Mean Reversion Expectations and the 1987 Stock Market Crash: An Empirical Investigation

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Abstract

After the stock market crash of 1987, Fischer Black proposed a model in which he explained the crash by inconsistencies in the formation of expectations of mean reversion in stock returns. Following this explanation, a model that allows for mean reversion in stock returns is estimated on daily stock index data around the crash of 1987. The results strongly support Black's hypothesis. Simulations show that on Friday Oct 16, 1987, a crash of 20 percent or more had a probability of more than seven percent. (*JEL* G10, C22)

I. Introduction

The report of the Brady Commission emphasized the role of portfolio insurance strategies in the stock market crash of 1987 (Brady et al. 1988). According to the Brady Report, dynamic hedging strategies were triggered on a large scale on Black Monday and led to a downward cascade. This interpretation was criticized because only a small fraction of the market volume was managed using dynamic hedging strategies and the elasticity of stock demand implied by the magnitude of the crash seemed unreasonable (Leland 1988, Rubinstein 1988, Brennan and Schwartz 1989).

A small number of studies interpreted portfolio insurance differently (Grossman 1988, Black 1988, Gennotte and Leland 1990, and Jacklin, Kleidon, and Pfleiderer 1992). According to the theory developed in this literature, the drop in the level of the stock market *prior* to the crash resulted in adjustments of the dynamic hedges that revealed the volume of portfolios under insurance. This share was much larger than the average investor expected. Therefore, during the bull market before the crash, more purchases than expected were made to mirror a put option and not because of fundamental information. This was in itself a piece of fundamental information. From this point of view, the crash is explained by the shattering of an illusion about the share of portfolios under dynamic hedging strategies. The fact that portfolio insurance is unobservable makes such an illusion possible. The transactions in stocks (that is, usually index futures) and bonds do not reveal that they are meant to synthesize an option.

Fischer Black (1988) connected this idea to the concept of mean reversion in stock returns. In his model, the underestimation of portfolio insurance translates to an underestimation of mean reversion. When the true size of insured portfolios becomes known, two things happen: The mean reversion parameter in the stock price model must be adjusted, and the stock price path since the beginning of the illusion must be corrected to where it would have been if the illusion had not occurred. This latter correction in the path is the stock market crash. The model is consistent with equilibrium since the change in the price process is caused by the change in expected mean reversion.

So far, the implications of this mean reversion explanation of the 1987 crash have not been explored empirically. The theory predicts that after the stock market crash, mean reversion must have been higher than before the crash. Also, before the crash two periods of different mean reversion should be identifiable, one of relatively higher mean reversion before the illusion set in and one of relatively lower mean reversion right before the crash.

The aim of this paper is to provide evidence from stock index data for Black's hypothesis. We specify a stock return model that allows for mean reversion and estimate it on daily S&P 500 index data around the stock market crash of 1987. The results strongly support Black's theory. For about five years after the crash, mean reversion was significantly higher than before the crash. During the period 1982–1986 which was identified in the Brady Report as the bull market that led up to the crash, a significantly higher mean reversion than during the year 1987 itself is measured. The report characterized the boom in 1987 as exaggerated. A generalized likelihood ratio test for a parameter change-point finds evidence of a change in the mean reversion regime in early 1987, supporting the segmentation of the Brady Report. Simulations of the model for the 1982–1986 and the January 1987–October 1987 periods result in a probability of more than seven percent for a crash of 20 percent or more. A correction of minus 10 percent or more had a probability of over 40 percent. These probability estimates can be obtained without having to assume heavy-tailed distributions for returns.

The paper is organized as follows. Section II briefly reviews the literature on mean reversion, outlines Black's explanation of the crash, and discusses the events of the week prior to the crash of 1987 in this light. Section III specifies and discusses the model and Section IV reports the results of its estimation. Section V briefly discusses the stock market crash of 1929. Section VI concludes.

II. A Mean Reversion Theory of Stock Market Crashes

A. *Mean Reversion in Stock Returns*

Unlike mean reversion in stock price volatility, mean reversion in stock returns is a controversial subject. De Bondt and Thaler (1985, 1987) show that underperforming stocks outperform the market in later periods, where "later" can mean anything between three and five years. Summers (1986) makes the point that available statistical methods have no power to discriminate between the null hypothesis of no mean reversion and a slowly mean reverting alternative. Early studies that find evidence of mean reversion in returns (Fama and French 1988, Poterba and Summers 1988, Jegadeesh 1990, Cutler, Poterba, and Summers 1991) often use variance ratio tests and autoregressions of return series to detect mean reversion. These methods tend to reject the null hypothesis of no mean reversion too often (Richardson and Stock 1989, Richardson 1993, Kim, Nelson, and Startz 1998, Kim and Nelson 1998). Also, segmenting the data into pre- and post-world-war-II periods significantly reduces the estimates of mean reversion (Kim, Nelson, and Startz 1991). Lo and MacKinlay (1988) find evidence against mean reversion in weekly data. However, De Bondts and Thaler's (1985) findings and their support in later studies (Jegadeesh and Titman 1993, 1995, Lakonishok, Shleifer, and Vishny 1994) remain largely undisputed. Balvers, Wu, and Gilliland (2000) find evidence for mean reversion in a panel across countries.

Different explanations are presented for mean reversion in returns, most prominently over- and underreaction to news and momentum trading (Lo and MacKinlay 1990, De Long et al. 1990, Barberis, Shleifer, and Vishni 1998, Hong and Stein 1999, Gatev and Ross 2000), and time-varying required returns as a result of consumption smoothing (Black 1990, Cechetti, Lam, and Mark 1990).

Lo and Wang (1995) study option valuation with mean reverting stock returns.

Following Black (1988, 1990), we consider a stock market with mean reversion in returns. Investors are assumed to form expectations of mean reversion. The next subsections give an outline of Black's explanation of the 1987 stock market crash and connect this explanation to the model introduced in Section III.

B. *Mean Reversion Expectations*

Consider the situation in Figure 1 where at time t a positive change is observed and consider an individual investor who expects that returns will revert quickly. If λ is some parameter in the return generating process that controls the reversion, her a-priori expectations can be represented by, say, the parameter value λ_0, which stands for a fast reversion. The investor takes the expectations of other market participants into account, however, so she has to have an update scheme. A natural way to update the expectation of mean reversion is to observe the market after a change in the return. Therefore, between times t and $t+h$ she does not make any transactions but observes the behavior of the other market participants in order to form an expectation which is some weighted average of her a-priori expectation indicated by λ_0 and the observed market behavior. If the market indicates a reversion expectation like the one represented by λ_2, the investor recognizes that her a-priori expectation was too high relative to the market and consequently adjusts it to λ_1, for instance.

FIGURE 1 ABOUT HERE.

We assume there are participants who act autonomously, i.e. who do not wait for others to act between times t and $t+h$. These may be institutional investors with predefined investment strategies which, explicitly or implicitly, induce certain reversion expectations. The parameter value λ_2 can be understood as the mean of their expectations.

Alternatively, there may be investors who have similar expectations but who are less risk averse than the individual investor considered first. They may implement a strategy on the spot market as if returns followed λ_2 while being hedged against the case that returns revert faster than according to λ_0. This would allow them to participate in gains arising from slow reversion while at the same time hedging against the risk of faster reversion than λ_0. Fischer Black describes these investors as having more flexible tastes for risk than the non-autonomous investor (Black 1988, p. 272).

Such a hedge can be implemented easily. Each reversion parameter $\lambda_{0,1,2}$ corresponds to a certain index (or stock) price at any given time. For example, consider time t^* in Figure 1 as the investment horizon. Let $S(t, \lambda_i)$ denote the index price at time t corresponding to reversion parameter λ_i. Then, in t^* the relation $S(t^*, \lambda_0) < S(t^*, \lambda_1) < S(t^*, \lambda_2)$ holds because at that time, λ_2 implies a higher return than λ_1 and λ_0, which in turn implies higher respective prices. The investor could buy a put option at strike price $S(t^*, \lambda_0)$ with maturity t^*. She could then control her positions as if she expected the price to behave according to λ_2. If the price dropped below $S(t^*, \lambda_0)$ at time t^*, her exposure would be restricted to $S(t^*, \lambda_2) - S(t^*, \lambda_0)$.

The more risk averse investor could hedge against the possibility that the price falls below her a-priori level $S(t^*, \lambda_0)$ just as well. Her exposure would be $S(t^*, \lambda_1) - S(t^*, \lambda_0)$ which is less than $S(t^*, \lambda_2) - S(t^*, \lambda_0)$, showing that her position is more risk averse.

In the case where a decline in returns below the reversion level is observed, an investor who expects a fast improvement of returns will hold her long positions to avoid realizing temporary losses or will buy more to exploit a cost-average effect. The investor will tend to close short positions to take advantage of the temporarily low prices. On the other hand, an investor who expects returns to revert slowly or to stay low for a while may want to sell her long positions in order to avoid possibly heavier losses in the future. She will keep or even expand short positions to take advantage of possible further downturns. Thus, after a fall in returns, the expectation of fast reversion implies higher buying pressure than does the expectation of slow reversion. After an increase in returns, the opposite is true. Expectations of fast reversion imply higher selling pressure than do expectations of slow reversion. This translation of expected mean reversion into actual mean reversion is important for the empirical test of Black's hypothesis. It enables us to consider actual mean reversion in stock price data, even though the explanation of the crash rests on shifts in expected mean reversion.

Consider an investor who wants to hedge against the possibility of fast reversion after a decline in returns. In order to take advantage of stable low prices, she could assume short positions as if she expects returns to behave according to λ_2 (which now means that returns improve slowly). At the same time she could buy a call option with strike price $S(t^*, \lambda_0)$ and maturity t^*. The price relation at time t^* would be $S(t^*, \lambda_0) > S(t^*, \lambda_1) > S(t^*, \lambda_2)$. If prices recovered quickly, the investor would have to cover at higher prices than she got when taking the short position. If the price at time t^* rose above $S(t^*, \lambda_0)$, her exposure would be restricted to $S(t^*, \lambda_0) - S(t^*, \lambda_2)$.

As before, an individual investor who assumes an a-priori reversion of λ_0 could wait until time $t + h$ to observe the market behavior. If she saw a reversion of λ_2 in the market, she would choose a weighted mean, for instance λ_1, just as in the case of an increase in returns. This would imply smaller long positions and larger short positions than according to λ_0. If she hedged against prices higher than her a-priori level $S(t^*, \lambda_0)$, her exposure would be $S(t^*, \lambda_0) - S(t^*, \lambda_1)$ which is lower than $S(t^*, \lambda_0) - S(t^*, \lambda_2)$, the exposure of the investor considered before. Thus, after decreases in returns as well as after increases, it is risk-averse to expect fast mean reversion.

C. *Mean Reversion Illusions and Disillusions*

Assume that those investors who are risk-tolerant and buy an option while speculating on slow reversion publically reveal of what they are doing. Then, when a risk-averse investor forms her expectations between times t and $t + h$, she will not only look at the market but will also look at these public records. She will recognize that the mean reversion expectations of the risk-tolerant investors who are already active on the market are similar to her a-priori expectations, but that they are hedged and take riskier positions in the spot market. Her perception of the market's expected mean reversion will be higher and thus her own posterior expectation, the weighted average of her a-priori expectation and her market perception, will be higher as well.

The put-call ratio is a proxy for these imaginary public records. If investors look at a stable high market after an increase in returns to form mean reversion expectations, they can conclude from a high put-call ratio that the market's expected mean reversion is greater than indicated by stock sales alone. Conversely, if the market stays low after a decrease in returns, a low put-call ratio indicates the same.

If the group of risk-tolerant investors chooses to synthesize the options contracts, they hold hedge portfolios consisting of stocks (or futures) and bonds. Investors cannot observe that the transactions related to these portfolios are designed to mirror an option and hence there is no record at all. In this case, the risk-averse investor will base her expectations only on stock sales. If the market stays high after an increase in returns or low after a decrease, the investor will systematically underestimate the market's expected mean reversion. The information that the risk-tolerant investors are not confident of a low reversion but hedged against a high one is hidden.

A crash is possible if the underestimation of mean reversion is a mass phenomenon and not confined to a single investor. This may occur because the expectations of the group of risk-tolerant investors are not, or are only rudimentarily observable. For example, consider the extreme case where, except for the small group of risk-tolerant investors, all others are risk-averse. The risk-averse investors wait between t and $t+h$ to observe the market and are not able to infer the true expectations of the active, risk-tolerant investors. The position of the risk-tolerant investors accounting for both their purchases and their short sales from the portfolio that replicates the put option will be positive after a positive change. This assertion is shown in the Appendix (Lemma A1).

Assume that the λ_2-reversion in Figure 1 is the result of the risk-tolerant group's consolidated purchases and short sales. It is the actual mean reversion that results as the average of the expected mean reversion of all decision makers in the risk-tolerant group. Their sales and purchase decisions translate their expected mean reversion into actual mean reversion. Next, when the risk-averse investors observe λ_2, they adjust their expectation from λ_0 to λ_1. That is, every single investor adjusts her expectations without knowing about the others. The move from the mean λ_0 to the mean λ_1 is the result of all these adjustments. Denote the proportion of the transactions by the risk-averse majority by $\alpha \in [0,1]$ and the proportion of the transactions by the autonomous, risk-tolerant group by $\beta \in [0,1]$. The market consists of these two groups only, i.e. $\alpha + \beta = 1$. Then, the actual mean reversion that will be effective on the market after $t+h$ is given by $\lambda = \alpha\lambda_1 + \beta\lambda_2$. The theory outlined here is valid as long as $0 < \lambda < \lambda_0$, i.e. the actual mean reversion is slower than the average a-priori expected mean reversion.

The situation described so far will be called *mean reversion illusion* in this paper. The mean reversion λ prevalent on the market is lower than it would have been if the group of risk-averse investors had seen the hedge activity of the autonomous group correctly. This misconception is disclosed when the true expectations of the autonomous group and their hedge positions become known. At this point, every single investor readjusts her expectations. The disillusion can also set in if the majority becomes aware of its majority, that is, when it becomes known that a large number of investors had expected a faster reversion but adjusted to a slower one after observing the activities of a small group. These two disillusions are independent: the information about the true expectations of the risk-tolerant group does not imply that many others followed them. Conversely, the information that a majority with expectations of fast mean reversion followed a minority with seemingly low expectations does not reveal anything about the true expectations of the minority.

FIGURE 2 ABOUT HERE.

Accordingly, a stock market crash will be defined as the *mean reversion disillusion*. If one of the two possible disillusions occurs at time t_c, it will become apparent that the market assumed an incorrect reversion since time t. That is, the price process followed an incorrect path between t and t_c. "Incorrect" means that it did not reflect the average a-priori mean reversion expectations of the market participants accurately. The process has to be set into a position as if the illusion had not happened. This means that the incorrect path has to be offset, and stock prices must adjust to the level consistent with the true mean reversion expectations. The crash is thus not just a readjustment in one parameter. Instead, it is this readjustment *plus* a discontinuous correction for the difference in the paths induced by λ and λ_0 between t and t_c. The magnitude of the crash therefore depends on the "depth" of the illusion $(\lambda - \lambda_0)$ and its duration $(t_c - t)$. Figure 2 illustrates the point.

The sequence of events is (1) disclosure of the true mean reversion expectations of the risk-tolerant group, (2) adjustment of mean reversion expectations of the risk-averse group, (3) investment decisions of the risk-averse group according to the new mean reversion expectations. Because the new information reveals that the actual and the expected mean reversion did not properly reflect the average a-priori expected mean reversion for some time $t_c - t$, the investment decisions not only reflect a parameter adjustment but a switch to a different trajectory of the stock price process. The change in expected mean reversion causes a change in the actual mean reversion, therefore the hypothesis is consistent with general equilibrium.

The argument is symmetric: It may be that during the mean reversion illusion, the price process follows a path below the one given by the higher a-priori mean reversion. When the disillusion occurs, this will cause an upward jump. The magnitude of upward jumps is more restricted than that of downward jumps for two reasons. Firstly, the majority of investors is likely to be risk-averse and therefore more inclined to sell in the case of a crash than to buy in the case of an upward jump. Secondly, most investors have no large cash position that can be unloaded onto the market in such an instance. They have to shift investments and restructure portfolios, which leads to a delay between the decision to buy and the actual purchase. However, in the case of a downward jump, investors who have decided to sell will do so instantaneously.

D. *Mean Reversion Disillusion and October 19, 1987*

If errors in the perception of mean reversion expectations played a role in the stock market crash of 1987, this would imply that there was an illusion and later a disillusion about the market's average a-priori mean reversion expectation. In the notation of Figure 2, we are looking for the points t and t_c and the related events. We cannot expect any particular news event to cause the illusion at time t, so it will be difficult to identify t. The point t_c is the point immediately before the crash.

The disillusion is caused by a piece of information that is relevant for mean reversion expectations and that surprises the public. Following the argument outlined in Section C that a mean reversion illusion is particularly likely to occur if hedges can be implemented that cannot be observed by other market-participants, we are searching for the disclosure of large hedge positions. According to the hypothesis, this implies that a group of active market participants was expecting a faster reversion than the average investor perceived and that this active group was hedged against this fast reversion.

The three days prior to October 19, 1987, are of prime interest in this respect. From Wednesday, October 14, to Friday, October 16, the U.S. stock market lost more than ten percent. The Dow Jones Industrial Average fell from 2,508 at closing on Tuesday to 2,246 at closing on Friday, the S&P 500 from 314 to 282 over the same time. The loss on Wednesday was three percent, on Thursday two percent, and on Friday five percent.

These drops can be attributed to fundamental reasons, namely to announcements concerning the simultaneous budget and trade balance deficits and to the House Ways and Means Committee's plans to eliminate tax benefits for takeovers. On Wednesday, October 14, the U.S. government announced that the trade deficit was about ten percent higher than expected. The dollar fell sharply in reaction, and this led to an expected decrease in foreign investment. At the same time, it became known that the House Ways and Means Committee filed legislation concerning takeovers (Brady et al. 1988, p. III-2f). Mitchell and Netter (1989) observed that the losses on the stock markets in reaction to these news items were largely confined to the U.S. market. This indicates that the losses where the result of revisions in fundamentals.

Portfolio insurance companies reacted by increasing their cash positions through sales of index futures. They sold 530 million dollars on Wednesday, 965 million dollars on Thursday, and 2.1 billion dollars on Friday, the latter being eleven percent of the total daily sales on the futures market (Brady et al. 1988, p. III-16). By the end of the week it became apparent to market participants that these sales were, by far, not sufficient to adjust the portfolio insurance positions adequately. The Brady Report mentions another eight billion dollars that were expected to be sold on the futures market. The implied volume of equities under portfolio insurance, 60 to 90 billion dollars, seems to have surprised the market. This may have been the event that disclosed the true a-priori mean reversion expectations of the market participants (Brady et al. 1988, p. 29).

The Brady Report explained the crash in part by the mere existence of portfolio insurance and associated program trading that cascaded in the crash. The view proposed here is quite different; it follows Black (1988). In this view, the unexpectedly high portfolio insurance volumes were *fundamental* information, not just a technical issue. They revealed that during the boom of 1987 a mean reversion illusion had occurred.

III. A Mean Reversion Model for Stock Returns

An intuitive way to think about mean reversion in stock returns is to assume that the returns process reacts to any deviation from its long-term mean. If the return is above the mean in one period, there is a force that pushes it downwards in following periods, if the return is below the mean, it is pushed upwards. Following Fama and French (1988), Jegadeesh (1990), and Metcalf and Hassett (1995), we use a diffusion model that contains a mean reversion term for the purpose of estimation using stock market data.

The mean return induces a certain stock price, denoted by ϑ_t, which can be interpreted as an estimator of the fundamental value of the stock or stock index. It is

$$\vartheta_t = S_0 e^{\mu\, t},$$

where S denotes the stock price. Consider the return process given by

$$\frac{dS_t}{S_t} = \mu dt + \lambda \frac{\vartheta_t - S_t}{S_t} dt + \sigma dW_t. \tag{1}$$

Here, the term $(\vartheta - S)/S$ measures the deviation of the return process from the long-term mean μ. The parameter $\lambda \geq 0$ controls the speed with which the return is pushed back to the mean μ. The average mean reversion time is $1/\lambda$ units of time. W_t is standard Brownian Motion. It is shown in the Appendix (Lemma A2) that the expected value of the price process satisfying (1) is

$$\mathbb{E} S_t = S_0 e^{\mu\, t} = \vartheta_t.$$

The process satisfying

$$d \log S_t = \tilde{\mu} dt + \lambda (\log \tilde{\vartheta}_t - \log S_t) dt + \sigma dW_t, \tag{2}$$

where $\tilde{\mu} = \mu - \sigma^2/2$ and $\tilde{\vartheta}_t = S_0 e^{\tilde{\mu} t}$ is a first-order approximation to (1). This is shown in the Appendix (Lemma A3). The solution to model (2),

$$\log S_t = \log S_0 + \tilde{\mu}\, t + \sigma \int_0^t e^{-\lambda(t-u)} dW_u, \tag{3}$$

is an Ornstein-Uhlenbeck process. Hence, (2) is a Vasicek-type model for log-prices (Vasicek 1977). The unconditional distribution of the log-price process is given by

$$\log S_t \sim \mathcal{N} \left(\tilde{\mu}\, t + \log S_0, \frac{\sigma^2}{2\lambda} \right); \tag{4}$$

the process is non-stationary. The higher the mean reversion λ, the smaller the variance because the process will stay within a narrower corridor around its mean with the same probability. For purposes of estimation, the conditional distribution of the log-returns $\log S_{t+1} - \log S_t$ given information through date t is relevant. It can be read directly from the model (2):

$$(\log S_{t+1} - \log S_t) \sim \mathcal{N} \left(\tilde{\mu} + \lambda (\log \tilde{\vartheta}_t - \log S_t), \sigma^2 \right). \tag{5}$$

To estimate the model, maximize the log-likelihood

$$L(\theta, \{S_t\}_{t=1,\dots,T}) = -\frac{T}{2}\log\sigma^2 - \frac{1}{2}\sum_{t=1}^{T}(r_{t+1} - \tilde{\mu} - \lambda(\log\tilde{\vartheta}_t - \log S_t))^2. \tag{6}$$

Here, T denotes the number of observations, $\theta = (\tilde{\mu}, \lambda, \sigma)'$ is the parameter vector, $r_t = \log S_t - \log S_{t-1}$ denotes the logarithmic returns, $\tilde{\vartheta}_t = S_0 e^{\tilde{\mu}\, t}$ as above. The model is estimated by numerical maximization of (6) using the 'dfpmin' routine from Press et al. (2002) and the 'fminunc' routine from the MATLAB optimization toolbox. Both implement a Quasi-Newton method with line search using analytical gradients and numerical Hessians. The derivatives are readily calculated from (6).

The unconditional distribution of the log-returns is given by

$$(\log S_{t+1} - \log S_t) \sim \mathcal{N}\left(\tilde{\mu}, \frac{\sigma^2}{2\lambda}(e^{\lambda} - 1) + \frac{\sigma^2}{2\lambda}e^{-2\lambda t}(1 - e^{-\lambda})\right),$$

so that for $t \to \infty$, the stationary distribution is obtained as

$$(\log S_{t+1} - \log S_t) \overset{t\to\infty}{\sim} \mathcal{N}\left(\tilde{\mu}, \frac{\sigma^2}{2\lambda}(e^{\lambda} - 1)\right).$$

Thus, the maximum likelihood estimates of $\theta = (\tilde{\mu}, \lambda, \sigma)'$ will be asymptotically normal and the usual statistical inference of maximum likelihood estimation applies.

The model can be criticized for many reasons. It is not a model of efficient markets; the stock price process (3) is not a martingale. Within this model framework, we will be able to provide an explanation of stock market crashes at the cost of having to accept local non-efficiency.

The mean return $\tilde{\mu}$ is not trivial to estimate (Merton 1980). Hence, the samples considered must be chosen carefully to make sure that the estimated mean return is relevant to the analysis. We will use a change-point detector and segmentations proposed in the Brady Report for this purpose.

The discrete time equivalent of the model is a first order autoregressive process, and we estimate it locally on different segments of our data set. Recently, Hillebrand (2004a, 2004b) has shown that neglecting parameter changes in autoregressive models results in strong biases in the estimates of the autoregressive parameters towards unit roots. Therefore, a local approach and a change-point detection study as carried out in Section IV is preferable to global estimation.

One might argue that when $\vartheta_t = S_0 e^{\mu t}$ is an estimator of the fundamental value, it does not make sense for an asset to trade far above or below this value. In other words, a non-negligible distance $\log \vartheta_t - \log S_t$ should not occur. White (1990) observed for the case of the 1929 stock market crash that during the boom that preceded the crash, fundamentals were very difficult to evaluate. This was mainly because many new companies entered the stock market that had virtually no dividend history. A similar case can be made for the internet boom at the turn of the century. The quality of an estimator for the fundamental value that uses any type of historical long term mean is questionable in situations like that. It seems reasonable, therefore, to allow a process to deviate from the estimated mean process.

IV. Mean Reversion and the Stock Market Crash of 1987 in Market Data

The data are daily closing prices of the S&P 500 index ranging from January 4, 1982 to December 30, 1991, which provides 2563 observations. The series was obtained from Datastream. All holidays that repeat the price of the previous day were deleted.

TABLE 1 ABOUT HERE.

TABLE 2 ABOUT HERE.

We expect movements in the parameter of actual mean reversion corresponding to the mean reversion illusion and disillusion around the stock market crash of 1987. As outlined in Section II, the investment decisions of market participants translate their mean reversion expectations into actual mean reversion. When the risk-averse majority learned during the week prior to the crash that they had adjusted their expected mean reversion to an understated mean reversion (expected and actual) of the risk-tolerant group, the risk-averse group adjusted their expectations to a much faster reversion and subsequently sold stock to move the price process to a level that reflected the average a-priori expected mean reversion. After the crash, the investment decisions reflected the new mean reversion expectations and thereby translated the faster expected reversion into faster actual reversion.

We will look at the disillusion, that is, the crash itself first. The hypothesis is that after the crash, we should see a faster mean reversion than before the crash.

A. *The Mean Reversion Disillusion*

The observations October 16, 1987, to October 26, 1987, are eliminated from the returns and the price series of the S&P 500. This excludes the crash itself and the large moves that followed from the estimation of the mean reversion after the crash. Then, model (2) is estimated for the 100, 200, ..., 1000 observations before and after the crash. (More precisely, before and after the gap.) Table 1 reports the estimations.

Campbell, Lo and MacKinlay (1997, p. 66) report significantly positive estimates of the first order autocorrelation coefficient of stock returns, which is the discrete time analogue to $-\lambda$ in model (2). That is, if there is mean reversion in the market, we would expect the first order autocorrelation coefficient to be negative. The simple autocorrelation does not account for a non-zero mean in returns, however, as model (2) does, and therefore the positive autocorrelation can be interpreted as an aversion from the zero mean. Estimations of model (2) on a rolling window on the return series of the Dow Jones Industrial Average between 1901 and 2003 (provided by Dow Jones & Company), which are not reported here for reasons of brevity, resulted in consistently positive estimates of λ, that is, mean reversion. Therefore, we can make a case for a one-sided test situation. However, all results are strong enough to reject the null hypotheses of the two-sided tests, so these are reported in Table 1.

The findings clearly support the hypothesis. Up to 700 points before and after the crash, there is an increase in mean reversion. All estimations of the mean reversion parameter λ in these samples are significant on the two-sided 0.95 confidence level, and four out of seven are significant on the two-sided 0.99 confidence level. As the sample size increases from sample to sample, the mean return is measured differently each time. Except for the 100 and 400 points samples, a slightly higher mean return before the crash than afterwards is measured.

These estimates are not independent; therefore model (2) is also estimated on the corresponding opposite intervals of length 200. That is, the samples are *crash-1000* to *crash-800* and *crash+800* to *crash+1000*, then *crash-800* to *crash-600* and *crash+600* to *crash+800*, and so on up to *crash-200* to *crash* and *crash* to *crash+200*. Table 2 reports the estimates. The mean return concept applied here is a moving 200-day average without overlap. The estimates of the first row are identical to those of the second row of Table 1; the other estimates are not comparable to those of Table 1. With the single exception of the intervals corresponding to $n_i = 800$, these estimates also support the hypothesis.

B. *The Mean Reversion Illusion*

As outlined in Section II, one of the defining characteristics of a mean reversion illusion is that mean reversion expectations can be implemented without being noticed by other market participants, for example, by the use of synthesized options. Furthermore, the fundamental value of the assets in question is hard to evaluate in this situation. Therefore, the beginning of the illusion is not as obvious to locate as the disillusion.

In the notation of Figure 2, we search for the time t. That is, we search for a point in the return series before the crash where mean reversion expectations change from relatively high to relatively low. Since expectations cannot be measured, actual mean reversion is used as a proxy. According to the hypothesis, the segment with slower mean reversion should lead up to the crash.

The Brady Report locates the beginning of the bull market that led up to the crash in 1982. The contributing factors are described as *"... continuing deregulation of the financial markets; tax incentives for equity investing; stock retirements arising from mergers, leveraged buyouts and share repurchase programs; and an increasing tendency to include 'takeover premiums' in the valuation of a large number of stocks"* (Brady et al. 1988, p. 9, I-2). The level of the U.S. stock market by the end of 1986 is described as high, but not unprecedented, in terms of price-earnings ratios. The appreciation from January 1987 through August 1987, however, *"... challenged historical precedent and fundamental justification"* (Brady et al. 1988, p. 9, I-2).

Using this segmentation as a guideline, model (2) is estimated on the segments 01/02/82–12/30/86 and 01/02/87–10/15/87. That is, we set t = January 2, 1987. Model (2) is then estimated on the 1987-segment with the mean return fixed at the estimate from the period 1982–1986. Figure 3 illustrates the estimations. The estimate of the mean reversion parameter on the 1982–1986 segment is significant at the one-sided 0.95 significance level. On the 1987 segment, mean reversion is negligible. Thus, following the segmentation suggested by the Brady Report, we find a significant change between the segments. This method is somewhat arbitrary, in particular with regard to the appropriateness of the 1982-86 mean during 1987. Therefore, we seek support for the segmentation suggested by the Brady Report using a statistically rigorous change-point detector.

FIGURE 3 ABOUT HERE.

FIGURE 4 ABOUT HERE.

In order to detect a change in the parameter vector of model (2), a Generalized Likelihood Ratio (GLR) scheme is applied as a change-point detector (Lai 1995). Let $S = \{S_t\}_{t\in\{1,\ldots,T\}}$ be the time series of index prices. The GLR scheme sets a change-point at

$$(7) \qquad \inf_{t\in\{1,\ldots,T\}} \left\{ \max_{1\le k\le t} \sup_{\theta\in\Theta} \left[\sum_{i=k}^{t} \log \frac{f_\theta(S_i|S_1,\ldots,S_{i-1})}{f_{\theta_0}(S_i|S_1,\ldots,S_{i-1})} \right] > c \right\},$$

where T is the number of observations, Θ is the open parameter set, f_θ is the probability density given the parameter vector θ, θ_0 is the parameter vector of the null hypothesis, and c is an a-priori constant. There is no analytical expression or distribution result for the critical value c, so that it must be found by simulation methods. The problem is substantially simplified by the fact that we search for a single change-point.

Problem (7) is decomposed into the following steps. On a baseline segment of the first m points of the series model (2) is estimated, providing the null hypothesis $\hat\theta_0 = (\hat\mu_0, \hat\lambda_0, \hat\sigma_0)'$. Then, (2) is estimated on every single subseries $\{S_1,\ldots, S_j\}$, $j = m+1,\ldots, T$, resulting in a series of $\hat\theta_j$ maximizing the likelihood functions (6) of the subseries. From this series of parameter estimates, the probability densities $f_{\hat\theta_j}(S_j|S_1,\ldots,S_{j-1})$ for every $j = m+1,\ldots, T$ are obtained and

$$Z_j := \log \frac{f_{\hat\theta_j}(S_j|S_1,\ldots,S_{j-1})}{f_{\hat\theta_0}(S_j|S_1,\ldots,S_{j-1})}$$

is stored. From the resulting series $\{Z_j\}_{j\in\{m+1,\ldots,T\}}$, the statistics series

$$(8) \qquad \xi_t = \max_{m+1\le k\le t} \sum_{j=k}^{t} Z_j,\ t = m+1,\ldots, T$$

is calculated. As we search for a single change-point, it makes sense to plot the $\{\xi_t\}$ series. Figure 4 shows the series when the baseline distribution is estimated on the S&P 500 observations January 2, 1982, through December 30, 1985. The series is then calculated for the observations January 2, 1986 through October 15, 1987. Using this method, the estimated mean is allowed to change with every single observation that is added. The estimated parameters move away from the estimated baseline parameters at two distinct speeds as the sample size increases. This is the interpretation of the two trends in the series that can be distinguished in Figure 4. The trend break is at the turn of the years 1986 to 1987. This supports the segmentation suggested by the Brady Report.

Critical values c for ξ_t are obtained in a simulation: 1,000 time series are generated according to model (2) with the parameters obtained from the estimation of the baseline sample period January 2, 1982, through December 30, 1985. This baseline sample consists of 1,012 observations. The sample period January 2, 1986, through October 15, 1987, for which the detector series ξ_t in Figure 4 is plotted, consists of 454 observations. Therefore, for each of the 1000 simulated time series we generate 1466 observations. On the first 1,012 observations of each series, model (2) is estimated. Then, for each series the detector statistic ξ_t is calculated for the remaining 454 observations, yielding 454,000 observations of the detector statistic. The significance levels reported in Figure 4 are the quantiles of these 454,000 observations.

C. *Estimations of the Size of the Crash*

Given these estimations of the mean reversion illusion and using only data available on October 16, 1987, what would have been the estimate of the magnitude of a possible crash? More precisely, given that the mean reversion illusion occurred at the beginning of the year 1987, roughly 200 days before October 16, and given that the mean reversion disillusion occurs on October 16, what is the distance in the paths of the price series that must be offset? With view to Figure 2, we are interested in the distance in the trajectories that is shaded black, measured at the point immediately before the crash. Note that we do not estimate the time of the crash, the disillusion is assumed to happen on October 16. for whatever reason.

To answer this question, model (2) is simulated with the estimated parameters reported in Figure 3. Ten thousand paths of length 200 of model (2) under the parameter regime obtained from the 1982–1986 segment are generated, using the value 246.45 of the S&P 500 on January 2, 1987, as the starting point. If a mean reversion illusion occurred in January 1987, it lasted for about 200 days up to October 16, 1987. That is, without the illusion the process would have continued for another 200 days under the old regime. The simulation thus gives an estimate of the distribution of the index value $S_{\text{no illusion}}(200)$ on October 16, 1987, without mean reversion illusion. The actual value of the S&P 500 at the closing of October 15, 1987, was 298.08. The sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(298.08)$ is an estimate of the distribution of the magnitude of the crash.

Table 3 shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(298.08)$ in Panel A. Although the distribution is negatively skewed, there is still a substantial probability for an upward jump because even under the regime with stronger mean reversion there are a number of paths that end up above 298.08 after 200 days. The probability of a crash of minus 20 percent or more is greater than seven percent. The probability of a correction of minus ten percent or more is greater than 40 percent.

To put the endpoint of 298.08 into perspective, model (2) is simulated for 10,000 sample paths under both parameter regimes, the 1982–1986 period ($S_{\text{no illusion}}$) and the 1987 period (S_{illusion}). Table 3 shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(S_{\text{illusion}}(200))$ in Panel B. Even after only 200 days the difference in the mean reversion parameter λ results in substantial distances in the trajectories and thus substantial probabilities for large jumps when a mean reversion disillusion occurs. The probability of a crash of more than 20 percent is higher than 10 percent under this distribution.

TABLE 3 ABOUT HERE.

These sample distributions are calculated under the assumption that if the mean reversion illusion had not occurred, the path of the underlying white noise process could have been different from the one that was realized between January 2, 1987, and October 15, 1987. One might argue that the arrival of fundamental information that makes up the noise part would have been the same in either case. Under this assumption, we can reconstruct the white noise process between January 2, 1987, and October 15, 1987, from model (2) by

$$\hat{\varepsilon}_t = \frac{1}{\hat{\sigma}}\left[\hat{\mu} + \hat{\lambda}\log\hat{\vartheta}_t + (1-\hat{\lambda})\log S_t - \log S_{t-1}\right]$$

using the parameter estimates from the 1987 segment.

Plugging $\hat{\varepsilon}_t$ back into the model with the parameters from the 1982–1986 segment generates a point estimate for $S_{\text{no illusion}}(200)$ and thus a point estimate for the magnitude of the crash. According to this method, we have $S_{\text{no illusion}}(200) = 273.78$ and thus,

$$\log(S_{\text{no illusion}}(200)) - \log(298.08) = -0.085,$$

a correction of minus 8.5 percent.

Thus, we have three different forecasts of the magnitude of the stock market crash using only data available on October 16, 1987. A point estimate using the estimated white noise path from model (2) predicts a -8.5 percent correction. Two density estimates, one with the endpoint of the price process fixed at the actual value on October 16, and one where this endpoint is simulated as well, result both in a 7.5 percent probability of a crash of 20 percent or more.

TABLE 4 ABOUT HERE.

V. A Note on the Stock Market Crash of 1929

The stock market crash of 1929 can not be explained by a mean reversion illusion and disillusion. The knowledge about the hedge portfolio of the Black-Scholes analysis was not available and, therefore, it was not possible to implement mean reversion expectations in the same way as 1987. There were a number of coarser dynamic hedging strategies, for example, stop-loss orders, but these were easy to observe. Estimation of model (2) in analogy to Table 2 on the daily closing prices of the Dow Jones Industrial Average shows that there was no change in mean reversion before and after the crash. The results are reported in Table 4. The observations October 26, 1929, through December 17, 1929, were deleted from the series because it took almost two months before the Dow returned to normal daily changes. The results of Table 4 are not qualitatively sensitive to the choice of this gap, and similar results were obtained for only ten deleted days. Except for the single instance corresponding to $n_i = 400$, the change in the mean reversion parameter λ does not have the right sign to support any mean reversion explanation. The estimates are much less significant than for the crash in 1987, significance is found at the one-sided 0.95 level in only two instances *before* the crash.

VI. Conclusions

Errors in the perception of mean reversion expectations can cause stock market crashes. This view was proposed by Black (1988) to explain the stock market crash of 1987. It is closely related to the models proposed by Grossman (1988), Gennotte and Leland (1990), and Jacklin, Kleidon, and Pfleiderer (1992). These studies rejected the widely held view that the mere existence of portfolio insurance and cascading program trading caused the crash. Instead, they proposed that the disclosure of large hedge positions came as a surprise and as a piece of fundamental information to the market.

When the average a-priori expectation of mean reversion is relatively high but market participants can hedge against a fast reversion and these hedge positions are not public information, a situation may occur that is called *mean reversion illusion* in this paper. In that case, a large group A of investors adapts their high a-priori mean reversion expectations to the low expectations that they infer from the market behavior. Investors A do not know that the mean reversion expectations of those investors B already active on the market are about as high but hedged, for example, by synthesized put options that cannot be distinguished from stock sales and purchases caused by fundamental information. Investors A adapt their expectations to those that they believe are B's, and the stock price process behaves according to a lower mean reversion. If the true a-priori expectations of group B become known, for instance, because a surprisingly large hedge position is disclosed, *mean reversion disillusion* for group A sets in. It is now clear that the stock price process followed a path that did not properly reflect the average a-priori mean reversion expectations of all market participants. The process has to be set into a position as if the illusion had not happened. This crash is a correction in both trajectories and process parameters, and the change in trajectories can be of substantial magnitude.

We explore Black's hypothesis empirically. Estimating an Ornstein-Uhlenbeck model for stock returns with mean reversion on daily data of the S&P 500 index, the stock market crash of 1987 is examined in detail. Using the periods of the "sound" bull market 1982–1986 and the exaggeration in 1987 identified in the report of the Brady Commission, it is shown that mean reversion in 1987 was much slower than during the period 1982–1986. A generalized likelihood ratio test shows a significant change in the parameters of the model in early 1987. This supports the segmentation in the Brady Report and the hypothesis of a mean reversion illusion.

After the 1987-crash, mean reversion was significantly stronger than before. This supports the hypothesis that in fact a mean reversion disillusion occurred.

The cause of the disillusion can be identified as the surprisingly large volumes of equities under portfolio insurance, which became known during the week prior to the crash.

Simulations of the model with the estimated parameters of the two segments 1982–1986 and 1987 show that a crash of 20 percent or more had a probability of more than seven percent. A correction of minus 10 percent or more had a probability of more than 40 percent.

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Notes

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LEMMA A1: *The result of the market transactions of investors buying a stock and simultaneously replicating a put option on it is positive. That is, the purchases are greater than the sales.*

Proof. This will be shown here for the case of a European put option. According to the Black-Scholes model, the replicating portfolio of a European put on one share of the underlying stock consists of a short position of $|\Delta(t)|$. Δ is the sensitivity of the option to changes in the price of the underlying given by

$$\Delta(t) = \Phi(d_1) - 1 < 0,$$
$$d_1 = \frac{\log \frac{S}{X} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Φ is the cumulative distribution function of the standard normal distribution, S is the stock price, X is the exercise price of the put option, r is the risk-free interest rate, $T-t$ is the time to maturity and σ^2 is the variance of the stock price.

The proceeds from the short position are invested and gain the risk-free interest rate r. Assume that the investor hedges every single stock that she buys. Her position $P(t)$ is (in terms of inventories)

$$P(t) = (S - S\Delta(t)e^{rt}) \cdot n,$$

where n denotes the number of shares. The assertion made here is equivalent to

$$\frac{1}{n}P(t) > 0 \quad \Longleftrightarrow \quad S > S\Delta(t)e^{rt}.$$

Now, it is obvious that

$$1 + e^{-rt} > \Phi(d_1),$$

as the exponential function is strictly positive on $\mathbb{R}$ and $\Phi(d_1) \in [0,1]$ as it is a probability. It follows that

$$1 > (\Phi(d_1) - 1)\, e^{rt} \quad \Longrightarrow \quad 1 > \Delta(t)e^{rt}.$$

Multiplying with $S > 0$ proves the assertion. □

LEMMA A2: *The expected value of the price process solving model (1) is given by* $\vartheta_t = S_0 e^{\mu t}$.

Proof. Rewrite (1) to

$$dS_t = (\mu - \lambda) S_t dt + \lambda \vartheta_t dt + \sigma S_t dW_t,$$

and solve the associated homogeneous equation

$$dX_t = (\mu - \lambda) X_t dt + \sigma X_t dW_t,$$

to obtain $X_t = \exp[(\mu - \lambda - \sigma^2/2)t + \sigma W_t]$. Then the solution to (1) is given by

$$\begin{aligned} S_t &= X_t \left(S_0 + \int_0^t (X_u)^{-1} \lambda \vartheta_u du \right) \\ &= S_0 \exp\left[\left(\mu - \lambda - \frac{\sigma^2}{2} \right) t + \sigma W_t \right] \left(1 + \lambda \int_0^t \exp\left[\left(\lambda + \frac{\sigma^2}{2} \right) u - \sigma W_u \right] du \right) \end{aligned}$$

Taking expectations, we obtain

$$\begin{aligned} \mathbb{E} S_t &= S_0 e^{(\mu - \lambda - \frac{\sigma^2}{2})\, t} \left(\mathbb{E} e^{\sigma W_t} + \lambda \int_0^t e^{(\lambda + \frac{\sigma^2}{2})\, u} \mathbb{E} e^{\sigma (W_t - W_u)} du. \right) \\ &= S_0 e^{\mu t - \lambda t} + S_0 e^{\mu t - \lambda t} \lambda \int_0^t e^{\lambda u} du \\ &= S_0 e^{\mu\, t}. \end{aligned}$$

□

LEMMA A3: *Model (2) is a first-order approximation to model (1).*

Proof. The mean reversion term in the model (1) can be rewritten as

$$\lambda \frac{\vartheta_t - S_t}{S_t} dt = \lambda \left(\frac{\vartheta_t}{S_t} - 1 \right) dt.$$

Denote $r := \vartheta_t / S_t - 1$, then

$$1 + r = \frac{\vartheta_t}{S_t}$$

and as $\log(1+r) \doteq r$ we have a first-order equivalent representation

$$\lambda \frac{\vartheta_t - S_t}{S_t} dt \doteq \lambda \log \frac{\vartheta_t}{S_t} dt = \lambda(\log \vartheta_t - \log S_t) dt.$$

From Ito's Lemma, we have

$$d \log S_t = \frac{dS_t}{S_t} - \frac{\sigma^2}{2} dt.$$

Define $\tilde{\mu} = \mu - \sigma^2/2$ and $\tilde{\vartheta}_t := S_0 \exp(\tilde{\mu}\, t)$. Then there is a first-order equivalent of the model (1) given by (2):

$$\log S_t = \log S_0 + \tilde{\mu}\, t + \lambda \int_0^t (\log \tilde{\vartheta}_u - \log S_u) du + \sigma W_t. \tag{A1}$$

□

Table 1. Estimation of model (2) on sample periods before and after the 1987 stock market crash. The observations from October 16, 1987, through October 26, 1987, were deleted from the series. The numbers in parentheses are quasi-maximum-likelihood standard errors according to White (1982). The estimations of the mean returns and standard deviations are significant according to all common confidence levels with the single exception of the mean return of 100 days before the crash. For the mean reversion parameter λ, estimates that are significant according to the two-sided 0.95 confidence level are marked with a single asterisk, the double asterisk denotes significance according to the two-sided 0.99 confidence level. Mean reversion clearly increased after the crash.

	n days before Oct. 16, 1987			**n days after Oct. 26, 1987**		
n	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\sigma}$
100	0.000483	**0.010565**	0.009632	0.001401	**0.16636****	0.016239
	(0.000613)	(0.011637)	(0.000827)	(0.000114)	(0.061999)	(0.001703)
200	0.001172	**0.005814**	0.010373	0.000851	**0.077153***	0.013997
	(0.000401)	(0.005729)	(0.000581)	(7.6e-5)	(0.03205)	(0.001075)
300	0.001004	**0.024993**	0.010127	0.000683	**0.052885***	0.012292
	(0.000133)	(0.015156)	(0.000592)	(4.9e-5)	(0.021259)	(0.000859)
400	0.000672	**0.016753**	0.009907	0.000713	**0.052938****	0.011204
	(0.000110)	(0.008603)	(0.000496)	(3.1e-5)	(0.019714)	(0.000722)
500	0.001062	**0.013118**	0.009516	0.000769	**0.044117****	0.010935
	(0.000103)	(0.008098)	(0.000425)	(2.9e-5)	(0.014459)	(0.000699)
600	0.000834	**0.015902***	0.009034	0.000715	**0.033246****	0.01056
	(6.4e-5)	(0.006800)	(0.000379)	(2.8e-5)	(0.011698)	(0.000614)
700	0.000951	**0.017962***	0.008701	0.000663	**0.023374***	0.010297
	(4.7e-5)	(0.007320)	(0.000344)	(3.2e-5)	(0.009914)	(0.000554)
800	0.000801	**0.015205***	0.008523	0.000541	**0.008240**	0.010536
	(4.3e-5)	(0.0059002)	(0.000314)	(8.1e-5)	(0.005264)	(0.000496)
900	0.000728	**0.010165***	0.008546	0.000563	**0.009867***	0.010475
	(5.1e-5)	(0.004539)	(0.000289)	(5.5e-5)	(0.005016)	(0.000451)
1000	0.000578	**0.003738**	0.008442	0.000547	**0.009902***	0.010221
	(9.6e-5)	(0.002642)	(0.000269)	(4.5e-5)	(0.004796)	(0.000420)

Table 2. Estimation of model (2) on sample periods before and after the 1987 stock market crash. The observations from October 16, 1987, through October 26, 1987, were deleted from the series. The numbers in parentheses are quasi-maximum-likelihood standard errors according to White (1982). The estimations of the mean returns and standard deviations are significant according to all common confidence levels. For the mean reversion parameter λ those estimates that are significant according to the two-sided 0.95 confidence level are marked with a single asterisk, the double asterisk denotes significance according to the two-sided 0.99 confidence level.

		day n_i through day n_{i-1}			**day n_{i-1} through day n_i**		
	n_i	**before Oct. 16, 1987**			**after Oct. 26, 1987**		
i	$(n_0 = 1)$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\sigma}$
1	200	0.001172	**0.005814**	0.010373	0.000851	**0.077153***	0.013997
		(0.000401)	(0.005729)	(0.000581)	(7.6e-5)	(0.032050)	(0.001075)
2	400	0.000672	**0.016753**	0.0099065	0.000713	**0.052938****	0.011204
		(0.000110)	(0.008603)	(0.000496)	(3.1e-5)	(0.019714)	(0.000722)
3	600	0.000833	**0.015902***	0.009034	0.000714	**0.033246****	0.010560
		(6.4e-5)	(0.006800)	(0.000379)	(2.8e-5)	(0.011698)	(0.000614)
4	800	0.000801	**0.015205***	0.008523	0.000541	**0.008240**	0.010536
		(4.3e-5)	(0.005900)	(0.000314)	(8.1e-5)	(0.005264)	(0.000496)
5	1000	0.000578	**0.003738**	0.008442	0.000546	**0.009902***	0.010221
		(9.6e-5)	(0.002642)	(0.000268)	(4.5e-5)	(0.004796)	(0.000420)

Table 3. Panel A shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(298.08)$, the latter value is that of the S&P 500 at the close of October 15, 1987. This gives an estimate of the distribution of the magnitude of the crash. The probability of a downward jump of 20 percent or more was more than seven percent. Panel B shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(S_{\text{illusion}}(200))$ when 10,000 Brownian sample paths of length 200 are evaluated under both regimes, that of the 1982–1986 period ($S_{\text{no illusion}}$) and that of the 1987 period (S_{illusion}). This shows that the difference in the mean reversion parameter leads to substantial probabilities for large moves when a mean reversion disillusion occurs.

Panel A		Panel B	
r_i	$\mathbb{P}(r_i - 0.10 \leq r < r_i)$	r_i	$\mathbb{P}(r_i - 0.10 \leq r < r_i)$
		-0.5	0.0009
		-0.4	0.0053
-0.3	0.0029	-0.3	0.0221
-0.2	0.0753	-0.2	0.0751
-0.1	0.3652	-0.1	0.1572
0	0.4332	0	0.2333
0.1	0.1160	0.1	0.2297
0.2	0.0072	0.2	0.1687
0.3	0.0001	0.3	0.0775
		0.4	0.0244
		0.5	0.0052
		0.6	0.0006

Table 4. Estimation of model (2) on sample periods before and after the 1929 stock market crash. The observations from October 26, 1929, through December 17, 1929, were deleted from the series. The numbers in parentheses are quasi maximum likelihood standard errors according to White (1982). The estimations of the mean returns and standard deviations are significant according to all common confidence levels except for the mean return estimation of the 200 days before and after the crash. For the mean reversion parameter λ those estimates that are significant according to the one-sided 0.95 confidence level are marked with a single asterisk. The mean reversion theory cannot explain the crash of 1929.

		day n_i through day n_{i-1}			**day n_{i-1} through day n_i**		
	n_i	**before Oct. 26, 1929**			**after Dec. 17, 1929**		
i	$(n_0 = 1)$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\sigma}$
1	200	9.6e-5	**0.014254**	0.013919	-0.000230	**0.012919**	0.015425
		(0.000546)	(0.014218)	(0.001097)	(0.000550)	(0.010533)	(0.000980)
2	400	0.001116	**0.003449**	0.012798	-0.001015	**0.008857**	0.016447
		(0.000364)	(0.003398)	(0.000739)	(0.000293)	(0.005738)	(0.000634)
3	600	0.000990	**0.009946**	0.011623	-0.001654	**0.004993**	0.021101
		(0.000149)	(0.007427)	(0.000558)	(0.000496)	(0.005851)	(0.001057)
4	800	0.000950	**0.010573**∗	0.010688	-0.001663	**0.008240**	0.023915
		(8.3e-5)	(0.006111)	(0.000471)	(0.000273)	(0.003719)	(0.000906)
5	1000	0.000751	**0.005945**∗	0.010145	-0.001415	**0.005034**	0.025005
		(8.6e-5)	(0.003410)	(0.000401)	(0.000224)	(0.003759)	(0.000881)

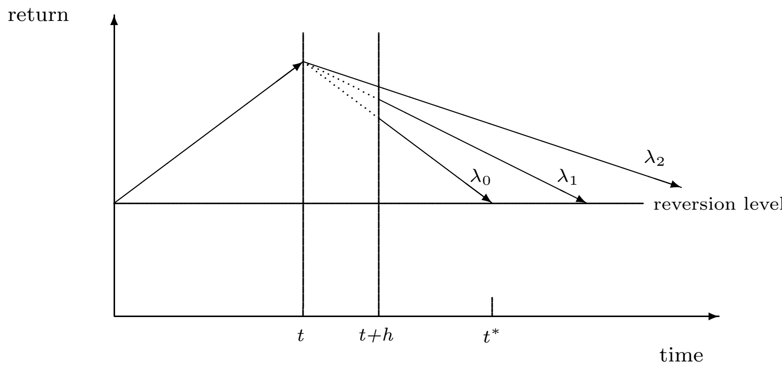

Figure 1. The development of mean reversion expectations.

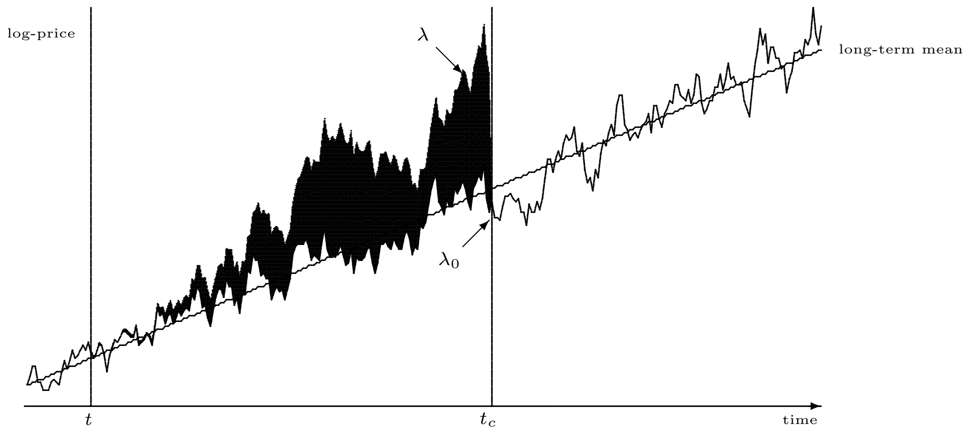

Figure 2. The mean reversion illusion between times t and t_c and the resulting difference in mean reversion velocities λ and λ_0 drive the log-price process above the λ_0-level. The difference in the trajectories is shaded black and gives the potential crash at every point in time.

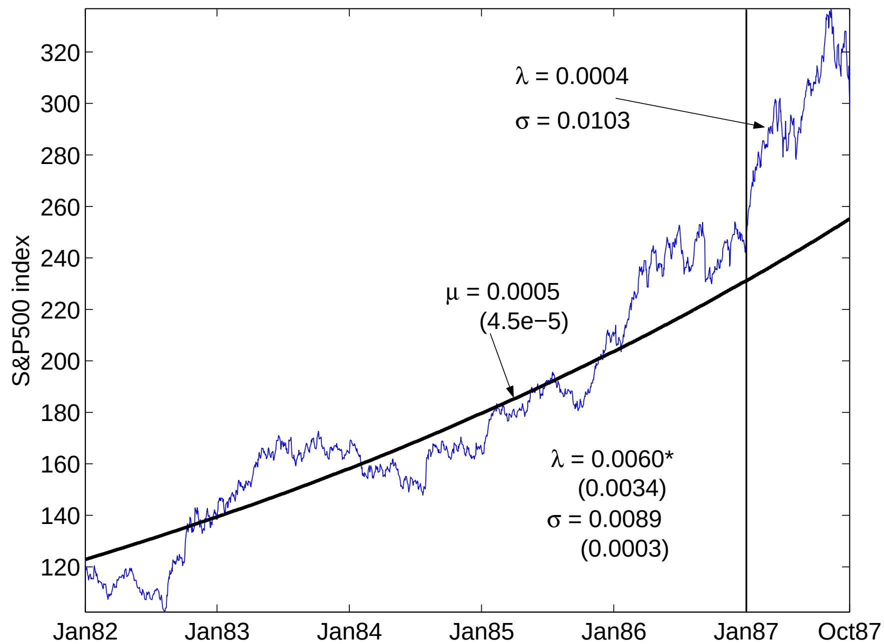

Figure 3. The bull market January 1982 to October 15, 1987 as seen in the S&P 500. Using the segmentation of the Brady Report, we estimate model (2) on the period January 1982 to December 1986 and January 1987 to October 15, 1987. The numbers in parentheses are standard errors according to White (1982). Assuming that the mean of the 1982-86 period did not change in 1987, we find a significantly weaker mean reversion in 1987.

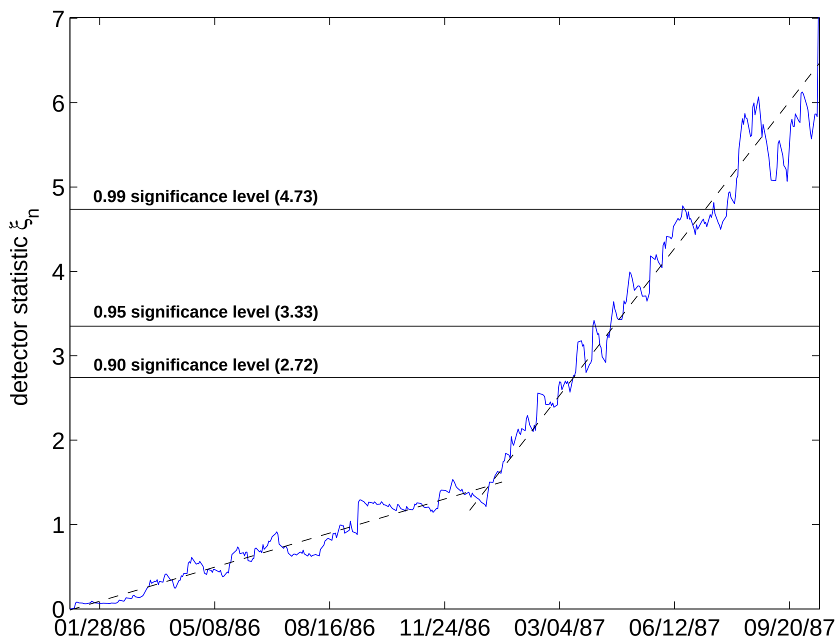

Figure 4. Change-point detector statistic series $\{\xi_n\}$ as given by Equation (8). The baseline parameter vector θ_0 was estimated on the segment January 2, 1982 through December 30, 1985. The detector statistics series was then calculated for the observations January 2, 1986 through October 15, 1987. Two distinct trends can be observed in the statistic. This means that the estimated parameters move away from the estimated baseline parameters by two distinct speeds as the sample size increases. The trend break is almost exactly at the turn of the years 1986 to 1987, in line with the periods as given by the Brady Report. The significance levels were obtained by simulation of the statistic.