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Asset Price Dynamics in a Financial Market with Heterogeneous Trading Strategies and Time Delays

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Abstract

In this paper we present a continuous time dynamical model of heterogeneous agents interacting in a financial market where transactions are cleared by a market maker. The market is composed of fundamentalist, trend following and contrarian agents who process information from the market with different time delays. Each class of investor is characterized by path dependent risk aversion. We also allow for the possibility of evolutionary switching between trend following and contrarian strategies. We find that the system shows periodic, quasi-periodic and chaotic dynamics as well as synchronization between technical traders. Furthermore, the model is able to generate time series of returns that exhibit statistical properties similar to those of the S&P500 index, which is characterized by excess kurtosis, volatility clustering and long memory.

Keywords: Dynamic asset pricing; Heterogeneous agents; Complex dynamics; Strange attractors; Chaos; Intermittency; Stock market dynamics; Synchronization.

JEL classification: G11, G12, G14

Asset Price Dynamics in a Financial Market with Heterogeneous Trading Strategies and Time Delays

Giuseppe Garofalo• and Alessandro Sansone••

1. Introduction

In recent years there has been a growing disaffection with the standard economic paradigm of efficient markets and rational expectations. In an efficient market, asset prices are the outcome of the trading of rational agents, in the sense that they forecast the expected price by exploiting all the information available and know that other traders are rational. As pointed out by Fama (1970), if market were not efficient, there would be profit opportunities which would be exploited by the trading of rational agents. This implies that prices must equal the fundamental prices, given by the expected discounted dividend streams, and therefore changes in prices are only caused by changes in the fundamental value. In real markets, however, traders have different information on traded assets and process information differently, therefore the assumption of homogeneous rational traders may not be appropriate. In addition to this, the efficient market hypothesis motivates the use of random walk increments in financial time series modeling: if news about fundamentals are normally distributed, the returns on an asset will be normal as well. However the random walk assumption does not allow the replication of some stylized facts of real financial markets, such as volatility clustering, excess kurtosis, autocorrelation in square and absolute returns, bubbles and crashes. Recently a large number of models that take into account heterogeneity in financial markets has been proposed. The typical agents considered in these model are basically fundamentalists, who believe that prices tend to equal the fundamental value of an asset, and technical traders, who predict future prices by extrapolating past patterns in the time series. Recent contribution to this literature include Beja and Goldman (1980); Day and Huang (1990); Caginalp and Ermentrout (1990, 1991); Chiarella (1992); Sethi (1996); Gaunersdorfer (2000); Gaunersdorfer and Hommes (2005); Chiarella, Dieci and Gardini (2002, 2005); Franke and Sethi (1998); Westerhoff (2003, 2004a, 2004b). Brock (1997), Brock and Hommes (1997, 1998, 2001a) have introduced the important concept of financial markets as adaptive belief systems, in the sense that agents switch prediction rule among different predictors according to a fitness function that depends on the realized profits of a given prediction strategy. Chiarella and He (2001) analyze asset price and wealth dynamics in the framework of Brock and Hommes (1998) and Levy and Levy (1996) without switching among different predictors. Such a model is extended by adding a switching rule between momentum and contrarian strategies by Chiarella and He (2002) in the context of a

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Walrasian scenario and Chiarella and He (2003) and He (2003) in a market maker scenario. Brock, Hommes and Wagener (2005) analyze the limit evolution of Brock and Hommes (1998) when the strategies are distributed according to a continuous distribution; Thurner, Dockner and Gaunersdorfer (2002) analyze a market composed of a continuum of fundamentalists who show delays in information processing. These models allow for the formation of speculative bubbles, which may be triggered by news about fundamentals and reinforced by technical trading. Because of the presence of nonlinearities according to which different investors interact with one another, these models are capable of generating stable equilibria, periodic, quasi-periodic dynamics and strange attractors.

This paper builds on the model of Thurner, Dockner and Gaunersdorfer (2002), henceforth TDG, which is inspired by the Nosè (1984a,b, 1991) and Hoover (1985) models of thermodynamics and analyzes a financial market in which there are only fundamental investors who trade according to the mispricing of the asset with delays which are uniformly distributed from initial to current time. We generalize TDG by introducing a continuum of technical traders who behave as either trend followers or contrarians and a switching rule between these technical trading rules. As for the fundamentalists, technical traders react with uniformly distributed delays to the information that they receive from the market. We do not assume the existence of a Walrasian auctioneer, but allow for transactions to be made in a condition of disequilibrium by assuming the existence of a market maker who takes an offsetting long or short position so as to clear the market and set the price according to the direction and magnitude of excess demand. We analyze how the interaction of different types of investors with path dependent risk aversion determines the dynamics and the statistical properties of the system as the proportion of fundamentalists, the growth rate of the fundamental, the speeds of reaction of the market participants and the intensity of switching between technical trading strategies are changed. In particular, the system is characterized by strange attractors that are capable of giving rise to time series of returns featuring stylized facts of real financial markets such as excess kurtosis, volatility clustering and long memory, even in a purely deterministic framework.

The paper is organized as follows. In *Section 2* a continuous time model of heterogeneous agent trading with different frequencies is outlined. *Section 3* analyzes the statistical properties of the model-generated time series when the parameter values are chosen as to produce time series similar to those of the S&P500 index. *Section 4* examines how the interaction of different investors determines the price dynamics of the time series and the effects of changing the proportion of the fundamentalists, the growth rate of the fundamental value, the price adjustment from the market maker, the speed of expected price adjustment from the fundamentalists and the extrapolation speed of technical traders. We will also analyze the introduction of a switching rule between trend followers and contrarians. *Section 5* concludes.

2. The model

Let us consider a security continuously traded at price $P(t)$. Assume that this security is in fixed supply, so that the price is only driven by excess demand. Following TDG, let us assume that the excess demand $D(t)$ is a function of the current price and the fundamental value $F(t)$. Differently from the standard financial economic literature, we assume that transactions are not made at equilibrium prices, but that a market maker takes a long position whenever the excess demand is negative and a short position whenever the demand excess is positive so as to clear the market. The market maker adjusts the price in the direction of the excess demand with speed equal to λ^M. The instantaneous rate of return is:

$$\frac{\dot{P}(t)}{P(t)} = \lambda^M D(P(t), F(t)); \lambda^M > 0 \tag{1}$$

the fundamental value is assumed to grows at constant rate *g*, therefore:

$$\frac{\dot{F}(t)}{F(t)} = g \tag{2}$$

The market is composed of an infinite number of investors, who choose among three different investment strategies. Let us assume that a fraction α of investors follows a fundamentalist strategy and a fraction $(1-\alpha)$ follows a technical analysis strategy. The fraction of technical analysts is in turn composed of a fraction β of trend followers and a fraction $(1-\beta)$ of contrarians. Let $D^F(t)$, $D^{TF}(t)$ and $D^C(t)$ be respectively the demands of fundamentalists, trend followers and contrarians rescaled by the proportions of agents who trades according to a given strategy. The excess demand for the security is thus given by:

$$D(t) = \alpha D^F(t) + (1-\alpha)[\beta D^{TF}(t) + (1-\beta) D^C(t)]; \alpha, \beta \in [0,1] \tag{3}$$

Each trader operates with a delay equal to τ, that is, the demand of a particular trader at time t depends on her decision variable at time $t-\tau$. Time delays are uniformly distributed in the interval $[0, t]$.

Fundamentalists react to the differences between price and fundamental value. The total demand of fundamentalists operating with delay τ is:

$$D^{F\tau}(t) = \lambda^{F\tau} \log\left[\frac{F(t-\tau)}{P(t-\tau)}\right]; \lambda^{F\tau} > 0 \tag{4}$$

where $\lambda^{F\tau}$ is a parameter that measures the speed of reaction of fundamental traders; we assume that $\lambda^{F\tau} = \lambda^F$ throughout the paper. This demand function implies that the fundamentalists believe that the price tends to the fundamental value in the long run and reacts to the percentage mispricing of the asset in symmetric way with respect to underpricing and overpricing.[1]

If time delays are uniformly distributed, the market demand of fundamentalists is given by:

$$D^F(t) = \lambda^F \int_0^t \log\left[\frac{F(t-\tau)}{P(t-\tau)}\right] d\tau; \lambda^F > 0 \tag{5}$$

time differentiation yields:

$$\dot{D}^F(t) = \lambda^F \log\left[\frac{F(t)}{P(t)}\right]; \lambda^F > 0. \tag{6}$$

[1] TDG utilize $D^{F\tau}(t) = \lambda^F [F(t-\tau) - P(t-\tau)]$ as functional form for the demand of fundamentalists. We rather utilize function (4) because we consider more plausible that fundamentalists react to mispricing in percentage terms. Of course if $F(t-\tau)$ and $P(t-\tau)$ are in logarithm terms, the fundamentalist demand of TDG is equivalent to (4).

Following TDG, let us modify equation (6) by introducing the variable $\varsigma(t)^F$ and adding a term $-\varsigma^F(t)D^F(t)$ to the right hand side:[1]

$$\dot{D}^F(t) = \lambda^F \log\left[\frac{F(t)}{P(t)}\right] - \varsigma^F D^F(t); \lambda^F > 0. \tag{7}$$

According to the sign of ς^F, if there is an excess demand, the term $-\varsigma^F(t)D^F(t)$ either drives it towards zero (if $\varsigma(t)^F$ is positive) or foster it (if $\varsigma(t)^F$ is negative). The variable $\varsigma(t)^F$ may be interpreted as an indicator of the risk that traders bear and their risk aversion (if $\varsigma(t)^F$ is negative, traders become risk-seekers). The dynamics for $\varsigma(t)^F$ are given by:

$$\dot{\varsigma}^F(t) = \delta^F [D^F(t)^2 - V^F]; \delta^F > 0 \tag{8}$$

where V^F is a factor controlling the variance. Throughout the paper, we will assume that V^F is given. The economic motivation of equation (8) is that, the larger an open position on the asset, the more risk averse the investors become.

Let us consider now the behavior of technical traders. As for the fundamentalists, their time delays are uniformly distributed in the interval $[0, t]$. A trader operating with delay τ utilizes the percentage return that occurred at time $t - \tau$ in a linear prediction rule in order to form an expectation of future returns. The demands of trend followers and contrarians operating with delay τ are respectively:

$$D^{TF\tau}(t) = \lambda^{TF\tau}\left[\frac{\dot{P}(t-\tau)}{P(t-\tau)}\right]; \lambda^{TF\tau} > 0 \tag{9}$$

$$D^{C\tau}(t) = \lambda^{C\tau}\left[\frac{\dot{P}(t-\tau)}{P(t-\tau)}\right]; \lambda^{C\tau} < 0 \tag{10}$$

throughout the paper we will assume that $\lambda^{TF\tau} = \lambda^{TF}$ and $\lambda^{C\tau} = \lambda^C$. By integrating (9) and (10) with respect to τ and time differentiating we get the time derivatives of the total demands of technical analysts, which are:

$$\dot{D}^{TF}(t) = \lambda^{TF}\left[\frac{\dot{P}(t)}{P(t)}\right]; \lambda^{TF} > 0 \tag{11}$$

$$\dot{D}^{C}(t) = \lambda^{C}\left[\frac{\dot{P}(t)}{P(t)}\right]; \lambda^{C} < 0. \tag{12}$$

As for the fundamentalists, we add now the terms $-\varsigma^{TF}(t)D^{TF}(t)$ and $-\varsigma^C(t)D^C(t)$ in order to take into account the risk and risk attitude of chartists. Time derivatives of their total demands are therefore:

$$\dot{D}^{TF}(t) = \lambda^{TF}\left[\frac{\dot{P}(t)}{P(t)}\right] - \varsigma^{TF}(t)D^{TF}(t); \lambda^{TF} > 0 \tag{13}$$

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[1] TDG introduce the variable ξ, which is linear transformation of $D^F(t)$, and utilize it instead of $D^F(t)$. We will continue to utilize the variable $D^F(t)$ without any loss of generality.

$$\dot{D}^C(t) = \lambda^C \left[\frac{\dot{P}(t)}{P(t)}\right] - \varsigma^C(t) D^C(t); \lambda^C < 0. \quad (14)$$

Following TDG, the dynamics for $\varsigma^{TF}(t)$ and $\varsigma^C(t)$ are:

$$\dot{\varsigma}^{TF}(t) = \delta^{TF}[D^{TF}(t)^2 - V^{TF}]; \delta^{TF} \geq 0 \quad (15)$$

$$\dot{\varsigma}^C(t) = \delta^C[D^C(t)^2 - V^C]; \delta^C \geq 0 \quad (16)$$

We will now consider the fraction α as given, whereas the fraction of trend followers β may be path dependent. In fact β is considered as an endogenous variable because both trend followers and contrarians follow technical analysis strategies and therefore may be likely to switch them if one brings about higher returns. We assume that the more profitable is a strategy, the more investors will choose that strategy. The difference in the absolute return at time *t* between the two strategies is given by $\dot{P}(t)[D^{TF}(t) - D^C(t)]$. The use of absolute returns as a measure of evolutionary fitness stems from the absence of wealth in the model, therefore it is not possible to calculate the percentage return of a strategy. Moreover, β must be bounded in the interval $[0,1]$ and we assume that it tends to move towards 0.5 if both the strategies lead to equal profits. These assumptions can be taken into account if we assume this functional form for the time derivative of $\beta(t)$:

$$\dot{\beta}(t) = \cot[\pi\beta(t)] + z\dot{P}(t)[D^{TF}(t) - D^C(t)]; z \geq 0 \quad (17)$$

where the first term keeps the fraction of trend followers bounded in the interval [0,1] and z is a parameter that measure the speed of switching between the technical strategies. The proportion tends to 0.5 if the two strategies are characterized by the same absolute return. Therefore, the dynamics are ruled by the following nine ordinary differential equation system:

$$\begin{aligned}
&\dot{P}(t) = \lambda^M P(t)\left[\alpha D^F(t) + (1-\alpha)\left[\beta(t) D^{TF}(t) + (1-\beta(t)) D^C(t)\right]\right]; \alpha \in [0,1]; \lambda^M > 0 \\
&\dot{D}^F(t) = \lambda^F \log\left[\frac{F(t)}{P(t)}\right] - \varsigma^F D^F(t); \lambda^F > 0 \\
&\dot{D}^{TF}(t) = \lambda^{TF}\left[\frac{\dot{P}(t)}{P(t)}\right] - \varsigma^{TF}(t) D^{TF}(t); \lambda^{TF} > 0 \\
&\dot{D}^C(t) = \lambda^C\left[\frac{\dot{P}(t)}{P(t)}\right] - \varsigma^C(t) D^C(t); \lambda^C < 0 \\
&\dot{\varsigma}^F(t) = \delta^F\left[D^F(t)^2 - V^F\right]; \delta^F > 0 \\
&\dot{\varsigma}^{TF}(t) = \delta^{TF}\left[D^{TF}(t)^2 - V^{TF}\right]; \delta^{TF} > 0 \\
&\dot{\varsigma}^C(t) = \delta^C\left[D^C(t)^2 - V^C\right]; \delta^C > 0 \\
&\dot{\beta}(t) = \cot[\pi\beta(t)] + z\dot{P}(t)\left[D^{TF}(t) - D^C(t)\right]; z \geq 0 \\
&\dot{F}(t) = gF(t)
\end{aligned} \quad (18)$$

If *z=0* or if the proportion of trend followers and contrarians is taken as a constant, then the system may be made stationary by defining the variable $M(t) \equiv \frac{F(t)}{P(t)}$. In this case *System 18* becomes:

$$\begin{aligned}
&\dot{D}^F(t) = \lambda^F \log[M(t)] - \varsigma^F D^F(t); \lambda^F > 0 \\
&\dot{D}^{TF}(t) = \lambda^{TF} r - \varsigma^{TF}(t) D^{TF}(t); \lambda^{TF} > 0 \\
&\dot{D}^C(t) = \lambda^C r - \varsigma^C(t) D^C(t); \lambda^C < 0 \\
&\dot{\varsigma}^F(t) = \delta^F \left[D^F(t)^2 - V^F\right]; \delta^F > 0 \\
&\dot{\varsigma}^{TF}(t) = \delta^{TF} \left[D^{TF}(t)^2 - V^{TF}\right]; \delta^{TF} > 0 \\
&\dot{\varsigma}^C(t) = \delta^C \left[D^C(t)^2 - V^C\right]; \delta^C > 0 \\
&\dot{M}(t) = g - r
\end{aligned} \tag{19}$$

where r is defined as $r \equiv \lambda^M [\alpha D^F(t) + (1-\alpha)[\beta D^{TF}(t) + (1-\beta) D^C(t)]]; \alpha \in [0,1]; \lambda^M > 0$. *Equations (1)* and *(3)* imply that r is the rate of return on the asset. *System 19* has equilibrium points only for a zero-Lebesgue measure parameter set. Indeed, if the system is on an equilibrium point, $\dot{\varsigma}^F(t) = \dot{\varsigma}^{TF}(t) = \dot{\varsigma}^C(t) = 0$ and the equilibrium demands are:

$$D^F = \pm\sqrt{V^F}\,; D^{TF} = \pm\sqrt{V^{TF}}\,; D^C = \pm\sqrt{V^C} \tag{20}$$

Moreover, the rate of return is equal to the growth rate of the fundamental, so that, plugging the equilibrium demands into the equation for r, we obtain that the following equality relation between parameters must hold for the system to have equilibrium points:

$$g = \lambda^M \left(\pm \alpha\sqrt{V^F} + (1-\alpha)\left(\pm \beta\sqrt{V^{TF}} \pm (1-\beta)\sqrt{V^C}\right)\right) \tag{21}$$

3. Statistical properties

In this section, we analyze the statistical properties of the simulated time series, which have been generated by integrating the system up to time 7529 and recording the price at integer times starting from $t = 4000$ in order to allow the system to get sufficiently close to the asymptotic dynamics and to have time series as long as the daily time series of the S&P500 index between 1 January 1990 and 31 December 2003. The system has been integrated by utilizing the default method of Mathematica 5, which switches between BFG and Adams algorithms depending on the stiffness of the system. No stochastic elements are added, because we are interested in analyzing how the interaction among different investors, whose risk aversions are time varying, may reproduce the stylized fact observed in real financial markets: volatility clustering, fat tails, no autocorrelation between returns and long memory. Thus the features of system-generated time series are endogenous and originate from the nonlinear structure of the systems. The model displays statistical properties similar to those of the index S&P500 using various parameter values. In Table 1 there are reported the mean, maximum, minimum, variance, skewness, kurtosis and the results of the Jarque-Bera test of the daily returns on the S&P500 and on the time series generated by the differential equation system with parameters and initial values reported respectively on *Table 2* and *Table 3* and identified as *Example 1* and *Example 2*. We have also reported in *Table 1* the value of the largest Lyapunov exponent for *Example 1*.[1] The distribution functions, autocorrelations of returns and square returns up to lag fifty are illustrated in *Figure 1* and *Figure 2*.

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[1] The Lyapunov exponent of *Example* 2 is not been reported because the trajectories of the price and fundamental are unbounded and the system cannot be made stationary by performing a change of variables, therefore the Lyapunov exponent would be meaningless.

	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
S&P 500	0.000375309	0.0573148	-0.0686674	0.000110923	-0.0163294	6.49388	1794.62	
Example 1	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1908.44	0.241898
Example 2	0.000366283	0.0563845	-0.0550595	0.0000852793	0.0880204	6.2641	1325.25	

Table 1: Statistics for S&P500*, Example 1* and *Example 2* and Lyapunov exponent of *Example 1*.

	λ^M	λ^F	λ^{TF}	λ^C	α	δ^F	δ^{TF}	δ^C	V^F	V^{TF}	V^C	g	z
Example 1	60	95/15	0.25	-0.22	0.4	240000	240000	240000	1/54000	1/54000	1/54000	0.000319	0
Example 2	55	5	0.24	-0.25	0.4	240000	216000	216000	1/90000	1/90000	1/90000	0.000319	4

Table 2: Parameter values of *Example 1* and *Example 2*.

	P	F	D^F	D^{TF}	D^C	ς^F	ς^{TF}	ς^C	β
Example 1	1.1	1	$\lambda^F * \log[G(0)/P(0)]$	0	0	1	1	1	0.5
Example 2	1.1	1	$\lambda^F * \log[G(0)/P(0)]$	0	0	1	1	1	0.5

Table 3: Initial values of *Example 1* and *Example 2*.

The growth rate of the fundamental, g, is equal to the mean growth rate of S&P500, which in turn has been calculated as the rate that in a continuously compounded capitalization regime implies the same return on the index on the overall period. Since the price moves around the fundamental, the means of the simulated time series match that of the S&P500. The other parameter values have been chosen so as to give rise to statistics similar to those of the S&P500 index. In TDG, the variable that accounts for the variance is V^F, in this model variance control is much more complicated, as there exist three different types of investment strategies, each characterized by a potentially different value of V.

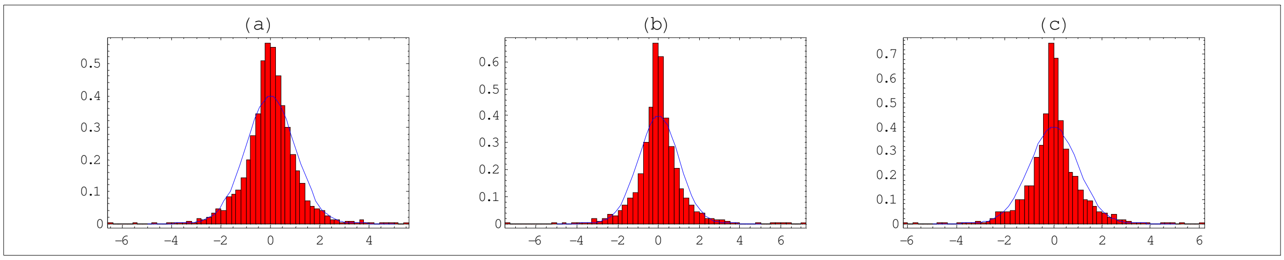

Figure 1: Standardized distributions of returns (red histograms) and standard normal distribution (blue lines) for S&P500 index from 1 January 1990 to 31 December 2003 (*a*) , Example 1 (*b*) and Example 2 (*c*).

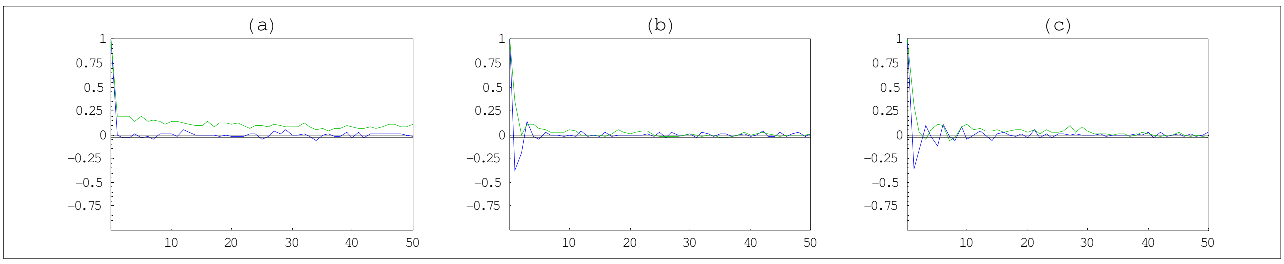

Figure 2: Autocorrelations of returns (blue lines) and square returns (green lines) for S&P500 index from 1 January 1990 to 31 December 2003 (*a*) , Example 1 (*b*) and Example 2 (*c*).

We have considered the case where trend followers and contrarians have the same values of V and δ, whereas fundamentalists may be characterized by different values, due to the smaller difference between two technical strategies than between technical and fundamentalist strategies. V^F, V^{TF}, V^C are constants because otherwise the unconditional variance would be in turn variable. For instance, the TDG setting with our specifications for the demand functions, $V = \varepsilon F$, would give rise to time series whose variance increases over time. Such a behavior is not typical of real time series, whose variances tend to be constant, unless there occur structural changes, or anyway do not follow well defined trends. V^F, V^{TF}, V^C affect not only the variance, but skewness and kurtosis as well, and the relation is not monotonic. They may even bring about a global bifurcation of the system. As pointed out by TDG, kurtosis and volatility clustering are due to the delayed reaction of investors that determines price overshooting. In a multi-agent modeling, such a process is fostered by the interaction among investors who are heterogeneous not only as concerns the time that they need to process information from the market, but also the strategies that they use to predict future prices. Real time series show little or no autocorrelation in returns and significant autocorrelations in square or absolute returns, which decay according to power laws, because of volatility clustering. Time series are also characterized by long memory and nonlinear structure. The model by TDG displays negative first order autocorrelations, close to 0.5, because of the presence of only fundamentalists that tends to drive the price back to its long period fundamental value. The introduction of trend followers should cause this autocorrelation to fall because price overshooting is more likely to occur. The action of contrarians should have less predictable effects on the autocorrelation, as these investors may offset both fundamentalists and trend followers. However the simulations give rise to significant autocorrelation that nevertheless decays very quickly. The significance of autocorrelations is due to the absence of medium and long term trends. The autocorrelations of square returns instead decay much faster than those of S&P500, because of the fact that the price moves around the exponential fundamental trend in the long run. Changes in the speed of switching between technical strategies may affect qualitatively the system dynamics, and, even in the case where the dynamics remain qualitatively unchanged, they may determine large variations in the statistical properties, even if the proportion between trend followers and contrarians remains close to 0.5. In the simulations that we have run, the smaller variance in *Example 2* is mostly due to the introduction of switching between technical strategies rather than to the decrease in V^F, V^{TF}, V^C. Kurtosis tends to rise as λ^{TF} and λ^C rise, whereas variance and skewness do not show a clear dependence on such parameters. Skewness tends to be slightly positive, conversely to the time series of the S&P500 index, which instead show a slightly negative skewness. Positive skewness is due to the exponentially growing fundamental value that determines that large price overshooting is on average positive. Price overshooting, which also determines kurtosis in returns, is induced by both the delayed reaction of investors and the interaction between fundamentalists and trend followers, as the latter may reinforce a trend triggered by the former. On the long run, fundamentalists cause the price to growth. Contrarians' trading may not be sufficient to offset trend followers', and moreover it may happen that the demands of both trend followers and contrarians have the same sign, because of the delay in investors' reactions and the different dynamics of risk attitudes. The mean returns on time scales of 1,5,10 and 15 days are shown in *Figure 3*. It is apparent that returns cluster together on all the time scales, confirming an underlying long memory process. Such characteristics are typical of multifractal process. Let us consider a stochastic process $x(t)$ and define the increment between t and $t+\Delta t$ in the following way:

$$x(t,\Delta t) = x(t+\Delta t) - x(t) \ ; \ 0 \le t \le T \ . \tag{22}$$

Let us assume now that increments are stationary and the distribution of $x(0,\Delta t)$ is invariant with respect to time shifts. According to Mandelbrot, Fisher and Calvet (1997), a multifractal process is a continuous time process with stationary increments which satisfy:

$$E[|x(t,\Delta t)|^q] = c(q)(\Delta t)^{\tau(q)+1} \tag{23}$$

for each $t, \Delta t$ on which x is defined and for each $q \in [0, q_{\max})]$ such that $E[|x(t,\Delta t)|^q] < \infty$, where E is the expectation operator. The scaling function $\tau(q)$ determines the variations in expected value as time scale changes. Mandelbrot, Fisher and Calvet (1997) prove that scaling functions remain unchanged only for bounded time intervals, that is, multifractal processes must show transitions in their scaling properties or crossovers. Taking the logarithms of equation (23) we get:

$$\log E[|x(t,\Delta t)|^{q}] = \log[c(q)] + (\tau(q)+1)\log[\Delta t] \qquad (24)$$

If $x(t) = \log P(t) - \log P(0)$ then $x(t,\Delta t)$ approximates the return on the time series $P(t)$ in the interval $[t, t+\Delta t]$. The plots of $\log E[|x(t,\Delta t)|^{q}]$ of the S&P500 and the simulated time series with respect to $\log[\Delta t]$ for q+1=1,1.5,2,2.5,3 are drawn in *Figure 4*. Since we are interested only in the scaling function $\tau(q)$ and not in the intercept, the values have been normalized by subtracting $\log E[|x(t,\log[10])|^{q}]$. Time intervals ranges from 1 to 100 days. There is no apparent crossover up to a time scale of 100 days in the time series of S&P500, thus confirming its multifractal nature. In the simulations, crossover occurs for values of t between e^{3} and e^{4} and the fluctuations are more erratic than those of S&P500. Such a behavior underlines the capability of the model to generate dynamics typical of a multifractal process, however the exponential growth in the fundamental value implies an exponential long run growth in the expected returns, which in turn implies that crossover occurs for smaller time intervals than those of real time series.

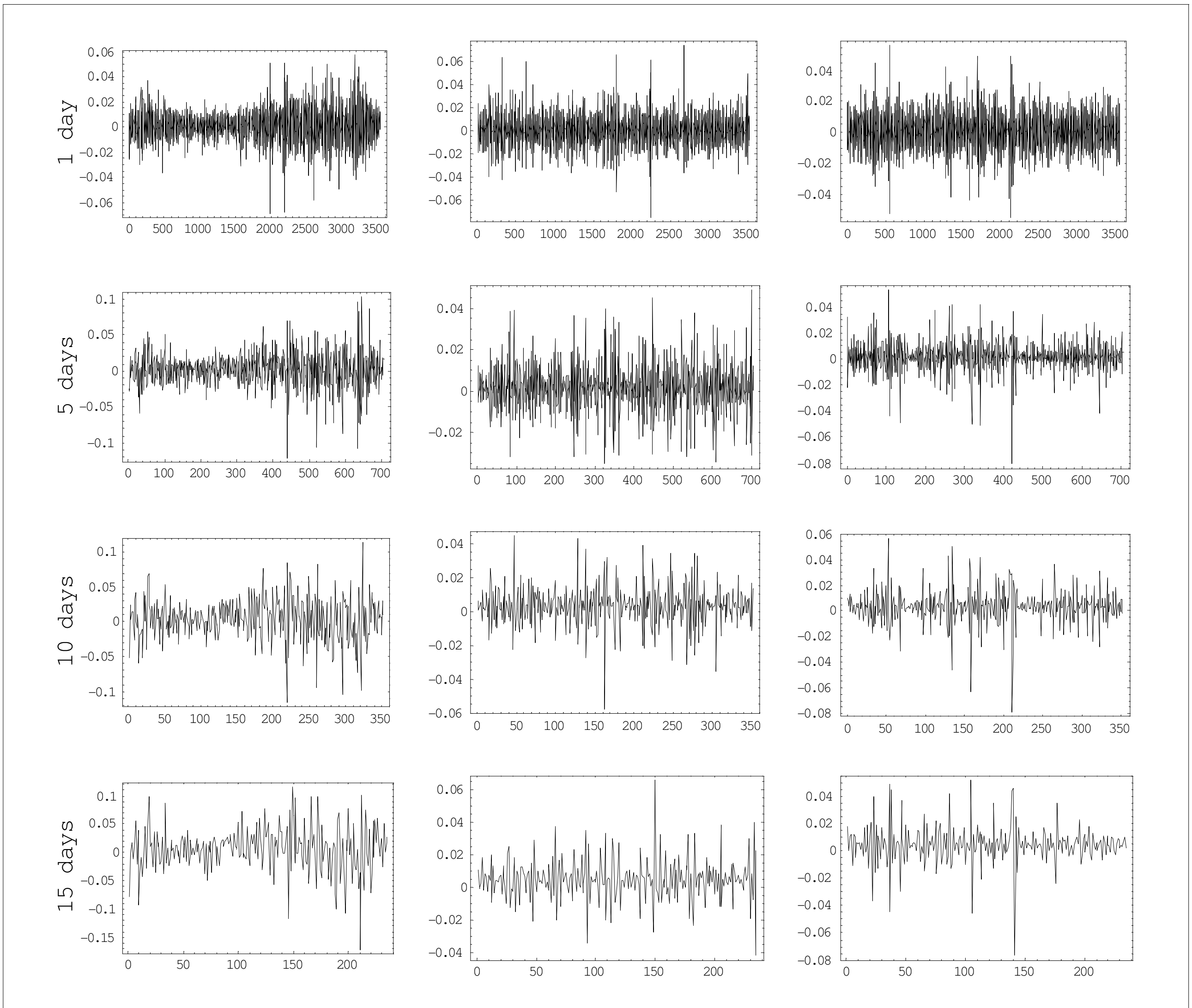

Figure 3: Time series of returns on time scales of 1, 5, 10 and 15 days on S&P500 index (left), Example 1 (middle), Example 2 (right).

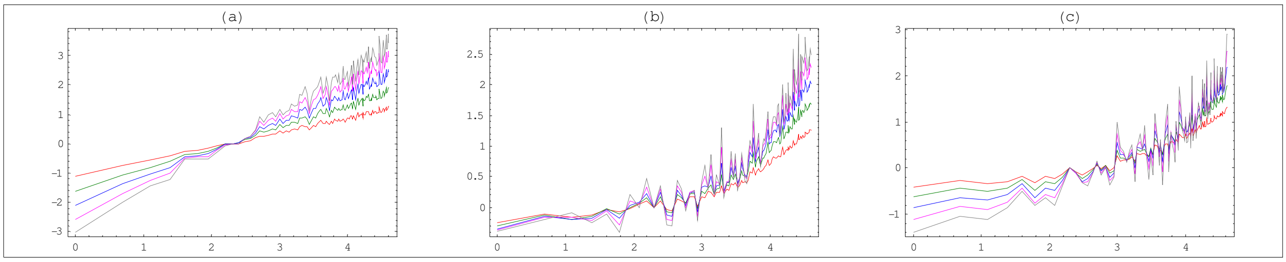

Figure 4: Plots of $\log E[|x(t,\Delta t)|^q]$ against $\log[\Delta t]$ on S&P500 index (a), Example 1 (b), Example 2 (c) for q=1,1.5,2,2.5,3 respectively in red, green, blue, purple and grey.

4. Comparative dynamics

In this section we will first analyze the dynamics for the system identified as *Example 1*, the dynamics for the other example being very similar, and then we will study the variations in dynamics as parameters change. In *Figure 5* there are depicted the time series in the interval $[7029,7529]$ of prices, returns, proportion of trend followers out of the total of technical traders (it is constant because z=0), demands, risk attitudes and the projection of the phase space on the planes $[D^F,\varsigma^F]$, $[D^{TF},\varsigma^{TF}]$, $[D^C,\varsigma^C]$, $[D^F,D^{TF}]$, $[D^F,D^{TF}]$, $[D^{TF},D^C]$. *Tables 4-10* show the mean, maximum, minimum, variance, skewness, kurtosis, Jarque-Bera and Lyapunov exponent for different parameters values. From the graph of the price, two type of trends are apparent: there is an upward long period trend that matches the dynamics of the fundamental value and is due to the trading of fundamentalists and upward and downward short period trends that oscillate around the long run trend and are instead due to the trading of technical analysts as well as to the delayed reaction of the fundamentalists. Short period cycles are characterized by considerable variations in both frequency and amplitude. Such a variability is due to the heterogeneity in strategies and time horizons and is reinforced by the variability in risk attitudes. The demands of technical traders switch between positive and negative phases, differently from the fundamentalist demand, which instead tends to move around zero. The average demand of fundamentalists is slightly positive, because of the upward trend in the fundamental value. The presence of long phases of positive and negative demands of technical traders, together with the dynamics for the risk aversion may determine very large price oscillations in both directions. In fact, long phases of positive demand provoke considerable increases in price, associated with strong sales from the fundamentalists. The increase in the fundamental value triggers a stock price increase due to the purchases by fundamentalists, which is reinforced by the action of trend followers, whereas contrarians tend to sell the stock. The opposite behavior of trend followers and contrarians is shown on the projection of the phase space on the space of technical analysts' demands: the attractor is stretched along the bisector between the first and third orthant. The demand of fundamentalists has smaller oscillations in the periods where the risk aversion is high, because a high risk aversion induces the fundamentalists not to open large positions if the stock is mispriced. Whereas the risk aversion of fundamentalists follows well defined trends and is on average positive, those of technical traders tends to oscillate around zero. As such, technical traders switch between phases in which they are risk averse and phases in which are risk seekers. The dynamics for the risk attitudes may be explained in the following way: let us assume that the price is rising and the demand of trend followers is positive and greater than $\sqrt{V^{TF}}$. *Equation 15* implies that their risk aversion rises as well. The increase in price reduces the demand of fundamentalists and contrarians, but reinforces that of trend followers, which on the other hand tends to fall because of the increase in their risk aversion. Once the price falls, the demand of trend followers approaches zero (eventually becoming negative) and, as a consequence, their risk aversion falls. The dynamics are also the same in the case where the cycle is triggered by fundamentalists or contrarians. The only difference is that the demand of these investors will eventually change sign independently of their risk attitudes whereas the demand of trend followers are self-fulfilling because the price movements they induce in turn reinforce the demands, given the risk aversion. The projections of the attractor on the planes $[D^F, \varsigma^F]$, $[D^{TF}, \varsigma^{TF}]$, $[D^C, \varsigma^C]$ show the interactions between demands and risk attitudes. The different shape of the projection on the plane $[D^F, \varsigma^F]$ is due to greater amplitude and lower frequency of the dynamics of fundamentalists' risk aversion than those of the other investors. In any case, however, risk attitudes may vary considerably even during phases in which the demands are almost steady. Indeed it is sufficient that the absolute value of the demand of investors type i remains for a long time respectively above $\sqrt{V^i}$ to get a considerable change in risk aversion. The time derivatives of the risk attitudes tend to reach their lower bounds, which are respectively equal to $-\delta^F V^F$, $-\delta^{TF} V^{TF}$ and $-\delta^C V^C$, only when the demands are very close to zero.

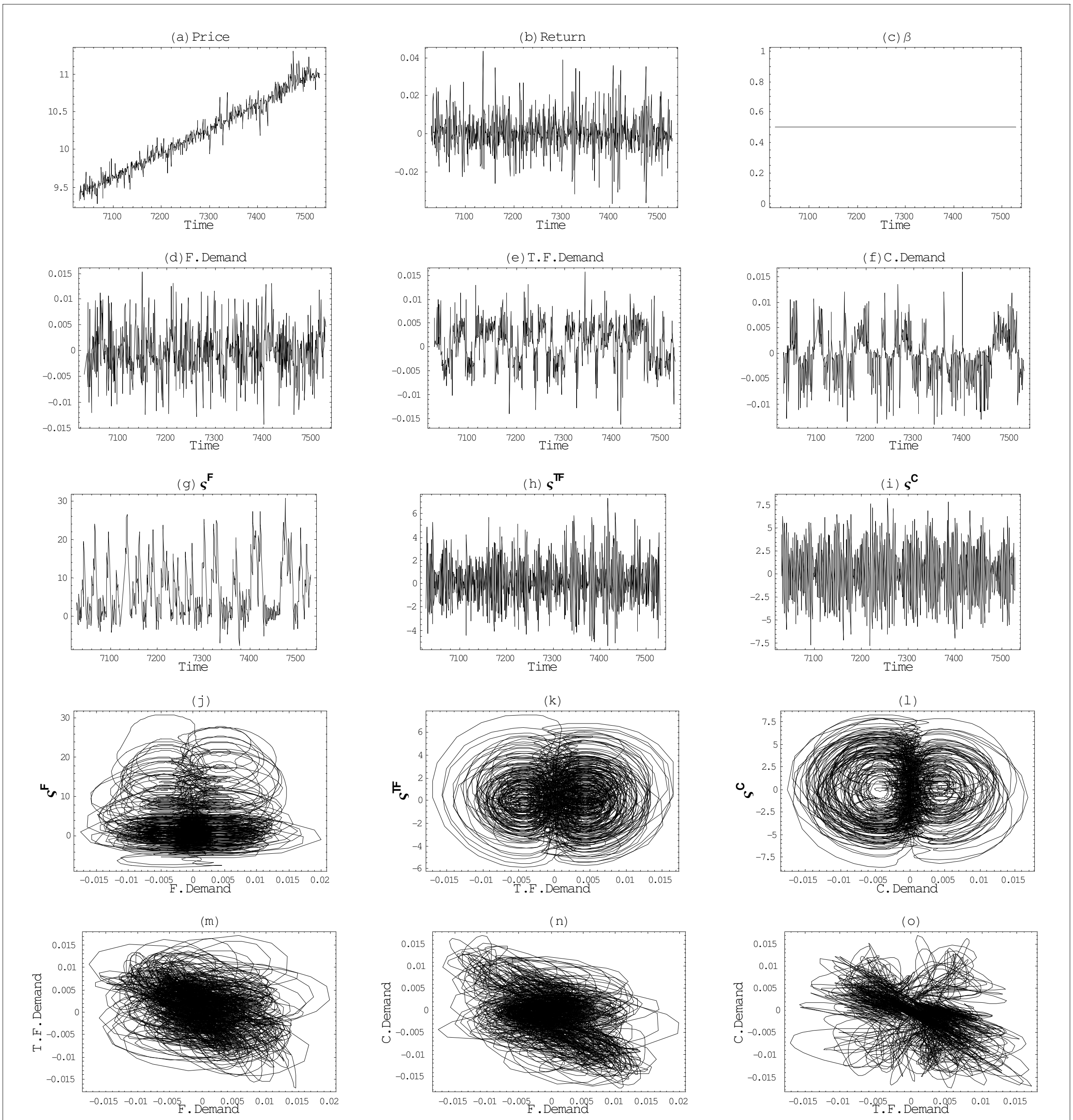

Figure 5: Time series of prices (*a*), returns (*b*), proportion of trend followers of technical traders (*c*), demand of fundamentalists (*d*), trend followers (*e*), contrarians (*f*), risk aversion of fundamentalists (*g*), trend followers (*h*), contrarians (*i*), projections of the phase space on the planes $[D^F, \varsigma^F]$ (*j*), $[D^{TF}, \varsigma^{TF}]$ (*k*), $[D^C, \varsigma^C]$ (*l*), $[D^F, D^{TF}]$ (*m*), $[D^F, D^{TF}]$ (*n*), $[D^{TF}, D^C]$ (*o*) for the time interval [7029,7529].

4.1. Effects of changing the proportion of fundamentalists and technical traders. In order to analyze the effect of the proportion of fundamentalists and technical traders, we select values of α ranging from 0 to 1 and with a difference of 0.1 between a simulation and the next. If there are no fundamentalists or if their proportion is only ten percent, the price goes to infinity, because technical trading drives the price away from the fundamental.[1] If $\alpha = 0.2$ the fundamentalists are able to steer the price to the fundamental value, but prices are subject to large oscillations induced by technical traders. Such oscillations become larger and larger as time goes on. In fact larger departures from the fundamental value are needed for the fundamentalists to bring the price back close to the fundamental value. The dynamics for the fundamentalist demand differ considerably from the baseline case where $\alpha = 0.4$, in fact the departure from the fundamental value brings about long phases in which the fundamentalists go either long or short on the asset, determining in this way an increase in their risk aversion. This in turn implies a lower capability of offsetting technical traders. The overall demand of the latter presents long phases in which the demand is either positive or negative, phases in which it changes sign quickly and phases where the demands of contrarians and trend followers offset each other. This latter feature is called synchronization in the dynamical systems literature. During phases of synchronization the system reduces by one dimension. When the technical demand is equal or close to zero, fundamentalists bring the price back close to the fundamental value. As a consequence of the fact that the total demand does not change sign for long periods, the price tends to follow a monotonic trajectory when it is far from the fundamental and to oscillate as it gets close to it. Thus, the synchronization of technical traders determines an intermittent behavior in the system with regular monotonic phases interrupted by chaotic bursts. The time series of fundamentalist and technical demands are depicted in *Figure 6*. If α is equal to 0.3 the proportion of fundamentalist is sufficiently high as to prevent technical trading from bringing about larger and larger departures from the fundamental value. The oscillations have anyway larger amplitudes than in the case where $\alpha = 0.4$, and this in turn determines an increase in the variance and a decrease in the kurtosis. If fundamentalists account for half of the investors, the demand of technical traders is generally lower than in the baseline case because fundamental trading prevents strong changes in the price. This leaves little room for a persistent phase of fundamentalist demand and therefore fundamentalists are more likely to became risk seekers. The higher proportion of fundamentalists determines a more regular behavior of the system, as denoted by the decrease in kurtosis. If the fraction of fundamentalists is equal to or greater than sixty percent, the system no longer converges to a strange attractor. Furthermore, the only attracting invariant set is a quasi-periodic attractor, as denoted by the values of the Lyapunov exponents. Moreover, as the proportion of fundamentalists in the market increases, the amplitude of the oscillations reduces. If there are only fundamentalists the attractor becomes strange again and the Lyapunov exponent rises up to 0.523002, which would indicate a highly chaotic system. However the rise in the Lyapunov exponent is due to the increase in the amplitudes of the oscillations that in turn are due to the overreaction induced by the delayed reaction of fundamentalists, which brings price above (below) the fundamental price when the security is originally underpriced (overpriced).

4.2. Effects of changing the growth rate of the fundamental value. Increases in g cause a stronger activity of the fundamentalists on the market. The price tends to remain close to the fundamental value and the amplitude of the price oscillations is smaller, therefore the variance decreases as g increases. If g is four times greater than in the baseline case the action of fundamentalists is so strong as to break the strange attractor into a limit cycle. If g is five times greater, the system converges to a quasi-periodic attractor. The attractor is a limit cycle for g equal to or greater than six times the baseline case.

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[1] The price goes to zero with other parameter values. What matters here is that the price does not match the fundamental in the long run.

α	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
0.2	0.00210084	0.281998	-0.223493	0.00347306	0.657885	4.55161	608.396	0.0283678
0.3	0.000902394	0.161515	-0.115126	0.00117017	0.151231	3.91785	137.289	0.268056
0.4	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1907.9	0.241898
0.5	0.00034002	0.0309093	-0.0298682	0.0000438907	0.164723	5.05137	634.548	0.175174
0.6	0.000505747	0.0351398	-0.032607	0.000360944	0.0181459	1.57432	298.98	0.0989546
0.7	0.000542628	0.0235851	-0.0309212	0.000432451	0.0149926	1.50448	328.907	0.0288602
0.8	0.000425532	0.0235851	-0.0225013	0.000227566	0.0137007	1.50864	327.061	0.0345303
0.9	0.000323392	0.00685721	-0.00633042	0.0000171097	0.0046185	1.51802	322.865	0.0124616
1	0.000502513	0.108746	-0.0974863	0.000367779	0.397363	15.0274	21357.8	0.523002

Table 4: Mean, maximum, minimum, variance, skewness, kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* as α varies from 0.2 to 1.

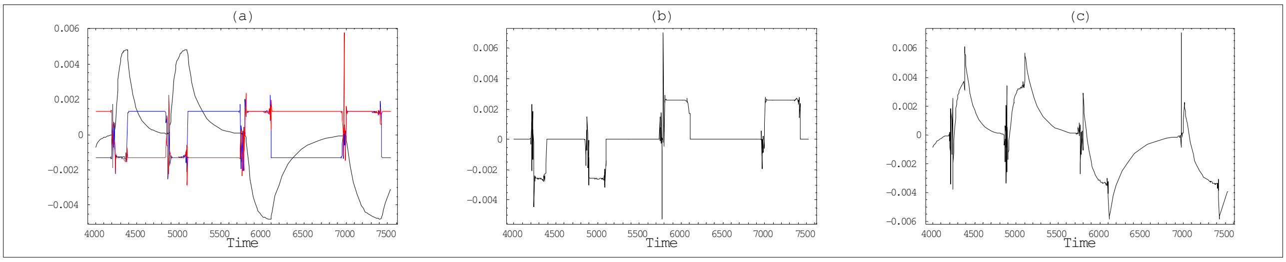

Figure 6: In *Panel a* there are represented the total demands of fundamentalist (in black), trend followers (in blue) and contrarians (in red) respectively given by αD^F, $(1-\alpha)\beta D^{TF}$ and $(1-\alpha)(1-\beta)D^C$ when $\alpha = 0.2$ and $\beta = 0.5$. In *Panel b* there is depicted the total technical demand, given by the sum of the demands of trend followers and contrarians. Total excess demand, given by equation (3), is depicted in *Panel c*. Time interval ranges from 4000 to 7529.

$\frac{g}{0.000319}$	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
0	0.0000537886	0.0495222	0.0452839	0.000118037	0.027382	5.16913	692.095	0.252298
1	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1907.9	0.241898
2	0.000684626	0.0610451	-0.0564503	0.0000919609	0.109867	5.82433	1179.69	0.224348
3	0.000994219	0.0414887	-0.0391781	0.0000699423	0.124856	5.70547	1085.14	0.204809
4	0.00129316	0.0191797	-0.0148457	0.0000362081	0.342639	3.63877	129.012	0.00105954
5	0.00159735	0.00478863	-0.00147119	$(2.93407)10^{-6}$	0.0557558	1.81467	208.364	0.00214585
6	0.00191871	0.00426463	-0.000423662	$(2.72064)10^{-6}$	-0.00054509	1.50511	328.501	0.00123471

Table 5: Maximum, minimum, variance, skewness, kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* as g varies from 0 to 6·0.000319.

4.3. Effects of changing the speed of adjustment of the market maker. A higher value of the speed of reaction of the market maker determines a greater response of the price to a given excess demand and this in turn brings about an increase in the variance. This in turn determines a greater disorder in the system. For instance, if $\lambda^M = 20; 40$ the trajectories are periodic, if $\lambda^M = 10; 30$ the attractor is strange but more tidy than in the standard case. Indeed, the Lyapunov exponents are respectively equal to 0.0021286 and 0.0012897 and the return distributions are approximately normal. In *Figures 7* and *8* there are reported the phase plots respectively for λ^M equal to 20 and 30.

4.4. Effects of changing the speed of expected price adjustment of fundamentalists. Increasing the speed reaction of fundamentalists brings about a decrease in the variance because the price tends to stay close to the fundamental. The system undergoes a transition as the parameter λ^F is increased, that is, the dynamics shows a cyclical behavior after a transient chaotic phase. This kind of transition, called attractor destruction, is a type of crisis-induced intermittency and has been investigated by Grebogi, Ott, Romeiras and Yorke (1986) and Grebogi, Ott, Romeiras and Yorke (1987). However, for large values of λ^F the attractor becomes strange again; if $\lambda^F = 40$ the Lyapunov exponent is 0.127318, that is, the system is weakly chaotic due to the overreaction of fundamentalists. This case is similar in some respects to that where there are only fundamentalists on the market, indeed kurtosis rises up to 10.1876. Because of the presence of technical traders, which are affected by the changes in prices triggered by the fundamentalists, it is not possible to determine what the dynamics eventually are as the reaction speed of the fundamentalists is further increased. For instance, if $\lambda^F = 190$ the dynamics are periodic, but if $\lambda^F = 300$ the attractor is strange, with a Lyapunov exponent of 0.240876, and is characterized by an intermittent behavior.

λ^M	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
10	0.000330484	0.0206878	-0.0201229	0.0000253034	0.0387064	3.48532	35.505	0.183651
20	0.000360739	0.0176783	-0.0168995	0.0000794659	0.170052	1.97156	172.483	0.0021286
30	0.000365441	0.0331703	-0.0319962	0.000081857	0.017598	3.02064	0.24473	0.159328
40	0.000357116	0.014606	-0.0172128	0.0000779788	-0.206538	1.9594	184.261	0.0012897
50	0.000361973	0.0197733	-0.0142528	0.0000844143	0.241145	2.0067	179.23	0.0014726
60	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1907.9	0.241898
70	0.000381041	0.0793545	-0.0628766	0.000128775	0.145983	7.13087	2520.95	0.233348
80	0.000405412	0.0669915	-0.0570457	0.000172349	0.157421	4.88388	536.273	0.235278
90	0.00041944	0.0631928	-0.0533371	0.000204906	0.0882721	4.03829	163.054	0.257029
100	0.000425154	0.0978468	-0.0698592	0.000234512	0.160142	4.68113	430.533	0.250071

Table 6: Maximum, Minimum, Variance, Skewness, Kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* as λ^M varies from 10 to 100.

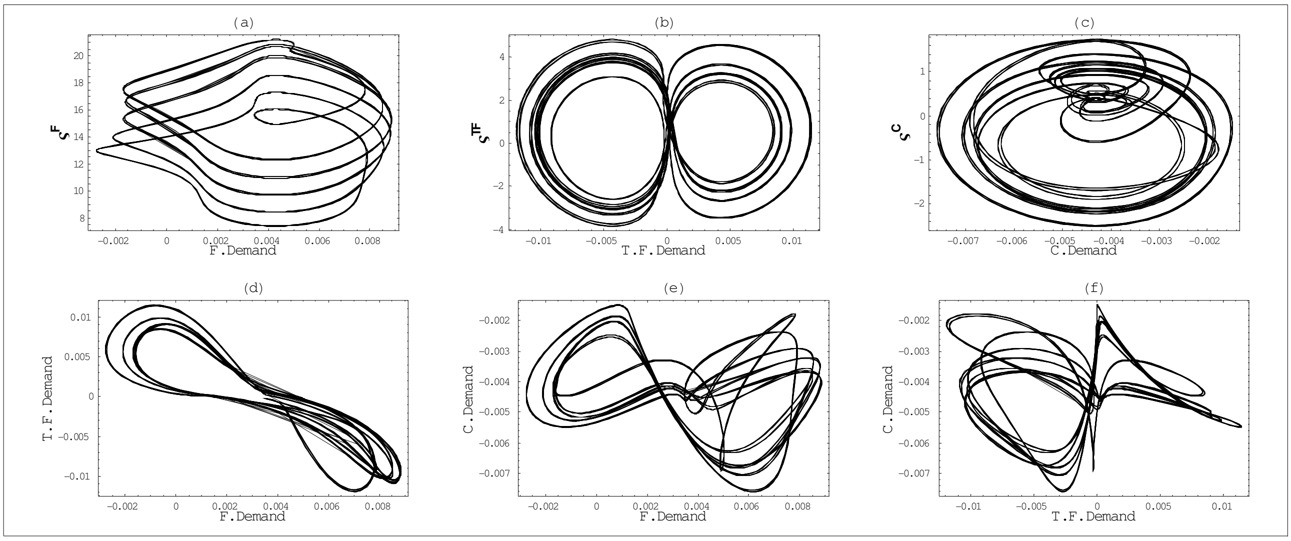

Figure 7: Projections of the phase space on the planes $[D^F, \varsigma^F]$ (*a*), $[D^{TF}, \varsigma^{TF}]$ (*b*), $[D^C, \varsigma^C]$ (*c*), $[D^F, D^{TF}]$ (*d*), $[D^F, D^{TF}]$ (*e*), $[D^{TF}, D^C]$ (*f*) when $\lambda^M = 20$.

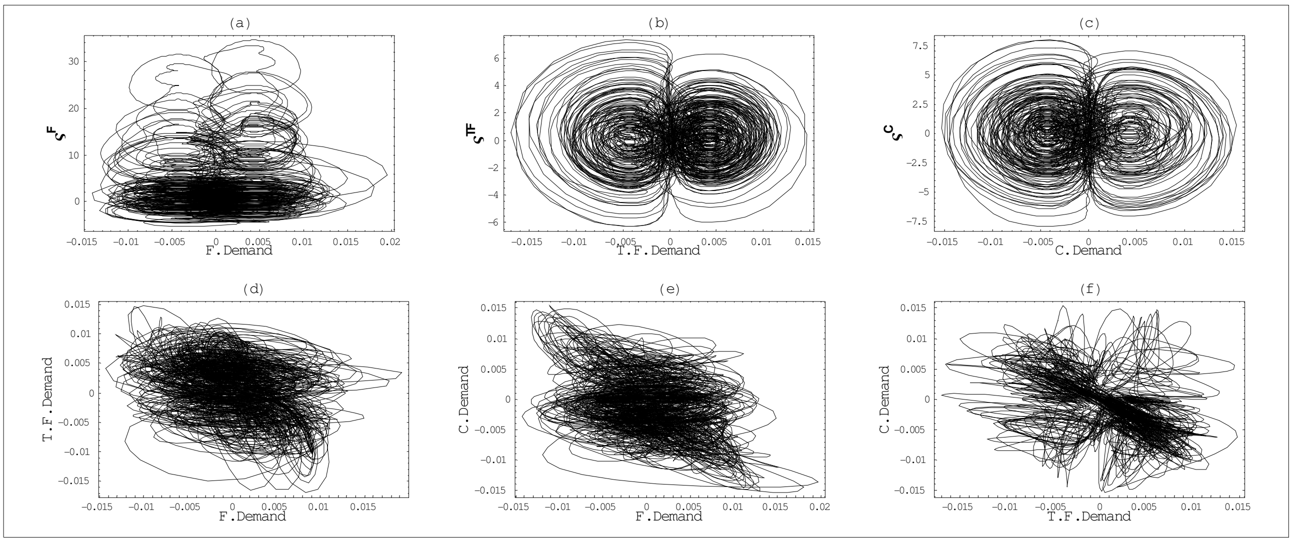

Figure 8: Projections of the phase space on the planes $[D^F, \varsigma^F]$ (*a*), $[D^{TF}, \varsigma^{TF}]$ (*b*), $[D^C, \varsigma^C]$ (*c*), $[D^F, D^{TF}]$ (*d*), $[D^F, D^{TF}]$ (*e*), $[D^{TF}, D^C]$ (*f*) when $\lambda^M = 30$.

λ^F	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
19/15	0.00055803	0.0825522	-0.0761281	0.000483397	0.101521	3.75793	90.5049	0.260045
38/15	0.00048943	0.0905994	-0.0713038	0.000324588	0.0888249	3.6847	73.5548	0.20373
57/15	0.000434872	0.054647	-0.054509	0.000234301	0.0296036	3.25936	10.4036	0.243968
76/15	0.000398758	0.0697828	-0.0714307	0.000157999	0.174213	5.02767	622.226	0.247527
95/15	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1907.9	0.241898
114/15	0.000370917	0.0694991	-0.0494394	0.000104576	0.236946	6.56116	1897.24	0.232318
133/15	0.000368247	0.0610275	-0.0395701	0.0000912715	0.256507	3.44351	67.603	0.00113966
152/15	0.000368016	0.0736272	-0.0701349	0.0000930663	0.213145	4.98602	606.522	0.00244454
171/15	0.000356914	0.0190932	-0.0147682	0.0000847965	0.163268	1.78405	233.018	0.00236142
190/15	0.000352543	0.0593143	-0.0400121	0.0000674469	0.145138	6.65448	1975.6	0.064413
40	0.000327483	0.0215197	-0.0206138	0.0000167665	0.0946897	10.1879	7602.37	0.127318
190	0.000373124	0.0427878	-0.0428233	0.000112505	0.158359	4.63448	407.577	0.0739194
300	0.000472087	0.0845286	-0.0675717	0.000316402	0.273594	4.19152	252.783	0.240876

Table 7: Maximum, Minimum, Variance, Skewness, Kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* as λ^F varies from 19/15 to 190/15 and for λ^F equal to 30, 190, 300.

4.5. Effects of changing the extrapolation speed of trend followers and contrarians. From the values of the Lyapunov exponent, it is apparent that for low values of λ^{TF} and λ^C the system converges to a limit cycle. The dynamics may explode or converge to a price equal to zero if contrarians are much more reactive than trend followers, as in the cases where $\lambda^{TF} = 0.16; 0.17; 0.19$. This result is due to the risk aversion dynamics that cause the demands of trend followers and contrarians to have the same sign, because contrarians become risk seekers or not sufficiently risk averse to offset the trend followers. The price diverges to infinity or converges to zero when the demand of technical traders remains positive or negative (in these cases the statistics are meaningless and therefore are not reported in *Table 8* and *9*). When the system converges to a strange attractor, the statistics do not show a clear dependence on λ^{TF} and λ^C. Skewness tends to be slightly positive, differently from the time series of the S&P 500 index, which is instead slightly negative skewed. Positive skewness is due to the short term overshooting, as explained in *Section 3*. Overshooting, which causes also kurtosis in the time series, is induced by both the delayed reaction of investors and the interactions between fundamentalists and trend followers, since the latter may reinforce a trend triggered by the action of the former and contrarian trading is not sufficient to offset the trend followers. If we increase the reactivity of technical traders, the system becomes more regular, as trend followers and contrarians tend to balance each other. The dynamics are less regular if we only increase the reaction parameter of trend followers, because they prevail over contrarians.

4.6. Effects of switching between trend following and contrarian strategies. So far we have dealt with a model where the proportion between trend followers and contrarians are kept constant. If $z > 0$ such proportions become path dependent. The higher the value of *z*, the higher the fraction of trend followers because this strategy is generally more profitable than the contrarian one, since price grows in the long run. This higher presence of trend chasers may render the system chaotic.

λ^{TF}	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
0.15	0.000384673	0.0187929	-0.0242905	0.000135443	-0.355236	2.13223	184.895	0.00263052
0.16	-	-	-	-	-	-	-	-
0.17	-	-	-	-	-	-	-	-
0.18	0.000372227	0.0173602	-0.0222974	0.000117359	-0.290264	2.03177	187.349	0.0010877
0.19	-	-	-	-	-	-	-	-
0.20	0.000378055	0.0598837	-0.0553325	0.000117933	0.133162	6.00388	1336.85	0.201067
0.21	0.00038489	0.122849	-0.0810554	0.000131612	0.387327	10.6243	8633.27	0.221105
0.22	0.000369102	0.0272894	-0.0259532	0.0000990937	0.037827	3.71217	75.3983	0.209684
0.23	0.000367604	0.0593017	-0.0495785	0.0000994084	0.146344	6.06788	1396.14	0.238593
0.24	0.000366941	0.0481003	-0.0573432	0.0000990906	-0.00316219	5.81994	1168.96	0.237215
0.25	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1907.9	0.241898

Table 8: Maximum, Minimum, Variance, Skewness, Kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* as λ^{TF} varies from 0.15 to 0.25.

λ^C	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
-0.15	0.000369556	0.0213667	-0.0158433	0.000106152	0.179352	1.91581	191.707	0.0021656
-0.16	0.000372044	0.0211957	-0.015605	0.000102849	0.181869	1.92979	187.815	0.0024273
-0.17	0.000370366	0.0210591	-0.0156312	0.000099644	0.186281	1.95433	181.137	0.0026694
-0.18	0.000380905	0.0478136	-0.0481585	0.000121981	0.0891421	4.88654	527.85	0.230824
-0.19	0.000377074	0.0465356	-0.0424717	0.000115556	0.0318894	4.71706	433.996	0.226468
-0.20	0.000376226	0.0505584	-0.0478036	0.000109272	0.105942	5.21538	728.061	0.246516
-0.21	0.00036901	0.0523215	-0.0468544	0.0000991786	0.11308	5.38548	844.025	0.233281
-0.22	0.000369353	0.0587311	-0.0709184	0.000105104	0.0690029	6.59998	1907.9	0.241898
-0.23	0.000369951	0.0864171	-0.0585479	0.000102825	0.138732	6.8382	2176.89	0.213399
-0.24	0. 0.000376719	0.0723912	-0.0645418	0.000114519	0.047148	7.25154	2658.42	0.227939
-0.25	0.000396688	0.0706271	-0.0654394	0.000159497	0.263336	15.3175	22343.7	0.158465

Table 9: Maximum, Minimum, Variance, Skewness, Kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* as λ^C varies from -0.15 to -0.25.

Let us consider the case with constant proportion where $\lambda^{TF} = 0.16$ and $\lambda^C = -0.15$. The system converges towards a limit cycle. If $z = 50$ the system, after an initial chaotic phase, until $t \approx 1000$, approximates a regular orbit very similar to the limit cycle obtained with constant proportion and eventually becomes chaotic as $t \approx 6600$. Indeed, the dynamics approximate a limit cycle as long as the proportion remains close to 0.5. The phase space projections of the system in the time interval $[2000,2200]$ are represented in *Figure 9.* The fraction of trend followers is on average equal to 0.503711 and tends to oscillate between 0.48 and 0.54 with a variance of 0.000248758. If $z = 100$ there are larger oscillations in the composition of technical analysts. Indeed, while the mean of the fraction of trend followers remains slightly over half (0.508117), the variance increases up to 0.00264788.[1] The higher proportion of trend followers causes greater departures from the fundamental value triggering a reaction by all types of investors. Such dynamics bring about an increase in the variance and the kurtosis of returns. If z is increased up to 150 and subsequently up to 500, the oscillations in the proportion between technical traders become larger and the variance of returns increases further, while kurtosis decreases because the increase in variance determines that some returns previously in the tails of the distribution now approach the center.

z	Mean	Maximum	Minimum	Variance	Skewness	Kurtosis	Jarque-Bera	Lyapunov exponent
0	0.000419108	0.0204648	-0.0259058	0.000187242	-0.184689	1.7386	253.951	0.00137831
50	0.000420392	0.0568036	-0.0711078	0.000191746	-0.16719	2.8211	21.1409	
100	0.000467982	0.0851743	-0.164496	0.000304584	-0.035824	8.06507	3772.03	
150	0.000565746	0.126586	-0.125605	0.000493903	0.122439	6.1863	1501.23	
500	0.000994454	0.154838	-0.128476	0.00136727	0.222942	4.19228	238.189	

Table 10: Maximum, Minimum, Variance, Skewness, Kurtosis, Jarque-Bera and Lyapunov exponent for *Example 1* for z equal to 0, 50, 100, 150, 500.

5. Conclusion

In this paper we have outlined a continuous time deterministic model of a financial market with heterogeneous interacting agents. The dynamical system shows periodic, quasi-periodic and strange attractors, and is able to generate some stylized facts present in real markets, even in a purely deterministic setting: excess kurtosis, volatility clustering and long memory. We have indeed tuned the parameters in order to produce artificial time series with statistical properties similar to those of the daily time series of S&P500 index between 1 January 1990 and 31 December 2003. Since the fundamental value grows exponentially as time goes on, large price overshooting is on average positive, thus giving rise to positive skewness. Mean, variance and kurtosis tend to match quite close those of the S&P500, whereas skewness and autocorrelation patterns are somewhat affected by the long run exponentially increasing fundamental value and price. Furthermore, because of the

[1] The mean and variance of trend followers are computed in the time interval from 4000 and 7529, as the statistics of time series of returns.

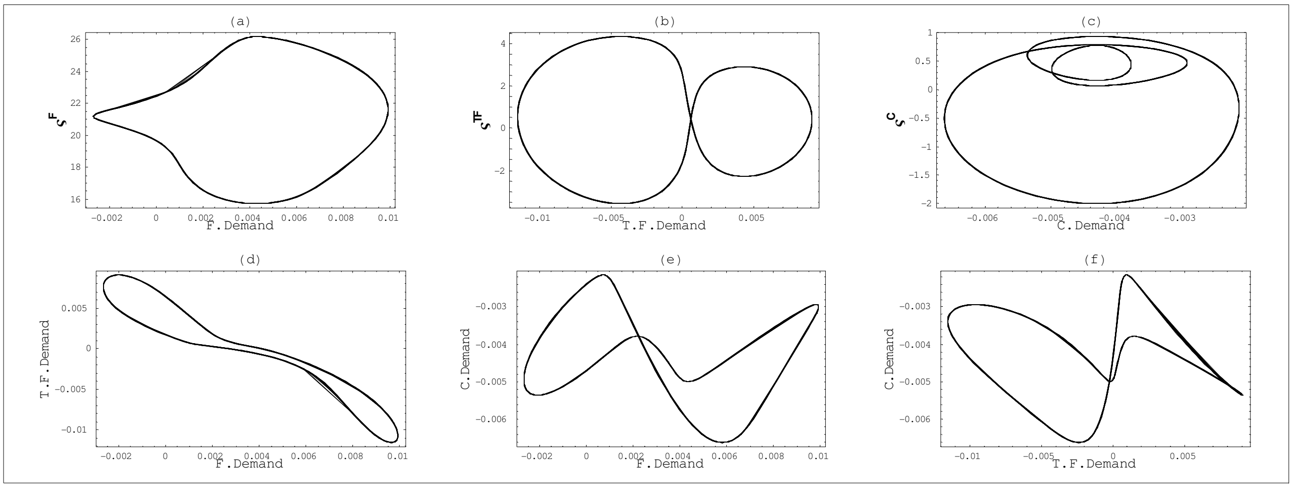

Figure 9: Projections of the phase space on the planes $[D^F, \varsigma^F]$ (*a*), $[D^{TF}, \varsigma^{TF}]$ (*b*), $[D^C, \varsigma^C]$ (*c*), $[D^F, D^{TF}]$ (*d*), $[D^F, D^{TF}]$ (*e*), $[D^{TF}, D^C]$ (*f*) when $\lambda^{TF} = 0.16$, $\lambda^C = -0.15$ and *z*=50 in the time interval [2000,2200].

absence of chaotic long term trends and cycles, the model is not able to completely cancel the autocorrelations of returns and to give rise to square autocorrelations that decay according to a power law. Nevertheless the introduction of technical traders allows for a reduction in the autocorrelation with respect to TDG, which is characterized only by fundamentalist agents and shows a very high negative first order autocorrelation, because fundamentalists tend to drive the price back to fundamental too quickly. Even in the case where fundamentalists are the only agents present in the market, they are unable to drive the price back to the fundamental on a steady state trajectory, because of both the increasing risk aversion as they trade in order to profit out of a mispricing and the delays in processing the information from the market. Moreover, the increase in the fundamentalist reaction speed on the one hand may destroy the strange attractor giving rise to a chaotic transient, on the other may even increase the disorder in the system, as pointed out by the values of the Lyapunov exponent, because the fundamentalists trigger a strong response of technical traders. It may also be possible that, when the fraction of fundamentalists is low, trend followers and contrarians give rise to synchronization in the system, bringing about a dramatic change in the dynamics. In this case, the system exhibits the phenomenon of intermittency, that is, regular phase interrupted by chaotic bursts in the dynamics. The introduction of an evolutionary switching between technical traders leads to an increase in the volatility and in the kurtosis, provided that the speed of switching is not too high because otherwise the increase in the variance makes it less likely that returns will fall in the tails of the distributions.

There are many ways to extend the model. While in the present paper the fundamental value is assumed to grow exogenously at a rate *g*, further research will introduce a feedback between price and fundamental, that is, a feedback between real and financial parts of the model. Another extension will consider time delays distributed according to distributions that give more importance to more recent observations as well as technical traders who take into account the whole history of past prices. Such extensions should produce time series with long run chaotic dynamics displaying more realistic statistical properties, mainly in terms of autocorrelation patterns and long memory.

References

Beja, A. and Goldman, M. (1980), *On the dynamic behavior of prices in disequilibrium,* Journal of Finance **35**, 235-247.

Brock, W. A. (1997), *Asset price behavior in complex environment*, in: Arthur, W. B., Durlauf, S. N. and Lane D.A., eds., The Economy as an Evolving Complex System II, Addison-Wesley, Reading, MA, 385-423.

Brock, W. A., Hommes C. H. (1997), *A rational route to randomness*, Econometrica **65**, 1059-1095.

Brock, W.A. and Hommes, C. H. (1998), *Heterogeneous beliefs and routes to chaos in a simple asset pricing model*, Journal of Economic Dynamics and Control **22**, 1235-1274.

Brock, W.A. and Hommes, C. H. (2001), *Heterogeneous beliefs and routes to complex dynamics in asset pricing models with price contingent contracts*, Working Paper, CeNDEF, University of Amsterdam.

Brock, W. A., Hommes C. H., Wagener, F. O. O. (2005), *Evolutionary dynamics in markets with many traders type*, Journal of Mathematical Economics **41**, 7-42.

Caginalp, G. and Ermentrout G. B. (1990), *A kinetic thermodynamics approach to psychology of fluctuations in financial markets*, Applied Mathematics Letters **3**, 17-19.

Caginalp, G. and Ermentrout G. B. (1991), *Numerical studies of differential equations related to theoretical financial markets*, Applied Mathematics Letters **4**, 35-38.

Chiarella, C. (1992), *The dynamics of speculative behaviour*, Annals of Operations Research **37**, 101-123.

Chiarella, C., Dieci, R. and Gardini L. (2002), *Speculative behaviour and complex asset price dynamics: a global analysis*, Journal of Economic Behaviour and Organization **49**, 173-197.

Chiarella, C., Dieci, R. and Gardini L. (2005), *The dynamics interaction of speculation and diversification*, Applied Mathematical Finance **12**, 17-52.

Chiarella, C. and He, X.-Z. (2001), *Asset pricing and wealth dynamics under heterogeneous expectations*, Quantitative Finance **1**, 509-526.

Chiarella, C. and He, X.-Z. (2002), *Heterogeneous beliefs, risk and learning in a simple asset pricing model*, Computational Economics **19**, 95-132.

Chiarella, C. and He, X.-Z. (2003), *Heterogeneous beliefs, risk, and learning in simple asset-pricing model with a market maker*, Macroeconomic Dynamics **7**, 503-536.

Day, R. and Huang, W. (1990), *Bulls, bears and market sheep*, Journal of Economic Behavior and Organization **14**, 299-329.

Franke R. and Sethi R. (1998), *Cautious trend-seeking and complex asset price dynamics*, Research in Economics **52**, 61-79.

Gaunersdorfer, A. (2000), *Endogenous fluctuations in a simple asset pricing model with heterogeneous agents*, Journal of Economic Dynamics and Control **24**, 799-831.

Gaunersdorfers, A. and Hommes, C.H. (2005), *A nonlinear structural model for volatility clustering*, in: Teyssière G. and Kirman A., eds., Long Memory in Economics, Spring-Verlag, forthcoming.

Grebogi C., Ott E. and Yorke J. A. (1986), *Critical exponent of chaotic transients in nonlinear dynamical systems*, Physical Review Letters **57**, 1284-1287.

Grebogi C., Ott E., Romeiras F. and Yorke J. A. (1987), *Critical exponent for crisis-induced intermittency*, Physical Review A **36**, 5365-5380.

He, X.-Z. (2003), *Asset pricing, volatility and market behavior: a market fraction approach*, Working paper, Research Papers Series **95**, Quantitative Finance Research Centre, University of Technology, Sydney.

Hoover, W.G. (1985), *Canonical dynamics: equilibrium phase-space distributions*, Physical Review A **31**, 1695-1697.

Levy, M., Levy, H. (1996), *The danger of assuming homogeneous expectations*, Financial Analysts Journal **52**, 65-70.

Mandelbrot, B. B., Fisher, A. and Calvet, L. (1997), *A multifractal model of asset returns*, Working paper, Cowles Foundation Research Paper **1164**, Yale University.

Nosè, S. (1984a), *A unified formulation of the constant temperature molecular dynamics method*, The Journal of Chemical Physics **81**, 511-519.

Nosè, S. (1984b), *A molecular dynamics method for simulation in the canonical ensemble*, Molecular Physics **52**, 255-268.

Nosè, S. (1991), *Molecular dynamics simulations,* Progress of Theoretical Physics Supplement **103**, 1-49.

Sethi, R. (1996), *Endogenous regime switching in speculative markets*, Structural Change and Economic Dynamics **7**, 99-118.

Westerhoff, F. (2003), *Bubbles and crashes: optimism, trend extrapolation and panic*, International Journal of Theoretical and Applied Finance **6**, 829-833.

Westerhoff, F. (2004a), *Greed, fear and stock market dynamics*, Physica A: Statistical Mechanics and its Application **343C**, 635-642.

Westerhoff, F. (2004b), *Market depth and price dynamics: a note*, International Journal of Modern Physics C **15**, 1005-1012.

Thurner, S., Dockner, E.J., Gaunersdorfer, A. (2002), *Asset Price Dynamics in a model of investors operating on different time horizons*, Working paper, University of Vienna.

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