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On the viscosity solutions of a stochastic differential utility problem

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Abstract

We prove existence, uniqueness and gradient estimates of stochastic differential utility as a solution of the Cauchy problem for the following equation in $\mathbb{R}^3$:

$$\partial_{xx}u + u\partial_y u - \partial_t u = f(\cdot, u),$$

where f is Lipschitz continuous. We also characterize the solution in the vanishing viscosity sense.

Keywords: Bernstein's method; Burgers' equation; Stochastic differential utility

1. Introduction

In this paper we consider the following Cauchy problem:

$$u_{xx}(z) + u(z)u_y(z) - u_t(z) = f(z, u(z)), \quad z \equiv (t, x, y) \in]0, T] \times \mathbb{R}^2, \tag{1}$$

$$u(0, \cdot) = g \quad \text{in } \mathbb{R}^2, \tag{2}$$

where, as usual, $u_x = \partial_x u$ and we assume f: $]0, T] \times \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ globally Lipschitz continuous. This problem has been recently considered in

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mathematical finance. Antonelli et al. [2] introduced a new model for agents' decision under risk, in which the utility function is the solution to (1)–(2). We mention that (1) also arises when studying nonlinear physical phenomena such as the combined effects of diffusion and convection of matter (cf. [13]).

Here we prove the existence of a viscosity solution u of (1)–(2) in the sense of the User's guide [11], and we characterize it in the vanishing viscosity sense. In other words, we show that u is the limit, uniform on compacts of $[0,T]\times\mathbb{R}^2$ as $\varepsilon\to 0^+$, of the family (u^ε) of solutions to the regularized Cauchy problem

$$v_{xx}+\varepsilon^2 v_{yy}+vv_y-v_t=f(\cdot,v)\quad \text{in }]0,T]\times\mathbb{R}^2, \tag{3}$$

$$v(0,\cdot)=g\quad \text{in } \mathbb{R}^2. \tag{4}$$

This result allows to study the properties of u in the framework of Sobolev spaces and it has been used in the recent papers by Citti et al. [8,9] to investigate the regularity of u. In particular, in [9], conditions are given for u to be smooth.

Before stating our main theorem, we introduce some notations. We set

$$\bar{T}=2(4k_1+\max\{1,2k_2\})^{-1}, \tag{5}$$

where k_1 is the Lipschitz constant of $f=f(t,x,y,v)$ w.r.t. the variables y,v and k_2 is the Lipschitz constant of $g=g(x,y)$ w.r.t. y. We aim to prove the following.

Theorem 1.1. *Let* $0<T<\bar{T}$. *There exists a unique viscosity solution* u *of problem* (1)–(2) *such that*

$$\begin{aligned}|u(t_1,x_1,y_1)-u(t_1,x_2,y_2)|&\leqslant C_0(|x_1-x_2|+|y_1-y_2|),\\ |u(t_1,x_1,y_1)-u(t_2,x_1,y_1)|&\leqslant C_0(1+|(x_1,y_1)|)|t_1-t_2|^{\frac{1}{2}}\end{aligned} \tag{6}$$

for every $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$, $t_1,t_2\in[0,T]$, *where* C_0 *is a positive constant which depends on* k_1 *and* k_2. *For every* $\varepsilon\in]0,1[$, *the regularized problem* (3)–(4) *has a unique classical solution* u^ε *for which* (6) *holds with* C_0 *independent of* ε. *Moreover*, (u^ε) *converges to* u *as* ε *goes to zero*, *uniformly on compacts of* $[0,T]\times\mathbb{R}^2$.

In spite of the similar terminology, the concepts of viscosity and vanishing viscosity solution are not, in general, equivalent. For first-order problems, a connection between these two notions has been shown by Crandall et al. [10] and Lions [22]. In the case of linear degenerate elliptic PDEs, the relationship with the notion of distributional solutions has been studied by Lions [22] and Ishii [18]. We also refer to Bardi and Capuzzo Dolcetta [3].

Due to the global estimate (6), the uniqueness part of Theorem 1.1 is not unexpected. The uniqueness of viscosity solutions to fully nonlinear second-order PDEs has been investigated by several authors including Crandall, Ishii, Jensen, Lions, Nunziante, Souganidis, Trudinger (see, e.g., [11,17,20,22,25,29]). These results require some structural conditions on the equation which do not fit for (1).

One of the main characteristics of Eq. (1) is the mixed parabolic–hyperbolic feature due to the lack of diffusion in the y-direction. We remark explicitly that (1) includes the Burgers' equation in the case $g = g(y)$ and $f = 0$. It is classical to prove the existence of solutions of this kind of problems, by adding a vanishing diffusion term as in (3), trying to obtain ε-uniform estimates of u^ε. This can be usually achieved by the Bernstein's method [5], differentiating the equation and by using the maximum principle to estimate the gradient of u^ε. Yet this method or more sophisticated versions of it (cf. Barles [4]) do not seem to work in our setting since the nonlinearity in (1) is not monotone and we allow growths at infinity. From a PDE viewpoint, these features seem to be non-standard. Moreover, since (1) is a degenerate second-order equation, regularity results proved by Caffarelli and Cabre [7], Trudinger [28], Ishii and Lions [19], Bian and Dong [6], Wang [31,32] do not apply.

Here we present a probabilistic technique which appears to be natural for the problem. We construct an appropriate system of stochastic differential equations that are related to our PDE. By proving the existence and uniqueness for the stochastic system, we deduce the existence of the solution u and the estimate on the gradient. More precisely, we consider a complete probability space $(\Omega, \mathscr{F}, P)$, on which two independent one-dimensional standard Brownian motions B, W are defined. We endow this space with the family of σ-algebras $\{\mathscr{F}_t\}_{t\in[0,T]} \subseteq \mathscr{F}$ generated in the following manner:

$$\mathscr{N} = \{P\text{-null sets of } \mathscr{F}\}, \quad \mathscr{G}_t = \sigma(W_s, B_s, s \leqslant t),$$
$$\mathscr{F}_t^+ = \bigcap_{s>t} \mathscr{G}_s, \quad \mathscr{F}_t = \sigma(\mathscr{F}_t^+ \cup \mathscr{N}).$$

In this way $\{\mathscr{F}_t\}_{t\in[0,T]}$ is a filtration ($\mathscr{F}_s \subseteq \mathscr{F}_t$ for $s \leqslant t$) that satisfies the "usual hypotheses" (cf. [27]). Chosen a constant $\varepsilon \in [0, 1[$, we consider the following forward–backward system:

$$Y_t^\varepsilon = y_0 + \int_0^t V_s^\varepsilon \, ds + \varepsilon W_t, \tag{7}$$

$$V_t^\varepsilon = E(g(B_T, Y_T^\varepsilon) - \int_t^T f(s, B_s, Y_s^\varepsilon, V_s^\varepsilon) \, ds | \mathscr{F}_t). \tag{8}$$

We say that (7)–(8) is solvable if there exists a pair of *adapted* and integrable processes $(Y^\varepsilon, V^\varepsilon)$ that verify the equations P—a.s. We stress that even under global Lipschitz assumptions, the solution of (7)–(8) may not exist globally in time. Various authors [14,16,23,26] studied conditions to have existence and uniqueness in an arbitrary time interval. Those methods do not apply in our case. Indeed, the first two results are based on monotonicity conditions of the coefficients that are not verified here, while the monotonicity conditions introduced by Pardoux and Tang [26] impose an analogous restriction of the time interval. The method adopted by Ma et al. [23], based on the PDE correspondence, instead is applicable only within the framework of Ladyzhenskaya et al. [21] for semilinear and quasilinear parabolic PDEs.

Correspondingly, it is well-known that, even for smooth initial datum g, the solution of (1)–(2) may develop discontinuities in finite time. In the framework of scalar conservation laws, this problem is usually overcome by interpreting the equation in the distributional sense. For instance, we refer to Escobedo et al. [13] for a non-local existence and uniqueness theory for (1)–(2) with bounded and integrable data. In a more general setting, existence and uniqueness results go back to Vol'pert and Hudjaev [30].

On the other hand, we stress that the assumption on the linear growth of g is a real obstruction for the global existence of the solution, as the following example shows.

Example 1.1. In (7)–(8), let us take $f = 0$, $g(x,y) = x + y$ and assume that there exists an integrable solution (Y, V) (by integrable we mean at least $E(|Y_t| + |V_t|) < +\infty$ for each $t \in [0,T]$). By construction, $V_t = E(B_T + Y_T|\mathscr{F}_t) = B_t + E(Y_T|\mathscr{F}_t)$ is a martingale, hence it has constant expectation $E(V_t) = C$ for all $t \in [0,T]$. Consequently, the following holds:

$$C = E(V_t) = E(Y_T) = y + \int_0^T E(V_s)\,ds \quad \Rightarrow \quad C = \frac{y}{1-T},$$

which is defined only if $T \neq 1$ (actually only for $T < 1$).

Analogously, problem (3)–(4), for $\varepsilon \geqslant 0$, becomes

$$\begin{aligned} &u_{xx} + \varepsilon^2 u_{yy} + u u_y - u_t = 0 \quad \text{in }]0,T] \times \mathbb{R}^2, \\ &u(0,x,y) = x + y \quad \text{in } \mathbb{R}^2 \end{aligned}$$

with solution $u(t,x,y) = \frac{x+y}{1-t}$ which blows up as $t \to \bar{T} = 1^-$. Roughly speaking, through the classical Hopf transformation [15], the linear growth of the initial datum for Eq. (1) corresponds to the rate of growth of e^{y^2} for the heat equation.

On the other hand, if $g(x,y) = -x - y$, we still have $\bar{T} = 1$ in (5), so that Theorem 1.1 misses the global existence of the solution $u(t,x,y) = -\frac{x+y}{1+t}$.

The paper is organized as follows. In Section 2, we prove the existence of a solution $(Y^\varepsilon, V^\varepsilon)$ of (7)–(8). In Section 3, we show that the flows of solutions associated to $(Y^\varepsilon, V^\varepsilon)$ define a deterministic function u^ε satisfying (6). In Section 4, we prove that u^ε is a viscosity solution of a backward Cauchy problem related to (3)–(4). In Section 5, a comparison principle for viscosity solutions is established and the proof of Theorem 1.1 is concluded.

2. Existence

In this section we prove the existence and uniqueness of the solution to the stochastic differential system (7)–(8) associated to (1)–(2). From now on, we shall denote by $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$ and by

$$\mathscr{L}^2 = \left\{ X \text{ adapted, progressively measurable processes :} \right.$$
$$\left. \left[E\left(\int_0^T |X_s|^2 \, ds \right) \right]^{\frac{1}{2}} < \infty \right\},$$
$$\underline{S}^2 = \left\{ X \text{ semimartingales : } \left[E\left(\sup_{0 \leqslant t \leqslant T} |X_t|^2 \right) \right]^{\frac{1}{2}} < +\infty) \right\}.$$

We refer the reader to [27] for details about the theory of semimartingales and to [1,24,26] for more information about forward–backward stochastic differential equations.

We recall that k_1 denotes the Lipschitz constant of $f = f(t, x, y, v)$ w.r.t. the variables y, v and k_2 the Lipschitz constant of $g = g(x, y)$ w.r.t. y.

Proposition 2.1. *Let the foregoing hypotheses hold and let* $(k_1 \vee 1 + k_2)T < 1$ *and* $\varepsilon \in [0, 1[$. *Then there exists a unique solution to* (7)–(8) *in* $\mathscr{L}^2 \times \mathscr{L}^2$.

Proof. Let us consider the following operator:

$$\Lambda(Y, V)_t \equiv \begin{pmatrix} F(Y, V)_t \\ G(Y, V)_t \end{pmatrix} \equiv \begin{pmatrix} y + \int_0^t V_s \, ds + \varepsilon W_t \\ E\left(g(B_T, F(Y, V)_T) - \int_t^T f(s, B_s, Y_s, V_s) \, ds | \mathscr{F}_t \right) \end{pmatrix}.$$

Then $\Lambda : \mathscr{L}^2 \times \mathscr{L}^2 \to \mathscr{L}^2 \times \mathscr{L}^2$, as the following shows:

$$\begin{aligned} E\left(\int_0^T (|F_t| + |G_t|)^2 \, dt \right) \leqslant E\Bigg(\int_0^T \Bigg(& (k_2 + 1)|y| + \varepsilon |W_t| + \varepsilon k_2 |W_T| \\ & + |g(B_T, 0)| + \int_0^T |f(s, B_s, 0, 0)| \, ds \\ & + (k_1 + k_2 + 1) \int_0^T (|Y_s| + |V_s|) \, ds \Bigg)^2 dt \Bigg) < \infty \end{aligned}$$

because of the Lipschitz hypotheses and Jensen inequality.

The space $\mathscr{L}^2 \times \mathscr{L}^2$ is a Banach space and under our conditions, the operator Λ is a contraction. Indeed for any choice of

$(Y^2, V^2), (Y^1, V^1) \in \mathscr{L}^2 \times \mathscr{L}^2$, using the global Lipschitz conditions, we have

$$|F(Y^2, V^2)_t - F(Y^1, V^1)_t| \leqslant \int_0^t |V_s^2 - V_s^1|\, ds,$$

$$\begin{aligned}&|G(Y^2, V^2)_t - G(Y^1, V^1)_t|\\ &\quad\leqslant E\Bigg((|g(B_T, F(Y^2, V^2)_T) - g(B_T, F(Y^1, V^1)_T)| \\ &\qquad + \int_t^T |f(s, B_s, Y_s^2, V_s^2) - f(s, B_s, Y_s^1, V_s^1)|\, ds|\mathscr{F}_t\Bigg).\end{aligned}$$

Using the first inequality in the second and summing the two together, we obtain

$$\begin{aligned}&|F(Y^2, V^2)_t - F(Y^1, V^1)_t| + |G(Y^2, V^2)_t - G(Y^1, V^1)_t|\\ &\quad\leqslant (k_1 \vee 1 + k_2) E\left(\int_0^T (|Y_s^2 - Y_s^1| + |V_s^2 - V_s^1|)\, ds|\mathscr{F}_t \right).\end{aligned}$$

Therefore, integrating on Ω and from 0 to T, applying Jensen inequality, we may conclude

$$\begin{aligned}&\|\Lambda(Y^2, V^2) - \Lambda(Y^1, V^1)\|_{\mathscr{L}^2 \times \mathscr{L}^2}\\ &\quad\leqslant (k_1 \vee 1 + k_2) T \|(Y^2, V^2) - (Y^1, V^1)\|_{\mathscr{L}^2 \times \mathscr{L}^2},\end{aligned}$$

that is to say Λ is a contraction, by virtue of our assumption. □

We denote by $(Y^\varepsilon, V^\varepsilon)$ the adapted solution of (7)–(8). The bound on the norm of $(Y^\varepsilon, V^\varepsilon)$ in $\mathscr{L}^2 \times \mathscr{L}^2$ can be made independent of ε. As a matter of fact

$$|Y_t^\varepsilon| \leqslant |y| + \int_0^t |V^\varepsilon|\, ds + \varepsilon |W_t|,$$

$$\begin{aligned}|V_t^\varepsilon| \leqslant E\Bigg(&|g(B_T, 0)| + k_2 \left[|y| + \int_0^T |V^\varepsilon|\, ds + \varepsilon |W_T| \right]\\ &+ \int_t^T \{|f(s, B_s, 0, 0)| + k_1(|Y_s^\varepsilon| + |V_s^\varepsilon|)\}\, ds|\mathscr{F}_t \Bigg).\end{aligned}$$

Since $\varepsilon < 1$, the above inequalities imply

$$\begin{aligned}|Y_t^\varepsilon| + |V_t^\varepsilon| \leqslant E\Bigg(&(k_1 \vee 1 + k_2) \int_0^T (|Y_s^\varepsilon| + |V_s^\varepsilon|)\, ds\\ &+ (k_2 + 1)|y| + |W_t| + k_2 |W_T|\\ &+ |g(B_T, 0)| + \int_0^T |f(s, B_s, 0, 0)|\, ds|\mathscr{F}_t \Bigg), \end{aligned} \tag{9}$$

squaring both sides, employing Schwartz inequality in the form $(\alpha + \beta)^2 \leqslant (1 + \frac{1}{a})\alpha^2 + (1 + a)\beta^2$ for a suitably large $a > 0$ and integrating from 0 to T, we get

$$E\left(\int_0^T (|Y_t^\varepsilon|+|V_t^\varepsilon|)^2\,dt\right)$$
$$\leqslant \frac{(1+a)}{1-(1+\frac{1}{a})(k_1\vee 1+k_2)^2T^2}E\left((k_2+1)^2|y|^2+\left(\frac{1}{2}+k_2\right)\right.$$
$$\left.\times\, T^2+T|g(B_T,0)|^2+T^2\int_0^T |f(s,B_s,0,0)|^2\,ds\right). \qquad (10)$$

Plugging this inequality back into (9) and using Doob's inequality for submartingales, we also obtain

$$E\left(\sup_{0\leqslant t\leqslant T}(|Y_t^\varepsilon|+|V_t^\varepsilon|)^2\right)$$
$$\leqslant C\left(k_1,k_2,T,a,y,g,f,B,W,\frac{1}{1-(1+\frac{1}{a})(k_1\vee 1+k_2)^2T^2}\right),$$

which is independent of ε.

3. Continuity

Let $(Y^\varepsilon, V^\varepsilon)$ be the adapted solution of (7)–(8) whose existence has been proved in the previous section. It is to be remarked that, by the martingale representation theorem, the backward component of our system may be rewritten as

$$V_t^\varepsilon = g(B_T, Y_T) - \int_t^T f(s, B_s, Y_s^\varepsilon, V_s^\varepsilon)\,ds$$
$$-\int_t^T H_s^\varepsilon\,dB_s - \int_t^T Z_s^\varepsilon\,dW_s \qquad (11)$$

with predictable processes H^ε and Z^ε such that

$$E\left(\int_0^T [(H_s^\varepsilon)^2+(Z_s^\varepsilon)^2]\,ds\right)<+\infty.$$

With this representation, the continuity in t of the process V^ε follows directly, since for any $t_1\leqslant t_2$, we have

$$V_{t_2}^\varepsilon - V_{t_1}^\varepsilon = \int_{t_1}^{t_2} f(s,B_s,Y_s^\varepsilon,V_s^\varepsilon)\,ds + \int_{t_1}^{t_2} H_s^\varepsilon\,dB_s + \int_{t_1}^{t_2} Z_s^\varepsilon\,dW_s. \qquad (12)$$

The processes $H^\varepsilon, Z^\varepsilon$ are in general unknown, but if the coefficients f, g are differentiable in the spatial variables, by using Malliavin Calculus techniques, one may have an explicit representation of H, Z.

Since we are in a Brownian environment and the functions g and f are deterministic, the solution processes $Y^\varepsilon, V^\varepsilon$ are Markovian, hence by exploiting the Blumenthal's 0–1 law, one can show that the associated flows of solutions (cf. [24])

$$\begin{aligned} B_s^{t,x} &= x + B_s - B_t, \\ Y_s^{\varepsilon,t,x,y} &= y + \int_t^s V_r^{\varepsilon,t,x,y}\,dr + \varepsilon(W_s - W_t), \\ V_s^{\varepsilon,t,x,y} &= E\left(g(B_T^{t,x}, Y_T^{\varepsilon,t,x,y}) - \int_s^T f(r, B_r^{t,x}, Y_r^{\varepsilon,t,x,y}, V_r^{\varepsilon,t,x,y})\,dr | \mathscr{F}_s \right) \end{aligned} \tag{13}$$

define a deterministic function

$$u^\varepsilon(t,x,y) = V_t^{\varepsilon,t,x,y}, \quad (t,x,y) \in [0,T] \times \mathbb{R}^2. \tag{14}$$

In the following proposition, we prove a uniform Hölder estimate of u^ε.

Proposition 3.1. *Under the above hypotheses,* u^ε *verifies estimate* (6), *i.e.* u^ε *is globally Lipschitz in* x, y *and Hölder of order* $\frac{1}{2}$ *in* t *with constant* C_0 *independent of* $\varepsilon \in [0,1[$.

Proof. Let us consider $t_1, t_2 \in [0,T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and consider the associated flows. Without loss of generality, we may assume that $t_1 \leqslant t_2$ and we extend naturally the flows to the whole interval, that means

$$(B_s^{t_i,x_i}, Y_s^{\varepsilon,t_i,x_i,y_i}, V_s^{\varepsilon,t_i,x_i,y_i}) = (B_{t_i}^{t_i,x_i}, Y_{t_i}^{\varepsilon,t_i,x_i,y_i}, V_{t_i}^{\varepsilon,t_i,x_i,y_i})$$

for any $s \leqslant t_i$, $i = 1,2$. We want to estimate $|V_{t_2}^{\varepsilon,t_2,x_2,y_2} - V_{t_1}^{\varepsilon,t_1,x_1,y_1}|$. We adopt the notation $X^i = X^{t_i,x_i,y_i}$ for any indexed process that appears in the expressions and we denote by k_0 the Lipschitz constant of f and g w.r.t. the first spatial variable. For any $t \in [0,T]$, we have

$$|B_t^2 - B_t^1| \leqslant |x_2 - x_1| + |B_{t_2 \vee t} - B_{t_2} - B_{t_1 \vee t} + B_{t_1}|,$$

$$|Y_t^2 - Y_t^1| \leqslant |y_2 - y_1| + \int_{t_2}^{t_2 \vee t} |V_s^2 - V_s^1|\,ds + \int_{t_1 \wedge t}^{t_2 \wedge t} |V_s^1|\,ds + \varepsilon |W_{t_2 \vee t} - W_{t_2} - W_{t_1 \vee t} + W_{t_1}|,$$

$$\begin{aligned} |V_t^2 - V_t^1| \leqslant E\Bigg(& |g(B_T^2, Y_T^2) - g(B_T^1, Y_T^1)| \\ & + \int_{t_2 \vee t}^T |f(s, B_s^2, Y_s^2, V_s^2) - f(s, B_s^1, Y_s^1, V_s^1)|\,ds \\ & + \int_{t_1 \vee t}^{t_2 \vee t} |f(s, B_s^1, Y_s^1, V_s^1)| ds \ |\mathscr{F}_t \Bigg). \end{aligned}$$

Summing the two components Y, V and squaring both sides we obtain

$$(|Y_t^2 - Y_t^1| + |V_t^2 - V_t^1|)^2$$

$$
\begin{aligned}
&\leqslant \Bigg\{ E\Bigg(((k_1 \vee 1) + k_2) \int_0^T (|Y_s^2 - Y_s^1| + |V_s^2 - V_s^1|)\, ds \\
&\quad + (k_2 + 1)|y_2 - y_1| + k_0|x_2 - x_1| + \varepsilon |W_{t_2 \vee t} - W_{t_2} - W_{t_1 \vee t} + W_{t_1}| \\
&\quad + k_2 \varepsilon |W_{t_2} - W_{t_1}| + k_0(1 + T)|B_{t_2} - B_{t_1}| + \int_{t_1 \vee t}^{t_2 \vee t} |f(s, B_s, 0, 0)|\, ds \\
&\quad + k_2 \int_{t_1}^{t_2} |V_s^1|\, ds + \int_{t_1 \wedge t}^{t_2 \wedge t} |V_s^1|\, ds \\
&\quad + k_1 \int_{t_1 \vee t}^{t_2 \vee t} (|Y_s^1| + |V_s^1|)\, ds | \mathscr{F}_t \Bigg) \Bigg\}^2 .
\end{aligned}
$$

Chosen $a > 0$, using Schwartz inequality as before and integrating on $[0, T]$, we have

$$
E\left(\int_0^T (|Y_t^2 - Y_t^1| + |V_t^2 - V_t^1|)^2\, dt \right) \leqslant \frac{T(1 + a)E(A^2)}{1 - (1 + \frac{1}{a})(k_1 \vee 1 + k_2)^2 T^2},
$$

where A is a random variable such that

$$
\begin{aligned}
E(A^2) \leqslant\ & E\big(k_0^2(1 + T)^2 |x_2 - x_1|^2 + (k_2 + 1)^2 |y_2 - y_1|^2 \\
& + k_0^2 (1 + T)^2 |B_{t_2} - B_{t_1}|^2 + (k_2 + 1)^2 |W_{t_2} - W_{t_1}|^2 \\
& + (t_2 - t_1) \int_{t_1}^{t_2} |f(s, B_s, 0, 0)|^2\, ds \\
& + (k_2 + (k_1 \vee 1))^2 (t_2 - t_1) \int_{t_1}^{t_2} (|Y_s^1| + |V_s^1|)^2\, ds \Big) \\
\leqslant\ & C_1(|t_2 - t_1| + |x_2 - x_1|^2 + |y_2 - y_1|^2),
\end{aligned}
$$

where

$$
C_1 = C_1\left(x_1, y_1, k_0, k_1, k_2, T, \frac{1}{1 - (1 + \frac{1}{a})((k_1 \vee 1) + k_2)^2 T^2} \right)
$$

and we used (10), the fact that $\varepsilon < 1$ and the properties of Brownian motions. Proceeding as before, we can obtain a similar estimate in the $\underline{S}^2 \times \underline{S}^2$ norm

$$
\begin{aligned}
& E\left(\sup_{0 \leqslant t \leqslant T} (|Y_t^2 - Y_t^1| + |V_t^2 - V_t^1|)^2 \right) \\
& \quad \leqslant C_2(|t_2 - t_1| + |x_2 - x_1|^2 + |y_2 - y_1|^2)
\end{aligned}
$$

for some

$$
C_2 = C_2\left(x_1, y_1, k_0, k_1, k_2, T, \frac{1}{1 - (1 + \frac{1}{a})((k_1 \vee 1) + k_2)^2 T^2} \right).
$$

Since the last estimate holds uniformly in t, it is true also for t_1, hence we obtain estimates (6). $\square$

4. Existence of a viscosity solution

In this section we show, by using Itô's formula on the test functions, that u^ε, defined in (14), is a viscosity solution of the backward Cauchy problem

$$\frac{1}{2}v_{xx} + \frac{\varepsilon^2}{2}v_{yy} + vv_y + v_t = f(\cdot, v) \quad \text{in }]0, T] \times \mathbb{R}^2, \tag{15}$$

$$v(T, \cdot) = g \quad \text{in } \mathbb{R}^2. \tag{16}$$

It is then clear that, by a straightforward transformation, we also prove the existence part and estimate (6) in Theorem 1.1. Indeed, it suffices to solve the forward backward SDE related to $\tilde{g}, \tilde{f}, \tilde{T}$ satisfying the above assumptions and then impose

$$\tilde{g}(x, y) = g(2x, 2y), \quad \tilde{f}(t, x, y, v) = 2f(2(T-t), 2x, 2y, v),$$

$$\tilde{u}^\varepsilon(t, x, y) = u^\varepsilon(2(T-t), 2x, 2y), \quad \tilde{T} = \frac{T}{2}.$$

Proposition 4.1. *Let* $\varepsilon \in [0, 1[$. *The function* u^ε *in* (14) *is a viscosity solution of problem* (15)–(16).

Proof. Since in the previous section we already proved the continuity of the function u^ε, it now remains only to prove that it is both a viscosity subsolution and supersolution. Since the technique is truly the same, we only show the subsolution case.

By the Markov property and the pathwise uniqueness of the solution, it is possible to show that a.s. $V_s^{\varepsilon,t,x,y} = u^\varepsilon(s, B_s^{t,x}, Y_s^{\varepsilon,t,x,y})$, for any $s \in [t, T]$.

Let us consider a point $(t, x, y) \in [0, T] \times \mathbb{R}^2$ and a function $\varphi \in C^{1,2}$, with bounded derivatives, such that

$$0 = u^\varepsilon(t, x, y) - \varphi(t, x, y)$$

is a global maximum for $u^\varepsilon - \varphi$ (without loss of generality we can assume this maximum to be zero). This means that for any $\mathscr{F}_t$-stopping time τ, necessarily

$$u^\varepsilon(\tau, B_\tau^{t,x}, Y_\tau^{\varepsilon,t,x,y}) - \varphi(\tau, B_\tau^{t,x}, Y_\tau^{\varepsilon,t,x,y}) \leqslant 0. \tag{17}$$

For ease of writing, in the following we omit the superscripts of u, B, Y and V. Since φ is regular we may apply Itô's formula in the interval $[t, \tau]$, with τ stopping time. By the independence of B and W we obtain

$$\begin{aligned}\varphi(\tau, B_\tau, Y_\tau) = \varphi(t, x, y) &+ \int_t^\tau \varphi_t(r, B_r, Y_r)\, dr + \int_t^\tau \varphi_x(r, B_r, Y_r)\, dB_r \\ &+ \frac{1}{2}\int_t^\tau \varphi_{xx}(r, B_r, Y_r)\, dr + \int_t^\tau \varphi_y(r, B_r, Y_r) u(r, B_r, Y_r)\, dr \\ &+ \int_t^\tau \varepsilon\varphi_y(r, B_r, Y_r)\, dW_r + \frac{\varepsilon^2}{2}\int_t^\tau \varphi_{yy}(r, B_r, Y_r)\, dr.\end{aligned}$$

On the other hand, by the martingale representation (11), keeping in mind that by the uniqueness of paths $V_r = u(r, B_r, Y_r)$, we have

$$
\begin{aligned}
u(t,x,y) = V_t = V_\tau - \int_t^\tau f(r, B_r, Y_r, V_r)\,dr - \int_t^\tau H_r\,dB_r - \int_t^\tau Z_r\,dW_r \\
= u(\tau, B_\tau, Y_\tau) - \int_t^\tau f(r, B_r, Y_r, V_r)\,dr \\
- \int_t^\tau H_r\,dB_r - \int_t^\tau Z_r\,dW_r.
\end{aligned}
$$

Substituting the last two equalities in (17), we conclude

$$
\begin{aligned}
0 &\geqslant u(\tau, B_\tau, Y_\tau) - \varphi(\tau, B_\tau, Y_\tau) \\
&= u(t,x,y) - \varphi(t,x,y) \\
&\quad - \int_t^\tau \left(\varphi_t + \frac{1}{2}\varphi_{xx} + \frac{\varepsilon^2}{2}\varphi_{yy} + \varphi_y u - f(\cdot, u)\right)(r, B_r, Y_r)\,dr \\
&\quad + \int_t^\tau \left(H_r - \frac{1}{2}\varphi_x(r, B_r, Y_r)\right)dB_r \\
&\quad + \int_t^\tau \left(Z_r - \frac{\varepsilon^2}{2}\varphi_y(r, B_r, Y_r)\right)dW_r.
\end{aligned}
$$

By assumption $u(t,x,y) - \varphi(t,x,y) = 0$ and taking expectations in the previous inequality the martingale parts give no contribution, so we can summarize the inequality by writing

$$
E\left(\int_t^\tau \Phi(r, B_r, Y_r)\,dr\right) \geqslant 0, \tag{18}
$$

where

$$
\Phi = \frac{1}{2}\varphi_{xx} + \frac{\varepsilon^2}{2}\varphi_{yy} + \varphi_y u + \varphi_t - f(\cdot, u).
$$

To say that u is a subsolution of (15)–(16) means that we must verify that $\Phi(t,x,y) \geqslant 0$, since the equality at T is automatically verified, because of the definition of V.

By contradiction we assume there exists an $\delta_0 < 0$ such that $\Phi(t,x,y) < \delta_0$ and we define the stopping time

$$
\tau_1 = \inf\left\{ r > t\colon\ \Phi(r, B_r, Y_r) \geqslant \frac{\delta_0}{2} \right\} \wedge T.
$$

By construction $\tau_1 > t$ a.s. Inequality (18) holds for any stopping time, therefore also for τ_1 and we have

$$
0 > \frac{\delta_0}{2} E(\tau_1 - t) \geqslant E\left(\int_t^{\tau_1} \Phi(r, B_r, Y_r)\,dr\right) \geqslant 0,
$$

which is a clear contradiction. Hence we proved that u is a subsolution of (15)–(16).

Analogously, we can prove that u is a viscosity supersolution of (15)–(16) and complete the proof. □

5. Uniqueness of the viscosity solution

In this section we prove a comparison principle for viscosity solutions and Theorem 1.1. We introduce some notations that will be used in the sequel. We denote $h=(x,y)$, $D_h=(\partial_x,\partial_y)$ and by D_h^2 the Hessian matrix w.r.t. the spatial variables. Moreover, $\mathscr{P}$ denotes the parabolic semijet (see [11, Section 8]). We first state a preliminary lemma whose proof will be omitted.

Lemma 5.1. *Let O be an open subset of $\mathbb{R}^3$ and $z_0=(t_0,h_0)\in O$.*

If $w: O\to\mathbb{R}$ and $H\in C^2(O,]0,+\infty[)$, then $(a,p,X)\in\bar{\mathscr{P}}_O^{2,+}w(z_0)$ if and only if

$$(aH+wH_t, pH+wD_hH, HX+2p\otimes D_hH+wD_h^2H) \times (z_0)\in\bar{\mathscr{P}}_O^{2,+}wH(z_0), \tag{19}$$

where $(p_1,p_2)\otimes(q_1,q_2)$ denotes the matrix

$$\begin{pmatrix} p_1q_1 & \frac{p_1q_2+p_2q_1}{2} \\ \frac{p_1q_2+p_2q_1}{2} & p_2q_2 \end{pmatrix}.$$

An analogous statement holds if $\bar{\mathscr{P}}^{2,+}$ is replaced by $\bar{\mathscr{P}}^{2,-}$.

We next prove a comparison result.

Proposition 5.1. *Let $\varepsilon\in[0,1[$. If u is a subsolution and v is a supersolution of problem* (3)–(4) *such that they both verify the Hölder estimate* (6), *then $u\leqslant v$.*

Proof. We set $S_\varrho=]0,\varrho[\times\mathbb{R}^2$ and we consider the function

$$H(t,h)=\exp\left(\frac{|h|^2}{1-(2\varrho)^{-1}t}+\sigma t\right), \qquad (t,h)\in\overline{S_\varrho}. \tag{20}$$

Since

$$\frac{H_{xx}+\varepsilon^2H_{yy}+(u+v)H_y-H_t}{H} = \frac{4(x^2+\varepsilon^2y^2)}{(1-(2\varrho)^{-1}t)^2}+\frac{2}{1-(2\varrho)^{-1}t} + \frac{2y(u+v)}{1-(2\varrho)^{-1}t}-\frac{x^2+y^2}{2\varrho(1-(2\varrho)^{-1}t)^2}-\sigma$$

and u,v verify estimate (6), it is possible to choose sufficiently large positive constants ϱ^{-1},σ such that, for every $\varepsilon\in[0,1[$,

$$\sup_{S_\varrho} \frac{H_{xx} + \varepsilon^2 H_{yy} + (u+v)H_y - H_t}{H} + k_1 < 0, \tag{21}$$

where k_1 is the Lipschitz constant of $f = f(t,x,y,v)$ w.r.t. the variables y, v. We prove that $u \leqslant v$ in S_ϱ. By contradiction, we suppose that there exists $\bar{z} \in S_\varrho$ such that $u(\bar{z}) - v(\bar{z}) > 0$.

We consider the following functions defined on $[0,\varrho[\times\mathbb{R}^2$:

$$w = \frac{u}{H} - \frac{\delta}{\varrho - t}, \quad \omega = \frac{v}{H} + \frac{\delta}{\varrho - t}$$

and we choose $\delta > 0$ suitably small so that $w(\bar{z}) - \omega(\bar{z}) > 0$. We have

$$\lim_{|h|\to\infty} (w-\omega)(t,h) = -\frac{2\delta}{\varrho - t} < 0 \tag{22}$$

and

$$\lim_{t\to\varrho^-} (w-\omega)(t,h) = -\infty \quad \text{uniformly in } h \in \mathbb{R}^2. \tag{23}$$

By a standard argument, we double the number of spatial variables and we consider the function

$$\Phi_\alpha(t,h,h') = w(t,h) - \omega(t,h') - \frac{\alpha}{2}|h-h'|^2, \quad \alpha > 0.$$

Let $(t_\alpha, h_\alpha, h'_\alpha)$ be a maximum point of Φ_α in $[0,\varrho[\times\mathbb{R}^2$. Such a maximum exists in view of (22)–(23). Moreover, we have

$$0 < w(\bar{z}) - \omega(\bar{z}) \leqslant \Phi_\alpha(t_\alpha,h_\alpha,h'_\alpha) \leqslant \sup_{S_\varrho}(w-\omega) < +\infty. \tag{24}$$

By Lemma 3.1 in [11], we have

$$\lim_{\alpha\to\infty} \alpha|h_\alpha - h'_\alpha|^2 = 0, \tag{25}$$

so that, by (22) and (24), there exists a compact subset M of $\mathbb{R}^2$ such that $h_\alpha, h'_\alpha \in M$ for every $\alpha > 0$. Hence we may suppose that there exists the limit

$$\lim_{\alpha\to\infty} (t_\alpha,h_\alpha,h'_\alpha) = (t_0,h_0,h_0) \in [0,\varrho] \times \mathbb{R}^2 \times \mathbb{R}^2.$$

If $t_0 = 0$, then $\Phi_\alpha(t_\alpha,h_\alpha,h'_\alpha) \to -2\delta\varrho^{-1}$ and this contradicts (24). Hence $t_\alpha > 0$ if α is large. Analogously, by (23) and (24), $t_0 < \varrho$. Then Lemma 3.1 in [11] yields

$$\lim_{\alpha\to\infty} \Phi_\alpha(t_\alpha,h_\alpha,h'_\alpha) = w(t_0,h_0) - \omega(t_0,h_0) = \sup_{[0,\varrho[\times\mathbb{R}^2}(w-\omega). \tag{26}$$

Thus, we may apply Theorem 8.3 in [11] to infer that there exist $a \in \mathbb{R}$ and some matrices X^w, Y^ω such that

$$(a, \alpha(h_\alpha - h'_\alpha), X^w) \in \bar{\mathscr{P}}^{2,+}_{S_\varrho} w(t_\alpha,h_\alpha),$$

$$(a, \alpha(h_\alpha - h'_\alpha), Y^\omega) \in \bar{\mathscr{P}}^{2,-}_{S_\varrho} \omega(t_\alpha,h'_\alpha)$$

and

$$X^w \leqslant Y^\omega. \tag{27}$$

Since

$$u = \left(w + \frac{\delta}{\varrho - t}\right)H, \quad v = \left(\omega - \frac{\delta}{\varrho - t}\right)H,$$

by Lemma 5.1, we deduce that

$$(d_t^u, (d_x^u, d_y^u), X^u) \in \bar{\mathscr{P}}^{2,+}_{S_\varrho} u(t_\alpha, h_\alpha),$$
$$(d_t^v, (d_x^v, d_y^v), Y^v) \in \bar{\mathscr{P}}^{2,-}_{S_\varrho} v(t_\alpha, h'_\alpha),$$

where

$$d_t^u = \left(\left(a + \frac{\delta}{(\varrho - t)^2}\right)H + \frac{u}{H}H_t\right)(t_\alpha, h_\alpha),$$
$$(d_x^u, d_y^u) = \left(\alpha(h_\alpha - h'_\alpha)H + \frac{u}{H}D_h H\right)(t_\alpha, h_\alpha),$$
$$X^u = \left(X^w H + 2\alpha(h_\alpha - h'_\alpha)\otimes D_h H + \frac{u}{H}D_h^2 H\right)(t_\alpha, h_\alpha)$$

and

$$d_t^v = \left(\left(a - \frac{\delta}{(\varrho - t)^2}\right)H + \frac{v}{H}H_t\right)(t_\alpha, h'_\alpha),$$
$$\left(d_x^v, d_y^v\right) = \left(\alpha\left(h_\alpha - h'_\alpha\right)H + \frac{v}{H}D_h H\right)(t_\alpha, h'_\alpha),$$
$$Y^v = \left(Y^\omega H + 2\alpha(h_\alpha - h'_\alpha)\otimes D_h H + \frac{v}{H}D_h^2 H\right)(t_\alpha, h'_\alpha).$$

Next, since u is a subsolution of (1)–(2), we get

$$f(\cdot,\cdot,u)(t_\alpha, h_\alpha) - (X_{11}^u + \varepsilon^2 X_{22}^u + u(t_\alpha, h_\alpha)d_y^u - d_t^u)$$
$$+\ u(t_\alpha, h_\alpha)d_y^v \leqslant u(t_\alpha, h_\alpha)d_y^v \tag{28}$$

or, by using the expressions above,

$$\frac{f(\cdot,\cdot,u)}{H}(t_\alpha, h_\alpha) - \left[X_{11}^w + \varepsilon^2 X_{22}^w + 2\alpha\left(x_\alpha - x'_\alpha\right)\frac{H_x}{H}(t_\alpha, h_\alpha)\right.$$
$$\left. +\ \alpha\left(y_\alpha - y'_\alpha\right)\left(2\varepsilon^2\frac{H_y}{H}(t_\alpha, h_\alpha) + u(t_\alpha, h_\alpha)\right) - a - \frac{\delta}{(\varrho - t_\alpha)^2}\right]$$
$$-\frac{u}{H^2}(t_\alpha, h_\alpha)\left[H_{xx}(t_\alpha, h_\alpha) + \varepsilon^2 H_{yy}(t_\alpha, h_\alpha) + uH_y(t_\alpha, h_\alpha)\right.$$
$$\left. +\ H(t_\alpha, h_\alpha)\frac{vH_y}{H}(t_\alpha, h'_\alpha) - H_t(t_\alpha, h_\alpha) + \alpha(y_\alpha - y'_\alpha)H(t_\alpha, h_\alpha)H(t_\alpha, h'_\alpha)\right]$$
$$\leqslant u(t_\alpha, h_\alpha)d_y^v. \tag{29}$$

On the other hand, since v is a supersolution of (1)–(2), we have

$$f(\cdot,\cdot,v)(t_\alpha,h'_\alpha) - (Y^v_{11} + \varepsilon^2 Y^v_{22} + v(t_\alpha,h'_\alpha)d^v_y - d^v_t) + u(t_\alpha,h_\alpha)d^v_y \geqslant u(t_\alpha,h_\alpha)d^v_y, \tag{30}$$

that is,

$$\frac{f(\cdot,\cdot,v)}{H}(t_\alpha,h'_\alpha) - \left[Y^\omega_{11} + \varepsilon^2 Y^\omega_{22} + 2\alpha(x_\alpha - x'_\alpha)\frac{H_x}{H}(t_\alpha,h'_\alpha) + \alpha(y_\alpha - y'_\alpha)\left(2\varepsilon^2\frac{H_y}{H}(t_\alpha,h'_\alpha) + u(t_\alpha,h_\alpha) + v(t_\alpha,h'_\alpha)\right) - a + \frac{\delta}{(\varrho - t_\alpha)^2}\right] - \frac{v}{H^2}(t_\alpha,h'_\alpha)[H_{xx}(t_\alpha,h'_\alpha) + \varepsilon^2 H_{yy}(t_\alpha,h'_\alpha) + (u(t_\alpha,h_\alpha) + v(t_\alpha,h'_\alpha))H_y(t_\alpha,h'_\alpha) - H_t(t_\alpha,h'_\alpha)] \geqslant u(t_\alpha,h_\alpha)d^v_y. \tag{31}$$

Finally, we deduce from (29), (31) and (27) that, for $\alpha > 0$,

$$I_\alpha + J_\alpha \geqslant \frac{2\delta}{(\varrho - t_\alpha)^2} > 0, \tag{32}$$

where

$$I_\alpha = \frac{f(\cdot,\cdot,v)}{H}(t_\alpha,h'_\alpha) - \frac{f(\cdot,\cdot,u)}{H}(t_\alpha,h_\alpha) + 2\alpha\left\langle h_\alpha - h'_\alpha, \left(\frac{H_x}{H}(t_\alpha,h_\alpha) - \frac{H_x}{H}(t_\alpha,h'_\alpha), \varepsilon^2\frac{H_y}{H}(t_\alpha,h_\alpha) - \left(\varepsilon^2\frac{H_y}{H} + \frac{v}{2}\right)(t_\alpha,h'_\alpha)\right)\right\rangle$$

and

$$J_\alpha = \frac{u}{H^2}(t_\alpha,h_\alpha)\Bigg[H_{xx}(t_\alpha,h_\alpha) + \varepsilon^2 H_{yy}(t_\alpha,h_\alpha) + uH_y(t_\alpha,h_\alpha). + H(t_\alpha,h_\alpha)\frac{vH_y}{H}(t_\alpha,h'_\alpha) - H_t(t_\alpha,h_\alpha) + \alpha(y_\alpha - y'_\alpha)H(t_\alpha,h_\alpha)H(t_\alpha,h'_\alpha)\Bigg] - \frac{v}{H^2}(t_\alpha,h'_\alpha)[H_{xx}(t_\alpha,h'_\alpha) + \varepsilon^2 H_{yy}(t_\alpha,h'_\alpha) + (u(t_\alpha,h_\alpha) + v(t_\alpha,h'_\alpha))H_y(t_\alpha,h'_\alpha) - H_t(t_\alpha,h'_\alpha)].$$

As α goes to infinity, by the Lipschitz continuity of f, we have

$$I_\alpha \to \frac{f(\cdot,\cdot,v)}{H}(t_0,h_0) - \frac{f(\cdot,\cdot,u)}{H}(t_0,h_0) \leqslant k_1\frac{u-v}{H}(t_0,h_0)$$

and

$$J_\alpha \to \frac{u-v}{H}(t_0,h_0)\frac{H_{xx} + \varepsilon^2 H_{yy} + (u+v)H_y - H_t}{H}(t_0,h_0).$$

Since $\frac{u-v}{H}(t_0,h_0) > 0$, by (21), we have a contradiction. Thus we have proved that $u \leqslant v$ in S_ϱ. Repeating this procedure finitely many times, we conclude the proof. □

We end up with the proof of Theorem 1.1.

Proof of Theorem 1.1. Existence, estimate (6) and uniqueness of the solution follow from Propositions 4.1, 3.1 and 5.1, respectively.

If $\varepsilon > 0$, then u^ε is a solution of (1)–(2) in the classical sense. Indeed, let us fix $R > 0$ and denote

$$S = \{(x, y, t) \mid x^2 + y^2 < R^2,\ t \in]0, T[\},$$
$$\tilde{\partial} S = \partial S \cap \{t < T\}.$$

By the Hölder continuity of u^ε and since $\varepsilon > 0$, it is well-known (cf., e.g., [21]) that there exists a function $v \in C^{1+\frac{\alpha}{2}, 2+a}(S) \cap C(S \cup \tilde{\partial} S)$ classical solution of the linear Cauchy–Dirichlet problem

$$\frac{1}{2} v_{xx} + \frac{\varepsilon^2}{2} v_{yy} + u^\varepsilon v_y - v_t = f(\cdot, u^\varepsilon) \quad \text{in } S,$$
$$v|_{\tilde{\partial} S} = u^\varepsilon|_{\tilde{\partial} S}.$$

By the comparison principle for viscosity solutions [11, Theorem 8.2], we have $u^\varepsilon = v$ in S. The thesis follows since R is arbitrary.

We also remark that, if f is a smooth function and $\varepsilon > 0$, then a bootstrap argument shows that $u^\varepsilon \in C^\infty$.

Finally, we prove that u is a vanishing viscosity solution in the sense that u is the limit of u^ε, uniform on compacts as $\varepsilon \to 0^+$. We first remark that a weaker result can be directly obtained from the Hölder estimate (6) for u^ε. Indeed, Ascoli–Arzela's Theorem and Cantor's diagonal argument yield the existence of a sequence of solutions (u^{ε_n}) convergent uniformly on compacts of $[0, T] \times \mathbb{R}^2$ to a function v. Since the convergence is uniform, it is quite standard (cf., e.g., [22]) to prove that v is a viscosity solution of (1)–(2) satisfying (6). Therefore, by uniqueness, v coincides with u.

With a bit more effort, we prove the first, stronger assertion. Since the technique is the same of Proposition 5.1, we only sketch the proof. We fix $\varrho > 0$ suitably small so that the function H in (20) is such that

$$\hat{k} \equiv \sup_{\varepsilon \in]0,1[} \sup_{S_\varrho} \frac{H_{xx} + (u^\varepsilon + u) H_y - H_t}{H} + k_1 < 0. \tag{33}$$

We have to show the following:

$$\forall R,\ \gamma > 0,\ \exists \varepsilon_0 > 0 \quad \text{s.t. } |u^\varepsilon(z) - u(z)| \leqslant \gamma, \quad \forall z \in [0, \varrho[\times B(0, R),$$
$$\varepsilon \in]0, \varepsilon_0[,$$

where $B(0, R)$ denotes the Euclidean ball in $\mathbb{R}^2$. By contradiction, we assume that for some $R, \gamma > 0$ and every $\varepsilon > 0$ there exists $z^\varepsilon \in [0, \varrho[\times B(0, R)$ such that $(u^\varepsilon - u)(z^\varepsilon) > \gamma$. We consider the following functions defined on $[0, \varrho[\times \mathbb{R}^2$:

$$w^\varepsilon = \frac{u^\varepsilon}{H} - \frac{\delta}{\varrho - t}, \quad \omega = \frac{u}{H} + \frac{\delta}{\varrho - t}$$

and we choose $\delta > 0$ suitably small and independent of ε, so that

$$w^\varepsilon(z^\varepsilon) - \omega(z^\varepsilon) > 0. \tag{34}$$

Proceeding as in the proof of Proposition 5.1, we may prove the existence of a global maximum $(t_0^\varepsilon, h_0^\varepsilon)$ of $w^\varepsilon - \omega$ (see (26)). By (34), since

$$\lim_{|h| \to \infty} (w^\varepsilon - \omega)(t, h) = -\frac{2\delta}{\varrho - t} < 0,$$

uniformly in $\varepsilon > 0$ and δ is independent of ε, we infer that

$$\sup_{\varepsilon \in]0,1[} |h_0^\varepsilon| < \infty. \tag{35}$$

Then, as in (32), we obtain the following inequality:

$$I_\alpha^\varepsilon + J_\alpha^\varepsilon \geqslant \frac{2\delta}{(\varrho - t_\alpha)^2} > 0, \tag{36}$$

where

$$\begin{aligned} I_\alpha^\varepsilon = & \frac{f(\cdot,\cdot,u)}{H}(t_\alpha, h'_\alpha) - \frac{f(\cdot,\cdot,u^\varepsilon)}{H}(t_\alpha, h_\alpha) \\ & + 2\alpha \left\langle h_\alpha - h'_\alpha, \left(\frac{H_x}{H}(t_\alpha, h_\alpha) - \frac{H_x}{H}(t_\alpha, h'_\alpha), \varepsilon^2 \frac{H_y}{H}(t_\alpha, h_\alpha) \right.\right. \\ & \left.\left. - \frac{1}{2} u^\varepsilon (t_\alpha, h'_\alpha) \right) \right\rangle, \end{aligned}$$

and

$$\begin{aligned} J_\alpha^\varepsilon = & \frac{u^\varepsilon}{H^2}(t_\alpha, h_\alpha) \Big[H_{xx}(t_\alpha, h_\alpha) + \varepsilon^2 H_{yy}(t_\alpha, h_\alpha) + u^\varepsilon H_y(t_\alpha, h_\alpha) \\ & + H(t_\alpha, h_\alpha) \frac{u H_y}{H}(t_\alpha, h'_\alpha) - H_t(t_\alpha, h_\alpha) \\ & + \alpha(y_\alpha - y'_\alpha) H(t_\alpha, h_\alpha) H(t_\alpha, h'_\alpha) \Big] - \frac{u}{H^2}(t_\alpha, h'_\alpha)[H_{xx}(t_\alpha, h'_\alpha) \\ & + u^\varepsilon(t_\alpha, h_\alpha) + u(t_\alpha, h'_\alpha) H_y(t_\alpha, h'_\alpha) - H_t(t_\alpha, h'_\alpha)]. \end{aligned}$$

We remark explicitly that $(t_\alpha, h_\alpha, h'_\alpha)$ depends on ε. By the Lipschitz continuity of f, we have

$$\lim_{\alpha \to +\infty} I_\alpha \leqslant k_1 \frac{u^\varepsilon - u}{H}(t_0^\varepsilon, h_0^\varepsilon)$$

and

$$\lim_{\alpha \to +\infty} J_\alpha = \left(\frac{u^\varepsilon - u}{H} \frac{H_{xx} + (u^\varepsilon + u)H_y - H_t}{H} + \varepsilon^2 \frac{u^\varepsilon H_{yy}}{H^2} \right)(t_0^\varepsilon, h_0^\varepsilon).$$

Therefore, by (33) and setting

$$\check{k} = \sup_{S_\varrho} \left| \frac{u^\varepsilon H_{yy}}{H^2} \right| < \infty,$$

we get, as $\alpha \to +\infty$ in (36),

$$0 \leqslant \hat{k} \left(\frac{u^\varepsilon - u}{H} \right) (t_0^\varepsilon, h_0^\varepsilon) + \varepsilon^2 \check{k} < \frac{\hat{k}\gamma}{H(t_0^\varepsilon, h_0^\varepsilon)} + \varepsilon^2 \check{k}.$$

By (35), this obviously contradicts the fact that $\varepsilon > 0$ is arbitrarily small. □

6. Uncited Reference

[12]

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