Time-Changed Lévy Processes and Option Pricing[*]

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Abstract

As is well known, the classic Black-Scholes option pricing model assumes that returns follow Brownian motion. It is widely recognized that return processes differ from this benchmark in at least three important ways. First, asset prices jump, leading to non-normal return innovations. Second, return volatilities vary stochastically over time. Third, returns and their volatilities are correlated, often negatively for equities. We propose that *time-changed Lévy processes* be used to simultaneously address these three facets of the underlying asset return process. We show that our framework encompasses almost all of the models proposed in the option pricing literature. Despite the generality of our approach, we show that it is straightforward to select and test a particular option pricing model through the use of characteristic function technology.

1. Introduction

The shortest path between two results in the real domain passes through the complex domain. – Jacques Hadamard

It is widely recognized that the key to developing successful strategies for managing risk and pricing assets is to parsimoniously describe the stochastic process governing asset dynamics. Brownian motion has emerged as the benchmark process for describing asset returns in continuous time. However, many studies of the time series of asset returns and derivatives prices conclude that there are at least three systematic and persistent departures from this benchmark for both the statistical and risk-neutral process. First, asset prices jump, leading to non-normal return innovations. Second, return volatility varies stochastically over time. Third, returns and their volatilities are correlated, often negatively for equities.

The purpose of this paper is to explore the use of **time-changed Lévy processes** as a way to simultaneously and parsimoniously capture all three of these stylized facts. Roughly speaking, a Lévy process is a continuous time stochastic process with stationary independent increments, analogous to iid innovations in a discrete setting. Important examples of Lévy processes include the drifting Brownian motion underlying the Black and Scholes (1973) model and the compound Poisson process underlying the jump diffusion model of Merton (1976). While a Brownian motion generates normal innovations, non-normal innovations can be generated by a pure jump Lévy process. To capture the stylized evidence on stochastic volatility, we apply a stochastic time change to the Lévy process. This amounts to stochastically altering the clock on which the Lévy process is run. Intuitively, one can regard the original clock as the calendar time and the new random clock as the business time. A more active business day implies a faster business clock. Randomness in business activity generates randomness in volatility. To capture the correlation between returns and their volatilities, we let innovations in the Lévy process be correlated with innovations in the random clock on which it is run. When this correlation is negative, the clock tends to run faster when the Lévy process falls. This captures the "leverage effect" first discussed by Black (1976).[1]

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[1]The term "leverage effect" has become generic in describing the negative correlation between stock returns and their volatilities. However, various other explanations have also been proposed in the economics literature, e.g., Haugen, Talmor, and Torous (1991), Campbell and Hentschel (1992), Campbell and Kyle (1993), and Bekaert and Wu (2000).

Our proposal to use time-changed Lévy processes unifies two large strands of the vast option pricing literature. The first strand follows Merton (1976) in using compound Poisson processes to model jumps, and Heston (1993) in using a mean-reverting square root process to model stochastic volatility. Prominent examples in this first strand of research include, among others, Andersen, Benzoni, and Lund (2002), Bakshi, Cao, and Chen (1997), Bates (1996, 2000), and Pan (2002). All of these works can be regarded as examples from the affine jump diffusion framework of Duffie, Pan, and Singleton (2000), where the asset return and variance are driven by a finite number of potentially correlated state variables. A Poisson-type jump component can be incorporated into any of these state variables. The arrival rate can be an affine function of the state variables and hence can be stochastically time-varying. Duffie, Pan, and Singleton (2000) illustrate how to value many state contingent claims in this affine framework.

While the affine framework represents an important theoretical advance, a limitation arises due to the exclusive use of compound Poisson processes to model jumps. These processes generate a finite number of jumps within a finite time interval, and have accordingly been referred to as finite activity jump processes. The observation that asset prices actually display many small jumps on a fine time scale has lead to the development of a second strand of option pricing literature. This second strand considers more general jump structures, which permit an infinite number of jumps to occur within any finite time interval. Examples of infinite activity jump models include the inverse Gaussian model of Barndorff-Nielsen (1998), the generalized hyperbolic class of Eberlein, Keller, and Prause (1998), the variance-gamma (VG) model of Madan, Carr, and Chang (1998), the generalization of VG in Carr, Geman, Madan, and Yor (2002), and the finite moment log-stable model of Carr and Wu (2002). Empirical work by these authors is generally supportive of the use of infinite activity processes as a way to model returns in a parsimonious way. The recognition that volatility is stochastic has lead to further extensions of infinite activity Lévy models by Barndorff-Nielsen and Shephard (2001) and by Carr, Geman, Madan, and Yor (2001). However, these models all assume that changes in volatility are independent of asset returns, despite the well-documented evidence on the leverage effect.

The use of time-changed Lévy processes can extract the best features in the above two literature streams since it generalizes both streams simultaneously. In particular, our framework generalizes the affine Poisson jump-diffusion economy of Duffie, Pan, and Singleton (2000) by relaxing the affine

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requirement and by allowing more general specifications of the jump structure. We also generalize the stochastic volatility Lévy models by letting changes in volatility be arbitrarily correlated with asset returns. Hence, by regarding both literature streams from a more general perspective, we are able to capture both high jump activity and the leverage effect.

Since the pioneering work of Heston (1993), both literature strands have focused on the use of characteristic functions for understanding the proposed processes. The characteristic function is a complex-valued function which is in a one-to-one correspondence with the probability density function. It is well known that when the risk-neutral probability density function (PDF) of the underlying is known in closed form, option prices can be obtained by a single integration of their payoff against this PDF. Analogously, when the characteristic function of the underlying is known in closed form, option prices can also be obtained by a single integration. The numerical valuation is sufficiently fast that a wide variety of models can be tested empirically.

Bakshi and Madan (2000) provide an economic foundation for characteristic functions by considering complex exponentials as an alternative set of basis functions for spanning the payoff universe. In this interpretation, the characteristic function is the price of a claim whose payoff is given by a sinusoid, just as the risk neutral density is the price of a claim with a delta function payoff. For many interesting random variables, the characteristic function is simpler analytically than the probability density function. Accordingly, this paper focusses on developing analytic expressions for the characteristic function of a time-changed Lévy process.

The characteristic function of a random return is defined as the expected value of the complex exponential of the return. When the return is given by a stochastic process evaluated on a stochastic clock, deriving the characteristic function involves integrating over the two sources of randomness. We show that the key to obtaining the characteristic function in our general setting is to employ the powerful tool of a measure change. This measure change simplifies the expectation operation into an integration over a single source of uncertainty, thereby reducing the problem to one that has already been solved in the finance literature.

Measure changes have already seen wide application in option pricing theory. For example, measure changes are used to switch from statistical to risk-neutral measure, and they are further used to

change numeraires to dramatically simplify calculations (see Margrabe (1978), Geman, El Karoui, and Rochet (1995), Schroder (1999), Benninga, Björk, and Wiener (2001)). As an example of the latter use, consider valuing an option on a foreign stock. A priori, there are two sources of risk, namely the stock price in the foreign currency and the exchange rate. However, by valuing the option in the foreign economy, the valuation problem reduces to taking the expected value over just the terminal stock price.

The same dimension reduction arises in our use of a measure change to determine the characteristic function of a time-changed Lévy process. However, as the characteristic function is defined on the complex plane, our new measure must also be complex-valued. The reason behind this result is explored in detail in the next section. The introduction of the effective use of a complex-valued measure is the main methodological contribution of this paper. Besides providing a formal mathematical proof of our main result, we deliberately select some familiar examples from the literature in order to confirm the validity of our novel approach. We find that in every case, our approach agrees with the known results previously obtained by solving partial differential equations.

In many respects, the sophistication of a field is measured by the extent to which it is willing to use abstract methods to solve concrete problems. Complex analysis has been widely used for decades in many fields outside of finance and is just beginning to see use inside the field. Just as the risk-neutral measure effectively removes the complications arising from correlation between the pricing kernel and the payoff, our use of a complex-valued measure removes the complications arising in determining the characteristic function when there is correlation between the Lévy process and the random time. Using the same logic that leads to the term risk-neutral measure, we refer to our new measure as the **leverage-neutral** measure. Under the complex-valued leverage-neutral measure, the expectation can be performed as if there is no leverage effect. Just as the effects of risk-aversion are embedded in risk-neutral probabilities, leverage effects are embedded in our leverage-neutral measure.

Given the characteristic function of a stochastic process, Heston (1993) shows how to numerically value standard European options by using Lévy's inversion formula for the distribution function. By analytically relating the Fourier transform of an option price to its characteristic function, Carr and Madan (1999) show how the fast Fourier transform (FFT) can alternatively be used to speed up the calculation. In contrast to Heston (1993), their approach uses generalized Fourier transforms, which requires that the argument of the characteristic function be evaluated in a particular domain of the

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complex plane. In this paper, we extend the formulation in Carr and Madan (1999) to a wide variety of contingent claims. Our work complements recent work on applying FFT technology to spread options by Dempster and Hong (2000) and to Asian options by Benhamou (2000). We show that the choice of domain for the argument of the characteristic function depends on the exact structure of the state-contingent payoff. We identify the admissible domains for a wide variety of state-contingent claims, thereby reaping the significant computational benefits of FFT technology for valuation and estimation.

The structure of the paper is as follows. The next section presents the fundamental theorem simplifying the calculation of the characteristic function of the time-changed Lévy process. Section 3 shows how time-changed Lévy processes can be used to model the uncertainty of the economy. Section 4 provides extensive examples of Lévy processes, random time changes, and feasible pairings of them. Section 5 shows how FFT technology can be used to efficiently value many state-contingent claims from knowledge of the characteristic function. Section 6 briefly summarizes the paper and suggests avenues for future research.

2. Time-changed Lévy processes

Consider a d-dimensional real-valued stochastic process $\{X_t | t \geq 0\}$ with $X_0 = 0$ defined on an underlying probability space $(\Omega, \mathcal{F}, P)$ endowed with a standard complete filtration $\mathbf{F} = \{\mathcal{F}_t | t \geq 0\}$. We assume that X is a Lévy process with respect to the filtration $\mathbf{F}$. That is, X_t is adapted to $\mathcal{F}_t$, the sample paths of X are right-continuous with left limits, and $X_u - X_t$ is independent of $\mathcal{F}_t$ and distributed as X_{u-t} for $0 \leq t < u$. By the Lévy-Khintchine Theorem (see Bertoin (1996), page 12), the characteristic function of X_t has the form,

$$\phi_{X_t}(\theta) \equiv E\left[e^{i\theta^\top X_t}\right] = e^{-t\Psi_x(\theta)}, \quad t \geq 0, \tag{1}$$

where the *characteristic exponent* $\Psi_x(\theta)$, $\theta \in \mathbb{R}^d$, is given by:

$$\Psi_x(\theta) \equiv -i\mu^\top \theta + \frac{1}{2}\theta^\top \Sigma \theta + \int_{\mathbb{R}_0^d} \left(1 - e^{i\theta^\top x} + i\theta^\top x 1_{|x|<1}\right) \Pi(dx). \tag{2}$$

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The Lévy process X is specified by the vector $\mu \in \mathbb{R}^d$, the positive semi-definite matrix Σ on $\mathbb{R}^{d \times d}$, and the Lévy measure Π defined on $\mathbb{R}_0^d$ ($\mathbb{R}^d$ less zero). The triplet (μ, Σ, Π) is referred to as the *Lévy characteristics* of X. Intuitively, the first member of the triplet describes the constant drift of the process. The second member describes the constant covariance matrix of the continuous components of the Lévy process. Finally, the third member of the triplet describes the jump structure. In particular, the Lévy measure Π describes the arrival rates for jumps of every possible size for each component of X.

To value options, we extend the characteristic function parameter θ to the complex plane, $\theta \in \mathcal{D} \subseteq \mathbb{C}^d$, where $\mathcal{D}$ is the set of values for θ for which the expectation in (1) is well defined. When $\phi_{X_t}(\theta)$ is defined on the complex plane, it is referred to as the *generalized* Fourier transform (see Titchmarsh (1975)).

Next, let $t \to T_t (t \geq 0)$ be an increasing right-continuous process with left limits such that for each fixed t, the random variable T_t is a *stopping time* with respect to $\mathbf{F}$. Suppose, furthermore, that T_t is finite P-a.s. for all $t \geq 0$ and that $T_t \to \infty$ as $t \to \infty$. Then the family of stopping times $\{T_t\}$ defines a *random time change*. Without loss of generality, we further normalize the random time change so that $E[T_t] = t$. With this normalization, the family of stopping times is an unbiased reflection of calendar time.

Finally, consider the d-dimensional process Y obtained by evaluating X at T, i.e.,

$$Y_t \equiv X_{T_t}, \quad t \geq 0.$$

We propose that this process describe the underlying uncertainty of the economy. For example, in the one dimensional case, we may take Y as describing the returns on the asset underlying an option. Obviously, by specifying different Lévy characteristics for X_t and different random processes for T_t, we can generate a plethora of stochastic processes from this setup.

In principle, the random time T_t can be modeled as a nondecreasing semimartingale,

$$T_t = \alpha_t + \int_0^t \int_0^\infty y\mu(dt, dy), \tag{3}$$

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where α_t is the locally deterministic component and μ denotes the counting measure of the jumps of the semimartingale. For simplicity, we suppress jumps in time and focus on a locally deterministic time change. This simplification allows us to characterise the random time in terms of its local intensity $v(t)$,

$$T_t = \alpha_t = \int_0^t v(s_-)ds,$$

where $v(t)$ is the *instantaneous (business) activity rate*. Intuitively, one can regard t as the calendar time and T_t as the business time at calendar time t. A more active business day, captured by a higher activity rate, generates higher volatility for the economy. The randomness in business activity generates randomness in volatility. In particular, changes in the business activity rate can be correlated with innovations in X_t, due to leverage effects for example.

Note that although T_t has been assumed to be continuous, the instantaneous activity rate process $v(t)$ can jump. However, it needs to be nonnegative in order that T_t not decrease. Also note that in this paper, the term "volatility" is used generically to describe the uncertainty surrounding financial activities in the underlying economy. It is not used as a statistical term for the standard deviation of returns. In fact, when X_t is a Brownian motion, the activity rate is proportional to the instantaneous *variance* rate of the Brownian motion. When X_t is a pure jump Lévy process, $v(t)$ is proportional to the Lévy density of the jumps.

Many well known option pricing models arise as special cases of our framework. For example, the stochastic volatility model of Heston (1993) can be generated by randomly time-changing a Brownian motion and by specifying the activity rate as a mean-reverting square root process. The affine jump-diffusion economy of Duffie, Pan, and Singleton (2000) can be generated when the jump components in X_t are compound Poisson jumps and when the stochastic process for the activity rate $v(t)$ satisfies the affine constraints.[2] Our framework allows more general jump structures in both X_t and v_t. Furthermore, the dynamics for the activity rate process v_t does not need to be restricted to have an affine specification.

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[2]The drift vector and covariance matrix of the diffusion are affine in the state vector $[X_t, v_t]^\top$. When compound Poisson jumps are present in the activity rate process, the Poisson intensity is also affine in the state vector.

Since the time-changed process $Y_t \equiv X_{T_t}$ is a stochastic process evaluated at a stochastic time, its characteristic function involves expectations over two sources of randomness,

$$\phi_{Y_t}(\theta) \equiv E e^{i\theta^\top X_{T_t}} = E\left[E\left[e^{i\theta^\top X_u}|T_t = u\right]\right]. \tag{4}$$

If the random time T_t is independent of X_t, the randomness due to the Lévy process can be integrated out using equation (1),

$$\phi_{Y_t}(\theta) = E e^{-T_t \Psi_x(\theta)} = \mathcal{L}_{T_t}(\Psi_x(\theta)). \tag{5}$$

Thus under independence, the characteristic function of Y_t is just the Laplace Transform of T_t evaluated at the characteristic exponent of X. Hence, the characteristic function of Y_t can be expressed in closed form if the characteristic exponent $\Psi_x(\theta)$ of X_t and the Laplace transform for T_t are both available in closed form. In principle, the characteristic exponent can be computed from the Lévy-Khintchine Theorem in (2). To obtain the Laplace transform in closed form, consider its specification in terms of the activity rate v_t,

$$\mathcal{L}_{T_t}(\lambda) \equiv E\left[\exp\left(-\lambda \int_0^t v(s_-)ds\right)\right].$$

This formulation arises in the bond pricing literature if we regard $\lambda v(t)$ as the instantaneous interest rate. Furthermore, the instantaneous interest rate and the instantaneous activity rate are both required to be nonnegative and can be modeled by similar processes. Thus, one can adopt the vast literature in term structure modeling for the purpose of modeling the instantaneous activity rate $v(t)$.

Our primary objective is to generalize the reduction in (5) of the characteristic function to a bond pricing formula to the case where the Lévy process and time change are correlated. This generalization would allow us to easily capture the well-known leverage effect. Before presenting the formal theorem on this generalization, it is useful to consider the special case when the Lévy process has a symmetric distribution about zero, e.g., a standard Brownian motion. The corresponding characteristic function of such a symmetric Lévy process is real, so is its characteristic exponent. Now consider a time change on this symmetric Lévy process. If the time change is independent of the Lévy innovation, the distribution of the time changed process remains symmetric. Its characteristic function remains real and can be solved via iterated expectation as in (4) and the bond pricing analog in (5).

Alternatively, one can introduce asymmetry to the distribution of the time-changed Lévy process by introducing correlation between the time change and the Lévy innovation. Then, the characteristic function of this distribution must have a non-zero imaginary part due to the asymmetry. Our objective is to generate this non-zero imaginary part while still using the bond pricing analog:

$$\phi_{Y_t}(\theta) = E\left[\exp\left(-\Psi_x(\theta)\int_0^t v(s)ds\right)\right], \tag{6}$$

where both the characteristic exponent $\Psi_x(\theta)$ and the stochastic time T are real valued. We achieve this objective by taking the above expectation under a complex-valued measure. When the real-valued random variables are averaged using complex weights rather than real ones, the resulting characteristic function becomes complex-valued, as is required to correspond with an asymmetric distribution.

The following theorem shows that the use of a complex-valued measure is the key to determining the characteristic function of the time-changed process. This theorem is the main contribution of this paper. It shows that a complex-valued measure can be used to reduce the problem of finding the characteristic function of Y_t in the original economy into the problem of finding it in an artificial economy that is devoid of the leverage effect. From (5), we see that this calculation in turn reduces to determining the Laplace Transform of the random time in the leverage-neutral economy. As the Laplace transform calculation itself reduces to a bond pricing formula, we can find characteristic functions for a wide array of processes resulting from pairing Lévy processes with correlated time changes.

Theorem 1 *The problem of finding the generalized Fourier transform of the time-changed Lévy process $Y_t \equiv X_{T_t}$ under measure P reduces to the problem of finding the Laplace transform of the random time under the complex-valued measure $Q(\theta)$, evaluated at the characteristic exponent $\Psi_x(\theta)$ of X_t,*

$$\phi_{Y_t}(\theta) \equiv E\left[e^{i\theta^\top Y_t}\right] = E^\theta\left[e^{-T_t\Psi_x(\theta)}\right] \equiv \mathcal{L}_{T_t}^\theta\left(\Psi_x(\theta)\right), \tag{7}$$

where $E[\cdot]$ and $E^\theta[\cdot]$ denote expectations under measures P and $Q(\theta)$, respectively. The new class of complex-valued measures $Q(\theta)$ are absolutely continuous with respect to P and are defined by

$$\left.\frac{dQ(\theta)}{dP}\right|_t \equiv M_t(\theta),$$

with

$$M_t(\theta) \equiv \exp\left(i\theta^\top Y_t + T_t \Psi_x(\theta)\right), \quad \theta \in \mathcal{D}. \tag{8}$$

Note that the last two equalities in (7) extend the notions of expected value and Laplace transform beyond their usual domain. As indicated, the "expected value" $E^\theta\left[e^{-T_t \Psi_x(\theta)}\right]$ is to be computed under the complex measure $Q(\theta)$ rather than the usual real one. Furthermore, the "Laplace Transform" $\mathcal{L}_{T_t}^\theta$ is not the usual Laplace Transform of T_t due to the dependence of the measure Q on θ.[3] The superscript θ is used to indicate this extended Laplace Transform, which can still be interpreted as a bond pricing formula.

To prove the theorem, we first need to prove that $M_t(\theta), \theta \in \mathcal{D}$, defined in (8) is a *P*-martingale with respect to the filtration generated by the process $\{(Y_t, T_t) : t \geq 0\}$.

Lemma 1 *For every $\theta \in \mathcal{D}$, $M_t(\theta)$ in (8) is a complex-valued P-martingale with respect to the filtration generated by the process $\{(Y_t, T_t) : t \geq 0\}$.*

Proof. First, define

$$Z_t(\theta) \equiv \exp\left(i\theta^\top X_t + t\Psi_x(\theta)\right). \tag{9}$$

Given that $\Psi_x(\theta)$ is finite by definition, $E\left[|Z_t(\theta)|\right]$ is finite since

$$E\left[|Z_t(\theta)|\right] \leq \exp\left(t\Psi_x(\theta)\right).$$

Now for $0 \leq s < t$,

$$E\left[e^{i\theta^\top (X_t - X_s) + \Psi_x(\theta)(t-s)} \big| \mathcal{F}_s\right] = e^{-\Psi_x(\theta)(t-s) + \Psi_x(\theta)(t-s)} = 1.$$

Hence, $Z_t(\theta)$ is a complex-valued *P*-martingale with respect to $\{\mathcal{F}_t | t \geq 0\}$.

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[3]We thank a referee for pointing this out.

Next, for every fixed $t \geq 0$, T_t is a stopping time which is finite P-a.s. By the optional stopping theorem, $M_t(\theta) \equiv Z_{T_t}(\theta)$ is also a complex-valued martingale with respect to the filtration generated by the process $\{(Y_t, T_t) : t \geq 0\}$. ∎

$Z_t(\theta)$ is the familiar Wald martingale.[4] We extend the real-valued exponential family of martingales defined on Lévy processes in (Küchler and Sørensen 1997, page 8) to the complex plane. Similarly, $M_t(\theta) = Z_{T_t}(\theta)$ can be regarded as a complex extension of the time-changed exponential martingale in (Küchler and Sørensen 1997, page 230).

Given that $M_t(\theta)$ is a well-defined complex-valued P-martingale, the proof of Theorem 1 is straightforward.

Proof. (*Theorem 1*)

$$
\begin{aligned}
E\left[e^{i\theta^\top Y_t}\right] &= E\left[e^{i\theta^\top Y_t + T_t \Psi_x(\theta) - T_t \Psi_x(\theta)}\right] \\
&= E\left[M_t(\theta)e^{-T_t \Psi_x(\theta)}\right] = E^\theta\left[e^{-T_t \Psi_x(\theta)}\right] = \mathcal{L}_{T_t}^\theta\left(\Psi_x(\theta)\right).
\end{aligned}
$$

∎

Our theorem generalizes the previous results on an independent time change to the case where the Lévy process and the time change can be correlated. When T_t is independent of X_t, our result reduces to the previous one. The reason is that T_t follows the same process under the two measures P and $Q(\theta)$ and hence $\mathcal{L}_{T_t} \equiv \mathcal{L}_{T_t}^\theta$. In other words, if the original economy is devoid of the leverage effect, no measure change is required. When the original economy does possess the leverage effect, our complex-valued measure change simplifies the calculation by absorbing the effects of the correlation into the measure. One can then perform the expectation under this new measure as if the economy is devoid of the leverage effect. In analogy with the terminology underlying the risk-neutral measure, we christen this new complex-valued measure as the **leverage-neutral measure**.

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[4]See Harrison (1985), page 7, Karlin and Taylor (1975), page 243, and Bertoin (1996), page 40 for example.

3. Asset Pricing under Time-Changed Lévy Processes

We use the time-changed Lévy process, $Y_t \equiv X_{T_t}$, to model the uncertainty of the economy. In this section, we illustrate how asset returns can be modeled as time-changed Lévy processes, how market prices of risk can be defined on such processes, and how these definitions of risk premia link the objective dynamics of Y_t to its risk-neutral dynamics.

3.1. Asset price modeling

As one application, we can use the time-changed Lévy process as the driver of asset return processes. Specifically, let S_t denote the price at time t of an asset, e.g., a stock or a currency. Then, we can specify the price process as an exponential affine function of the uncertainty Y_t,

$$S_t = S_0 e^{\vartheta^\top Y_t}, \tag{10}$$

where S_0 denotes the price at time 0, which we assume is known and fixed. Let $s_t \equiv \ln(S_t/S_0)$ denote the log return of the asset. Then, by Theorem 1, the generalized Fourier transform of s_t is given by

$$\phi_{s_t}(u) \equiv E\left[e^{ius_t}\right] = E\left[e^{iu\vartheta^\top Y_t}\right] = \phi_{Y_t}(u\vartheta) = \mathcal{L}_{T_t}^{u\vartheta}(\Psi_x(u\vartheta)). \tag{11}$$

For option pricing, the asset price process is often specified directly under the risk-neutral measure, under which the instantaneous rate of return on an asset is determined by no arbitrage. Formally, it can be specified as,

$$S_t = S_0 e^{(r-q)t} e^{\vartheta^\top Y_t}, \quad \text{with} \quad E\left[e^{\vartheta^\top Y_t}\right] = 1, \tag{12}$$

where r is the continuously compounded riskless rate and q is the dividend yield in the case of a stock and foreign interest rate in the case of a currency, both of which are assumed constant. To assure no arbitrage, we restrict the specification of the time-changed Lévy process Y_t to guarantee that the last

term is an exponential martingale under the risk-neutral measure. Alternatively, for an arbitrary Y_t, we can replace the last term $e^{\vartheta^\top Y_t}$ by the following *Doléans-Dade exponential*,

$$\mathcal{E}\left(\vartheta^\top Y_t\right) = \exp\left(\vartheta^\top Y_t + T_t^\top \Psi_x(-i\vartheta)\right), \tag{13}$$

which has mean one by Lemma 1. The generalized Fourier transform of s_t under this specification can also be derived via Theorem 1,

$$\begin{aligned}
\phi_{s_t}(u) &\equiv E\left[e^{ius_t}\right] = E\left[\exp\left(iu(r-q)t + iu\vartheta^\top Y_t + iuT^\top \Psi_x(-i\vartheta)\right)\right] \\
&= e^{iu(r-q)t} \mathcal{L}_{T_t}^{u\vartheta}\left(\Psi_x(u\vartheta) - iu\Psi_x(-i\vartheta)\right).
\end{aligned} \tag{14}$$

3.2. Market price of risk

A current research trend is to perform integrated price series analysis of both derivative securities and their underlying assets. A key objective of such an analysis is to identify how the market prices different sources of risk. The literature has followed three different routes in analyzing market risk premia. The first approach starts with a specification of a general equilibrium and utility functions for agents. The functional form of the risk premia are derived from this equilibrium setting (see Bates (1996, 2000)). The second approach starts with a specification of the pricing kernel and links the pricing of the underlying asset and its derivatives via this pricing kernel. Under regularity conditions, no arbitrage implies the existence of at least one such kernel (Duffie (1992)). Examples of this line of research include Pan (2002) and Eraker (2001). The pricing kernel can be regarded as the reduced form of some general equilibrium and hence is more flexible in terms of its specification. Finally, the third approach takes flexibility to another level by nonparametrically estimating the pricing kernel under each state. Specifically, this approach first estimates the conditional density of the asset price under the objective measure using the time series data of the underlying asset returns, and then estimates the conditional density under the risk-neutral measure using option prices. The pricing kernel under each state is then given by the ratio of the two conditional densities. Examples along this line of research include Carr, Geman, Madan, and Yor (2001) and Engle and Rosenberg (1997).

When the sources of risk for an economy are governed by time-changed Lévy processes, the measure change from the objective measure to the risk-neutral measure can be conveniently defined by a set of exponential martingales. Formally, let ξ_t denote the pricing kernel, which relates future cash flows, $K_s, s \in (t, \mathcal{T}]$, to today's price, p_t, by,

$$p_t = E\left[\left.\int_t^{\mathcal{T}} \frac{\xi_s K_s ds}{\xi_t}\right| \mathcal{F}_t\right].$$

One can perform a multiplicative decomposition on the kernel,

$$\xi_t = \exp\left(-\int_0^t r_s ds\right) \cdot \mathcal{E}\left(-\gamma(Y_t)\right), \tag{15}$$

where the Doléans-Dade exponential $\mathcal{E}\left(-\gamma(Y_t)\right)$ can be interpreted as the *Radon-Nikodým derivative*, which takes us from the objective measure to the risk neutral measure. $\gamma(Y_t)$ is an $\mathcal{F}_t$-adapted process satisfying the usual regularity conditions and is often referred to as the *market price of risk* for the uncertainty of the economy, Y_t.

A particularly tractable specification for the market price of risk is given by the affine form,

$$\gamma(Y_t) = \gamma^\top Y_t, \quad \gamma \in \mathcal{D} \subset \mathbb{R}^d, \tag{16}$$

where $\mathcal{D}$ is a subset of the d-dimensional real space so that $\mathcal{E}\left(-\gamma(Y_t)\right)$ is well-defined. Then, the Radon-Nikodým derivative is given by the class of *Esscher transforms*. Measure changes under this specification take extremely simple forms. In Appendix A, we identify the characteristics of Y_t under measure changes defined by Esscher transforms. Furthermore, measure changes expressible as Esscher transforms are often supported by utility optimization (e.g., Keller (1997), Kallsen (1998)) or entropy minimization problems (Chan (1999)). See Kallsen and Shiryaev (2000) for an excellent analysis on measure changes defined by Esscher transforms and their economic underpinnings.

4. Specification Analysis

Through extensive examples, this section addresses the issue of specifying the Lévy process, the activity rate process, and the correlation between the two. For each Lévy process, we focus on the derivation of its characteristic exponent; for each activity rate process, we focus on the derivation of the Laplace transform of the random time; and finally, for each pairing of the two, we focus on the form of the complex-valued measure change and the Laplace transform of the random time under this new measure. By Theorem 1, a joint specification of the Lévy process and the activity rate process determines the characteristic function of the time-changed Lévy process. Whenever possible, we link each example to the existing literature to show how they can be arrived at from our general perspective. We also illustrate how these different specifications can be combined to generate a plethora of tractable specifications for the uncertainty of the economy.

4.1. The Lévy process and its characteristic exponent

Since a Lévy process is uniquely characterized by its triplet of Lévy characteristics (μ, Σ, Π), the Lévy process is determined by individual specification of the components of this triplet. The first component μ is the constant drift term. This component is often determined by no arbitrage or equilibrium pricing relations and thus depends on the specification of the other two elements of the triplet. The second component Σ denotes the constant covariance matrix of a vector diffusion martingale. The third component is the Lévy measure $\Pi(dx)$, which controls the arrival rate of jumps of size x. By definition, this third jump component is orthogonal to the second diffusion component. Since the properties of a diffusion component are well known, we will focus on the properties of the lesser known jump component.

A pure jump Lévy process can display either finite activity or infinite activity. In the former case, the aggregate jump arrival rate is finite, while in the latter case, an infinite number of jumps can occur in any finite time interval. Within the infinite activity category, the sample path of the jump process can either exhibit finite variation or infinite variation. In the former case, the aggregate absolute distance travelled by the process is finite, while in the latter case, it can be infinite over any finite time interval. Hence, there are three types of jump processes in all. For each type, we discuss the defining properties

and existing examples in the literature. For ease of notation, we focus on scalar processes and use $\pi(dx)$ to denote the Lévy measure of such a scalar process.

4.1.1. Finite activity Lévy jumps

A pure jump Lévy process exhibits finite activity if the following integral is finite:

$$\int_{\mathbb{R}_0} \pi(dx) = \lambda < \infty. \tag{17}$$

Intuitively speaking, a finite activity jump process exhibits a finite number of jumps within any finite time interval. The classical example of a finite activity jump process is the *compound Poisson jump* process of Merton (1976) (MJ). For such processes, the integral in (17) defines the *Poisson intensity*, λ. Conditional on one jump occurring, the MJ model assumes that the jump magnitude is normally distributed with mean α and variance σ_j^2. The Lévy measure of the MJ process is given by

$$\pi(dx) = \lambda \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x-\alpha)^2}{2\sigma_j^2}\right) dx. \tag{18}$$

Obviously, one can choose any distribution, $F(x)$, for the jump size under the compound Poisson framework and obtain the following Lévy measure,

$$\pi(dx) = \lambda dF(x).$$

The exact specification of the conditional jump distribution should be determined by the data. For example, Kou (1999) assumes a double-exponential conditional distribution for the jump size. The Lévy measure in this case is given by

$$\pi(dx) = \lambda dF(x) = \lambda \frac{1}{2\eta} \exp\left(-\frac{|x-k|}{\eta}\right) dx.$$

In another example, Eraker, Johannes, and Polson (2000) and Eraker (2001) incorporate compound Poisson jumps into the stochastic volatility process, assuming that volatility can only jump upward and that the jump size is controlled by a one-sided exponential density. The Lévy measure is given by

$$\pi(dx) = \lambda dF(x) = \lambda \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right) dx, \quad x > 0.$$

Based on the *Lévy-Khintchine formula* in (2), the characteristic exponent corresponding to these compound Poisson jump components is given by

$$\Psi(\theta) = \int_{\mathbb{R}_0} \left(1 - e^{i\theta x}\right) \lambda dF(x) = \lambda\left(1 - \phi(\theta)\right), \tag{19}$$

where $\phi(\theta)$ denotes the characteristic function of the jump size distribution $F(x)$,

$$\phi(x) \equiv \int_{\mathbb{R}_0} e^{i\theta x} dF(x).$$

4.1.2. Infinite activity Lévy jumps

Unlike a finite activity jump process, an *infinite activity* jump process can generate an infinite number of jumps within any finite time interval. The integral of the Lévy measure in (17) is no longer finite. Examples in this class include the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), the generalized hyperbolic class of Eberlein, Keller, and Prause (1998), the variance gamma (VG) model of Madan and Milne (1991) and Madan, Carr, and Chang (1998), its generalization to the CGMY model of Carr, Geman, Madan, and Yor (2002), and the finite moment log-stable (LS) model of Carr and Wu (2002).

We list the Lévy measures and characteristic exponents of each of these examples in Table 1. For comparison, we also list the Lévy measures and characteristic exponents of the finite activity jump examples. Finally, for completeness, we list the characteristic exponent of an arithmetic Brownian motion, which is the only purely continuous Lévy process. Note that for the infinite activity pure jump Lévy examples, the NIG model is a special case of the generalized hyperbolic class with $\lambda = -1/2$.

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Furthermore, under the following parameterization, the VG model can be regarded as a special case of the CGMY model:

$$C = \lambda, G = \mu_-/\nu_-, M = \mu_+/\nu_+, Y = 1.$$

Finally, the LS model can also be regarded as a special case of CGMY with $G = M = 0$ and also with $C = 0$ when $x > 0$.

The sample paths of an infinite activity jump process exhibit *finite variation* if the following integral is finite:

$$\int_{\mathbb{R}_0} (1 \wedge |x|)\,\pi(dx) < \infty, \tag{20}$$

and *infinite variation* if the integral is infinite. Nevertheless, the *quadratic variation* has to be finite for the Lévy measure to be well-defined,

$$\int_{\mathbb{R}_0} (1 \wedge x^2)\pi(dx) < \infty.$$

The sample paths of the generalized hyperbolic class exhibit finite variation. The sample paths of the CGMY process exhibits finite variation when $Y \leq 1$ and infinite variation when $Y \in (1,2]$. For the quadratic variation to be finite, we need $Y \leq 2$.

4.2. The activity rate process and the Laplace transform

Given some specification of the Lévy process, the next step is to specify the random time. Since the random time is given by the integral, $T_t = \int_0^t v(s_-)ds$, we determine this random time by specifying the activity rate process $v(t)$. Given a process for $v(t)$, the Laplace transform of T_t is given by,

$$\mathcal{L}_{T_t}(\lambda) \equiv E\left[\exp\left(-\lambda \int_0^t v(s_-)ds\right)\right].$$

This formulation is analogous to the pricing formula for a zero coupon bond if we regard $v(t)$ as the "instantaneous interest rate." In particular, both the instantaneous interest rate and the instantaneous activity rate are required to be positive. We can therefore adopt existing interest rate models to model the instantaneous activity rate $v(t)$. In particular, we illustrate how to apply two of the most tractable

classes of term structure models to the modeling of the activity rates and to the derivation of the Laplace transform of the random time.

4.2.1. Affine activity rate models

Let Z be a k-dimensional Markov process, which starts at z_0 and satisfies the following stochastic differential equation:

$$dZ_t = \mu(Z_t)dt + \sigma(Z_t)dW_t + qdJ(\gamma(Z_t)). \tag{21}$$

Here, W is a k-dimensional Wiener process and J is a Poisson jump component with intensity $\gamma(Z_t)$ and random jump magnitude q, characterized by its two-sided Laplace transform $\mathcal{L}_q(\cdot)$. Furthermore, we require that the $k \times 1$ vector $\mu(Z_t)$ and $k \times k$ matrix $\sigma(Z_t)$ satisfy some technical conditions, such that the stochastic differential equation has a strong solution. The instantaneous rate of activity $v(t)$ is assumed to be a function of the Markov process Z_t.

Definition 1 *In **affine activity rate models**, the Laplace transform of the random time, $T_t = \int_0^t v(s_-)ds$, is an exponential-affine function of the Markov process Z_t:*

$$\mathcal{L}_{T_t}(\lambda) \equiv E\left[e^{-\lambda T_t}\right] = \exp\left(-\mathbf{b}(t)^\top z_0 - c(t)\right), \tag{22}$$

where $\mathbf{b}(t) \in \mathbb{R}^k$ and $c(t)$ is a scalar.

The following proposition presents conditions which are sufficient for (22) to hold.

Proposition 1 *If the instantaneous activity rate $v(t)$, the drift vector $\mu(Z)$, the diffusion covariance matrix $\sigma(Z)\sigma(Z)^\top$, and the arrival rate $\gamma(Z)$ of the Markov process are all affine in Z, then the Laplace transform $\mathcal{L}_{T_t}(\lambda)$ is exponential-affine in z_0.*

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The above process and proposition generalize the original work of Duffie and Kan (1996) for the affine term structure models of interest rates. The extension to compound Poisson-type jumps (with time-varying Poisson jump intensity) is due to Duffie, Pan, and Singleton (2000). In particular, let

$$
\begin{aligned}
v(t) &= \mathbf{b}_v^\top Z_t + c_v, \quad \mathbf{b}_v \in \mathbb{R}^k, c_v \in \mathbb{R}, \\
\mu(Z_t) &= a - \kappa Z_t, \quad \kappa \in \mathbb{R}^{k \times k}, a \in \mathbb{R}^k, \\
\left[\sigma(Z_t)\sigma(Z_t)^\top \right]_{ii} &= \alpha_i + \beta_i^\top Z_t, \quad \alpha_i \in \mathbb{R}, \beta_i \in \mathbb{R}^k, \\
\left[\sigma(Z_t)\sigma(Z_t)^\top \right]_{ij} &= 0, \quad i \neq j, \\
\gamma(Z_t) &= a_\gamma + \mathbf{b}_\gamma^\top Z_t, \quad a_\gamma \in \mathbb{R}, \mathbf{b}_\gamma \in \mathbb{R}^k.
\end{aligned}
$$

Then the coefficients $\{\mathbf{b}(t), c(t)\}$ for the Laplace transform in (22) are determined by the following ordinary differential equations:

$$
\begin{aligned}
\mathbf{b}'(t) &= \lambda \mathbf{b}_v - \kappa^\top \mathbf{b}(t) - \beta \mathbf{b}(t)^2/2 - \mathbf{b}_\gamma (\mathcal{L}_q(\mathbf{b}(t)) - 1); \qquad (23) \\
c'(t) &= \lambda c_v + \mathbf{b}(t)^\top a - \mathbf{b}(t)^\top \alpha \mathbf{b}(t)/2 - a_\gamma (\mathcal{L}_q(\mathbf{b}(t)) - 1),
\end{aligned}
$$

with the boundary conditions: $\mathbf{b}(0) = 0$, and $c(0) = 0$. Closed-form solutions for the coefficients exist only under special cases, although they are easily computed numerically. A one-factor special case, where an analytical solution is available, is the square-root model of Cox, Ingersoll, and Ross (1985) for interest rates and Heston (1993) for stochastic volatility.

4.2.2. Affine activity rate models with more general jump specifications

Jumps in the above affine framework are confined to be finite activity compound Poisson type jumps. The jump intensity γ can depend on the state vector and hence be time-varying. In a one factor setting, we can adopt the following generalized version of the affine term structure model due to Filipovic

(2001), which allows a more flexible jump specification. Formally, we can characterize the activity rate process $v(t)$ as a Feller process with generator,

$$
\begin{aligned}
\mathcal{A}f(x) \;=\; & \frac{1}{2}\sigma^2 x f''(x) + (a' - \kappa x) f'(x) \\
& + \int_{\mathbb{R}_0^+} \big(f(x+y) - f(x) - f'(x)(1 \wedge y) \big)(m(dy) + x\mu(dy)),
\end{aligned}
\tag{24}
$$

where $a' = a + \int_{\mathbb{R}_0^+}(1 \wedge y) m(dy)$ for some constant numbers $\sigma, a \in \mathbb{R}^+$, $\kappa \in \mathbb{R}$, and nonnegative Borel measures $m(dy)$ and $\mu(dy)$ satisfying the following condition:

$$
\int_{\mathbb{R}_0^+}(1 \wedge y) m(dy) + \int_{\mathbb{R}_0^+}\big(1 \wedge y^2\big)\mu(dy) < \infty.
\tag{25}
$$

Note that the first line in (24) is due to the continuous part of the process and is equivalent to the Cox, Ingersoll, and Ross (1985) or Heston (1993) specification. The second line is due to the jump part of the process. All three components of the Lévy triplet depend linearly on the state variable x. Such processes are known as (stochastically continuous) conservative CBI processes (continuous state branching processes with immigration) and have been well studied by Kawazu and Watanabe (1971) among others. The condition in (25) says that the jump component dictated by the measure $m(dy)$ has to exhibit finite variation, while the jump component dictated by the measure $\mu(dy)$ only needs to exhibit finite quadratic variation. Hence, one can adopt any of the Lévy measure specifications in Table 1 for $\mu(dy)$, and any of the finite variation ones for $m(dy)$, with only one slight modification: arrival rates of negative jumps need to be set to zero.

Under such a specification, the Laplace transform of the random time is exponential affine in the current activity rate level v_0,

$$
\mathcal{L}_{T_t}(\lambda) = \exp\left(-b(t)v_0 - c(t)\right),
\tag{26}
$$

with the coefficients $[b(t), c(t)]$ given by the following ordinary differential equations:

$$
\begin{aligned}
b'(t) \;=\; & \lambda - \kappa b(t) - \frac{1}{2}\sigma^2 b(t)^2 + \int_{\mathbb{R}_0^+}\left(1 - e^{-yb(t)} - b(t)(1 \wedge y)\right)\mu(dy), \\
c'(t) \;=\; & ab(t) + \int_{\mathbb{R}_0^+}\left(1 - e^{-yb(t)}\right)m(dy),
\end{aligned}
\tag{27}
$$

with the boundary conditions: $b(0) = 0$, and $c(0) = 0$.

4.2.3. Quadratic activity rate models

In this subsection, we adopt the quadratic term structure model of Leippold and Wu (2002) for the purpose of modeling the instantaneous activity rate.

Definition 2 *In **quadratic activity rate models**, the Laplace transform of the random time is an exponential-quadratic function of the Markov process Z if:*

$$\mathcal{L}_{T_t}(\lambda) = E\left[\exp(-\lambda \int_0^t v(s)ds\right] = \exp\left[-z_0^\top A(t)z_0 - \mathbf{b}(t)^\top z_0 - c(t)\right], \tag{28}$$

with $A(t) \in \mathbb{R}^{k \times k}$, $\mathbf{b}(t) \in \mathbb{R}^k$, and $c(t) \in \mathbb{R}$.

The following proposition presents the sufficient conditions for obtaining quadratic activity rate models:

Proposition 2 *If the instantaneous rate of activity $v(t)$ is quadratic in Z, $\mu(Z)$ is affine in Z, and $\sigma(Z) = \sigma$ is a constant matrix, then the Laplace transform of the random time $\mathcal{L}_{T_t}(\lambda)$ is exponential-quadratic in Z.*

The proof follows Leippold and Wu (2002). Formally, under non-degeneracy conditions and a possible re-scaling and rotation of indices, we let

$$\begin{aligned}
\mu(Z) &= -\kappa Z_t, \quad \sigma(Z) = I, \quad \kappa, I \in \mathbb{R}^{k \times k}, \\
v(t) &= Z_t^\top A_v Z_t + \mathbf{b}_v^\top Z_t + c_v, \quad A_v \in \mathbb{R}^{k \times k}, \mathbf{b}_v \in \mathbb{R}^k, c_v \in \mathbb{R}.
\end{aligned}$$

For the Markov process to be stationary, we need all eigenvalues of κ to be positive. Furthermore, a sufficient condition for $v(t)$ to be positive is to let A_v be positive definite and $c_v > \frac{1}{4}\mathbf{b}_v^\top A_v \mathbf{b}_v$. Given the

above parameterization, the ordinary differential equations governing the coefficients in the Laplace transform are given by

$$
\begin{aligned}
A'(t) &= \lambda A_v - A(t)\kappa - \kappa^\top A(\tau) - 2A(t)^2; \\
\mathbf{b}'(t) &= \lambda \mathbf{b}_v - \kappa \mathbf{b}(t) - 2A(t)^\top \mathbf{b}(t); \\
c'(t) &= \lambda c_v + trA(t) - \mathbf{b}(t)^\top \mathbf{b}(t)/2,
\end{aligned}
\tag{29}
$$

subject to the boundary conditions: $A(0) = 0$, $\mathbf{b}(0) = 0$, and $c(0) = 0$.

Table 2 summarizes the Laplace transform of the random time under the three classes of activity rate processes. Obviously, any of these activity rate specifications can be combined with the Lévy process specifications in Table 1 in forming a time-changed Lévy process.

4.3. Correlation and the leverage-neutral measure

The observed negative correlation between returns and their volatilities in the equity market is usually referred to as the leverage effect. This leverage effect can be accommodated by allowing (negative) correlations between increments in the Lévy process and increments in the activity rate process. Recall that every purely continuous component is orthogonal to every pure jump component. Hence, if the Lévy process is purely continuous, non-zero correlation can only be induced by a continuous component in the activity rate process. Similarly, if the Lévy process is pure jump, non-zero correlation can only be induced by a jump component in the activity rate process. Furthermore, if the pure jump Lévy process has finite (resp. infinite) activity, non-zero correlation can only be induced by a finite (resp. infinite) activity jump component in the activity rate process. We use examples to illustrate each case. For each example, we demonstrate how to perform the complex-valued measure change using Proposition 4 in Appendix A. We then derive the characteristic function of the time-changed Lévy process. We deliberately select some familiar examples from the literature in order to confirm the validity of our novel approach. We find that in every case, our approach accords with the known results previously obtained by solving partial differential equations. To illustrate the versatility of our approach, we also present an example that is new to the literature.

4.3.1. Leverage via diffusions

Consider the case where the Lévy process is a standard Brownian motion $X_t = W_t$, and the instantaneous activity rate follows the mean-reverting square-root process of Heston (1993). The leverage effect can be accommodated by negatively correlating the Brownian motion driving X_t and the Brownian motion driving $v(t)$. This setup is summarized by the following specification under measure P:

$$
\begin{aligned}
X_t &= W_t; \\
dv(t) &= (a - \kappa v(t))\, dt + \eta \sqrt{v(t)}\, dZ_t; \\
E\left[dW_t dZ_t\right] &= \rho dt.
\end{aligned}
$$

By Theorem 1, the characteristic function of $Y_t \equiv X_{T_t}$ can be represented as the Laplace transform of T_t under a new complex-valued measure $Q(\theta)$:

$$
\phi_{Y_t}(\theta) \equiv E\left[e^{i\theta Y_t}\right] = \mathcal{L}_{T_t}^{\theta}\left(\frac{1}{2}\theta^2\right),
$$

where $\theta^2/2$ is the characteristic exponent of the underlying Lévy process $X_t = W_t$ (see the first entry in Table 1). The new measure $Q(\theta)$ is defined by the following exponential martingale,

$$
\left.\frac{dQ(\theta)}{dP}\right|_t = \exp\left(i\theta Y_t + \frac{1}{2}\theta^2 \int_0^t v(s)\, ds\right).
$$

A slight extension of Girsanov's theorem to complex-valued measures implies that under measure $Q(\theta)$, the diffusion part of $v(t)$ is unaltered, while the drift of $v(t)$ is adjusted to

$$
\mu_v(t)^Q = a - \kappa v(t) + i\theta \eta \rho v(t).
$$

Hence, under measure $Q(\theta)$, $v(t)$ satisfies the conditions in (23) (see the first entry in Table 2) for the affine class of activity rates with

$$
\begin{aligned}
b_v &= 1, \quad c_v = 0; \\
\kappa^Q &= \kappa - i\theta\eta\rho, \quad a^Q = a; \\
\alpha &= 0, \quad \beta = \eta^2, \quad \gamma = 0; \\
\lambda &= \Psi_x(\theta) = \theta^2/2.
\end{aligned}
\tag{30}
$$

Based on Proposition 1, the characteristic function of Y_t is exponential-affine in v_0,

$$
\phi_{Y_t}(\theta) = \exp\left(-b(t)v_0 - c(t)\right),
$$

where the parameters $[a(t), b(t)]$ are given by the ordinary differential equations in (23) with the substitutions given in (30). For this particular one-factor case, the ordinary differential equations can be solved analytically,

$$
\begin{aligned}
b(t) &= \frac{-\theta^2\left(1-e^{-\delta t}\right)}{(\delta+\kappa^Q)+(\delta-\kappa^Q)e^{-\delta t}}; \\
c(t) &= \frac{-a}{\eta^2}\left[2\ln\frac{\left(2\delta-\left(\delta-\kappa^Q\right)\left(1-e^{-t\delta}\right)\right)}{2\delta}+\left(\delta-\kappa^Q\right)t\right],
\end{aligned}
$$

where $\delta^2 = (\kappa^Q)^2 + \theta^2\eta^2$.

4.3.2. Leverage via compound Poisson jumps

Consider the case where the Lévy process X_t is a compound Poisson jump process,

$$
X_t = \sum_{j=1}^{N_t} q_j^1,
$$

where N_t denotes the number of jumps within the time interval $[0,t]$ and is governed by a Poisson distribution with a constant arrival rate of γ. The conditional jump size q_j^1 is assumed to be iid.

To incorporate the leverage effect, we assume that the activity rate $v(t)$ also has a compound Poisson jump component and that the arrival rate of jumps is controlled by the same Poisson distribution,

$$v(t) = \int_0^t (a - \kappa v(s))\, ds + \sum_{j=1}^{N_{T_t}} q_j^2,$$

where q_j^2 denotes the jump size in $v(t)$ conditional on a jump occurring. We incorporate a linear mean-reverting drift to capture the persistence of volatility, but the presence or absence of it is irrelevant to our analysis. Since the jumps in X and in v are governed by the same Poisson process, they jump at the same time. Note that N_t becomes N_{T_t} after time change. Conditional on a jump event occurring, we assume that the jump sizes of the two processes $\mathbf{q} \equiv \left[q^1, q^2 \right]^\top$ have a correlated joint distribution $F(dq)$. Let $\phi(u)$ denote the joint characteristic function of $\mathbf{q}$.

Then, by Theorem 1, we have

$$\phi_{Y_t}(\theta) = \mathcal{L}_{T_t}^\theta \left(\Psi_x(\theta) \right),$$

where the characteristic exponent for X_t is (equation 19),

$$\Psi_x(\theta) = \lambda \left(1 - \phi_1(\theta) \right),$$

where $\phi_1(\theta) \equiv E\left[e^{i\theta q^1} \right]$ denotes the marginal characteristic function of q^1. The new measure $Q(\theta)$ is hence given by

$$\left. \frac{dQ(\theta)}{dP} \right|_t = \exp\left(i\theta Y_t + \Psi_x(\theta) T_t \right).$$

Straightforward application of Girsanov's theorem extended to complex measure implies that

$$F^Q(dq) = e^{i\theta q^1} F(dq).$$

The marginal characteristic function of q^2 under $Q(\theta)$ is $\phi_2^Q(b) = \phi([\theta; b])$. Other parameters in $v(t)$ remain the same under the measure change. Hence, the process describing $v(t)$ remains in the affine class

under $Q(\theta)$. The characteristic function of Y_t is therefore exponential-affine in v_0 and the coefficients are given by the solutions to the ordinary differential equations:

$$
\begin{aligned}
b'(t) &= \Psi_x(\theta) - \kappa b(t) + \gamma(1 - \phi([\theta; -ib(t)])); \\
c'(t) &= ab(t),
\end{aligned}
$$

with $b(0) = c(0) = 0$. The above ordinary differential equations are directly adopted from (23) with $\lambda = \Psi_x(\theta)$, $c_v = \alpha = a_\gamma = \eta = 0$, $b_v = 1$, $b_\gamma = \gamma$.

4.3.3. Leverage via infinite activity jumps

This example is new to the literature. Consider the log-stable (LS) model of Carr and Wu (2002) as the underlying Lévy process,

$$
X_t = L_t^{\alpha, -1},
$$

where $L_t^{\alpha, -1}$ denotes a standard Lévy α-stable motion with tail index $\alpha \in (1, 2]$ and maximum negative skewness. Note that this process not only exhibits infinite activity, but also exhibits infinite variation.

To accommodate the leverage effect, we assume that the activity rate is driven by the same Lévy α-stable motion. Since $L_t^{\alpha, -1}$ only allows negative jumps while the activity rate must be positive, we incorporate its mirror image, $L_t^{\alpha, 1}$ into the activity rate process. Hence, whenever there is a negative jump of absolute size x in X_t, there is a simultaneous positive jump of proportional size in $v(t)$. Specifically, the activity rate process solves the following stochastic differential equation,

$$
dv(t) = (a - \kappa v(t)) \, dt + \beta^{1/\alpha} dL_{T_t}^{\alpha, 1}, \tag{31}
$$

which is uniquely characterized by its generator,

$$
\mathcal{A}f(v) = (a - (\kappa + \delta)v) f'(v) + \beta v \int_{\mathbb{R}_0^+} \left(f(v + y) - f(v) - f'(v)(1 \wedge y) \right) \mu(dy), \tag{32}
$$

where

$$
\mu(dx) = c|y|^{-\alpha - 1} dy, \quad c = -\sec\frac{\pi\alpha}{2}\frac{1}{\Gamma(-\alpha)}, \quad \delta = \frac{c}{\alpha - 1}.
$$

The parameter constraints on c and δ are imposed in order to generate a standardized α-stable Lévy motion with zero mean and unit dispersion. This is an special exmaple of the CBI process in (24).

From Table 1, we obtain the characteristic exponent of this α-stable Lévy motion,

$$\Psi_x(\theta) = -(i\theta)^\alpha \sec\frac{\pi\alpha}{2}, \quad \text{Im}(\theta) < 0.$$

The leverage-neutral measure $Q(\theta)$ is then defined by

$$\left.\frac{dQ(\theta)}{dP}\right|_t = \exp\left(i\theta L_{T_t}^{\alpha,-1} + \Psi_x(\theta)T_t\right).$$

Under this new measure, we have

$$
\begin{aligned}
\pi^\theta(dx) &= e^{i\theta}\pi(dx), \\
\delta^\theta &= \delta - \int_{\mathbb{R}_0^+}\left(1 - e^{i\theta x}\right)(1 \wedge x)\pi(dx).
\end{aligned}
$$

Then, the Laplace transform of T_t under measure $Q(\theta)$ is given by

$$\mathcal{L}_{T_t}^\theta\left(\Psi_x(\theta)\right) = \exp\left(-b(t)v_0 - c(t)\right) \tag{33}$$

where

$$
\begin{aligned}
b'(t) &= \psi_x(\theta) - \left(\kappa + \delta^\theta\right)b(t) + \beta\int_{\mathbb{R}_0^+}\left(1 - e^{-b(t)x} - b(t)(1 \wedge x)\right)\pi^\theta(dx) \\
&= \psi_x(\theta) - \kappa b(t) + \sec\frac{\pi\alpha}{2}\beta\left[(b(t) + i\theta)^\alpha - (i\theta)^\alpha\right]; \\
c'(t) &= ab(t),
\end{aligned}
$$

By Theorem 1, (33) represents the characteristic function of $Y_t \equiv X_{T_t}$. Note that this example converges to the diffusion example, after some reparameterizations, when α approaches two. In particular, the stable motion becomes a Brownian motion at $\alpha = 2$.

5. Valuing State-contingent Claims

Given the generalized Fourier transform of the state vector Y_t or the return s_t defined in (10) and (12), many state contingent claims can be valued efficiently via the FFT. Formally, consider a European-style state-contingent claim with the general payoff structure at maturity,

$$\Pi_Y(k; a, b, \vartheta, c) = \left(a + b e^{\vartheta^\top Y_t} \right) 1_{c^\top Y_t \leq k}, \tag{34}$$

where $Y \equiv X_{T_t}$ is the d-dimensional time-changed Lévy process. For example, if we assume that the price of an asset is given by $S_t = S_0 \exp(\vartheta^\top Y_t)$ as in (10), the payoff of a European call with strike price K is given by $\Pi(-\ln K/S_0; -K, S_0, \vartheta, -\vartheta)$, the payoff of a European put with strike price K is given by $\Pi(\ln K/S_0; K, -S_0, \vartheta, \vartheta)$, that of a covered call is $\max[S_t, K] = \Pi(-\ln K/S_0; 0, S_0, \vartheta, -\vartheta) + \Pi(\ln K/S_0; K, 0, 0, \vartheta)$, and finally, the payoff of a binary call is given by $\Pi(-\ln K/S_0; 1, 0, 0, -\vartheta)$. We can therefore write the payoffs of many European-style contingent claims in the form of (34) or a linear combination of it.

Let $G(k; a, b, \vartheta, c)$ denote the initial price of a state-contingent claim with the payoff in (34). For simplicity, we focus on determining the initial forward price of the claim and hence:

$$G(k; a, b, \vartheta, c) = E\Pi_Y(k; a, b, \vartheta, c), \tag{35}$$

where the expectation is taken under the forward measure. Note that we drop the maturity argument t as no confusion shall occur. We now show that the price G can be obtained by an extension of the FFT method of Carr and Madan (1999). For this purpose, let $\mathcal{G}(z; a, b, \vartheta, c)$ denote the generalized Fourier transform of $G(k; a, b, \vartheta, c)$ defined as,

$$\mathcal{G}(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{\infty} e^{izk} G(k; a, b, \vartheta, c) \, dk, \qquad z \in \mathcal{C} \subseteq \mathbb{C}. \tag{36}$$

Note that the transform parameter has been extended to the complex plane, as $\mathcal{C}$ denotes the complex domain of z where $\mathcal{G}(z; a, b, \vartheta, c)$ is well-defined.

Proposition 3 *The generalized Fourier transform of the price $G(k;a,b,\vartheta,c)$ defined in (36), when well-defined, is given by,*

$$\mathcal{G}(z;a,b,\vartheta,c) = \frac{i}{z}\left(a\phi_Y(zc) + b\phi_Y(zc - i\vartheta)\right).$$

The result is obtained via integration by parts,

$$
\begin{aligned}
\mathcal{G}(z;a,b,\vartheta,c) &= G(k;a,b,\vartheta,c)\frac{e^{izk}}{iz}\bigg|_{-\infty}^{+\infty} - \frac{1}{iz}\int_{-\infty}^{\infty} e^{izy}dG(k;a,b,\vartheta,c) \\
&= \frac{i}{z}\left(a\phi_Y(zc) + b\phi_Y(zc - i\vartheta)\right).
\end{aligned}
$$

The second line is obtained by applying Fubini's theorem and applying the result on the Fourier transform of a *Dirac* function. Furthermore, since $\lim_{k\to\infty}G(k;a,b,\vartheta,c) = \mathcal{G}_0 \neq 0$, the limit term is well-defined and vanishes only when $\operatorname{Im} z > 0$. Therefore, the extension of the Fourier transform to the complex domain is necessary for it to be well-defined. In general, the admissible domain $\mathcal{C}$ of z depends on the exact payoff structure of the contingent claim. Table 3 present the generalized Fourier transforms of various contingent claims and their respective admissible domains on z. These domains are derived by integrating by parts and by checking the boundary conditions as $y \to \pm\infty$.

Let $z = z_r + iz_i$, where z_r and z_i denote, respectively the real and imaginary part of z. Let $\mathcal{G}(z;a,b,\vartheta,c)$ denote the generalized Fourier transform of some contingent claim pricing function $G(k)$, which can be in any of the forms presented in Table 3. Given that $\mathcal{G}(z;a,b,\vartheta,c)$ is well-defined, the corresponding contingent claim pricing function $G(k)$ is obtained via the inversion formula:

$$G(k) = \frac{1}{2}\int_{iz_i-\infty}^{iz_i+\infty} e^{-izk}\mathcal{G}(z;a,b,\vartheta,c)dz.$$

The integration is performed along a straight line in the complex z-plane parallel to the real axis. z_i can be chosen to be any real number satisfying the restriction in Table 3. The integral can also be written as

$$G(k) = \frac{e^{z_ik}}{\pi}\int_0^{\infty} e^{-iz_rk}\varphi(z_r + iz_i)\,dz_r,$$

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which can be approximated on a finite interval by

$$G(k) \approx G^*(k) = \frac{e^{z_i k}}{\pi} \sum_{k=0}^{N-1} e^{-iz_r(j)k} \varphi(z_r(j) + iz_i) \Delta z_r, \qquad (37)$$

where $z_r(j)$ are the nodes of z_r and Δz_r is the spacing between nodes. Recall that the FFT is an efficient algorithm for computing the discrete Fourier coefficients. The discrete Fourier transform is a mapping of $\mathbf{f} = (f_0, ..., f_{N-1})^\top$ on the vector of Fourier coefficients $\mathbf{d} = (d_0, ..., d_{N-1})^\top$, such that

$$d_j = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-jk\frac{2\pi}{N}i}, \quad j = 0, 1, ..., N-1. \qquad (38)$$

FFT allows the efficient calculation of $\mathbf{d}$ if N is an even number, say $N = 2^m$, $m \in \mathbb{N}$. The algorithm reduces the number of multiplications in the required N summations from an order of 2^{2m} to that of $m2^{m-1}$, a very considerable reduction. By a suitable choice of Δz_r and a discretization scheme for k, we can cast the approximation in the form of (38) to take advantage of the computational efficiency of the FFT. For more details on the discretization scheme, see Carr and Madan (1999), who implement the FFT algorithm on the pricing of a European call.

6. Summary and Future Research

In this paper, we proposed a general option pricing framework that unifies the vast option pricing literature and captures the three key pieces of stylized evidence on financial securities: (1) jumps, (2) stochastic volatility, and (3) leverage effect. Under our framework, the uncertainty of the economy is governed by a time-changed Lévy process. The underlying Lévy process provides a flexible framework for generating jumps; the random time change captures stochastic volatility; and the leverage effect is introduced through the correlation between the Lévy innovation and the time change. Furthermore, by employing a complex-valued measure change, we can reduce the calculation of the characteristic function for a time-changed Lévy process into the calculation of the Laplace transform of the random time, which can then be solved under many instances via an analogy to the bond pricing literature. For many choices of Lévy processes and random times, we obtain the characteristic function in closed form and price contingent claims via an efficient FFT method. A primary direction for future research

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is to investigate the empirical performance of the large variety of new option pricing models generated by this framework. Another line of research is to explore other applications of the complex-valued measure in the frequency domain.

Appendix A. Measure Changes of Time-Changed Lévy Process Under Esscher Transforms

Monroe (1978) proves that every semimartingale Y_t can be written as a time-changed Brownian motion, where the random time T_t is a positive and increasing semimartingale. As an implication, every semimartingale can also be written as a time-changed Lévy process, $Y_t \equiv X_{T_t}$. Furthermore, every semimartingale Y_t, starting at zero ($Y_0 = 0$), can be uniquely represented in the form

$$Y_t = \alpha_t + Y_t^c + \int_0^t \int_{|y|>1} y \, d\mu + \int_0^t \int_{|y| \leq 1} y \, d(\mu - \nu), \tag{A1}$$

where α_t is a finite, increasing process adapted to $\mathcal{F}_t$, Y_t^c is a continuous martingale, μ is the counting measure of the semimartingale and ν is its compensator. Let $\beta = \langle Y_t^c, Y_t^c \rangle$ denote the quadratic variation of Y_t^c. The triplet (α, β, ν) is uniquely determined by Y_t and measure P. Hence, the components of this triplet are called the *local characteristics* of the semimartingale Y_t with respect to P (see Jacod and Shiryaev, 1987). For a Lévy process with Lévy characteristics (μ, Σ, Π), the local characteristics are given by $(\mu t, \Sigma t, \Pi(dx)dt)$.

Now consider measure changes defined by Esscher transforms of the time-changed Lévy process:

$$\left. \frac{dP(\vartheta)}{dP} \right|_t = \mathcal{E}\left(\gamma^\top X_{T_t} \right).$$

The next proposition expresses the local characteristics of $Y_t \equiv X_{T_t}$ under $P(\vartheta)$ in terms of the Lévy characteristics of X_t under P.

Proposition 4 *Suppose T_t is X-continuous, i.e., X is constant on all intervals $[T_{u-}, T_u]$, $u > 0$. Then the local characteristics of the process Y under $P(\vartheta)$ are given by*

$$
\begin{aligned}
\nu^\vartheta(dt, dx) &= dT_{t-} e^{\vartheta^\top x} \Pi(dx), \\
\beta^\vartheta &= \Sigma T_{t-}, \\
\alpha^\vartheta &= \left(\mu + \Sigma \vartheta - \int_{|x|<1} x \left(1 - e^{\vartheta^\top x} \right) \Pi(dx) \right) T_{t-},
\end{aligned}
$$

where (μ, Σ, Π) are the Lévy characteristics of X_t.

The proof of this proposition can found in Küchler and Sørensen (1997), page 230. We repeat it here for the reader's convenience.

Proof. The characteristic component of X under the new measure $P(\vartheta)$ is given by

$$\Psi_x^{\vartheta}(\theta) = \Psi_x(\theta + \vartheta) - \Psi_x(\vartheta). \tag{A2}$$

The Lévy characteristics of X under $P(v)$ follows directly from (A2):

$$\begin{aligned}
\Pi^{\vartheta}(dx) &= e^{\vartheta^{\top}x}\Pi(dx), \\
\Sigma^{\vartheta} &= \Sigma, \\
\mu^{\vartheta} &= \mu + \Sigma\vartheta - \int_{|x|<1} x\left(1 - e^{\vartheta^{\top}x}\right)\Pi(dx).
\end{aligned}$$

Under the assumption of the theorem, the local characteristics of Y are found from those of X by applying the random time transformation $\{T_{t-}\}$, using the results found in Chapter 10.1 of Jacod (1979). ∎

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References

Andersen, T. G., Benzoni, L., Lund, J., 2002. An empirical investigation of continuous-time equity return models. Journal of Finance 57, 1239–1284.

Bakshi, G., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. Journal of Finance 52, 2003–2049.

Bakshi, G., Madan, D., 2000. Spanning and derivative-security valuation. Journal of Financial Economics 55, 205–238.

Barndorff-Nielsen, O. E., 1998. Processes of normal inverse gaussian type. Finance and Stochastics 2, 41–68.

Barndorff-Nielsen, O. E., Shephard, N., 2001. Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics. Journal of the Royal Statistical Society-Series B 63, 167–241.

Bates, D., 1996. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. Review of Financial Studies 9, 69–107.

Bates, D., 2000. Post-'87 crash fears in the S&P 500 futures option market. Journal of Econometrics 94, 181–238.

Bekaert, G., Wu, G., 2000. Asymmetric volatilities and risk in equity markets. Review of Financial Studies 13, 1–42.

Benhamou, E., 2000. Fast fourier transform for discrete asian options. manuscript. London School of Economics London.

Benninga, S., Björk, T., Wiener, Z., 2001. On the use of numeraires in option pricing. manuscript. Tel-Aviv University Israel.

Bertoin, J., 1996. Lévy Processes. Cambridge University Press, Cambridge.

Black, F., 1976. Studies of stock price volatility changes. In: Proceedings of the 1976 American Statistical Association, Business and Economical Statistics Section American Statistical Association, Alexandria, VA.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–654.

Campbell, J. Y., Hentschel, L., 1992. No news is good news: An asymmetric model of changing volatility in stock returns. Review of Economic Studies 31, 281–318.

Campbell, J. Y., Kyle, A. S., 1993. Smart money, noise trading and stock price behavior. Review of Economic Studies 60, 1–34.

Carr, P., Geman, H., Madan, D., Yor, M., 2001. Stochastic volatility for Lévy processes. Mathematical Finance forthcoming.

Carr, P., Geman, H., Madan, D., Yor, M., 2002. The fine structure of asset returns: An empirical investigation. Journal of Business 75, 305–332.

Carr, P., Madan, D., 1999. Option valuation using the fast fourier transform. Journal of Computational Finance 2, 61–73.

Carr, P., Wu, L., 2002. Finite moment logstable process and option pricing. Journal of Finance forthcoming.

Chan, T., 1999. Pricing contingent claims on stocks driven by lévy processes. The Annals of Probability 9, 504–528.

Cox, J. C., Ingersoll, J. E., Ross, S. R., 1985. A theory of the term structure of interest rates. Econometrica 53, 385–408.

Dempster, M., Hong, S., 2000. Spread option valuation and the fast fourier transform. manuscript. University of Cambridge.

Duffie, D., 1992. Dynamic Asset Pricing Theory. Princeton University Press, Princeton, New Jersey.

Duffie, D., Kan, R., 1996. A yield-factor model of interest rates. Mathematical Finance 6, 379–406.

Duffie, D., Pan, J., Singleton, K., 2000. Transform analysis and asset pricing for affine jump diffusions. Econometrica 68, 1343–1376.

Eberlein, E., Keller, U., Prause, K., 1998. New insights into smile, mispricing, and value at risk: The hyperbolic model. Journal of Business 71, 371–406.

Engle, R. F., Rosenberg, J. V., 1997. Empirical pricing kernels. manuscript. New York University New York.

Eraker, B., 2001. Do stock prices and volatility jump? reconciling evidence from spot and option prices. manuscript. Duke University.

Eraker, B., Johannes, M., Polson, N., 2000. The impact of jumps in volatility and returns. manuscript. University of Chicago.

Filipovic, D., 2001. A general characterization of one factor affine term structure models. Finance and Stochastics 5, 389–412.

Geman, H., El Karoui, N., Rochet, J.-C., 1995. Changes of numéraire, changes of probability measure and option pricing. Journal of Applied Probability 32, 443–458.

Harrison, J. M., 1985. Brownian Motion and Stochastic Flow Systems. Wiley, New York.

Haugen, R. A., Talmor, E., Torous, W. N., 1991. The effect of volatility changes on the level of stock prices and subsequent expected returns. Journal of Finance 46, 985–1007.

Heston, S., 1993. Closed-form solution for options with stochastic volatility, with application to bond and currency options. Review of Financial Studies 6, 327–343.

Jacod, J., 1979. Calcul stochastique et problèmes de martingales. Springer-Verlag, Berlin.

Jacod, J., Shiryaev, A. N., 1987. Limit Theorems for Stochastic Processes. Springer-Verlag, Berlin.

Kallsen, J., 1998. Duality links between portfolio optimization and derivative pricing. Preprint 40 Mathematische Fakultät Universität Freiburg i. Br.

Kallsen, J., Shiryaev, A. N., 2000. The cumulant process, exponential martingales, and Esscher's change of measure. Manuscript. Institut für Mathematische Stochastik and University of Moscou.

Karlin, S., Taylor, H. M., 1975. A First Course in Stochastic Processes. Academic Press, New York.

Kawazu, K., Watanabe, S., 1971. Branching processes with immigration and related limit theorems. Theory of Probability and Its Applications 16, 36–54.

Keller, U., 1997. Realistic modelling of financial derivatives. Dissertation Mathematiscje Fakultät de Albert-Ludwigs-Universität Freibur im Breisgau.

Kou, S. G., 1999. A jump-diffusion model for option pricing with three properties: Leptokurtic feature, volatility smile, and analytical tractability. manuscript. Columbia University.

Küchler, U., Sørensen, M., 1997. Exponential Families of Stochastic Processes. Springer, New York.

Leippold, M., Wu, L., 2002. Asset pricing under the quadratic class. Journal of Financial and Quantitative Analysis June, forthcoming.

Madan, D., Carr, P., Chang, E., 1998. The variance gamma process and option pricing. European Financial Review 2, 79–105.

Madan, D., Milne, F., 1991. Option pricing with VG martingale components. Mathematical Finance 1, 39–56.

Margrabe, W., 1978. The value of an option to exchange one asset for another. Journal of Finance 33, 177–186.

Merton, R. C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, 125–144.

Monroe, I., 1978. Processes that can be embedded in Brownian motion. Annals of Probability 6, 42–56.

Pan, J., 2002. The jump-risk premia implicit in options: Evidence from an integrated time-series study. Journal of Financial Economics 63, 3–50.

Schroder, M., 1999. Changes of numeraire for pricing futures, forwards, and options. Review of Financial Studies 12, 1143–1163.

Titchmarsh, E. C., 1975. Theory of Fourier Integrals. Oxford University Press, London.

Table 1
Lévy Measures and Characteristic Exponents

Entries summarize the Lévy measure and its corresponding characteristic exponent for each Lévy component specification.

Lévy Components	Lévy Measures $\pi(dx)/dx$	Characteristic Exponent $\Psi(\theta)$
Pure Continuous Lévy component		
$\mu t + \sigma W_t$	—	$-i\mu\theta + \frac{1}{2}\sigma^2\theta$
Finite Activity Pure Jump Lévy components		
Merton (76)	$\lambda \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x-\alpha)^2}{2\sigma_j^2}\right)$	$\lambda\left(1 - e^{i\theta\alpha - \frac{1}{2}\sigma_j^2\theta}\right)$
Kou (99)	$\lambda \frac{1}{2\eta} \exp\left(-\frac{|x-k|}{\eta}\right)$	$\lambda\left(1 - e^{i\theta k}\frac{1-\eta^2}{1+\theta^2\eta^2}\right)$
Eraker (2001)	$\lambda\frac{1}{\eta}\exp\left(-\frac{x}{\eta}\right)$	$\lambda\left(1 - \frac{1}{1-i\theta\eta}\right)$
Infinite Activity Pure Jump Lévy components		
NIG	$e^{\beta x}\frac{\delta\alpha}{\pi|x|}K_1(\alpha|x|)$	$-\delta\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+i\theta)^2}\right]$
Hyperbolic	$\frac{e^{\beta x}}{|x|}\left[\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}|x|}}{\pi^2 y\left(J_{|\lambda|}^2(\delta\sqrt{2y})+Y_{|\lambda|}^2(\delta\sqrt{2y})\right)}dy\right.$ $\left.+1_{\lambda\geq0}\lambda e^{-\alpha|x|}\right)$	$-\ln\left[\frac{\sqrt{\alpha^2-\beta^2}}{\sqrt{\alpha^2-(\beta+i\theta)^2}}\right]^\lambda\left[\frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+i\theta)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right]$
CGMY	$\begin{cases} Ce^{-G|x|}|x|^{-Y-1}, & x<0, \\ Ce^{-M|x|}|x|^{-Y-1}, & x>0 \end{cases}$	$C\Gamma(-Y)\left[M^Y - (M-i\theta)^Y + G - (G+i\theta)^Y\right]$
VG	$\frac{\mu_\pm^2}{v_\pm}\frac{\exp\left(-\frac{\mu_\pm}{v_\pm}|x|\right)}{|x|}$ $(\mu_\pm = \sqrt{\frac{\alpha^2}{4\lambda^2}+\frac{\sigma_j^2}{2}}\pm\frac{\alpha}{2\lambda}, v_\pm = \mu_\pm^2/\lambda)$	$\lambda\ln\left(1 - iu\alpha + \frac{1}{2}\sigma_j^2 u^2\right)$
LS	$c|x|^{-\alpha-1}, \quad x<0$	$-c\Gamma(-\alpha)(i\theta)^\alpha$

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Table 2
Activity Rate Processes and Laplace Transforms of the Random Time

Under each class of activity rate processes, the entries summarize the specification of the activity rate and the corresponding Laplace transform of the random time.

Activity Rate Specification $v(t)$	Laplace Transform $\mathcal{L}_{T_t}(\lambda) \equiv E\left[e^{-\lambda T_t}\right]$

Affine: Duffie, Pan, Singleton (2000)

$$
\begin{aligned}
v(t) &= \mathbf{b}_v^\top Z_t + c_v, \\
\mu(Z_t) &= a - \kappa Z_t, \\
\left[\sigma(Z_t)\sigma(Z_t)^\top\right]_{ii} &= \alpha_i + \beta_i^\top Z_t, \\
\left[\sigma(Z_t)\sigma(Z_t)^\top\right]_{ij} &= 0, \quad i \neq j, \\
\gamma(Z_t) &= a_\gamma + \mathbf{b}_\gamma^\top Z_t.
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(-\mathbf{b}(t)^\top z_0 - c(t)\right), \\
&\mathbf{b}'(t) = \lambda \mathbf{b}_v - \kappa^\top \mathbf{b}(t) - \tfrac{1}{2}\beta \mathbf{b}(t)^2 \\
&\qquad - \mathbf{b}_\gamma \left(\mathcal{L}_q(\mathbf{b}(t)) - 1\right), \\
&c'(t) = \lambda c_v + \mathbf{b}(t)^\top a - \tfrac{1}{2}\mathbf{b}(t)^\top \alpha \mathbf{b}(t) \\
&\qquad - a_\gamma \left(\mathcal{L}_q(\mathbf{b}(t)) - 1\right), \\
&\mathbf{b}(0) = 0, c(0) = 0.
\end{aligned}
$$

Generalized Affine: Filipovic (2001)

$$
\begin{aligned}
\mathcal{A}f(x) &= \tfrac{1}{2}\sigma^2 x f''(x) + (a' - \kappa x) f'(x) \\
&\quad + \int_{\mathbb{R}_0^+} \left(f(x+y) - f(x) + f'(x)(1 \wedge y)\right) \\
&\qquad (m(dy) + x\mu(dy)), \\
a' &= a + \int_{\mathbb{R}_0^+} (1 \wedge y)\, m(dy), \\
&\int_{\mathbb{R}_0^+} \left[(1 \wedge y)\, m(dy) + \left(1 \wedge y^2\right) \mu(dy)\right] < \infty.
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(-b(t)v_0 - c(t)\right), \\
&b'(t) = \lambda - \kappa b(t) - \tfrac{1}{2}\sigma^2 b(t)^2 \\
&\qquad + \int_{\mathbb{R}_0^+} \left(1 - e^{-yb(t)} - b(t)(1 \wedge y)\right) \mu(dy), \\
&c'(t) = ab(t) + \int_{\mathbb{R}_0^+} \left(1 - e^{-yb(t)}\right) m(dy), \\
&b(0) = c(0) = 0.
\end{aligned}
$$

Quadratic: Leippold and Wu (2002)

$$
\begin{aligned}
\mu(Z) &= -\kappa Z, \quad \sigma(Z) = I, \\
v(t) &= Z_t^\top A_v Z_t + \mathbf{b}_v^\top Z_t + c_v.
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left[-z_0^\top A(t) z_0 - \mathbf{b}(t)^\top z_0 - c(t)\right], \\
&A'(t) = \lambda A_v - A(t)\kappa - \kappa^\top A(\tau) - 2A(t)^2, \\
&\mathbf{b}'(t) = \lambda \mathbf{b}_v - \kappa \mathbf{b}(t) - 2A(t)^\top \mathbf{b}(t), \\
&c'(t) = \lambda c_v + trA(t) - \mathbf{b}(t)^\top \mathbf{b}(t)/2, \\
&A(0) = 0, \mathbf{b}(0) = 0, c(0) = 0.
\end{aligned}
$$

Table 3
Fourier Transforms of Various Contingent Claims

$(\alpha, \beta, a, b$ are real constants with $\alpha < \beta$.)

Contingent Claim	Generalized transform $-iz\varphi(z)$	Restrictions on Im z
$G(k; a, b, \vartheta, c)$	$a\phi_Y(zc) + b\phi_Y(zc - i\vartheta)$	$(0, \infty)$
$G(-k; a, b, \vartheta, c)$	$a\phi_Y(-zc) + b\phi_Y(-zc - i\vartheta)$	$(-\infty, 0)$
$e^{\alpha k} G(k; a, b, \vartheta, c)$	$a\phi_Y((z - i\alpha)c) + b\phi_Y((z - i\alpha)c - i\vartheta)$	(α, ∞)
$e^{\beta k} G(-k; a, b, \vartheta, c)$	$a\phi_Y(-(z - i\beta)c) + b\phi_Y(-(z - i\beta)c - i\vartheta)$	$(-\infty, \beta)$
$e^{\alpha k} G(k; a_1, b_1, \vartheta_1, c_1)$ $+ e^{\beta k} G(-k; a_2, b_2, \vartheta_2, c_2)$	$a_1\phi_Y((z - i\alpha)c_1) + b_1\phi_Y((z - i\alpha)c_1 - i\vartheta_1)$ $+ a_2\phi_Y(-(z - i\beta)c_2) + b_2\phi_Y(-(z - i\beta)c_2 - i\vartheta_2)$	(α, β)