. 4 provides the standardized empirical ACFs (autocorrelograms) of equally - weighted CRSP daily and monthly stock returns indexes. The observation period for the daily index is July 1962 to December 1987, and January 1926 to December 1987 for the monthly index. Notice that these empirical ACFs are not as smooth and continuous as presented by the theoretical FBMs of Table 2, thus emphasizing the problem of identification of the proper difference exponent d from empirical ACFs. Theu also don't die off: a clear indication of the presence of long memory.*

[FIGURE 4 ABOUT HERE]

We'll now turn to Hust's Range - Scale Analysis, which is the basis for most of the recent efforts to measure homogeneous Hurst exponents to determine the degree of scaling in financial time series or rates of return or of implied volatility.

6 Range/Scale Analysis

To detect global, "strong,", or long - term time dependence, Mandelbrot (1965) suggested to use Hurst's "rescaled range", or R/S statistic, which Hurst (1951) had developed in his study of the Nile river discharges. As we will see, the Hurst statistic leads to the Hurst or $H-$ exponent. Although recently the $H-$ exponent has become quite popular in finance (*cf.* Peters, 1992), there are some reasons to consider this exponent as too limited to measure all forms of aperiodic cyclicities, in particular, with financial turbulence and chaos. There are already better defined exponents supported over larger domains, which cover more extreme cases, as we will discuss a bit later in this paper.

6.1 Hurst's Original Range/Scale Statistic

Definition 24 *(**Hurst's Range/Scale Statistic**) Consider a sequence of investment returns* $\{x(t)\}$ *and its empirical mean (= first cumulant = first moment)*

$$c_1 = m_1 = \frac{1}{T}\sum_{t=1}^{T} x(t) \tag{44}$$

and its empirical variance (= second cumulant)

$$\begin{aligned} c_2 &= m_2 - m_1^2 \\ &= \frac{1}{T}\sum_{t=1}^{T}[x(t) - m_1]^2 \end{aligned} \tag{45}$$

then Hurst's R/S statistic is defined by

$$RS_H(T) \equiv \frac{1}{c_2^{0.5}}\left[Max_{1\leq t\leq T}\sum_{t=1}^{\tau}[x(t) - m_1] - Min_{1\leq t\leq T}\sum_{t=1}^{\tau}[x(t) - m_1]\right] \geq 0 \tag{46}$$

The first term in brackets is the maximum (over interval τ) of the partial sums of the first τ deviations of $x(t)$ from the mean. Since the sum of all τ deviations of $x(t)$ from their mean is zero, this maximum is always nonnegative. The second term is the minimum (over interval τ) of this same sequence of partial sums; hence it is always non - positive. The difference of these two quantities, called the "range" is thus always nonnegative. This range is then scaled by the empirical standard deviation for the whole data set $c_2^{0.5}$.

6.2 Lo and MacKinlay's (1999) "Modification"

Lo and MacKinlay (1999) modify the rescaled range measure of Hurst, so that it becomes robust to short - term dependence, and derive its limiting distribution under both short - term and long - term dependence. In contrast to many other authors in the current literature, including Mandelbrot (1965, 1972), Mandelbrot and Taqqu (1979), Mandelbrot and Wallis (1969), Lo and MacKinlay also claim that, when they apply their modified R/S statistic to daily and monthly stock return indices over different periods and sub - periods, there is no evidence of long - term dependence, once the effects of short - term dependence are accounted for. Therefore, they suggest that the time series behavior of stock returns may be adequately captured by the more conventional (Markov) models of short - term dependence. However, the accumulated empirical evidence of the last decade we collected in Jamdee and Los (2004) contradicts their assertion and strongly shifts the balance of the empirical veidence in the direction earlier indicated by Mandelbrot *c.s.*

6.3 Homogeneous Hurst Exponent

The Hurst statistic provides us with a means to analyze the dependence characteristics of time series and to determine if they are serially, or globally dependent, since it delivers the Hurst exponent as a fractal dimension, Hölder, or Lipschitz irregularity coefficient (Mandelbrot, 1972).[8]

Definition 25 *The* ***Hurst exponent*** H *is defined as*

$$0 < H = \lim_{\tau \to \infty} \frac{\ln RS_H(\tau)}{\ln \tau} < 1 \tag{47}$$

For serially, or short - term, dependent time series, such as strong - mixing processes, $H \to 0.5$ when $\tau \to \infty$, but for globally dependent time series $H \to 0.5 + d$. In fact, the fractionally - differenced random processes satisfy the equality $H = 0.5 + d$. Thus, Mandelbrot (1965) suggests to plot $\ln RS(\tau)$ against $\ln \tau$ to compute H from the slope of the resulting plot. He calls any time series $x(t)$ for which shows the R/S statistic time - scaling, $RS_H(\tau) \propto \tau^H$: "*Hurst noise.*"

Example 26 *As Hurst (1951) showed, based on the water-level minima recorded in the period 622 - 1469, the annual water flow of the Nile river in Egypt shows a strong long - term persistence with* $H = 0.91$, *that requires unusually high barriers, such as the Aswan High Dam, to contain damage and rein in the floods. As Mandelbrot and Wallis (1969) showed, for the rivers Saint Lawrence in Canada, Colorado in the USA, and the Loire in France, the persistence is considerably lower with* $0.5 < H < 0.9$. *The river Rhine (at the Swiss - French - German triple point near Basel) is exceptional with a long - term exponent of* $H = 0.5$, *indicating that its water flow changes like white noise (Whitcher et al., 2002). In other words, the Rhine river tends to produce no major catastrophic floods.*

The ACF of the fractionally-differenced time series can now be written in terms of the $H-$exponent, since we can now substitute $d = H - 0.5$ into the previously defined ACF to get:

$$\begin{aligned} \gamma(\tau) &= \frac{\sigma_\varepsilon^2 \Gamma(2-2H)\Gamma(\tau + H - 0.5)}{\Gamma(H-0.5)\Gamma(1.5-H)\Gamma(\tau + 1.5 - H)} \\ &\sim \sigma_\varepsilon^2 \tau^{2H-2} \text{ as } \tau \to \infty \end{aligned} \tag{48}$$

where $H \in (0, 1)$.

[8] Hölder (1859 - 1937) was a German mathematician, who devised treatment of divergent series of arithmetic summations, which led to a regularity exponent now recognized to be similar to Hurst's. However, Hölder was thinking about microscopic (physics) phenomena, in contrast to Hurst, who thought about macroscopic (hydrological) phenomena. The Hölder - Hurst exponents are also called critical Lipschitz irregularity exponents.

7 Critical Color Categorization of Randomness

7.1 Blue, White, Pink, Red, Brown and Black Noise

Following, Schroeder (1991a, pp. 121 - 137) we can now present a colored categorization of randomness, or irregularity, by collecting the various descriptive exponents and relating them to each other. This comparison of exponents will facilitate the reading of a great variety of interdisciplinary research articles on phenomena of time dependence. There exists an intimate relationship between the concept of financial "randomness" based on incomplete markets and the concept of "irregularity" as defined by the mathematician Lipschitz.

Definition 27 *(1) When the Hurst exponent* $0 < H < 0.5$*, i.e.,* $-0.5 < d < 0$*, the time series of increments is called* ***antipersistent****. (2) When* $H = 0.5$*, i.e.,* $d = 0$*, the increments are independent or "white", and the time dependence of the series is* ***neutral*** *(or* ***neutrally persistent****). Examples are the increments of Random Walks or Arithmetic Brownian Motions (for speculative prices) and of Geometric Brownian Motion (for investment returns). The Brownian Motion series is once - integrated "white noise" and is called "brown" noise. Its ACF decays hyperbolically:*

$$\begin{aligned}\gamma(\tau) &= \frac{\sigma_\varepsilon^2 \Gamma(\tau)}{\Gamma(\tau+1)} \\ &= \frac{\sigma_\varepsilon^2 (\tau-1)!}{\tau!} \\ &= \sigma_\varepsilon^2 \tau^{-1}\end{aligned} \tag{49}$$

(3) When $0.5 < H < 1$*, i.e.,* $0 < d < 0.5$*, the time series of increments is called* ***persistent****.*

In the case of extreme anti - persistence, $H \downarrow 0$, so that the ACF of the time series decays faster than hyperbolically in a quadratic fashion:

$$\begin{aligned}\gamma(\tau) &= \frac{\sigma_\varepsilon^2 \Gamma(\tau-0.5)}{\Gamma(\tau+1.5)} \\ &= \frac{\sigma_\varepsilon^2 (\tau-1.5)!}{(\tau+0.5)!} \\ &= \frac{\sigma_\varepsilon^2}{(\tau+0.5)(\tau-0.5)} \\ &= \frac{\sigma_\varepsilon^2}{(\tau^2-0.25)} \\ &\approx \sigma_\varepsilon^2 \tau w^{-2} \text{ as } \tau \to \infty\end{aligned} \tag{50}$$

At the other extreme of Hurst's limited *randomness spectrum* $H \uparrow 1$, so that the ACF of the time series remains a flat constant and it never vanishes:

$$\gamma(\tau) = \frac{\sigma_\varepsilon^2 \Gamma(\tau + 0.5)}{\Gamma(\tau + 0.5)}$$
$$= \sigma_\varepsilon^2 \text{ a constant, as } \tau \to \infty \tag{51}$$

7.2 Irregularity Exponents

We can make a connection with fat-tailed (leptokurtic and platykurtic) stable distributions, once we realize that, for globally (long - term) dependent time series, for which the autocovariance function has the form

$$\gamma(\tau) = \left\{ \begin{array}{l} \tau^\lambda H(\tau) \text{ for } \lambda \in [-1, 0), \text{ or} \\ -\tau^\lambda H(\tau) \text{ for } \lambda \in (-2, -1] \end{array} \right\} \tag{52}$$

as the time-interval lengthens, $\tau \to \infty$, and $H(\tau)$ is any slowly varying function at infinity, the *dependence exponent* λ equals

$$\begin{aligned} \lambda &= 2d - 1 \\ &= \upsilon - 1 \\ &= 2H - 2 \\ &= \frac{2}{\alpha_Z} - 2 \\ &= 2\alpha_L - 2 \end{aligned} \tag{53}$$

where d is the *difference (order) exponent,* υ is the *spectral exponent*, H is the aforementioned *Hurst exponent,* α_Z is the *stability exponent* of the Zolotarev parametrization of the stable distributions, and α_L is the Lipschitz *regularity exponent.*[9] Thus, the randomness, or irregularity,

[9] Somewhat confusingly presented in the literature, the Zolotarev stability $\alpha_Z = 1/\alpha_L$, where α_L is the Lipschitz regularity exponent. In the literature, one often finds just α and it is not always clear if the author(s) mean(s) the Zolotarev stability exponent α_Z or the Lipschitz α_L. We hope that this comparison of the various critical exponents and the presentation of their relationships will lift the dense fog between the various scientific subdisciplines, in particular in finance, physics and engineering, which deal with essentially the same signal processing phenomena.

categorizations can be expressed in terms of each of these *critical exponents*. For completeness of definition: $\frac{\lambda}{2}$ is the so - called *time - scaling exponent*.

The complete spectrum of randomness, or irregularity, in terms of the five critical exponents equivalent to the Lipschitz regularity exponent is given in the following Table 3, which provides the essential relationships between the exponents of the first difference of Fractional Brownian Motion (*cf.* also Keshner 1982; Flandrin, 1989) 1982).

[TABLE 3 ABOUT HERE]

For example, for the Brownian Motion increments $\varepsilon(t)$, which are white noise:

$$\lambda = -1, d = 0, \upsilon = 0, H = 0.5, \alpha_Z = 2 \tag{54}$$

Thus, the time series of Brownian Motion increments is modelled by white noise:

$$\begin{aligned} x(t) &= (1 - L)^0 \varepsilon(t) \\ &= \varepsilon(t) \end{aligned} \tag{55}$$

Fractional integration of such white noise, when $d = 0.5$ and $H \uparrow 1$, results in a red noise series (Gilman, *et al.*, 1963):

$$x(t) = (1 - L)^{-0.5} \varepsilon(t) \tag{56}$$

One complete integer integration of the white noise, when $d = 1$, results in a brown noise series (= Brownian Motion)

$$x(t) = (1 - L)^{-1} \varepsilon(t) \tag{57}$$

Visual samples of time series of such white, red and brown noise are given by Fig. 5.

In the case of $0.5 < H < 1$, the vital property of the FBM is that the persistence of its increments extends forever: *it never dies out* and gives rise to the empirically observed *catastrophes*. The strength of such persistence is measured by the critical $H-$exponent.

[FIGURE 5 ABOUT HERE]

Example 28 *The rates of return $x(t)$ of the S&P500 stock market index show mild persistence with $H = 0.67$. Indeed, their graph is less irregular than that of ordinary Geometric Brownian*

Motion increments. Its fractional dimension D is thus between the dimension of a line, $D = 1$, and the dimension of a plane, $D = 2$:

$$1 < D = 2 - H = 1.33 < 2 \tag{58}$$

Curiously, the Dow Jones Industrials stock index does not show any persistence, according to Li (1991).

Example 29 *The fractional dimension of GBM increments, with $H = 0.5$, is*

$$D = 2 - H = 1.5 = \frac{3}{2} \tag{59}$$

The case where $0.5 < d < 1.5$, or, equivalently, $1 < \upsilon < 3$, which cannot be measured directly by the $H-$exponent, but only after one differentiation, has been called the *infrared catastrophe* (Wornell and Oppenheim, *1992).* It can be measured by the wavelet multiresolution analysis (MRA). More fractional integration, for example $d = 2$, results in heavily persistent, or pure *black noise*

$$x(t) = (1 - L)^{-2}\varepsilon(t) \tag{60}$$

As Schroeder (1991a, p. 122) comments:

> "Black - noise phenomena govern natural and unnatural catastrophes, like floods, droughts, bear markets, and various outrageous outages, such as those of electrical energy. Because of their black spectra, such disasters often come in clusters."

In contrast, the FBM increments with $0 < H < 0.5$ are antipersistent noise, hence they diffuse more quickly than the Brownian increments. The FBM increments continuously return to the point they came from.

Remark 30 *Notably this means that the Random Walk innovations $\varepsilon(t)$ are rather exceptional. They exhibit the same stability, $\alpha_Z = 2$, and (in -)dependence, $H = 0.5$, as Gaussian random variables, but do not necessarily have to be Gaussian! Furthermore, their ACF drops off geometrically with $\lambda = -1$. By measuring the financial - economic, e.g., stock price innovations to be close to Gaussian, Granger and Morgenstern (1963) and Granger (1966) inferred that such innovations had a typical spectral shape. However, their inference was erroneous, and there was nothing typical about that inferred shape, because it was biased by thinking exclusively in term of Gaussian innovations $\varepsilon(t) \sim N(0, \sigma_\varepsilon^2)$. For example, the covariance function of modern foreign exchange rates, like the Japanese Yen or the German Deutschemark, shows anti - persistence, i.e., a slower drop - off of the ACF than the "typical " spectral shape based on this assumption of Gaussian i.i.d. innovations.*

7.3 Stability Spectra

It is very important to understand that the Hurst exponent H is a rather limited measure of randomness and distributional stability with a very limited measurement domain, and that the α_Z-stability exponent, and the $\upsilon-$spectral exponent have much more extensive measurement domains. This becomes clear, when we geometrically visualize the mathematical relationships, the constraints, and the respective domains of the various critical irregularity exponents in Fig. 6.

[FIGURE 6 ABOUT HERE]

The implied equality $\alpha_Z = \frac{1}{H}$ does not hold for all values of α_Z, since the Hurst exponent, per definition, $0 < H < 1$, implies that $1 < \alpha_Z < \infty$, while parametrized stable distributions are usually defined only for the limited domain $0 < \alpha_Z \leq 2$. Apparently there exist empirical *ultra - stable* distributions (not yet parametrized!) in the domain $2 \leq \alpha_Z < \infty$, since we find *in extremo* $\alpha_Z \uparrow \infty$ when $H \downarrow 0$ (and $d \uparrow 0.5$), which is *complete stability.* These distributions are the distributions of *singularities*, or *singularity spectra*, which can be characterized and measured by the stability exponent α_Z.

As is clearly visible in Fig. 6, when the Hurst exponent vanishes, $H \downarrow 0$, the Zolotarev stability exponent becomes infinite, $\alpha_Z \uparrow \infty$. In other words, for very small values of the Hurst exponent , $H \downarrow 0$, we acquire very uncertain measurements regarding Zolotarev's stability exponent α_Z.

In addition, there are now theoretically defined, parametrized stable distributions where $0 < \alpha_Z < 1$, which can also not be measured by the Hurst $H-$exponent directly, but can be measured by α_Z, if we can compute α_Z in some other fashion. These are the *ultra - unstable* distributions. But, empirically, there appears to be a physical turbulence barrier at $\alpha_Z = 2/5$. In other words, there appears not to exist any empirical α_Z such that $0 < \alpha_Z < 2/5$, even though there are theoretical Zolotarev - parametrized distributions defined for such α_Z values. Again, this is an area open for further theoretical and empirical research.

In conclusion, the best domain for using the $H-$exponent to compute the stability α_Z-exponent is in the Gaussian neighborhood of $H = 0.5$, where $\alpha_Z = 2$. Still, it is important to recognize that there exists a *stability spectrum* of randomness, or irregularity, completely specified by the stability exponent α_Z.

Remark 31 *Of course, one can still use the $H-$exponent for measuring infrared and black catastrophes, by measuring the $H-$exponent after proper integer - differentiation. For example, we hypothesize that $x(t)$ is pure black noise and has a spectral exponent $\upsilon = 4$, then differentiation two full times ($d = 2$) should theoretically result in white noise series with a flat spectrum, $\upsilon = 0$, so that $H = 0.5$. When we empirically measure, for example, $H = 0.2 \rightarrow \upsilon = -0.6$, then the original series must have a spectral coefficient of $\upsilon = -0.6 + 4 = 3.4$ and not 4.*

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9 Tables

For Parameter:	**Limit Kurtosis**
$a_0 = 1, a_1 = 0$	3 (= Gaussian process)
$a_0 = a_1 = 0.5$	9
$a_0 = 0.45, a_1 = 0.55$	23

Table 1: ARCH(1) Limit Kurtosis

	$d = -\frac{1}{3}$	$d = \frac{1}{3}$	$AR(1)$, $a_1 = 0.5$
Lag τ	$\gamma(\tau)$	$\gamma(\tau)$	$\gamma(\tau)$
1	-0.250	0.500	0.500
2	-0.071	0.400	0.250
3	-0.036	0.350	0.125
4	-0.022	0.318	0.063
5	-0.015	0.295	0.031
10	-0.005	0.235	0.001
25	-0.001	0.173	2.98×10^{-8}
50	-3.24×10^{-4}	0.137	8.88×10^{-16}
100	-1.02×10^{-4}	0.109	7.89×10^{-31}

Table 2: ACFs of Long and Short Memory Series

EXPONENTS: COLOR:	**Dependence** λ	**Difference** d	**Spectral** υ	**Hurst** H	**Stability** α_Z
Blue noise	$\lambda \downarrow -2$	$d = -0.5$	$\upsilon = -1$	$H \downarrow 0$	NA
Antipersistence	$-2 < \lambda < -1$	$-0.5 < d < 0$	$-1 < \upsilon < 0$	$0 < H < 0.5$	NA
White noise	$\lambda = -1$	$d = 0$	$\upsilon = 0$	$H = 0.5$	$\alpha_Z = 2$
Persistence (Pink)	$-1 < \lambda < 0$	$0 < d < 0.5$	$0 < \upsilon < 1$	$0.5 < H < 1$	$1 < \alpha_Z < 2$
Red noise	$\lambda \uparrow 0$	$d = 0.5$	$\upsilon = 1$	$H \uparrow 1$	$\alpha_Z = 1$
Brown noise	NA	$d = 1$	$\upsilon = 2$	NA	$\alpha_Z = 2/3$
Black noise	NA	$1 \leq d \leq 2$	$2 < \upsilon \leq 4$	NA	$2/5 \leq \alpha_Z < 2/3$

Table 3: Equivalence of Various Critical Irregularity Exponents

10 Figures

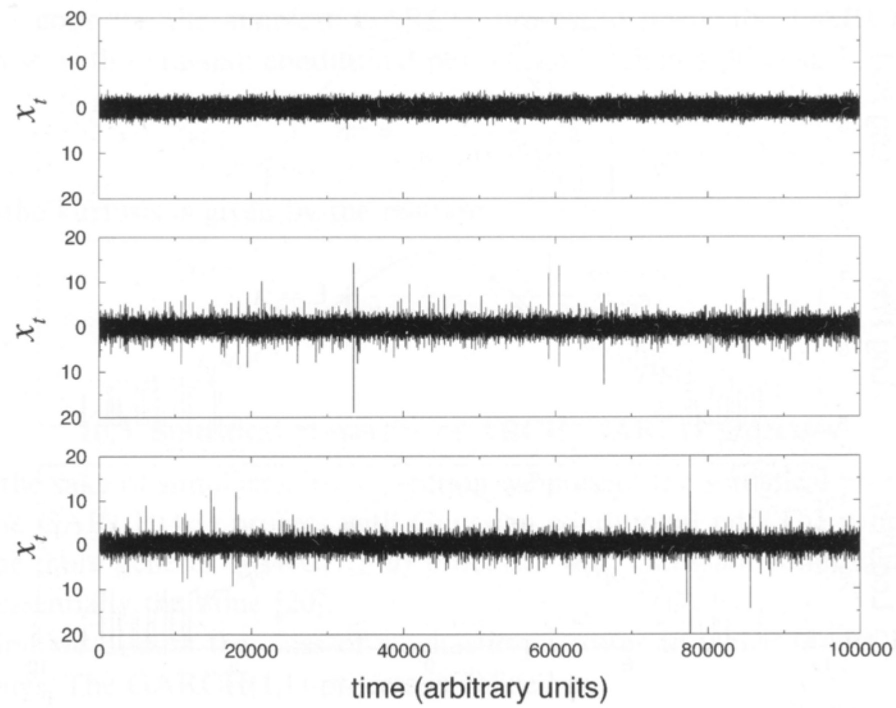

Figure 1: Successive increments of ARCH(1) simulations with the same unconditional variance ($\sigma^2 = 1$). Events outside three standard deviations are almost absent when $\kappa = 3$ (top: $\alpha_0 = 1, \alpha_1 = 0$). They are present when $\kappa = 9$ (middle: $\alpha_0 = \alpha_1 = 0.5$), and are more intense when $\kappa = 12$ (bottom: $\alpha_0 = 0.45, \alpha_1 = 0.55$)

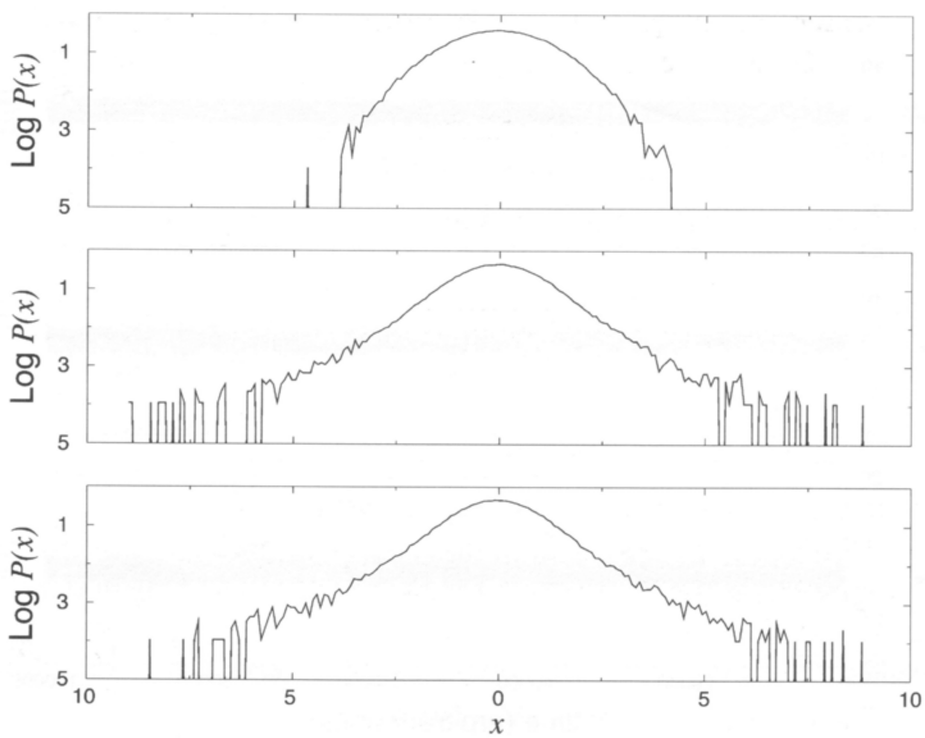

Figure 2: Logarithmic probability density function of the successive increments shown in Fig. 1. The p.d.f. is Gaussian when $\kappa = 3$ (top) and is leptokurtic when $\kappa = 9$ or $\kappa = 23$ (middle and bottom).

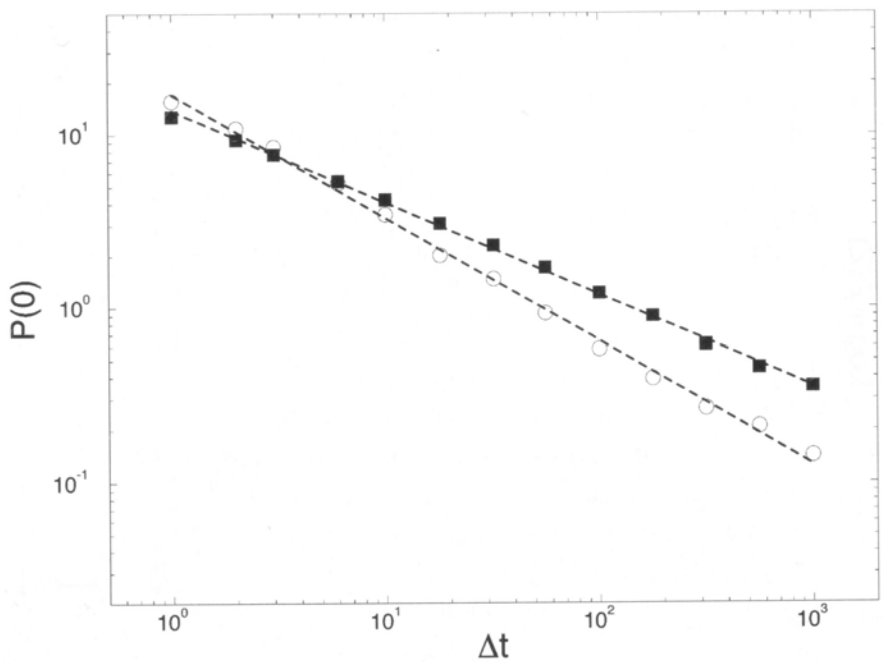

Figure 3: Comparison of the scaling properties of the unconditional p.d.f. of a GARCH(1, 1) stochastic process (black squares) with the ML estimated parameter values $a_0 = 2.30 \times 10^{-5}$, $a_1 = 0.09105$ and $b_1 = 0.9$ with the scaling properties of the p.d.f. of the S&P500 high - frequency data (white circles), which close to that of a Gaussian p.d.f.. The scaling of the GARCH(1, 1) process fails to describe the empirical behavior in the S&P500 high - frequency data

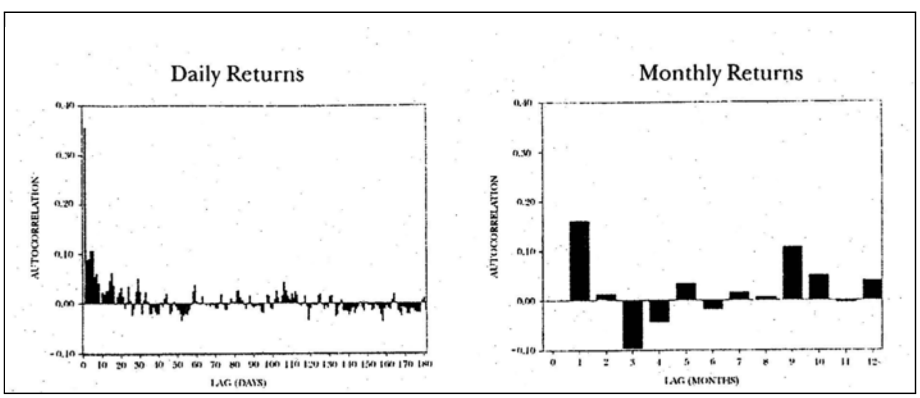

Figure 4: Autocorrelograms of equally - weighted CRSP daily (Jul 1962 - Dec 1987) and monthly (Jan 1926 - Dec 1987) stock return indices.

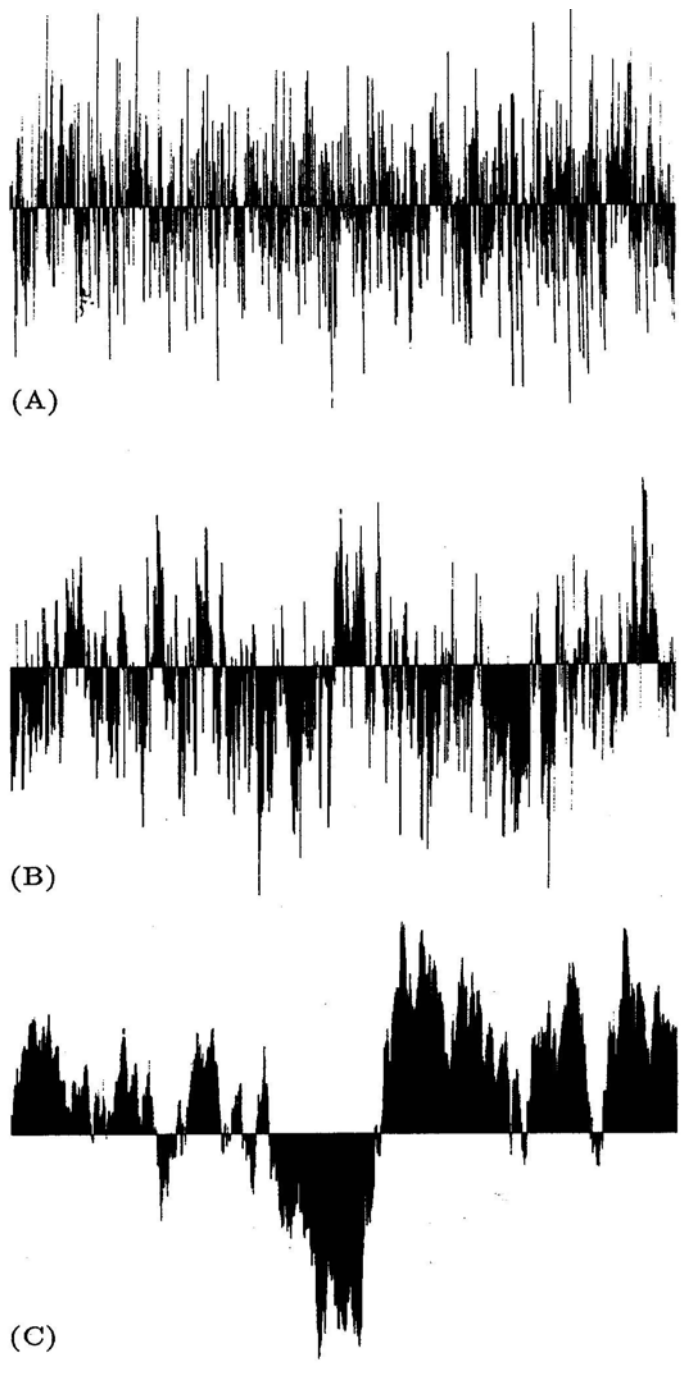

Figure 5: Sample of (a) white noise with $P(\omega) = \omega^{-0}$ power spectrum; (b) pink noise with $P(\omega) = \omega^{-1}$ power spectrum; and (c) brown noise with $P(\omega) = \omega^{-2}$ power spectrum.

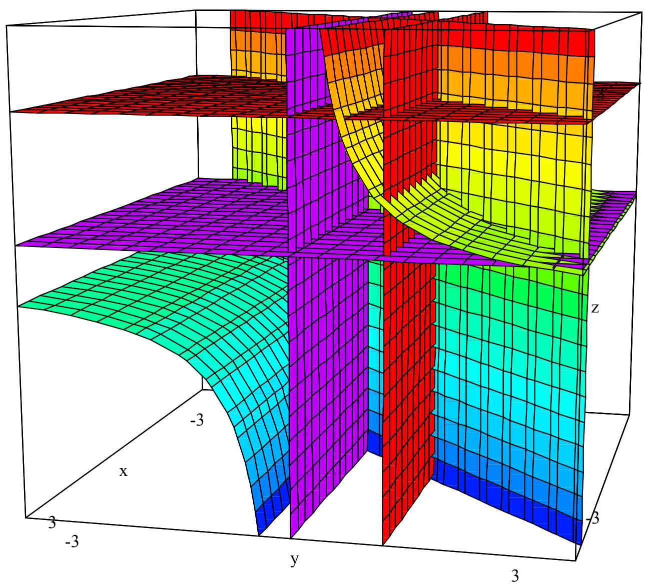

Figure 6: Relations between and constraints on d, H, and α_Z. The axes measure $x = d$, $y = H$, $z = \alpha_Z$.