Chaotic expansion of powers and martingale representation

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Abstract. This paper extends a recent martingale representation result of [N-S] for a Lévy process to filtrations generated by a rather large class of semimartingales. As in [N-S], we assume the underlying processes have moments of all orders, but here we allow angle brackets to be stochastic. Following their approach, including a chaotic expansion, and incorporating an idea of strong orthogonalization from [D], we show that the stable subspace generated by Teugels martingales is dense in the space of square-integrable martingales, yielding the representation. While discontinuities are of primary interest here, the special case of a (possibly infinite-dimensional) Brownian filtration is an easy consequence.

1. Introduction

Recently, [N-S] established a martingale representation property for the filtration generated by a Lévy process $X = (X_t)$ having an exponentially decaying law. They showed that every square-martingale $M \in \mathcal{H}^2$ has a representation as an infinite sum of the form $M = \sum_{n=1}^{\infty} \int H_n dN_n$ for certain pairwise strongly orthogonal martingales N_n.[1] The series convergence takes place in $\mathcal{H}^2$. The base martingales N_n are intrinsically associated to X, and, in their case, on a choice of an orthogonal polynomial. The result is an interesting contrast to the standard theory for filtrations generated by a finite-dimensional Brownian motion or Poisson process, where martingale representation takes the form of a finite sum.

Lévy processes are very interesting, but the concepts and techniques introduced in [N-S] appear of wider applicability. Chief among them are their notion of *Teugels Martingales* $X^{(n)}$, whose strong orthogonalization gives the base martingales N_n, a chaotic representation of n-th power X^n in terms of $X^{(i)}$, and the idea that polynomials in X_{t_j} are dense in the space of square integrable random variables, given a suitable growth condition on X.

In a recent expository article, [D] discusses several approaches and results on martingale representation, including those based on the Jacod-Yor Theorem, and an earlier general result in [D2] (and other cited references) for the filtration generated by a finite activity process. It appears that the [N-S] result is the first of its kind for an infinite activity process, let alone a discontinuous process of infinite variation, which Lévy processes often

[1]In this paper we use integer powers X^n frequently. To avoid confusion, we use subscripts to denote sequences of processes, such as H_n. If necessary, the time t-value is then denoted $H_n(t)$. (In the univariate case, we use the usual notation X_t.)

are. In connection to [N-S], [D] highlights the role played by the infinite direct sum of stable subspaces generated by a sequence of strongly orthogonal martingales in $\mathcal{H}^2$.

Our aim is to generalize the [N-S] results in two directions. First, we extend to processes X quite a bit more general than Lévy processes. These processes and their (generalized) Lévy measures $\nu = \nu(\omega, dt, dx)$ have moments of all order. Aside from stringent growth conditions, the main assumption is that $x^n * \nu$ be continuous and adapted to a Brownian filtration for all integers $n \geq 2$. In the Lévy case, $x^n * \nu$ is a constant times t. A more general example is a "Lévy processes with stochastic intensity λ_t", where ν takes the form $\nu = \lambda_t dt\, v(dx)$ for some ordinary Lévy measure v and a nonnegative Ito process (λ_t). Here, simply, $x^n * \nu = a_n \int_0^{\cdot} \lambda_t dt$, where $a_n = \int x^n v(dx)$; so $x^n * \nu$ are stochastic but proportional.

Secondly, we extend the univariate treatment of [N-S] to the multivariate case, indeed to the case where the underlying filtration is generated by a countable number of independent processes X_n of the above general type. The [N-S] approach to representation as a convergent series in $\mathcal{H}^2$ is ideal for this purpose. Such an extension is already of interest when the processes X_n are independent Brownian motions, extending the standard finite-dimensional result to yield a unique representation for every martingale $M \in \mathcal{H}^2$ as $M = \sum_{n=1}^{\infty} \int H_n dX_n$ for some predictable processes H_n satisfying $\sum_{n=1}^{\infty} \mathbb{E} \int_0^T |H_n(t)|^2 dt < \infty$.

With regard to the standard finite-dimensional Brownian case, as derived in texts such as [E], [K-S], [Ø], and [P] , [D] remarks that the approach of [Ø] appears the simplest. For the Brownian case, the Teugels martingales vanish, substantially simplifying the technique of [N-S]. In this case, the derivation in [N-S] becomes actually quite similar to that of [Ø]: both are based on denseness arguments, the former utilizing integer powers X^n and polynomials, the latter employing complex powers $e^{i\xi X}$ and the Fourier integral. It seems to us that, for the Brownian case, the technique of [N-S] is as simple, but more constructive.

We follow closely the approach and ideas of [N-S], aided also by an elaboration on strong orthogonalization in [D]. The more general development here calls for a somewhat different route at places, and furthering of some of the arguments and calculations in [N-S].

The next section establishes notation, culminating in definitions of "power brackets" $[X]^{(n)}$ and $\langle X\rangle^{(n)}$, and the Teugels martingales $X^{(n)} := [X]^{(n)} - \langle X\rangle^{(n)} = x^n * (\mu - \nu)$.[2] Section 3 sets forth the strategy, based on strong orthogonalization and a decomposition of $\mathcal{H}^2$ into an infinite orthogonal sum of stable subspaces, given a denseness hypothesis. Section 4 establishes some technical results based on the Burkholder-Davis-Gundy inequalities to ensure that various local martingales that later arise in the chaotic expansion as (iterated) stochastic integrals of Teugels martingales are in fact square-integrable martingales. Section 5 derives the needed L^2 denseness of polynomials for processes with an exponentially decreasing law. Section 6 presents an inductive chaotic expansion which basically shows (stopped) polynomials have representations as a sum of stochastic integrals of $X^{(i)}$ times functionals of the $\langle X\rangle^{(j)}$. These are put together in Section 7 to state and prove our main results. Section 8 is not needed for the main results, rather, by presenting an explicit chaotic expansion of powers X^n, it brings out the relevance of power brackets and provides motivation for the inductive definitions in Section 6. A final section concludes the paper.

[2]We use a different notation than [N-S]. Their equivalent of our $[X]^{(n)}$, $\langle X\rangle^{(n)}$, $X^{(n)}$ is $X^{(n)}$, $m_n t$, $Y^{(n)}$.

2. Notation and basic concepts

The notation below is for the most part standard, but we introduce some new ones too.

2.1. **Stochastic basis.** We fix throughout $0 < T \le \infty$ and a complete right-continuous filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F} = \mathcal{F}_T$.

We denote by $\mathbb{F}(X_n)_{n=1}^k$ the completed filtration generated by a finite or infinite sequence $(X_n)_{n=1}^k$, $1 \le k \le \infty$, of measurable processes X_n.

Let L^0 denote the set of $\mathcal{F}$-measurable real-valued functions on Ω. For $p > 0$, we denote

$$L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}) := \{\xi \in L^0 : \mathbb{E}\,|\xi|^p < \infty\}.$$

Of interest will be L^1, L^2, and random variables of finite moments, $L^* := \bigcap_{n=1}^\infty L^n$.

We denote by $\mathcal{M}$ the set of uniformly integrable martingales $M = (M_t)_{t\in[0,T]}$ with $M_0 = 0$. Note, $M \in \mathcal{M}$ is closed by M_T. As is well-known, as $t \to T_-$, M_t converges to $\mathbb{E}(M_T \mid \bigvee_{0\le t<T} \mathcal{F}_t)$ a.s. and in L^1. The localization of $\mathcal{M}$ is denoted $\mathcal{M}_{\text{loc}}$.

2.2. **Semimartingales.** Let $\mathcal{P}$ denote the set of predictable processes $H = (H_t)_{t\ge 0}$.[3]

If $X = (X_t)_{t\ge0}$ is a semimartingale, we abbreviate its bracket $[X, X]$ by $[X]$, and if H is a predictable X-integrable process, we denote the stochastic integral by[4]

$$\int H dX := H \cdot X = (\int_0^t H_s dX_s)_{t\ge0}.$$

Let $\mathcal{A}^+$ denote the set of adapted right-continuous increasing processes $A = (A_t)_{t\in[0,T]}$ such that $A_0 = 0$ and $A_T \in L^1$. Let $\mathcal{A} := \mathcal{A}^+ \ominus \mathcal{A}^+$ denote the set of adapted right-continuous processes of integrable variation. So, every $A \in \mathcal{A}$ has a unique decomposition $A = B - C$ for some $B, C \in \mathcal{A}^+$. Its total variation, denoted $\mathrm{Var}(A)$, then equals $B + C$.

Every $A \in \mathcal{A}$ has a unique Doob-Meyer decomposition $A = \widehat{A} + M$ with $\widehat{A} \in \mathcal{P} \cap \mathcal{A}$ and $M \in \mathcal{M}$. The compensator $\widehat{A}$ is increasing if A is so.

2.3. **Square-integrable martingales.** As customary, we denote this Hilbert space by

$$\mathcal{H}^2 := \{M \in \mathcal{M} : M_T \in L^2\} = \{M \in \mathcal{M}_{\text{loc}} : [M]_T \in L^1\}.$$

Let $M, N \in \mathcal{H}^2$. The compensators of $[M]$ and M^2 coincide, and is denoted $\langle M\rangle$.[5] We have $M^2 - [M]$, $[M] - \langle M\rangle \in \mathcal{M}$. One sets $\langle M, N\rangle := (\langle M + N\rangle - \langle M - N\rangle)/4$. (So, $\langle M, M\rangle = \langle M\rangle$.) The space $\mathcal{H}^2$ is endowed with the Hilbert norm[6]

$$\|M\|^2 := \mathbb{E}\, M_T^2 = \mathbb{E}\,[M]_T = \mathbb{E}\,\langle M\rangle_T. \quad (M \in \mathcal{H}^2)$$

Note, L^2 is isometric to $L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \bigoplus \mathcal{H}^2$.

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[3] By "$t \ge 0$" we mean $t \in [0, T]$ if $T < \infty$ and $t \in [0, \infty)$ if $T = \infty$.

[4] One has, $[X] = [X^c] + \sum_{s\le\cdot}(\Delta X)_s^2$, where X^c denotes the unique continuous local martingale such that $X_0^c = 0$ and $X - X^c$ is a purely discontinuous semimartingale. Also, $[X^c] = [X]^c = \langle X^c\rangle$.

[5] The definition of $\langle M\rangle$ extends to $\mathcal{H}^2_{\text{loc}}$ by localization. Then, we get $\mathcal{H}^2 = \{M \in \mathcal{H}^2_{\text{loc}} : \mathbb{E}\,\langle M\rangle_T < \infty\}$.

[6] We also have $\|M\|^2 = \sup_{0\le t<T} \mathbb{E}\, M_t^2 \ge \frac{1}{4}\,\mathbb{E}\, M_*^2$, where $M_* := \sup_{0\le t\le T} |M_t|$.

2.4. **Infinite direct sum of strongly orthogonal stable subspaces.** For $N \in \mathcal{H}^2$, set

$$L^2\langle N\rangle := \{H \in \mathcal{P} : \mathbb{E}\int_0^T H^2 d[N] < \infty\} = \{H \in \mathcal{P} : \mathbb{E}\int_0^T H^2 d\langle N\rangle < \infty\}.$$

Any $H \in L^2\langle N\rangle$ is N-integrable, $\int HdN \in \mathcal{H}^2$, and $\langle \int HdN\rangle = \int H^2 d\langle N\rangle$. Denote

$$\mathcal{S}(N) := \{\int HdN : H \in L^2\langle N\rangle\} \subset \mathcal{H}^2. \quad (N \in \mathcal{H}^2)$$

As is well known, the subspace $\mathcal{S}(N)$ is a (closed) *stable subspace* of $\mathcal{H}^2$.[7] Given a sequence $(N_i)_{i=1}^\infty$ of pairwise *strongly orthogonal* martingales $N_i \in \mathcal{H}^2$, we denote the direct sum[8]

$$\bigoplus_{i=1}^\infty \mathcal{S}(N_i) := \{\sum_{i=1}^\infty X_i : X_i \in \mathcal{S}(N_i) \text{ and } \sum_{i=1}^\infty \|X_i\|^2 < \infty\}$$

$$= \{\sum_{i=1}^\infty \int H_i dN_i : H_i \in \mathcal{P} \text{ and } \sum_{i=1}^\infty \mathbb{E}\int_0^T H_i^2 d\langle N_i\rangle < \infty\}.$$

As $\bigoplus_{i=1}^\infty \mathcal{S}(N_i)$ is a (countable) direct sum of orthogonal closed subspaces, it is a closed subspace of $\mathcal{H}^2$. (In fact, it is the stable subspace generated by $(N_i)_{i=1}^\infty$.[9])

2.5. **Power brackets.** For any semimartingale X, set $[X]^{(2)} := [X]$ and $[X]^{(n)} := \sum_{s\le \cdot}(\Delta X)_s^n$ for $3 \le n \in \mathbb{N}$. Note, $[X]^{(n+1)} = [X, [X]^{(n)}]$. Assume $\mathbb{E}\,[X]_T^{(2n)} < \infty$, i.e., $[X]^{(2n)} \in \mathcal{A}^+$, for all $n \in \mathbb{N}$. It is easy to see that $\mathrm{Var}([X]^{(m)}) \le [X]^{(m-1)} + [X]^{(m+1)}$ for any odd integer $m \ge 3$.[10] So, it follows $[X]^{(n)} \in \mathcal{A}$ for all $n \ge 2$. We denote the compensator of $[X]^{(n)}$ by $\langle X\rangle^{(n)}$. So, $\langle X\rangle^{(n)}$ is characterized as the unique predictable right-continuous finite variation process such that $[X]^{(n)} - \langle X\rangle^{(n)} \in \mathcal{M}$, and it is increasing when n is even.

[7]Recall, a stable subspace $\mathcal{K}$ is a *closed* subspace of $\mathcal{H}^2$ which is closed under stopping, or equivalently, closed under stochastic integration, i.e., $\mathcal{S}(M) \subset \mathcal{K}$ for every $M \in \mathcal{K}$.

[8]Recall, $M, N \in \mathcal{H}^2$ are strongly orthogonal if $\langle M, N\rangle = 0$. Then clearly, they are orthogonal in the Hilbert space sense, and every martingale in $\mathcal{S}(M)$ is strongly orthogonal to every martingale $\mathcal{S}(N)$.

[9]That is, $\bigoplus_{i=1}^\infty \mathcal{S}(N_i) =: \mathcal{K}$ is the smallest (the intersection of all) stable subspace(s) containing all N_i. Indeed, $\mathcal{K}$ is stable, for if $N = \sum_i X_i \in \mathcal{K}$ with $X_i \in \mathcal{S}(N_i)$ and T is stopping time, then the stopped process $N^T = \sum_i X_i^T \in \mathcal{K}$ as each $X_i^T \in \mathcal{S}(N_i)$. Further, any stable subspace that contains all N_i also contains $\mathcal{S}(N_i)$, and so contains the closure of linear span of the $\mathcal{S}(N_i)$, which closure clearly equals $\mathcal{K}$.

[10]Indeed, for odd $m \ge 3$, we have

$$\mathrm{Var}([X]^{(m)}) = \sum_{s\le\cdot} |\Delta X|_s^m = \sum_{s\le\cdot} 1_{|\Delta X|_s\le 1}|\Delta X|_s^m + \sum_{s\le\cdot} 1_{|\Delta X|_s>1}|\Delta X|_s^m$$

$$\le \sum_{s\le\cdot} 1_{|\Delta X|_s\le 1}|\Delta X|_s^{m-1} + \sum_{s\le\cdot} 1_{|\Delta X|_s>1}|\Delta X|_s^{m+1} \le [X]^{(m-1)} + [X]^{(m+1)}.$$

2.6. **Teugels martingales.** Assume $\mathbb{E}\,[X]^{(2n)}_T < \infty$ for all $n \in \mathbb{N}$. Following [N-S], we define the *Teugels martingales* $X^{(n)}$ *of order* $n \geq 2$ by ($X^{(1)}$ will be defined later)

$$X^{(n)} := [X]^{(n)} - \langle X \rangle^{(n)}, \quad n \geq 2.$$

As we saw, $X^{(n)} \in \mathcal{M}$, all n. (It is easy to see $X^{(n)} \in \mathcal{H}^2$ if all $\langle X \rangle^{(n)}$ are continuous.[11])

In order to relate to the Lévy measure notation adopted in [N-S], let $\mu = \mu(\omega, dt, dx)$ denote the integer-valued random measure associated to X and $\nu = \nu(\omega, dt, dx)$ be the compensator of μ.[12] Since, $x^2 * \mu = \sum_{s \leq \cdot} (\Delta X)^2_s$, we have $[X] = [X]^c + x^2 * \mu$ and $\langle X \rangle = [X]^c + x^2 * \nu$. So, $X^{(2)} := [X] - \langle X \rangle = x^2 * (\mu - \nu)$. Let $n \geq 3$. Above, we saw $[X]^{(n)}$ is of integrable total variation and denoted is compensator $\langle X \rangle^{(n)}$. But $x^n * \mu = [X]^{(n)}$; so $x^n * \nu$ is also the compensator of $[X]^{(n)}$. Therefore, $x^n * \nu = \langle X \rangle^{(n)}$, and we have,

$$X^{(n)} = x^n * \mu - x^n * \nu = x^n * (\mu - \nu), \quad n \geq 2.$$

3. Strong orthogonalization

Let $(M_i)_{i=1}^{\infty}$ be a sequence of martingales $M_i \in \mathcal{H}^2$. As in [D], we associate to it a sequence $(N_i)_{i=1}^{\infty}$ of pairwise strongly orthogonal martingales, which we call its *Strong Orthogonalization.* Set $N_1 := M_1$, and for $n \geq 2$ inductively define N_n as the orthogonal projection of M_n on the orthogonal complement of $\bigoplus_{i=1}^{n-1} \mathcal{S}(N_i)$. Note, this definition implies that N_i are pairwise strongly orthogonal and $\bigoplus_{i=1}^{n} \mathcal{S}(N_i)$ is a (closed) stable subspace.[13] For example, if M_i are correlated Brownian motions, then N_i will be independent Brownian motions.

Remark. For almost all paths ω, $d\langle M_i, N_j \rangle(\omega)$ is a measure on $[0, T]$ which is absolutely continuous with respect to the measure $d\langle N_j, N_j \rangle(\omega)$ on $[0, T]$. So, the Radon-Nikodym derivative $\frac{d\langle M_i, N_j \rangle}{d\langle N_j, N_j \rangle}$ is well-defined, and one easily verifies that

$$M_i = N_i + \sum_{j=1}^{i-1} \int \frac{d\langle M_i, N_j \rangle}{d\langle N_j, N_j \rangle} dN_j.$$

This leads to an alternative definition of N_i: set $N_1 := M_1$, and having defined N_j inductively for $j < i$, use the above equation to define N_i. Note, $N_2 = M_2 - \int \frac{d\langle M_1, M_2 \rangle}{d\langle M_1 \rangle} dM_1$.

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[11]Indeed, the continuity of $\langle X \rangle^{(n)}$ implies $[X^{(n)}] = [[X]^{(n)}] = [X]^{(2n)} \in L^1$; hence $X^{(n)} \in \mathcal{H}^2$.

[12]Following the notation in Chapter II of [J-S], for a random measure $\upsilon(\omega, dt, dx)$ and an optional function $W = W(\omega, t, x)$, we set $(W * \upsilon)_t := \int_{[0,t] \times \{x \neq 0\}} W(s, x) \upsilon(ds, dx)$. For a Lévy process, $\nu = dt\nu_0(dx)$ is time and state-independent. More general examples are processes with ν of form $\lambda_t dt \nu_0(dx)$, for some, say, Itô process (λ_t). These include Cox processes where $\nu_0(dx)$ is simply the Lévy measure of a Poisson process. As Cox processes are often thought of as "Poisson processes with stochastic intensity λ_t", the aforementioned more general examples may be thought of as "Lévy processes with stochastic intensity λ_t."

[13]These statements follow by a simple induction, using the fact if $\mathcal{K}$ is a stable subspace then its orthogonal complement is (a stable subspace and is) strongly orthogonal to $\mathcal{K}$.

Remark. For $1 \le k \le \infty$, $\bigoplus_{i=1}^{k} \mathcal{S}(N_i)$ is not only the stable subspace generated by $(N_i)_{i=1}^{k}$, but also the stable subspace generated by $(M_i)_{i=1}^{k}$.

We denote the linear span of $\mathcal{S}(M_i)$, $i = 1, 2 \cdots$, by[14]

$$\mathrm{Span}(\mathcal{S}(M_i))_{i=1}^{\infty} := \bigcup_{n=1}^{\infty} \mathcal{S}(M_1) + \cdots + \mathcal{S}(M_n).$$

The following is essentially a reformulation of the abstract martingale representation Theorem 3 of [D].[15] Our strategy will be to apply it the Teugels martingales $X^{(i)}$ as the M_i.

Proposition 3.1. *Let $(M_i)_{i=1}^{\infty}$ be a sequence of martingales in $\mathcal{H}^2$ such that* $\mathrm{Span}(\mathcal{S}(M_i))_{i=1}^{\infty}$ *is dense in $\mathcal{H}^2$. Then, $\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i)$, where $(N_i)_{i=1}^{\infty}$ is the strong orthogonalization of $(M_i)_{i=1}^{\infty}$. In other words, every martingale $M \in \mathcal{H}^2$ has a representation*

$$M = \sum_{i=1}^{\infty} \int H_i dN_i$$

(*as a convergent series in $\mathcal{H}^2$*) *for some predictable processes H_i satisfying*

$$\sum_{i=1}^{\infty} \mathbb{E}(\int_0^T H_i^2 d\langle N_i \rangle) = \sum_{i=1}^{\infty} \mathbb{E}(\int_0^T H_i^2 d[N_i]) = \|M\|^2 < \infty.$$

Moreover, if $(H_i')_{i=1}^{\infty}$ is another sequence with this property, then $\int |H_i' - H_i|^2 d\langle N_i \rangle = 0$ a.s., all i. In particular, the H_i are unique if $\langle N_i \rangle$ are strictly increasing.

Proof. Since $\bigoplus_{i=1}^{n} \mathcal{S}(N_i)$ is a stable subspace and contains M_n, we have $\mathcal{S}(M_n) \subset \bigoplus_{i=1}^{n} \mathcal{S}(N_i)$. Hence, $\mathrm{Span}(\mathcal{S}(M_i))_{i=1}^{\infty} \subset \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i)$. The first statement follows as the former is assumed dense and the latter is closed. The uniqueness statement follows because direct sum decomposition is unique; so, $\int H_i' dN_i = \int H_i dN_i$, implying $\int |H_i' - H_i|^2 d\langle N_i \rangle = 0$. □

Remark. The H_i are unique on the support of the measure measure $d\langle N_i, N_i \rangle$ (as measure on $[0, T]$ for each ω.) There, H_i in fact equals the Radon-Nikodym derivative $\frac{d\langle M, N_i \rangle}{d\langle N_i \rangle}$.

Remark. When $d\langle N_i, N_i \rangle = \lambda_i dt$ for some *positive* predictable processes λ_i, we can normalize by replacing N_i with $\int \lambda_i^{-1/2} dN_i$. The new N_i still satisfy $\langle N_i, N_i \rangle_t = t$, so the condition on the H_i simplify to $\sum_{i=1}^{\infty} \mathbb{E}(\int_0^T H_i^2 dt) < \infty$, as in [N-S]. This is possible in the

[14] In this paper, we denote the linear span of any subset $\mathcal{K}$ of a vector space by $\mathrm{Span}(\mathcal{K})$. So, $\mathrm{Span}(\mathcal{K})$ is a the set of (finite) linear combinations of elements of $\mathcal{K}$, i.e., the smallest (intersection of all) linear subspace(s) containing $\mathcal{K}$. If $\mathcal{K}_i$, $i \in I$, is a family of linear subspaces of a vector space, we denote their linear span $\mathrm{Span}(\mathcal{K}_i)_{i \in I}$. When the index set I is finite, we also denote $\mathrm{Span}(\mathcal{K}_i)_{i=1}^{n}$ by $\mathcal{K}_1 + \cdots + \mathcal{K}_n$. When $\mathcal{K}_i$ are orthogonal subspaces of a Hilbert space, we emphasize the orthogonality by writing $\mathrm{Span}(\mathcal{K}_i)_{i=1}^{n}$ as $\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_n$ or $\bigoplus_{i=1}^{n} \mathcal{K}_i$. Note, when I is countable, and $\mathcal{K}_i$ are closed, orthogonal subspaces, then their infinite direct sum $\bigoplus_{i=1}^{\infty} \mathcal{K}_i := \{\sum_{i=1}^{\infty} K_i : K_i \in \mathcal{K}_i; \sum_{i=1}^{\infty} \|K_i\|^2 < \infty\}$ equals the *closure* of $\mathrm{Span}(\mathcal{K}_i)_{i=1}^{\infty}$.

[15] Prop 3.1 yields the same conclusion as Theorem 3 of [D] if $\mathcal{H}^2$ is separable. For, if $(M_i)_{i=1}^{\infty}$ is a dense sequence in $\mathcal{H}^2$, then $\mathrm{Span}(\mathcal{S}(M_i))_{i=1}^{\infty}$ is also dense in $\mathcal{H}^2$, as it obviously contains all the M_i.

Lévy case, where the λ_i turn out to be positive constants. However, the condition does not hold in general (some N_i may even be zero); so, unlike [N-S], we do not normalize here.

Remark. It is easy show that the the strong orthogonalization of two sequences $(X_i)_{i=1}^\infty$ and $(M_i)_{i=1}^\infty$ of martingales in $\mathcal{H}^2$ coincide if $X_j := M_j + \sum_{i=1}^{j-1} \int H_{i,j} dM_j$ for some locally bounded predictable processes $H_{i,j}$.

4. Martingales and semimartingales of finite moments

Here we define a set $\mathcal{C}^*$ of semimartingales, to a subset of which our main results apply. Recall, $L^* := \bigcap_{n=1}^\infty L^n$. We begin with the definition of martingales of finite moments:

$$\mathcal{H}^\star := \{M \in \mathcal{H}^2 : M_T \in L^*\} = \{M \in \mathcal{M}_{\mathrm{loc}} : [M]_T \in L^*\}.$$

The equality is a direct consequence of the Burkholder-Davis-Gundy inequalities.[16]

Proposition 4.1. *Let $M, N \in \mathcal{H}^\star$. Then $\int M_- dN \in \mathcal{H}^\star \cap \mathcal{S}(N)$.*

Proof. Set $M_* = \sup_{t\in[0,T]} |M_t|$. By Schwartz inequality then Doob's maximal inequality,

$$\mathbb{E}\,[\int M_- dN]_T^n = \mathbb{E}\,(\int_0^T M_-^2 d[N])^n \leq \mathbb{E}\,(M_*^2 [N]_T)^n$$
$$\leq (\mathbb{E}\, M_*^{4n})^{\frac{1}{2}} (\mathbb{E}\,[N]_T^{2n})^{\frac{1}{2}} \leq (\frac{4n}{4n-1})^{2n} (\mathbb{E}\, M_T^{4n})^{\frac{1}{2}} (\mathbb{E}\,[N]_T^{2n})^{\frac{1}{2}} < \infty.$$

Hence $[\int M_- dN]_T \in L^*$. Thus the local martingale $\int M_- dN$ is in fact in $\mathcal{H}^\star \cap \mathcal{S}(N)$. ☐

Clearly, $[X]^{(2n)} \leq [X]^n$ for any semimartingales X and $n \in \mathbb{N}$.[17] So, if $M \in \mathcal{H}^\star$, then $[M]_T^{(2n)} \in L^*$ for all $n \in \mathbb{N}$. Recall, the Teugels martingale is now defined as $M^{(n)} := [M]^{(n)} - \langle M\rangle^{(n)} \in \mathcal{M}$. Our approach relies on $\langle M\rangle^{(n)}$ being continuous. We define

$$\mathcal{H}^* := \{M \in \mathcal{H}^\star : \langle M\rangle^{(n)} \text{ is continuous for all } n \geq 2\}.$$

For any $M \in \mathcal{H}^2$, we set

$$\mathcal{S}^*(M) := \mathcal{H}^* \cap \mathcal{S}(M).$$

Proposition 4.2. *Let $M, N \in \mathcal{H}^*$. Then $\int M_- dN \in \mathcal{S}^*(N)$.*

Proof. One readily shows by induction that $[\int M_- dN]^{(n)} = \int M_-^{2n} d[N]^{(n)}$. So $\langle \int M_- dN\rangle^{(n)} = \int M_-^{2n} d\langle N\rangle^{(n)}$, which is continuous. The desired result thus follows by Prop. 4.1. ☐

[16]The Burkholder-Davis-Gundy inequalities, as stated on page 175, section IV.5 of [P], states that for any local martingale M, finite stopping time T, and $p \geq 1$, there are constants c and C such that

$$\mathbb{E}\,(M_T^*)^p \leq c\,\mathbb{E}\,[M_T]^{p/2} \leq C\,\mathbb{E}\,(M_T^*)^p.$$

In this paper, T is deterministic, but is allowed to equal infinity. On page 190, [P] also states the first inequality for $T = \infty$. Above, $M_T^* := \sup_{t\leq T} |M_t|$. However, by Doob's maximal inequality, we can replace M_T^* simply by $|M_T|$ (with a larger constant C).

[17]Indeed, we obviously have $\sum_{s\leq t} (\Delta X_s)^{2n} \leq (\sum_{s\leq t} (\Delta X_s)^2)^n$.

The following consequence will be useful for multivariate representations.

Corollary 4.3. *Let $M', N' \in \mathcal{H}^2$. Let $M \in \mathcal{S}^*(M')$ and $N \in \mathcal{S}^*(N')$. Assume $[M', N'] = 0$. Then, $MN \in \mathcal{S}^*(M') \oplus \mathcal{S}^*(N')$.*

Proof. Clearly, $[M, N] = 0$. So, Prop 3.2 and integration by parts imply $MN \in \mathcal{S}^*(M) \oplus \mathcal{S}^*(N)$. But, $\mathcal{S}(M) \subset \mathcal{S}(M')$ and $\mathcal{S}(N) \subset \mathcal{S}(N')$ as $\mathcal{S}(M')$ and $\mathcal{S}(N')$ are stable subspaces. Hence, $MN \in \mathcal{S}^*(M') \oplus \mathcal{S}^*(N')$. □

The following result will guarantee that the stochastic integrals of the Teugels martingales in the chaotic expansions below will actually be martingales belonging to $\mathcal{H}^2$ (even to $\mathcal{H}^*$).

Proposition 4.4. *Let $M \in \mathcal{H}^*$. Then $M^{(n)} \in \mathcal{H}^*$ and $\langle M \rangle_T^{(n)} \in L^*$ for all $n \in \mathbb{N}$, where $M^{(1)} := M$. Moreover, $[M^{(n)}] = [M]^{(2n)}$ and $\langle M^{(n)} \rangle = \langle M \rangle^{(2n)}$.*

Proof. Recall, $[X, A] = 0$ for all semimartingales X and continuous finite variation semimartingales A. As $\langle X \rangle^{(n)}$ is assumed continuous, this implies $[M^{(n)}] = [M]^{(2n)}$. But, $[M]^{(2n)} \leq [M]^n$. Therefore $[M^{(n)}]_T \in L^*$. Thus $M^{(n)} \in \mathcal{H}^\star$. Hence $M_T^{(n)} \in L^*$, and $\langle M \rangle_T^{(n)} = [M^{(n)}]_T - M_T^{(n)} \in L^*$. Let $i \geq 2$. Clearly, $[M^{(n)}]^{(i)} = [M]^{(ni)}$. So $\langle M^{(n)} \rangle^{(i)} = \langle M \rangle^{(ni)}$ is continuous. Therefore $M^{(n)} \in \mathcal{H}^*$. □

As $\langle M \rangle^{(n)}$ is continuous if $M \in \mathcal{H}^*$, for all semimartingales X, $[M^{(n)}, X] = [[M]^{(n)}, X]$.

Proposition 4.5. *Let $M, N \in \mathcal{H}^*$. If $[M, N] = 0$, then $[M^{(i)}, N^{(j)}] = 0$ for all $i, j \in \mathbb{N}$.*

Proof. Note, for any two semimartingales X and Y, and $i + j \geq 3$, we have

$$[[X]^{(i)}, [Y]^{(j)}] = \sum_{\cdot \leq s} (\Delta X_s)^i (\Delta Y_s)^j =$$

$$\sum_{\cdot \leq s} (\Delta X_s \Delta Y_s)(\Delta X_s)^{i-1}(\Delta Y_s)^{j-1} = [[X, Y], [[X]^{(i-1)}, [Y]^{(j-1)}]].$$

This implies $[[X]^{(i)}, [Y]^{(j)}] = 0$ if $[X, Y] = 0$. The result follows by applying to M and N and invoking the remark preceding the proposition on continuity of $\langle M \rangle^{(i)}$ and $\langle N \rangle^{(j)}$. □

Let $\mathcal{A}^* \subset \mathcal{A}$ denote the set of *continuous* processes $A \in \mathcal{A}$ such that $\mathrm{Var}(A)_T \in L^*$. As $A_* := \sup_{t \in [0,T]} |A_t| \leq \mathrm{Var}(X)_T$, clearly then $A_t, A_* \in L^*$, all t.

Proposition 4.6. *Let $A, B \in \mathcal{A}^*$ and $M \in \mathcal{H}^*$. Then $AB \in \mathcal{A}^*$ and $\int AdM \in \mathcal{S}^*(M)$.*

Proof. Without loss of generality we may assume A and B are increasing. That $|A_T B_T| \in L^*$ then follows from Schwartz inequality. (Also $\int AdB \in \mathcal{A}^*$, as $|\int_0^T AdB| \leq |A_T B_T|$.) Similarly, $[\int_0^T AdM] \leq A_T^2 [M]_T$. So again by Schwartz inequality $\int AdM \in \mathcal{S}^*(M)$. □

We now define $\mathcal{C}^* := \mathcal{A}^* + \mathcal{H}^*$. So, any semimartingale $X \in \mathcal{C}^*$ has a decomposition $X = A + M$, necessarily unique, with $A \in \mathcal{A}^*$ and $M \in \mathcal{H}^*$. Note, $X_T \in L^*$. We denote this compensator A by $\langle X \rangle^{(1)}$ and this martingale M by $X^{(1)}$. So,

$$X = \langle X \rangle^{(1)} + X^{(1)}, \quad X \in \mathcal{C}^*, \, \langle X \rangle^{(1)} \in \mathcal{A}^*, \, X^{(1)} \in \mathcal{H}^*;$$

As $\langle X\rangle^{(1)}$ is continuous, $[X,Y]=[X^{(1)},Y]$ for any semimartingale Y. Hence $[X]^{(n)} = [X^{(1)}]^{(n)}$ for $n\ge 2$, implying $X^{(n)}=(X^{(1)})^{(n)}$.[18] Clearly, a process X belongs to $\mathcal{C}^*$ if and only if it is a special semimartingale, its compensator belongs to $\mathcal{A}^*$, $X_0=0$, and $[X]_T\in L^*$. The above propositions and the preceding remarks clearly yield

Corollary 4.7. *Let $X,Y\in\mathcal{C}^*$. Then $XY, \int X_- dY\in\mathcal{C}^*$ and $\int X_- dM\in\mathcal{S}^*(M)$ for any $M\in\mathcal{H}^*$. Moreover, $X^{(n)}\in\mathcal{H}^*$ and $\langle X\rangle^{(n)}\in\mathcal{A}^*$ for all $n\in\mathbb{N}$. Furthermore, if $[X,Y]=0$, then $[X^{(i)},Y^{(j)}]=0$ for all $i,j\in\mathbb{N}$.*

5. Exponentially decaying laws and L^2-denseness of polynomials

We first look at random variables, then processes. Define the subspace $L_*\subset L^*$ by

$$L_* := \{\xi\in L^0 : \mathbb{E}\,\exp(a|\xi|)<\infty \text{ for some } a>0\}.$$

Using Schwartz inequality, one easily verifies that L_* is indeed a linear subspace.[19]

Given a finite or infinite sequence $(\xi_i)_{i=1}^k$, $k\le\infty$ of random variables $\xi_i\in L^0$, we denote by $\mathcal{F}(\xi_i)_{i=1}^k$ the σ-algebra generated by the ξ_i. A *polynomial* in the ξ_i is a (finite) linear combination of products $\xi_{i_1}\cdots\xi_{i_m}$, with $m\ge 0$ ranging over non-negative integers, $i_j\in\mathbb{N}$, and $i_j\le k$ when $k<\infty$. (When $m=0$, the product is empty, and by convention equals 1). As the indices i_j need not be distinct, this includes the monomials $\xi_{i_1}^{n_1}\cdots\xi_{i_m}^{n_m}$, $n_i\in\mathbb{N}$.

Proposition 5.1. *Let $\xi_1,\cdots,\xi_n\in L_*$. Assume $\mathcal{F}=\mathcal{F}(\xi_i)_{i=1}^n$. Then the set of polynomials in ξ_i, i.e., the linear space* $\operatorname{Span}\{\xi_{i_1}\cdots\xi_{i_m}\}_{1\le i_1,\cdots,i_m\le n,\, m\ge 0}$*, is dense in L^2.*

Proof. Let $\varphi\in L^2$ satisfy $\mathbb{E}(\varphi\,\xi_{i_1}\cdots\xi_{i_m})=0$ for all $m\ge 0$ and multi-indices $(i_1,\cdots,i_m)\in\mathbb{N}^m$. (For $m=0$ this means $\mathbb{E}\,\varphi=0$.) It suffices to show $\varphi=0$. Let $C_0^\infty(\mathbb{R}^n)$ denote the set of complex-valued smooth functions of compact support on $\mathbb{R}^n$. As is well known, the set $\{f(\xi) : f\in C_0^\infty(\mathbb{R}^n) \text{ is real valued}\}$ is dense in L^2, where $\xi=(\xi_1\cdots\xi_n)$.[20] Therefore, it suffices to show $\mathbb{E}(\varphi f(\xi))=0$ for all $f\in C_0^\infty(\mathbb{R}^n)$. Define $u: C_0^\infty(\mathbb{R}^n)\to\mathbb{C}$ by $u(f):=\mathbb{E}(\varphi f(\xi))$. Then, u is distribution, i.e., a continuous linear functional on $C_0^\infty(\mathbb{R}^n)$ under the latter's usual Frechet topology. We must show $u=0$. For $x\in\mathbb{R}^n$, define $\widehat{u}(x)=\mathbb{E}(\varphi\exp(-\sqrt{-1}\,x\cdot\xi))$. Then, $\widehat{u}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and considered as such as a distribution, it is the Fourier transform of u in the sense of distribution. Hence, it suffices to show $\widehat{u}=0$.

As $|\xi_i|\in L_*$, we have $|\xi|\le|\xi_1|+\cdots+|\xi_n|\in L_*$. So, $\mathbb{E}\,\exp(a|\xi|)<\infty$ for some $a>0$. Using Schwartz inequality yields $\mathbb{E}|\varphi\exp(-iz\cdot\xi)|<\infty$ for $z\in\mathbb{C}^n$ with $|\mathrm{Im}(z)|<a/2$.

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[18]Indeed $[X]=[X,X^{(1)}]=[X^{(1)}]$, and for $n\ge 3$, using induction,

$$[X]^{(n)}=[X,[X]^{(n-1)}]=[X,[X^{(1)}]^{(n-1)}]=[X^{(1)},[X^{(1)}]^{(n-1)}]=[X^{(1)}]^{(n)}.$$

[19]Indeed, if $\xi=\xi_1+\xi_2$ with $\mathbb{E}\,\exp(a_i|\xi_i|)<\infty$, then $\mathbb{E}\,\exp(a|\xi|)<\infty$, where $a=\frac{1}{2}\min(a_1,a_2)$.

[20]Indeed, L^p can be identified with $L^p(\mathbb{R}^n,\mathcal{B},\mathbb{P}\circ\xi^{-1})$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$. Radon-integral theory then implies that compactly supported continuous functions of ξ are dense in L^p. But, such functions can be uniformly approximated by smooth functions of compact support, using convolution with a non-negative smooth function of small compact support and integral 1.

This implies that the function $z \mapsto \mathbb{E}(\varphi \exp(-\sqrt{-1} z \cdot \xi))$ is holomorphic on $|\mathrm{Im}(z)| < a/2$. It follows that $\widehat{u}$, which is the restriction of this function to $\mathbb{R}^n$, is real analytic. But,

$$\frac{\partial^m \hat{u}}{\partial x_{i_1} \cdots \partial x_{i_m}}(0) = (-\sqrt{-1})^m \mathbb{E}(\varphi\, \xi_{i_1} \cdots \xi_{i_m}) = 0,$$

for all $m \geq 0$ by assumption. Since $\widehat{u}$ is analytic, it follows $\widehat{u} = 0$, as desired. □

The result extends to infinite sequences by L^2-version of martingale convergence theorem:

Lemma 5.2. *Let* $\xi_i \in L^2$, $i = 1, 2, \cdots$. *Assume* $\mathcal{F} = \mathcal{F}(\xi_i)_{i=1}^\infty$. *Then* $\bigcup_{n=1}^\infty L^2(\Omega, \mathcal{F}(\xi_i)_{i=1}^n), \mathbb{P})$ *is dense in* L^2.

Proof. Set $\mathcal{F}_n := \mathcal{F}(\xi_i)_{i=1}^n$. Let $\theta \in L^2$. Set $\theta_n := \mathbb{E}[\theta \,|\, \mathcal{F}_n]$. By the martingale convergence theorem $\theta_n \to \theta$ a.s. and in L^1. Moreover, since $\theta \in L^2$, the convergence also takes place in L^2.[21] The desired result follows because θ_n belongs to $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ by construction. □

Proposition 5.3. *Let* $\xi_i \in L_*$, $i = 1, 2, \cdots$. *Assume* $\mathcal{F} = \mathcal{F}(\xi)_{i=1}^\infty$. *Then the set of polynomials in* ξ_i, *i.e., the linear space* $\mathrm{Span}\{\xi_{i_1} \cdots \xi_{i_m}\}_{(i_1, \cdots, i_m) \in \mathbb{N}^m,\, m \geq 0}$, *is dense in* L^2.

Proof. By Prop. 5.1, polynomials in $\xi_1, \cdots, \xi_n$ are dense in $L^2(\Omega, \mathcal{F}(\xi_i)_{i=1}^n, \mathbb{P})$. Since the latter's topology coincides with its relative topology as a subset of L^2, it follows that polynomials in $\xi_1, \xi_2, \cdots$ are dense in $\bigcup_{n=1}^\infty L^2(\Omega, \mathcal{F}(\xi_i)_{i=1}^n, \mathbb{P})$ in the L^2 topology. The desired result thus follows from the previous Lemma. □

We now extend these results to continuous-time stochastic processes, first univariate. Set

$$\mathcal{C}_* := \{\text{left or right continuous processes } X = (X_t)_{t \in [0,T]} \text{ such that } X_t \in L_* \text{ for all } t\}.$$

Proposition 5.4. *Let* $X \in \mathcal{C}_*$. *Assume* $\mathcal{F} = \mathcal{F}(X_t)_{t \in [0,T]}$. *Then the linear space of random variables* $\mathrm{Span}\{X_{t_1} \cdots X_{t_n}\}_{(t_1, \cdots, t_n) \in [0,T]^n,\, n \geq 0}$ *is dense in* L^2.

Proof. Let $(s_i)_{i=0}^\infty$ be a dense sequence in $[0, T]$, containing 0 and T. Set $\xi_i = X_{s_i}$. By right or left continuity of X, we have $\mathcal{F} = \mathcal{F}(\xi_i)_{i=1}^\infty$. Also, $\xi_i \in L_*$. The desired result therefore follows by Prop. 5.3. (More strongly, it follows that we may choose the t_i in $\{s_i\}_{i=0}^\infty$.) □

Remark. By not requiring the t_i to be distinct, we are including products of powers $X_{t_j}^{i_j}$. Indeed, $\mathrm{Span}\{X_{t_1} \cdots X_{t_n}\}_{(t_1, \cdots, t_n) \in [0,T]^n,\, n \geq 0} = \mathrm{Span}\{X_{t_1}^{i_1} \cdots X_{t_n}^{i_n}\}_{t_1 < \cdots < t_n \in [0,T],\, i_1, \cdots, i_n \in \mathbb{N},\, n \geq 0}$. Clearly, these also equal $\mathrm{Span}\{X_{t_1}^{i_1}(X_{t_2} - X_{t_1})^{i_1} \cdots (X_{t_n} - X_{t_{n-1}})^{i_n}\}_{t_1 < \cdots < t_n \in [0,T],\, i_1, \cdots, i_n \in \mathbb{N},\, n \geq 0}$. The latter is the form stated and used in [N-S]. Here, we use the simpler first form.

Proposition 5.5. *Let* $X_i \in \mathcal{C}_*$, $i = 1, 2, \cdots$. *Assume* $\mathcal{F} = \mathcal{F}(X_i(t))_{t \in [0,T],\, i \in \mathbb{N}}$. *Then the linear space* $\mathrm{Span}\{X_{i_1}(t_1) \cdots X_{i_n}(t_n)\}_{(t_1, \cdots, t_n) \in [0,T]^n,\, (i_1, \cdots, i_n) \in \mathbb{N}^n,\, n \geq 0}$ *is dense in* L^2.

[21] See, e.g., Theorem I.1.42 in [J-S]. The L^2-convergence can be seen directly as follows. Note,

$$\mathbb{E}[\theta_n^2] = \mathbb{E}[(\mathbb{E}[\theta \,|\, \mathcal{F}_n])^2] \leq \mathbb{E}[\mathbb{E}[\theta^2 \,|\, \mathcal{F}_n]] = \mathbb{E}[\theta^2].$$

Hence, $\theta_n \in L^2(\Omega, \mathcal{F}(\xi_1, \cdots, \xi_n), \mathbb{P})$. It remains to show $\mathbb{E}[(\theta_n - \theta)^2] \to 0$. Set $\varphi_n = (\theta_n - \theta)^2$. Then,

$$\mathbb{E}[\varphi_n] = \mathbb{E}[\theta^2] + \mathbb{E}[\theta_n^2] - 2\mathbb{E}[\theta_n]\mathbb{E}[\theta] \leq \mathbb{E}[\theta^2] + \mathbb{E}[\theta_n^2] + 2\sqrt{\mathbb{E}[\theta^2]}\sqrt{\mathbb{E}[\theta_n^2]} \leq 4\mathbb{E}[\theta^2].$$

Hence $\sup_n \mathbb{E}[\varphi_n] < \infty$. As $\varphi_n \to 0$ a.s. and $(\varphi_n)_{n=1}^\infty$ is a positive submartingale, it follows from the submartingale convergence theorem that $\mathbb{E}[\varphi_n] \to 0$, as desired.

Proof. Let $(s_i)_{i=0}^\infty$ be a dense sequence in $[0,T]$, containing 0 and T. Set $\xi_{ij} = X_j(s_i)$. By right or left continuity of X_i, we have $\mathcal{F} = \mathcal{F}(\xi_{ij})_{i,j=1}^\infty$. Also, $\xi_{ij} \in L_*$. Using any bijection of $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$, we may regard $(\xi_{ij})_{i,j=1}^\infty$ as one long sequence. The desired result therefore follows by Prop. 5.3. (More strongly, if follows that we may choose the t_i in $\{s_i\}_{i=0}^\infty$.) □

Remark. The above specializes to a finite dimensional version by letting all except a finite number of X_i be zero.

Remark. Since we are not requiring $i_1, \cdots, i_n$ to be distinct, we are including products of the form $X_i(t_1)\cdots X_i(t_n)$ for each i as well as products of such expressions over different i.

Although L^2 is isometric to $L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \oplus \mathcal{H}^2$, it is $\mathcal{H}^2$ that embodies the filtration structure, not L^2. For our purposes it is more convenient to cast the last two propositions in terms of $\mathcal{H}^2$. To this end, we utilize the following notation. For any $\xi \in L^1$, set

$$\overline{\mathbb{E}}(\xi \,|\, \mathbb{F}) := (\mathbb{E}(\xi \,|\, \mathcal{F}_t))_{t\in[0,T]} - \mathbb{E}(\xi \,|\, \mathcal{F}_0) \in \mathcal{M}.$$

Clearly, $\overline{\mathbb{E}}(\xi \,|\, \mathbb{F}) \in \mathcal{H}^*$ for $\xi \in L^*$. The previous two propositions respectively yield,

Corollary 5.6. *Let $X \in \mathcal{C}_*$. Assume $\mathbb{F} = \mathbb{F}(X)$. Then the linear subspace of martingales*

$$\mathrm{Span}\{\overline{\mathbb{E}}(X_{t_1}\cdots X_{t_n} | \,\mathbb{F})\}_{(t_1,\cdots,t_n)\in[0,T]^n,\, n\in\mathbb{N}}$$

is (*contained in $\mathcal{H}^*$ and*) *dense in $\mathcal{H}^2$.*

Corollary 5.7. *Let $X_i \in \mathcal{C}_*$, $i = 1, 2, \cdots$. Assume $\mathbb{F} = \mathbb{F}(X_i)_{i=1}^\infty$. Then the linear subspace*

$$\mathrm{Span}\{\overline{\mathbb{E}}(X_{i_1}(t_1)\cdots X_{i_n}(t_n) | \,\mathbb{F})\}_{(t_1,\cdots,t_n)\in[0,T]^n,\, (i_1,\cdots,i_n)\in\mathbb{N}^n,\, n\in\mathbb{N}}$$

is (*contained in $\mathcal{H}^*$ and*) *dense in $\mathcal{H}^2$.*

6. Inductive chaotic expansion of stopped polynomials

Throughout this section, let $X \in \mathcal{C}^*$. Let $\mathcal{A}_0^*(X)$ denote the set of simple functions, i.e., the linear span of (deterministic) processes of the form $1_{[0,t]}$, $0 < t \le T$. For $n \ge 1$, set

$$\mathcal{A}_n^*\langle X\rangle := \{A \in \mathcal{A}^* : A \text{ is adapted to } \mathbb{F}(\langle X\rangle^{(i)})_{i=1}^n\}.$$

Note, if $\langle X\rangle^{(i)}$ are deterministic (as in the Lévy case) then any $A \in \mathcal{A}_n^*\langle X\rangle$ is deterministic.

We next define a sequence of linear subspaces $(\mathcal{C}_i^*(X))_{i=0}^\infty$ of $\mathcal{C}^*$ and a sequence of linear subspaces of $(\mathcal{S}_i^*(X))_{i=1}^\infty$ of $\mathcal{H}^*$. We employ a joint inductive definition. Set $\mathcal{S}_0^*(X) := \mathbb{R}$,

$$\mathcal{S}_1^*(X) := \{\int A\, dX^{(1)} :\ A \in \mathcal{A}_0^*(X)\};$$

$$\mathcal{C}_1^*(X) := \mathcal{A}_1^*\langle X\rangle \ +\ \{AM : A \in \mathcal{A}_0^*(X);\ M \in \mathcal{S}_1^*(X)\}.$$

Note, $X \in \mathcal{C}_1^*(X)$. For $n \geq 2$, we define inductively,

$$\mathcal{S}_n^*(X) = \mathrm{span}\{\int Y_- dX^{(j)} : \; Y \in \mathcal{C}_i^*(X), \; i+j=n, \; 0 \leq i \leq n-1, \, 1 \leq j \leq n\};$$

$$\mathcal{C}_n^*(X) := \mathrm{span}\{AM : A \in \mathcal{A}_i^*\langle X\rangle ; \, M \in \mathcal{S}_j^*(X), \; i+j=n, \; 0 \leq i, j \leq n\}.$$

For example, $X^{(9)} + \int \langle X\rangle^{(6)} d\langle X\rangle^{(3)} + \langle X\rangle^{(2)} \int (\int \langle X\rangle^{(1)} dX^{(2)}) dX^{(4)} \in \mathcal{C}_9^*(X)$.

Section 8 below presents an explicit (huge) decomposition $X^n = \sum_k A_k M_k \in \mathcal{C}_n^*(X)$. The A_k will be iterated (multiple) Stieltjes integrals of $\langle X\rangle^{(i)}$, and the M_k will be iterated stochastic integrals of products of such forms A against the Teugels martingales $X^{(j)}$. However, what is important for our main results is not the explicit form, but two key properties of $\mathcal{C}_n^*$: it is closed under multiplication and under stopping at deterministic times. (The latter is clear.) The following is a simple consequence of Section 4.

Proposition 6.1. *We have* $\mathcal{S}_n^*(X) \subset \mathrm{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$ *and* $\mathcal{C}_n^*(X) \subset \mathcal{C}^*$*, all* $n \in \mathbb{N}$.

Proof. We use induction, case $n=1$ being clear. Let $n \geq 2$, $M \in \mathcal{S}_n^* := \mathcal{S}_n^*(X)$, and $Y \in \mathcal{C}_n^* := \mathcal{C}_n^*(X)$. By linearity we may assume $M = \int Z_- dX^{(j)}$ for some $Z \in \mathcal{C}_i^*$, $i+j=n$, $i<n$, and $Y = AN$ for some $A \in A_i^*$ and $N \in \mathcal{C}_j^*$, $i+j=n$. By induction, $Z \in \mathcal{C}^*$, and by Corollary 4.7, $X^{(j)} \in \mathcal{H}^*$. So by Corollary 4.7, $\int Y_- dX^{(j)} \in \mathcal{S}^*(X^{(j)})$. Therefore, $M \in \mathrm{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$. If $j=n$ by what was just shown and otherwise by induction, we have $N \in \mathcal{C}^*$. So, $Y = AN \in \mathcal{C}^*$ by Corollary 4.7. □

A principal and non-trivial property of $\mathcal{C}_n^*(X)$ is closedness under multiplication:

Proposition 6.2. *Let* $Y \in \mathcal{C}_n^*(X)$*,* $Z \in \mathcal{C}_m^*(X)$*,* $n, m \geq 0$*. Then* $YZ, \int Y_- dZ \in \mathcal{C}_{m+n}^*(X)$.[22]

Proof. We use induction on $n+m$. The case $n+m=1$ is trivial. Assume $n+m \geq 2$. Note, if $A \in \mathcal{A}_i^*$ and $B \in \mathcal{A}_j^*$, then $AB \in \mathcal{A}_{i \vee j}^*$. This shows we may assume $Y \in \mathcal{S}_n^*$ and $Z \in \mathcal{S}_m^*$. By linearity we may further assume $Y = \int Y'_- dX^{(j)}$ for some $Y' \in \mathcal{C}_i^*$ with $i+j=n$, $i \geq 0$, $j \geq 1$, and $Z = \int Z'_- dX^{(l)}$ for some $Z' \in \mathcal{C}_l^*$ with $l+k=m$, $k \geq 0$, $l \geq 1$.

By induction we have $YZ' \in \mathcal{C}_{n+m-l}^*$. Therefore, $\int Y_- dZ$ is a sum of forms $\int AM_- dX^{(l)}$ for some $A \in \mathcal{A}_a^*$, $M \in \mathcal{S}_b^*$ with $a+b+l = n+m$, $a, b \geq 0$. Clearly, $AM \in \mathcal{C}_{n+m-l}^*$; so $\int AM_- dX^{(l)} \in \mathcal{S}_{n+m}^*$. It follows $\int Y_- dZ \in \mathcal{S}_{n+m}^*$. Next, we show $[Y, Z] \in \mathcal{C}_{m+n}^*$. We have,

$$[Y,Z] = \int Y'_- Z'_- d[X^{(j)}, X^{(l)}] = \int Y'_- Z'_- d[X]^{(j+l)}$$

$$= \int Y'_- Z'_- dX^{(j+l)} + \int Y'_- Z'_- d\langle X\rangle^{(j+l)}$$

$$= \int Y'_- Z'_- dX^{(j+l)} + Y'Z'\langle X\rangle^{(j+l)} - \int \langle X\rangle^{(j+l)} d(Y'Z'),$$

[22]Moreover, by Itô's product rule we have, $[Y,Z] \in \mathcal{C}_{m+n}^*(X)$. The proof further shows, $YZ - [Y,Z]$, $[Y,Z] - \langle Y,Z\rangle \in \mathrm{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$, and $\langle Y,Z\rangle \in \mathcal{A}^*$.

the last step by integration by parts and continuity of $\langle X\rangle^{(j+l)}$. By induction $Y'Z' \in \mathcal{C}^*_{n+m-j-l}$. Hence, the first term is in $\mathcal{S}^*_{n+m}$, the second term is in $\mathcal{C}^*_{n+m}$, and the third term is a sum of forms $\int\langle X\rangle^{(j+l)}d(AM)$ (or simpler forms $\int\langle X\rangle^{(j+l)}d(A) \in \mathcal{A}^*_{n+m}$) for some $A \in \mathcal{A}^*_c$ and $M \in \mathcal{S}^*_d$ with $c+d+j+l = n+m$, $c \geq 0$, $d \geq 1$. Set $B := \int\langle X\rangle^{(j+l)}dA$. Then $B \in \mathcal{A}^*_{j+l+c}$. Integrating by parts twice (bracket vanishing by continuities of $\langle X\rangle^{(j+l)}$, B)

$$\int\langle X\rangle^{(j+l)}d(AM) = \int A\langle X\rangle^{(j+l)}dM + \int M_-\langle X\rangle^{(j+l)}dA$$
$$= \int A\langle X\rangle^{(j+l)}dM + \int M_-dB = \int A\langle X\rangle^{(j+l)}dM + BM - \int BdM.$$

All three terms are visibly in $\mathcal{C}^*_{n+m}$. Hence, $[Y,Z] \in \mathcal{C}^*_{n+m}$. We already showed $\int Y_-dZ$, and by symmetry $\int Z_-dY$, are in $\mathcal{C}^*_{n+m}$. Therefore, by Itô's product rule, so is YZ. ☐

In particular, $X^n \in \mathcal{C}^*_1(X)$ as $X \in \mathcal{C}^*_n(X)$. If $Y \in \mathcal{C}^*_n$ and $s \in [0,T]$, then clearly the stopped process $Y_{\cdot\wedge s} := (Y_{t\wedge s})_{t\in[0,T]}$ is also in $\mathcal{C}^*_n(X)$. Therefore the product $X_{\cdot\wedge t_1}\cdots X_{\cdot\wedge t_n} \in \mathcal{C}^*_n(X)$.

We illustrate the significance of this for the case when $\langle X\rangle^{(n)}$ are deterministic here, and for the stochastic case in Section 7.2. We begin with the univariate case.

Corollary 6.3. *If $\langle X\rangle^{(i)}$ are deterministic for all $i \in \mathbb{N}$ then for all $(t_1,\cdots,t_n) \in [0,T]^n$,*

$$\overline{\mathbb{E}}(X_{t_1}\cdots X_{t_n}\,|\,\mathbb{F}) \in \operatorname{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n.$$

Proof. Note, $X_{t_1}\cdots X_{t_n} = Y_T$, where $Y := X_{\cdot\wedge t_1}\cdots X_{\cdot\wedge t_n}$. So, it suffices to show $\overline{\mathbb{E}}(Y_T\,|\,\mathbb{F}) \in \operatorname{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$. By the previous proposition, $Y \in \mathcal{C}^*_n(X)$ because each $X_{\cdot\wedge t_i} \in \mathcal{C}^*_1(X)$. So, by linearity, we may assume $Y = AM$ for some $A \in A^*_i$ and $M \in \mathcal{S}^*_j$, $i+j = n$, $i,j \geq 0$. But, the assumption implies that A is deterministic. Therefore, $\overline{\mathbb{E}}(Y_T\,|\,\mathbb{F}) = A_T\overline{\mathbb{E}}(M_T\,|\,\mathbb{F}) = A_TM$. The desired result now follows from Prop. 6.1. ☐

The multivariate case combines a similar argument with Cor. 4.3 and Prop. 4.5 as follows.

Lemma 6.4. *Let $X,Y \in \mathcal{C}^*$. Assume $[X,Y] = 0$ and $\langle X\rangle^{(n)}$ and $\langle X'\rangle^{(n)}$ are deterministic for all $n \in \mathbb{N}$. Then, for any $Z \in \mathcal{C}^*_n(X)$ and $W \in \mathcal{C}^*_m(Y)$, we have $[Z,W] = 0$ and*

$$\overline{\mathbb{E}}(Z_TW_T\,|\,\mathbb{F}) = \overline{\mathbb{E}}(Z_T\,|\,\mathbb{F})\,\overline{\mathbb{E}}(W_T\,|\,\mathbb{F}) \in \operatorname{Span}(\mathcal{S}^*(X^{(i)}),\mathcal{S}^*(Y^{(j)}))_{1\leq i\leq n,\,1\leq j\leq m}.$$

Proof. By definition of $\mathcal{C}^*_n$ and linearity, we may assume $Z = AM$ for some $A \in \mathcal{A}^*_k(X)$ and $M \in \mathcal{S}^*_l(X)$ such that $k+l = n$, $k,l \geq 0$, and similarly, $W = BN$ for some $B \in \mathcal{A}^*_a(Y)$ and $N \in \mathcal{S}^*_b(X)$ such that $a+b = m$, $k,l \geq 0$. By Prop. 6.1, we have $M = \sum_{i=1}^l M_i$ and $N = \sum_{j=1}^b N_j$ for some $M_i \in \mathcal{S}^*(X^{(i)})$ and $N_j \in \mathcal{S}^*(Y^{(j)})$. And by Prop. 4.5 $[X^{(i)},Y^{(j)}] = 0$. Applying Cor. 4.3, with $M' = X^{(i)}$ and $N' = Y^{(j)}$, we see that $M_iN_j \in \mathcal{S}^*(X^{(i)}) \oplus \mathcal{S}^*(Y^{(j)})$. We conclude $MN \in \mathcal{K} := (\operatorname{Span}(\mathcal{S}^*(X^{(i)}),\mathcal{S}^*(Y^{(j)}))_{1\leq i\leq n,\,1\leq j\leq m}$. As $MN \in \mathcal{M}$, we have $\overline{\mathbb{E}}(M_TN_T\,|\,\mathbb{F}) = MN$. (Both martingales have the same terminal value.) Now, the assumption implies A and B are deterministic. Hence,

$$\overline{\mathbb{E}}(Z_TW_T\,|\,\mathbb{F}) = \overline{\mathbb{E}}(A_TM_TB_TN_T\,|\,\mathbb{F})$$

$$= A_T B_T \overline{\mathbb{E}}(M_T N_T \,|\, \mathbb{F}) = A_T B_T NM = A_T B_T \overline{\mathbb{E}}(M_T \,|\, \mathbb{F})\, \overline{\mathbb{E}}(N_T \,|\, \mathbb{F})$$
$$= \overline{\mathbb{E}}(A_T M_T \,|\, \mathbb{F})\, \overline{\mathbb{E}}(B_T N_T \,|\, \mathbb{F}) = \overline{\mathbb{E}}(Z_T \,|\, \mathbb{F})\, \overline{\mathbb{E}}(W_T \,|\, \mathbb{F}).$$

Since as we showed above $MN \in \mathcal{K}$, and A_T, B_T are deterministic, $A_T B_T NM \in \mathcal{K}$. □

A straightforward generalization using induction gives

Lemma 6.5. *Let $Y_1, \cdots, Y_m \in \mathcal{C}^*$. Assume $[Y_j, Y_k] = 0$ if $j \neq k$ and $\langle Y_j \rangle^{(k)}$ are deterministic all j, k. Let $Z_j \in \mathcal{C}^*_{n_j}(Y_k)$, $1 \leq j \leq m$. Then, $[Z_j, Z_k] = 0$ for $j \neq k$, and*

$$\overline{\mathbb{E}}(Z_1(T) \cdots Z_m(T) \,|\, \mathbb{F}) = \overline{\mathbb{E}}(Z_1(T) \,|\, \mathbb{F}) \cdots \overline{\mathbb{E}}(Z_m(T) \,|\, \mathbb{F}) \in \mathrm{Span}(\mathcal{S}^*(Y_j^{(k_j)}))_{1 \leq j \leq m,\, 1 \leq k_j \leq n_j}.$$

Corollary 6.6. *Let $X_i \in \mathcal{C}^*$, $i \in \mathbb{N}$. Assume $[X_i, X_j] = 0$ if $i \neq j$ and $\langle X_i \rangle^{(j)}$ are deterministic all $i, j \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, $(t_1, \cdots, t_n) \in [0, T]^n$, and $(i_1, \cdots, i_n) \in \mathbb{N}^n$,*

$$\overline{\mathbb{E}}(X_{i_1}(t_1) \cdots X_{i_n}(t_n) \,|\, \mathbb{F}) \in \mathrm{Span}(\mathcal{S}^*(X_i^{(j)}))_{i \in \mathbb{N},\, 1 \leq j \leq n}.$$

Proof. Let m be the number of (distinct) elements in the set $\{i_1, \cdots, i_n\}$. By a permutation if necessary, we may assume that $i_1 = \cdots = i_{n_1}$, $i_{n_1+1} = \cdots = i_{n_2}$, $\cdots$, $i_{n_{m-1}} = \cdots i_{n_m} = i_n$, with $n_1 + \cdots + n_m = n$, $n_j \geq 1$. (So non-distinct elements are put next to each other). Set $Y_j := X_{n_j}$, $j = 1, \cdots, m$. Define the product $Z_j(t) := Y_j(t \wedge t_{n_j}) \cdots Y_j(t \wedge t_{n_{j+1}-1})$. By Prop 6.2, $Z_j \in \mathcal{C}^*_{n_j}(Y_k)$. Moreover, clearly, $Z_1(T) \cdots Z_m(T) = X_{i_1}(t_1) \cdots X_{i_n}(t_n)$. The desired result now follows directly from the previous lemma. □

Remark. Cor. 6.6 generalizes Cor. 6.3: simply set $X_1 = X$ and $X_i = 0$ for $i \geq 2$.

7. Square-integrable martingale representation

We set $\mathcal{C} := \mathcal{C}^* \cap \mathcal{C}_*$. (Recall, $\mathcal{C}^*$ is the set of semimartingales of finite moments with continuous angel brackets, and $\mathcal{C}_*$ is the set of processes with exponentially decreasing law.)

7.1. Lévy and Infinite-dimensional Brownian filtrations.

We begin with an extension of the [N-S] result to Lévy processes which may be non-stationary.

Theorem 7.1. *Let $X \in \mathcal{C}$ be such that $\langle X \rangle^{(i)}$ are deterministic for all $i \geq 1$. Let $(N_i)_{i=1}^{\infty}$ denote the strong orthogonalization of $(X^{(i)})_{i=1}^{\infty}$. Assume $\mathbb{F} = \mathbb{F}(X)$. Then*

$$\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i).$$

Proof. Let $(t_1, \cdots, t_n) \in [0, T]^n$, $n \in \mathbb{N}$. By Cor. 6.3, $\overline{\mathbb{E}}(X_{t_1} \cdots X_{t_n} \,|\, \mathbb{F}) \in \mathrm{Span}(\mathcal{S}(X^{(i)}))_{i=1}^n$. Hence by Cor. 5.6, $\mathrm{Span}(\mathcal{S}(X^{(i)}))_{i=1}^n$ is dense in $\mathcal{H}^2$. Prop. 3.1 now yields the result. □

Remark: A curious consequence is that the continuous martingale part X^c is in $\bigoplus_{i=1}^{\infty} \mathcal{S}(N_i)$. It somehow indicates that the discontinuous martingale part can be recovered in the limit from stochastic integral of $X^{(n)}$, $n \geq 2$. This is readily seen when X is a linear combination of n independent Poisson processes. Then, in fact, $X^{(1)} \in \text{Span}\{X^{(2)}, \cdots, X^{(n+1)}\}$.

For a Brownian motion or a Poisson process the result simplifies to $\mathcal{H}^2 = \mathcal{S}(X^{(1)})$.

We need the Brownian case in our main results. Let us define a *Brownian martingale* as a continuous martingale such that $\langle B \rangle$ is deterministic. The law of B is then Gaussian, implying $B \in \mathcal{C}$. Clearly $B^{(n)} = 0$ for $n \geq 2$, as B is continuous. When $T < \infty$, a Brownian motion is a Brownian-martingale. In general, if W is a Brownian motion, then $\int HdW$ is a Brownian martingale for any *deterministic* process $H \in \mathcal{S}(W)$, i.e., with $\int_0^T H_t^2 dt < \infty$. Any Brownian martingale B with strictly increasing $\langle B \rangle$ is of this type.[23]

By a *Poisson-martingale* we mean a martingale $P \in \mathcal{C}$ such that $\langle P \rangle$ is deterministic and $P^{(2)} = P$. Clearly then, $P^{(n)} = P$, $\langle P \rangle^{(n)} = \langle P \rangle$, and $[P]^{(n)} = [P]$ for all $n \geq 2$. A non-stationary compensated Poisson process P with intensity (λ_t) is a Poisson martingale if $\int_0^T \lambda_t dt < \infty$. Then, $\langle P \rangle = \int \lambda dt$. The stationary case of constant λ implies $T < \infty$.

As Brownian and Poisson martingales both satisfy $N_n = 0$ for $n \geq 2$, Theorem 7.1 yields

Corollary 7.2. *Let B be either a Brownian martingale or a Poisson martingale. Assume $\mathbb{F} = \mathbb{F}(B)$. Then $\mathcal{H}^2 = \mathcal{S}(B)$.*

We now turn to multivariate Lévy filtrations. The argument is similar, but the statement utilizes a notation of iterated countable direct sums which we first explain. Suppose we have a doubly indexed family $(\mathcal{K}_{ij})_{i,j \in \mathbb{N}}$ of closed, pairwise orthogonal subspaces K_{ij} of $\mathcal{H}^2$. If we choose any bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, we can identify this family with a sequence of closed orthogonal subspaces, and then take their direct sum. Clearly, the resulting subspace is independent of the choice of bijection. We denote this direct sum by

$$\bigoplus_{i,j=1}^{\infty} \mathcal{K}_{ij} := \{\sum_{i,j=1}^{\infty} N_{ij} : N_{ij} \in \mathcal{K}_{ij};\ \sum_{i,j=1}^{\infty} \|N_{ij}\|^2 < \infty\} \subset \mathcal{H}^2.$$

The order of the summation is irrelevant: we can interchange sums and write $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} N_{ij}$ $= \sum_{i,j=1}^{\infty} N_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} N_{ij}$. Each inner sum is in $\mathcal{H}^2$. This, corresponds to writing,

$$\bigoplus_{j=1}^{\infty} \bigoplus_{i=1}^{\infty} \mathcal{K}_{ij} = \bigoplus_{i,j=1}^{\infty} \mathcal{K}_{ij} = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} \mathcal{K}_{ij}.$$

Theorem 7.3. *Let $X_i \in \mathcal{C}$, $i \in \mathbb{N}$ be such that $[X_i, X_j] = 0$ for $i \neq j$ and $\langle X_i \rangle^{(j)}$ are deterministic for all $i, j \in \mathbb{N}$. Assume $\mathbb{F} = \mathbb{F}(X_i)_{i=1}^{\infty}$. Then*

$$\mathcal{H}^2 = \bigoplus_{i,j=1}^{\infty} \mathcal{S}(N_{ij}),$$

where, for each i, the sequence $(N_{ij})_{j=1}^{\infty}$ is the strong orthogonalization of $(X_i^{(j)})_{j=1}^{\infty}$.

[23]Indeed, then $B = \int HdW$, where $W := \int KdB$, $K := \sqrt{d\langle B \rangle / dt}$, and $H := 1/K$.

Proof. Let $(t_1, \cdots, t_n) \in [0,T]^n$ and $(i_1, \cdots, i_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$. By Corollary 6.6, we have $\overline{\mathbb{E}}(X_{i_1}(t_1) \cdots X_{i_n}(t_n)|\,\mathbb{F}) \in \mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^{\infty}$. Hence by Corollary 5.7, $\mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^{\infty}$ is dense in $\mathcal{H}^2$. The desired result now follows by Prop. 3.1, applied to the doubly indexed sequence of martingales $(X_i^{(j)}))_{i,j=1}^{\infty}$. □

As a consequence we obtain an infinite-dimensional extension of the standard finite-dimensional martingale representation theorems for Brownian motions and Poisson processes.

Corollary 7.4. *Let $(B_i)_{i=1}^{\infty}$ be sequence martingales such that $[B_i, B_j] = 0$ for $i \neq j$, and for each i, B_i is either a Brownian or a Poisson martingale. Assume $\mathbb{F} = \mathbb{F}(B_i)_{i=1}^{\infty}$. Then,*

$$\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i).$$

Moreover, if all B_i are Brownian martingales, then every martingale in $\mathcal{H}^2$ is continuous.

Proof. The first statement follows because $N_{ij} = 0$ for $j \geq 2$ and $N_{i1} = B_i$. As for the continuity statement, let $M \in \mathcal{H}^2$. Write $M = M^c + M^d$ for the continuous-discontinuous decomposition. If all B_i are continuous, then M^d is strongly orthogonal to all B_i, and hence also strong orthogonal to $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) = \mathcal{H}^2$, implying $M^d = 0$. □

Remark. The above specializes to the standard finite-dimensional case by taking all but a finite number of B_i equal to zero. Also, the assumption $[B_i, B_j] = 0$, $i \neq j$ can weakened to correlated Brownian motions (such as $[B_i, B_j] = \rho_{ij} t$). The conclusion is then expressed in terms of the orthogonalization of the B_i, which will be independent Brownian motions.

7.2. **The main result.** We now generalize the results of the previous section to stochastic $\langle X \rangle^{(n)}$, beginning with the univariate case.

Theorem 7.5. *Let $X \in \mathcal{C}$ and $(B_i)_{i=1}^{\infty}$ be a sequence of Brownian martingales such that $[B_i, B_j] = 0$ for $i \neq j$, $[X, B_i] = 0$ all i, and $\langle X \rangle^{(n)}$ is adapted to $\mathbb{F}(B_i)_{i=1}^{\infty}$ all n. Let $(N_i)_{i=1}^{\infty}$ denote the strong orthogonalization of $(X^{(i)})_{i=1}^{\infty}$. Assume $\mathbb{F} = \mathbb{F}(X, B_1, B_2, \cdots)$. Then*

$$\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) \oplus \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j).$$

Proof. Note, $[X^{(j)}, B_i] = 0$ for all i, j, for $j = 1$ by assumption, and for $j \geq 2$ because $X^{(j)}$ is purely discontinuous and B_i is continuous. This implies $[B_i, N_j] = 0$, which in turn implies implies $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$ and $\bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$ are orthogonal subspaces. Therefore, $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) + \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$ is a closed subspace of $\mathcal{H}^2$; so it suffices to show it is dense.

Corollary 5.7 applied to the sequence $(X, B_1, B_2, \cdots)$ implies that the linear span of martingales of the form $\overline{\mathbb{E}}((X_{t_1} \cdots X_{t_n})(B_{i_1}(s_1) \cdots B_{i_m}(s_m))|\,\mathbb{F})$ is dense in $\mathcal{H}^2$, as the indices run over $(t_1, \cdots, t_n) \in [0,T]^n$, $n \in \mathbb{N}$, and $(s_1, \cdots, s_m) \in [0,T]^m$, $(i_1, \cdots, i_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$. As in Prop. 3.1, we have, $\mathrm{Span}(\mathcal{S}(X^{(j)}))_{j=1}^{n} \subset \bigoplus_{j=1}^{n} \mathcal{S}(N_j) \subset \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$.

Therefore it is sufficient to show that

$$\overline{\mathbb{E}}((X_{t_1}\cdots X_{t_n})(B_{i_1}(s_1)\cdots B_{i_m}(s_m))|\,\mathbb{F}) \in \bigoplus_{i=1}^{\infty}\mathcal{S}(B_i) + \mathrm{Span}(\mathcal{S}(X^{(j)}))_{j=1}^{n}.$$

Set $Y := X_{\cdot\wedge t_1}\cdots X_{\cdot\wedge t_n}$. Note, $X_{t_1}\cdots X_{t_n} = Y_T$. Set $\varphi := B_{i_1}(s_1)\cdots B_{i_m}(s_m)$. We must show $\overline{\mathbb{E}}(\varphi Y_T|\,\mathbb{F}) \in \bigoplus_{i=1}^{\infty}\mathcal{S}(B_i) + \mathrm{Span}(\mathcal{S}(X^{(j)}))_{j=1}^{n}$. By Prop. 6.2, $Y \in \mathcal{C}_n^*(X)$. So, Y is a sum of terms of the form AM, where $A \in \mathcal{A}_l^*(X)$ and $M \in \mathcal{S}_k^*(X)$, $l+k=n$, $0 \le l,k \le n$. Note that φA_T is in L^* and is also $\mathcal{G} := \mathcal{F}(B_i)_{i=1}^{n}$-measurable because both φ and A_T have these two properties. Therefore, it is sufficient to show that for all $M \in \mathcal{S}_k^*(X)$, $k \le n$, and all $\mathcal{G}$-measurable $\xi \in L^*$, we have $\overline{\mathbb{E}}(\xi M_T|\,\mathbb{F}) \in \bigoplus_{i=1}^{\infty}\mathcal{S}(B_i) + \mathrm{Span}(\mathcal{S}(X^{(j)}))_{j=1}^{n}$.

Let $\mathbb{G} := \mathbb{F}(B_i)_{i=1}^{\infty}$. Set $N := \overline{\mathbb{E}}(\xi\,|\,\mathbb{G})$. Cor. 7.4, applied to the filtration $\mathbb{G}$ implies N is continuous and $N = \sum_{i=1}^{\infty}\int H_i dB_i$ for some $\mathbb{G}$-predictable processes H_i satisfying $\sum_{i=1}^{\infty}\mathbb{E}\int_0^T H_i(t)^2 d\langle B_i\rangle < \infty$. But, H_i are a-forteriori $\mathbb{F}$-predictable too. So, in fact, we have $N \in \bigoplus_{i=1}^{\infty}\mathcal{S}(B_i)$. In particular, N is also an $\mathbb{F}$-martingale. Hence $N = \overline{\mathbb{E}}(\xi\,|\,\mathbb{F})$.

Since N is continuous and $X^{(n)}$ are purely discontinuous for $n \ge 2$, we have $[N, X^{(n)}] = 0$. This is also true for $n = 1$, as $[X, B_i] = 0$ by assumption. Hence, $[N, M] = 0$. Since $N, M \in \mathcal{H}^* \subset \mathcal{H}^2$, it follows $NM \in \mathcal{M}$. Hence $\overline{\mathbb{E}}(\xi M_T\,|\,\mathbb{F}) = NM$, as both sides are martingales with the same value at T (namely ξM_T).

Now, $NM = \int NdM + \int M_- dN$. By Prop. 4.2, $\int M_- dN \in \mathcal{S}(N)$. Since $\bigoplus_{i=1}^{\infty}\mathcal{S}(B_i)$ is a stable subspace and contains N, it contains $\mathcal{S}(N)$. Therefore, $\int M_- dN \in \bigoplus_{i=1}^{\infty}\mathcal{S}(B_i)$. So, it remains to show $\int NdM \in \mathrm{Span}(\mathcal{S}(X^{(j)}))_{j=1}^{n}$. Since $M \in \mathcal{S}_k^*(X)$, it is a sum of terms of form $\int AdX^{(i)}$ with $A \in \mathcal{A}^*$ and $i \le k \le n$. But, by Cor. 4.7, $X^{(i)} \in \mathcal{H}^*$ and $AN \in \mathcal{C}^*$. Hence, by Cor. 4.7, $\int ANdX^{(i)} \in \mathcal{S}(X^{(i)}) \subset \mathrm{Span}(\mathcal{S}(X^{(j)}))_{j=1}^{n}$, as desired. □

Remark . Lévy case is special case: simply take $B_i = 0$ for all i. The Brownian case of Corollary 7.4 is also a special case: simply take $X = 0$.

Remark . Since B_i are continuous, the assumption $[X, B_i] = 0$ is equivalent to $[X^c, B_i] = 0$. It is easy to see that that this assumption can be weakened to the following: $X^c = M + N$ for some $M, N \in \mathcal{H}^2$ such that $[M, B_i] = 0$ for all i and N is adapted to $\mathbb{F}(B_i)_{i=1}^{\infty}$.

Remark . We assumed $X_0 = 0$ throughout. This assumption is relaxed simply by requiring $X - X_0 \in \mathcal{C}$ instead of $X \in \mathcal{C}$.

Open Question: Assume X^c is a Brownian motion, $\langle X\rangle^{(n)}$ are adapted to $\mathbb{F}(X^c)$, and $\mathbb{F} = \mathbb{F}(X)$. If $\langle X\rangle^{(n)}$ are deterministic, then, as previously remarked, $X^c \in \bigoplus_{j=1}^{\infty}\mathcal{S}(N_j)$. The question is to what extend this holds in general. It holds in the simple case where $X - X^c$ is a linear combination of independent Cox processes. When it holds, the conclusion of above theorem sharpens to $\mathcal{H}^2 = \bigoplus_{j=1}^{\infty}\mathcal{S}(N_j)$ from $\mathcal{H}^2 = \mathcal{S}(X^c) \oplus \bigoplus_{j=1}^{\infty}\mathcal{S}(N_j)$.

The above result extends to the multivariate case by arguments already visited. For completeness, we include the proof.

Theorem 7.6. *Let $X_i \in \mathcal{C}$, $i \in \mathbb{N}$. Let $(B_i)_{i=1}^{\infty}$ be a sequence of Brownian martingales. Assume $[X_i, X_j] = [B_i, B_j] = 0$ for $i \neq j$, and for all i, j, $[X_i, B_j] = 0$, and $\langle X_i \rangle^{(j)}$ are adapted to $\mathbb{F}(B_k)_{k=1}^{\infty}$. Assume further that $\mathbb{F} = \mathbb{F}(X_i, B_i)_{i=1}^{\infty}$. Then*

$$\mathcal{H}^2 = \bigoplus_{k=1}^{\infty} \mathcal{S}(B_k) \oplus \bigoplus_{i,j=1}^{\infty} \mathcal{S}(N_{ij}),$$

where, for each i, the sequence $(N_{ij})_{j=1}^{\infty}$ is the strong orthogonalization of $(X_i^{(j)})_{j=1}^{\infty}$.

Proof. As above, we have $[X_i^{(j)}, B_k] = 0$, all i, j, k, and by Prop 4.5, we also have $[X_i^{(j)}, X_k^{(l)}] = 0$, all i, j, k, l. Hence all B_k and N_{ij} are strongly orthogonal to each other. Therefore $\bigoplus_{k=1}^{\infty} \mathcal{S}(B_k) \oplus \bigoplus_{i,j=1}^{\infty} \mathcal{S}(N_{ij})$ is a closed subspace of $\mathcal{H}^2$, and it suffices to show it is dense.

Corollary 5.7 applied to the sequence $(X_i, B_i)_{i=1}^{\infty}$ implies that the linear span of martingales of the form $\overline{\mathbb{E}}((X_{j_1}(t_1) \cdots X_{j_n}(t_n))(B_{i_1}(s_1) \cdots B_{i_m}(s_m)) | \mathbb{F})$ is dense in $\mathcal{H}^2$, as the indices run over $(t_1, \cdots, t_n) \in [0, T]^n$, $(j_1, \cdots, j_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$, and $(s_1, \cdots, s_m) \in [0, T]^m$, $(i_1, \cdots, i_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$. As in Prop. 3.1, we have, $\mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{j=1}^n \subset \bigoplus_{j=1}^n \mathcal{S}(N_{ij})$. Therefore it suffices to show that

$$\overline{\mathbb{E}}((X_{j_1}(t_1) \cdots X_{j_n}(t_1))(B_{i_1}(s_1) \cdots B_{i_m}(s_m)) | \mathbb{F}) \in \bigoplus_{k=1}^{\infty} \mathcal{S}(B_k) + \mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n.$$

Set $Y_t := X_{j_1}(t \wedge t_1) \cdots X_{j_n}(t \wedge t_n)$, and $\varphi := B_{i_1}(s_1) \cdots B_{i_m}(s_m)$. As $X_{j_1}(t_1) \cdots X_{j_n}(t_1)) = Y_T$, we must show $\overline{\mathbb{E}}(\varphi Y_T | \mathbb{F}) \in \bigoplus_{k=1}^{\infty} \mathcal{S}(B_k) + \mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$. As in the proof of Cor. 6.6, we break $j_1, \cdots, j_n$ into distinct elements, which by a permutation we may assume are next to each other. As such, we can write $Y = Y_1 \cdots Y_l$, where each Y_i is of the form $X_{j_i}(t \wedge t_{k_1}) \cdots X_{j_i}(t \wedge t_{k_{j_i}})$. By Prop. 6.2, each $Y_i \in \mathcal{C}^*_{m_i}(X_i)$ for some $m_i \geq 1$ with $\sum_i m_i = n$. So, $Y_i \in \mathcal{C}^*_n(X_i)$. So, each Y_i is a sum of terms of the form $A_i M_i$, where $A_i \in \mathcal{A}^*_{l_i}(X_i)$ and $M_i \in \mathcal{S}^*_{k_i}(X_i)$, $l_i + k_i = n$, $0 \leq l_i, k_i \leq n$. Note that $\varphi A_1(T) \cdots A_l(T)$ is in L^* and is also $\mathcal{G} := \mathcal{F}(B_i)_{i=1}^n$-measurable because φ and all $A_i(T)$ have these two properties. Therefore, it is sufficient to show that for all $M_i \in \mathcal{S}^*_{k_i}(X_i)$, $k_i \leq n$, $i \leq l$ $(l \leq n)$ and all $\mathcal{G}$-measurable $\xi \in L^*$, we have $\overline{\mathbb{E}}(\xi M_1(T) \cdots M_l(T) | \mathbb{F}) \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) + \mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$.

Let $\mathbb{G} := \mathbb{F}(B_i)_{i=1}^{\infty}$. Set $N := \overline{\mathbb{E}}(\xi \,|\, \mathbb{G})$. As in the proof of Theorem 7.5, it follows that is N is continuous and is actually $\mathbb{F}$-martingale; so $N = \overline{\mathbb{E}}(\xi \,|\, \mathbb{F})$. As before, the continuity of N and the assumption imply that $[N, X_i^{(j)}] = 0$, all i, j. Hence, $[N, M_i] = 0$, all i. Moreover, as $[X_i^{(k)}, X_j^{(l)}] = 0$ by Prop 4.5 for $i \neq j$, we get $[M_i, M_j] = 0$ for $i \neq j$. As $N, M_i \in \mathcal{H}^*$, these imply that $M := M_1 \cdots M_l$ and NM are martingales. Hence, $\overline{\mathbb{E}}(\xi M_1(T) \cdots M_l(T) | \mathbb{F}) = NM$, as both sides are martingales with the same value at T.

Now, $NM = \int NdM + \int M_- dN$. By Prop. 4.2, $\int M_- dN \in \mathcal{S}(N)$. Since $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$ is a stable subspace and contains N, it contains $\mathcal{S}(N)$. Therefore, $\int M_- dN \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$. So, it remains to show $\int NdM \in \mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$. But, $\int NdM = \int NM_{2-} \cdots M_{l-} dM_1 + \cdots + \int NM_{1-} \cdots M_{l-1-} dM_l$. Since $M_i \in \mathcal{S}^*_{k_i}(X_i)$, it is a sum of terms of the form $\int A_i dX_i^{(j_i)}$ with $A_i \in \mathcal{A}^*$ and $j_i \leq k_i \leq n$. But, by Cor. 4.7, $X_i^{(j_i)} \in \mathcal{H}^*$ and also all the products $M_1\cdots M_{l-1}A_lN$, $\cdots$, $M_2\cdots M_lA_1N$ are in $\mathcal{C}^*$. Hence, by Cor. 4.7, $\int M_1\cdots M_{l-1}A_lNdX_l^{(j_l)} \in \mathcal{S}(X_l^{(j_l)}) \subset \mathrm{Span}(\mathcal{S}(X_l^{(j)}))_{j=1}^n$, $\cdots$, $\int M_2\cdots M_{l-1}A_lNdX_1^{(j_1)} \in \mathcal{S}(X_1^{(j_1)}) \subset \mathrm{Span}(\mathcal{S}(X_1^{(j)}))_{j=1}^n$. Hence, $\int NdM \in \mathrm{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$, as desired. □

8. Explicit chaotic expansion of powers

The following binomial expansion shows the relationship between integer powers and the power brackets. We set $[X]^{(1)} := X$ for any semimartingale. (Recall, $[X]^{(2)} := [X]$.)

Proposition 8.1. *Let X be a semimartingale with $X_0 = 0$. Then, for all $n \in \mathbb{N}$ we have,*

$$X^n = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i d[X]^{(n-i)}. \tag{8.1}$$

Proof. By Itô's formula, and binomial expansion of $X^n = (X_- + \Delta X)^n$, we have

$$X^n - n\int X_-^{n-1}dX - \frac{1}{2}n(n-1)\int X_-^{n-2}d[X]^c$$

$$= \sum_{s\le\cdot}(X_s^n - X_{s_-}^n - n\Delta X_s X_{s_-}^{n-1}) = \sum_{s\le\cdot}\sum_{i=0}^{n-2}\binom{n}{i}X_{s-}^i(\Delta X_s)^{n-i}.$$

For $i \le n-3$, $\sum_{s\le\cdot} X_{s-}^i(\Delta X_s)^{n-i} = \int X_-^i d[X]^{(n-i)}$. For $i = n-2$, the term $\int X_-^{n-2}d[X]^c$ combines with the term $\sum_{s\le\cdot} X_{s-}^{n-2}(\Delta X_s)^2$ to give $\int X_-^{n-2}d[X]$. The formula follows. □

Note, the leading term (corresponding to $i = 0$) is $[X]^{(n)}$.

We can substitute the same formula for X_-^i on the right-hand-side of Eq. (8.1) Repeating this procedure clearly leads to iterated integrals. We adopt the following notation. If H is a locally bounded predictable process, and X and Y are semimartingales, we denote

$$\int^- HdX := (\int HdX)_-, \quad \int\int^- HdXdY := \int(\int^- HdX)dY.$$

Note, $\int X_-dY = \int\int^- dXdY$ if $X_0 = 0$. For semimartingales $Y_1, \cdots, Y_n$ define inductively

$$\int\int^-\cdots\int^- HdY_1\cdots dY_{n-1}dY_n := \int(\int^-\cdots\int^- HdY_1\cdots dY_{n-1})dY_n.$$

We denote multi-indices by $I = (i_1, \cdots, i_p) \in \mathbb{N}^p$, and for integers $1 \le p \le n$, we set

$$\mathbb{N}_n^p := \{I = (i_1, \cdots, i_p) \in \mathbb{N}^p : i_1 + \cdots i_p = n\}, \quad p, n \in \mathbb{N}.$$

Proposition 8.2. *Let X be a semimartingale with $X_0 = 0$. Then, for all $n \in \mathbb{N}$ we have,*

$$X^n = \sum_{p=1}^n \sum_{I\in\mathbb{N}_n^p} \frac{n!}{i_1!\cdots i_p!}\int\int^-\cdots\int^- d[X]^{(i_1)}\cdots d[X]^{(i_{p-1})}d[X]^{(i_p)}.$$

Proof. Cases $n = 1, 2$ are clear, as the formula reads $X = \int d[X]^{(1)}$ and $X^2 = \int d[X]^{(2)} + 2\int [X]^{(1)}_{-} d[X]^{(1)}$. For $n \geq 3$, each summand in Eq. (8.1) involving X^i_-, $i \geq 2$, can be expanded by Eq. (8.1) itself. Substituting and regrouping yields,

$$X^n = [X]_n + \sum_{i=1}^{n-1} \binom{n}{i} \int [X]_{i-} d[X]_{n-i} + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \binom{n}{i} \binom{i}{j} \int \int^{-} X^j_- d[X]_{i-j} d[X]_{n-i}.$$

If $n = 3$, we are done. For $n \geq 4$, substituting for X^j_-, $j \geq 2$ from (8.1) and regrouping,

$$X^n = [X]_n + \sum_{i=1}^{n-1} \binom{n}{i} \int [X]_{i-} d[X]_{n-i} + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \binom{n}{i} \binom{i}{j} \int \int^{-} [X]_{j-} d[X]_{i-j} d[X]_{n-i}$$
$$+ \sum_{i=3}^{n-1} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} \binom{n}{i} \binom{i}{j} \binom{j}{k} \int \int^{-} \int^{-} X^k_- d[X]_{j-k} d[X]_{i-j} d[X]_{n-i}.$$

If $n = 4$, we are done. For $n \geq 5$, we continue substituting from (8.1) in this way, and clearly this procedure terminates by the n-th step, yielding then the desired result. □

Combing the two propositions, one finds a similar iterated integral formula for $[X^n, X^m]$.[24]

Substituting $[X]^{(i)} = \langle X \rangle^{(i)} + X^{(i)}$ into the term $\int \int^{-} \cdots \int^{-} d[X]^{(i_1)} \cdots d[X]^{(i_{p-1})} d[X]^{(i_p)}$, we get sums of expressions of form $\int \int^{-} \cdots \int^{-} dY^{(i_1)} \cdots dY^{(i_{p-1})} dY^{(i_p)}$, where each $Y^{(i)}$ can be either $\langle X \rangle^{(i)}$ or $X^{(i)}$. If $Y^{(i_p)}$ is $X^{(i_p)}$, then the quantity belongs to $\mathcal{S}^*_p(X)$. Otherwise, if $q < p$ is the largest integer such that $Y^{(i_q)}$ is $X^{(i_q)}$, then we are dealing with an expression of the form $\int \cdots \int M_- d\langle X \rangle^{(i_{q+1})} \cdots d\langle X \rangle^{(i_p)}$, where M of the form $M = \int Y_- dX^{(i_q)} \in \mathcal{S}^*_q(X)$, with $Y \in \mathcal{C}^*_{q-1}(X)$. In the proof of Proposition 6.2, we integrated by parts such expressions and used induction to show it belongs to $\mathcal{C}^*_p(X)$. The next result reports the explicit outcome of such repeated integration by parts, under a slightly more general setting, which applies to the present case with the A_j standing for the various $\langle X \rangle^{(i_j)}$.

Proposition 8.3. *Let M, $A_1, \cdots, A_n$ be semimartingales. Assume that $M_0 = 0$ (or all $A_i(0) = 0$) and all A_i are continuous and of finite variation. (So, $[A_i, M] = 0$.) Then*

$$\int \cdots \int M_- dA_1 \cdots dA_n = \sum_{p=0}^{n} \sum_{0=i_0<i_1<\cdots<i_p \leq n} (-1)^p \Big(\int A_{i_0,i_1} \cdots A_{i_{p-1},i_p} \, dM\Big) \, A_{i_p,n},$$

---

[24] Namely, using the two propositions and the easily verified fact that $[[X]^{(i)}, [X]^{(j)}] = [X]^{(i+j)}$, we get

$$[X^n, X^m] = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{p=1}^{i+j} \sum_{I \in \mathbb{N}^p_{i+j}} \binom{n}{i} \binom{m}{j} \frac{(i+j)!}{i_1! \cdots i_p!} \int \int^{-} \cdots \int^{-} \int^{-} d[X]^{(i_1)} \cdots d[X]^{(i_{p-1})} d[X]^{(i_p)} d[X]^{(n+m-i-j)}.$$

where $A_{i,j}$ for $0 \leq i < j \leq n$ is defined by $A_{i,i} = 1$, ($A_{i-1,i} = A_i$,) and

$$A_{i,j} := \int \cdots \int A_{i+1} dA_{i+2} \cdots dA_j. \quad (0 \leq i < j \leq n)$$

Proof. (*Outline.*) For $n = 1$ the formula reads $\int M_- dA_1 = MA_1 - \int A_1 dM$, which follows by integration by parts. For $n = 2$, we substitute this expression in $\int\int M_- dA_1 dA_2$. The first term $\int MA_1 dA_2$ is integrated by parts to give $M \int A_1 dA_2 - \int\int A_1 dA_2 dM$. The second term $-\int\int A_1 dM dA_2$ is likewise integrated by parts. The result is

$$\int\int M_- dA_1 dA_2 = M \int A_1 dA_2 - \int\int A_1 dA_2 dM - \int A_1 dM A_2 + \int A_1 A_2 dM.$$

For $n \geq 3$, one proceeds in a similar manner using integration by parts and induction.[25] □

Note, the term corresponding to $p = 0$ is $M \int \cdots \int A_1 dA_2 \cdots dA_n$, while that corresponding to $p = n$ is $(-1)^n \int A_1 \cdots A_n dM$. As an example, say $n = 12 + 1$, $p = 4$, and $(i_1, i_2, i_3, i_4) = (2, 6, 7, 10)$. Then the corresponding term is

$$\int (\int A_1 dA_2 (\int\int\int A_3 dA_4 dA_5 dA_6) A_7 \int\int A_8 dA_9 dA_{10})\, dM\, (\int\int A_{11} dA_{12} dA_{13}).$$

The explicit form of the $\mathbb{F}(\langle X \rangle^{(i)})_{i=1}^n$-adapted processes $A \in \mathcal{A}_n^*(X)$ appearing in the chaotic expansion of $X^n \in \mathcal{C}_n^*(X)$ is now clear: such A are products of iterated integrals of $\langle X \rangle^{(i)}$.

9. Concluding remarks

The martingale representation result of [D] for finite activity processes mentioned in the introduction is seemingly of a quite different form than that of [N-S] or those here. However, the two forms can be tentatively reconciled through the language of random measures. Recast in this terms, Theorem 9 of [D] basically states that in the finite activity case a martingale can be represented as $W * (\mu - \nu)$ for a suitable predictable function $W(\omega, t, x)$. The [N-S] series representation $\sum_{n=1}^\infty H_n dN_n$ can be heuristically brought to this same form, once we consider that the Teugels martingale are given by $x^i * (\mu - \nu)$ and their strong orthogonalization N_n are basically of the form $(\sum_{i=1}^n K_{ni}\, x^i) * (\mu - \nu)$ for some predictable (constant in the Lévy case) processes $K_{n,i}$. In a loose sense, this then gives a representation of the form $W * (\mu - \nu)$ with the predictable function W given be the formal power series $W = \sum_{i=1}^\infty L_i x^i$, where, formally, $L_i = \sum_{n=i}^\infty H_n K_{ni}$.

In closing, we pose an open question. We assumed throughout that angle brackets are continuous. This is a natural assumption and often met in practice. It is essentially a quasi-left-continuity assumption requiring all jumps be unpredictable. However, it may still be interesting to investigate the relaxation of this requirement within the present setting.

[25] We point out that the continuity and finite variation assumption on A_i can be relaxed to $[A_i, M] = 0$ at the expense of left limits in the expressions. We also note that this is really an ordinary calculus result.

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