A Note on Constructing 50-50 Step Probability Binomial Lattices to Replicate Wiener Diffusion

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May 17, 2003

JEL Subject Classifications: C63, G13

Keywords: Binomial Lattices, Wiener Processes, Option Valuation Methods

Abstract

Binomial lattices are sequences of discrete distributions commonly used to approximate the future value states of a financial claim, such as a stock price, when the instantaneous rate of return is assumed to be governed by a Wiener diffusion process. In that case, both pedagogical and professional conventions generally follow the lattice construction methodology used by Cox, Ross, and Rubinstein ("CRR") in their classical article. In some applications, it is more convenient to replace the "implied" branching probabilities of that construction with a more natural and tractable alternative: that is, with the probability of "up" and "down" branching being exactly one-half, or, vernacularly, with a "50-50 step" probability.

This elementary note reviews such an alternative formulation for constructing a binomial lattice, which can be viewed as simply entailing multiplicative shifts of every state value on a CRR-constructed binomial lattice. This transformation maintains (in fact, improves) the equivalence of the lattice values' moments to those arising from the replicated diffusion. The expression of that transform is derived, and the effect on the lattice values' moments and orders of convergence to the limit imposed by the continuous process are given. To show the absence of numerical effect, the values of some simple European options obtained from the two alternative binomial lattice constructions are compared against the limiting Black Scholes values.

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1 The Cox, Ross, and Rubinstein Method

1.1 Binomial lattices and Future Paths of Asset Value.

For purposes of simplified numerical valuation of options, binomial lattices are typically developed following the methods developed by Cox, Ross, and Rubinstein [1] when financial claims are assumed to depend upon log normally distributed future stock prices, which arise from the assumption that the instantaneous rate of return on the stock is governed by a simple Wiener diffusion.

That work proscribed the methodology for using a binomial lattice to value contingent claims and demonstrated how to construct such a binomial lattice. Formulae are given for the binomial probabilities of passing from one node to another, as well as for the successive asset values or cash flows assigned to the individual states in future time. With pervasive pedagogical acceptance and advancement, those formulae have become a de-facto industry standard for basic binomial option modelling, and are employed even in the valuation software of professional implementations.

The Cox, Ross, and Rubinstein ("CRR") methodology models the future asset values at time T with lattice "slice" $N = T/dt$. The slice is comprised of $N + 1$ binomially distributed outcomes. The alternative outcomes, i.e., the "states", represent the asset's possible relative values. By construction the asset value is the same for every identically indexed state on different slices. The CRR lattice values thus do not incorporate the drift in expected future asset values. Rather, the drift is incorporated by defining the binomial parameter as a transition probability, so that the expected asset value upon transition $-i.e.$, at the "next" slice$-$ equates to the expected asset value in the continuous case being modelled. A constant interval between the logs of asset values in any two successive states is chosen so the variance of the outcomes asymptotically approaches the asset values' variance in the continuous case as dt tends to zero. Together, these two conditions ensure that the lattice slices will each convergence in probability to the corresponding time's finite dimensional lognormal distribution of asset values, as generated by the underlying Wiener process.

The essence of the CRR construction is determining p that satisfies :

$$e^{rdt} = \left(pe^{\sigma\sqrt{dt}} \ + \ (1-p)e^{-\sigma\sqrt{dt}} \right) \tag{1}$$

[1] J. Cox, S. Ross, and M. Rubinstein. "Option pricing: A simplified approach". *J. Financial Economics*, 7:229$-$264, 1979.

Here dt is the step, expressed in the same calendar time unit as is σ and r. Typically, the unit is years, so dt is the inverse of the number of lattice steps per year.

Treated as an algebraic equivalence, then p is expressed as:

$$p \;=\; \left(\frac{e^{rdt} - e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}} \right) \tag{2}$$

Expressing this in terms of hyperbolic trig functions [2] shows that:

$$p \;=\; \frac{1}{2} \;\left(1 \;+\; \frac{e^{rdt}}{\sinh(\sigma\sqrt{dt}\;)} \;-\; \coth(\sigma\sqrt{dt}\;) \right) \tag{3}$$

The two expressions in equation 3 are indeterminate as the time difference tends to zero, but L'Hospital's Rule applied to them shows:

$$\lim_{dt \to 0} \; p \;=\; \frac{1}{2}. \tag{4}$$

Expanding in Taylor series around $dt = 0$ shows that limit, and that the parameter varies inversely with σ and directly with r:

$$p \;\approx\; \frac{1}{2} \left(1 \;+\; \frac{\left(r - \;\frac{\sigma^2}{2}\right) dt}{\sigma\sqrt{dt}} \right) + \mathbf{O}(dt^{3/2}) \tag{5}$$

1.2 Alternative enumeration of a lattice's "nodes" and the underlying binomial distribution.

Consider the very commonly applied case where the expected instantaneous return is constant at the level r, throughout future time. Divide a fixed time interval, T, into n dt discrete steps. The outcome states of each slice can be populated iteratively, after the first. Since r is fixed, then so too, p is constant. At slice n, there are (n+1) states. The value of the asset after b "upsteps" (each with probability p) and $n - b$ "downsteps" (each with probability $1 - p$) is independent of the sequence of the up and down events.

Relative to a unit initial asset value, the "net outcome" of a sequence of random steps can range from $e^{n\sigma\sqrt{dt}}$, at the greatest, differencing by two, through $e^{-n\sigma\sqrt{dt}}$.

[2] The elementary hyperbolic functions are $\sinh(x) \;=\; (1/2)\left(\, e^{x} \;-\; e^{-x} \,\right)$ and $\cosh(x) \;=\; (1/2)\left(\, e^{x} \;+\; e^{-x} \,\right)$. All other hyperbolic functions are defined in correspondence to the equivalent standard trig definition, for instance, coth(x) $=$ $\cosh(x)/\sinh(x)$.

Here the states' indices are distributed symmetrically around zero. The meaning of, for example, "slice 8, state -4" is that a path which reached this indexed state experienced six "down-ticks", and only two "up-ticks". This is sometimes referred to as a "natural", or "signed" enumeration. The "net displacement" is six (negative) plus two (positive). While this is a most intuitive way −and certainly the most familiar way− to enumerate states, there is an alternative, and equivalent enumeration convention.

The "analytic", or "based", enumeration defines the states of slice n ordinally from zero through n. Heuristically, rather than write of a state's value in terms of net displacement, now the state values are written in terms of the number of "up" events in every path that reaches the particular state. With this practice, denote the value at slice n, for all paths that possess b upward branchings, as:

$$\nu_{n,b} \;=\; e^{(2\ b\ -\ n\ )\ \sigma\sqrt{dt}} \tag{6}$$

The main purpose of this enumeration scheme is that the probabilities of a random path attaining the b-th state can be immediately expressed in terms of binomial coefficients, here, denoted by $C(n,b)$, and also in terms of the binomial probability parameter, p, as:

$$Pr\left[\nu_{n,b}\right] \;=\; C(n,b)\ p^b\ (1-p)^{n-b} \tag{7}$$

It follows that the relative asset values at the states of slice n of the lattice are distributed as a binomial variate with parameter n.

2 The 50-50 Branch Probability Construction

The methodology of lattice construction proposed by CRR can be summarized as follows, for a unit initial value.

1. Receive three input parameters of the continuous process to be replicated by the constructed lattice: the constant expected instantaneous rate of return, denoted by r, the variance of the instantaneous return, or diffusion, denoted by σ^2, and, the time step of the lattice, dt, all expressed in the same unit of time as is r.

2. Construct a binomial process for the return, such that the variance of change in return asymptotically equals $\sigma^2 dt$, the diffusion of the underlying Wiener process. To affect this, choose the two outcomes (after time step dt), as $\exp\left(\pm\ \sigma\sqrt{dt}\ \right)$.

3. Equate the expectation of the binomial value change over the time step with the expected value change of the process being modelled. Affect this by selection of the binomial distribution's parameter, p.

4. Interpret p as the probability that the path of price change is up, *i.e.*, that the relative value change is $\exp(\sigma\sqrt{dt}\ )$, and thus, finally, determine the value of p by Equation(2).

The CRR construction functionally determines a probability parameter, subject to the asymptotic equivalence of the first two moments of, on the one hand, the binomial lattice process, and, on the other hand, the underlying continuous process that the lattice will model. The third step in the method −equating the moment− is of special constructive importance and will be termed the "CRR principle" below.

The CRR principle proceeds by equating only the first moment; exact equivalence of variance does attain in the limit, but approximate equivalence is a consequence of the choice of the up- and down-states as $\exp(\pm\ \sigma\sqrt{dt}\ )$.

With regard to the algebra of the construction, if the probability parameter is fixed, the problem requires a new parameter to affect the moment equivalence. Accordingly, define a location parameter, θ_n, and let the probability parameter by fixed as $p = 0.5$. Then, the CRR principle calls for solving for θ_n in the relationship:

$$e^{r\ n\ dt} = e^{\ \theta_n dt} \sum_{b\ =\ 0}^{n} \left(\frac{C(n,b)}{2^n} e^{(2b\ -\ n)\ \sigma\sqrt{dt}} \right). \tag{8}$$

The expression on the right forms the expectation by combining the definitions in Equations (7) and (6), with $p = (1/2)$, and $C(n, b)$ denotes the binomial coefficient, vernacularly "n choose b".

Writing the sum of successive powers of a variate, weighted by binomial coefficients, as the binomial expansion of the n-th power of a polynomial, the summation (8) can be generated by expanding the expression and collecting like powers of n:

$$\begin{aligned} e^{r\ n\ dt} &= e^{\ \theta_n dt} \left(\frac{1}{2}\ e^{-\sigma\sqrt{dt}} \right)^n \left(1\ +\ e^{2\ \sigma\sqrt{dt}} \right)^n \\ e^{\ r\ n\ dt} &= e^{\ \theta_n dt} \left(\frac{1}{2}\ e^{-\sigma\sqrt{dt}}\ +\ \frac{1}{2}\ e^{\ \sigma\sqrt{dt}} \right)^n . \\ &= e^{\ \theta_n dt} \left[\cosh\ (\sigma\sqrt{dt}\ ) \right]^n \end{aligned} \tag{9}$$

Observe that expressions raised to the power n appear on both the left and right sides of equation (9). For $n = 1$, denote the parameter is $\tilde{\theta}$. Then, it follows that the location parameter is linear in n:

$$\theta_n = n\tilde{\theta}, \tag{10}$$

Solving (9) gives $\tilde{\theta}$ by the concise definition:

$$\tilde{\theta} = r - \frac{\ln \cosh(\sigma\sqrt{dt}\,)}{dt} \tag{11}$$

It is informative to expand the fractional term in a Taylor series, in the neighborhood of $dt = 0$. Then $\tilde{\theta}$ is observed to approach a familiar form as dt tends to zero:

$$\theta = r - \frac{1}{2}\sigma^2 - \frac{1}{12}\sigma^4\, dt + \mathbf{O}(dt^2) \tag{12}$$

The first two terms express the rate of drift of the logarithms of the relative values. Equation (12) further indicates that the convergence of θ to its limit is linear in dt, but quadratic in *variance*; this convergence is, generally, numerically very rapid, since σ is typically quite small (on the order of r).

If $\tilde{\theta}$ is applied to establish slice-state values on a binomial lattice, it will be termed the "50-50" lattice construction. Such construction will retain the asymptotic convergence properties imposed by the CRR principle. The next section summarizes and compares the moments of the binomial distributions under the two alternative constructions.

2.1 Analytic Comparison of the CRR and 50-50 Constructions

Either method of construction populates the states on every slice of a binomial lattice. Each of the two methods, however, provides a different relative value for the same-enumerated state, and different values for the asset rate of return from inception to that state, and different probabilities of attaining a particular enumerated state. Notwithstanding, however, they are virtually identical with respect to application.

Tables 1 and 2 indicates the essential similarity of the binomial distributions that arise from the two construction methods. The expressions for the moments can be obtained from the characteristic functions of the instantaneous return's distribution, and the exponential function of that variate's distribution. They can also be obtained by symbolic integration of expressions for the moments. The analtyic forms have been expressed in terms of hyperbolic trig functions whenever possible.

Table 1: Exact/Asymptotic Moments of Lattice Slice $n = t \,/\, dt$
For Unit-Based State Variable Values

CRR:	Mean	Exact	e^{rndt}
		$dt \to 0$	e^{rt} (always exact)
	Variance	Exact	$\left(2e^{rdt}\cosh(\sigma\sqrt{dt}\,) - 1\right)^n - e^{2nrdt}$
		$dt \to 0$	$e^{2rt}\left(e^{\sigma^2} - 1\right) + \mathbf{O}(dt)$
50-50:	Mean	Exact	e^{rndt}
		$dt \to 0$	e^{rt} (always exact)
	Variance	Exact	$e^{2rt}\left\{\left(2 - \mathrm{sech}^2(\sigma\sqrt{dt}\,)\right)^n - 1\right\}$
		$dt \to 0$	$e^{2rt}\left(e^{\sigma^2} - 1\right) + \mathbf{O}(dt)$

The tables also provide the the first terms of Taylor's series expansion of the corresponding analytic form around $dt = 0$. The limits of these expressions, *i.e.,* the expressions evaluated with $dt = 0$, are the exact moments for the distributions realized by the the underlying process, taken at time $t = ndt$.

The limiting representations in both tables shows that there is no important analytic difference between the moments of the two constructions, nor in their respective rates of convergence in dt to the continuous distribution's values. Thus, there is nothing gained or lost, in terms of fundamental efficacy, in applying one or the other of the constructions. The 50-50 construction is, however, in a sense more natural, and more analytically tractable, as evidenced by the differences in complexity of some of the expressions. For one thing, convergence in probability is more regular and tractable, and this could accordingly be exploited when it is required.

This section concludes with a simple demonstration of the practical equivalence of option values obtained from the alternatives.

Table 2: Exact/Asymptotic Moments of Lattice Slice $n = t\ /\ dt$
For $\ln(v_n)$ (Holding Period Return)

CRR:	Mean	Exact	$n\ \sigma\sqrt{dt}\ \left(e^{rdt}\ \text{csch}(\sigma\sqrt{dt}\ ) - \coth(\sigma\sqrt{dt}\ )\ \right)$
		$dt \to 0$	$\left(r - \frac{\sigma^2}{2}\right) t + \mathbf{O}(dt)$
	Variance	Exact	$2e^{rdt} n \sigma^2 dt \left(\text{csch}(\sigma\sqrt{dt}\ )\right)^2 \cdot \left(\cosh(\sigma\sqrt{dt}\ ) - \cosh(rdt)\right)$
		$dt \to 0$	$\sigma^2\ t + \mathbf{O}(dt)$
50-50:	Mean	Exact	$n\left(rdt - \ln\left(\cosh(\sigma\sqrt{dt}\ )\right)\right)$
		$dt \to 0$	$\left(r - \frac{\sigma^2}{2}\right) t + \mathbf{O}(dt)$
	Variance	Exact	$\sigma^2 n\ dt$
		$dt \to 0$	$\sigma^2\ t$ (always exact)

2.2 Results for three European call options.

For given values of underlying volatility, σ, riskless rate, r, and term of an option, a binomial lattice will give a numerical approximation value for a European call option, which converges with diminishing step size to the continuous-time Black Scholes value.

There is a simple relationship between the values taken on CRR and 50-50 constructed lattices with the same underlying parameters. If each state value $-$the relative value of the underlying stock$-$ is multiplied by $\exp(n\ dt\ \tilde{\theta})$, where the latter constant is defined in equation (11), then the transition probabilities applied to the lattice values are, everywhere, exactly one-half; the CRR lattice, in effect, has "become" a 50-50 lattice. Nothing else changes in the application.

Consider the numerical valuation of the following options.

$$
\begin{aligned}
\text{Number of Steps, } N &= 60 \text{ (Expiry in 5 years.)} \\
dt &= 1/12 \text{ (one month)} \\
\text{Riskless Rate, } r &= 0.05 \text{ (per annum)} \\
\text{Volatility, } \sigma &= 0.17 \\
\text{Initial Asset value } &= 1.00 \\
\text{Strike Prices } &= \{0.50, 1.00, 2.00\} \text{ in turn.}
\end{aligned}
$$

Table 3 presents the results from the exercise. The difference of each numerical result from the corresponding Black Scholes value is given along with the latter value. The differences are neither consistent nor, in general, favoring one approach over the other.

Table 3: Comparison of Option Values For Three Strike Prices

Strike Level	Black Scholes Value	CRR value	50-50 value
2.00	0.02819746	$-1.2\ E-03$	$-1.6\ E-04$
1.00	0.27214924	$-6.4\ E-04$	$+9.1\ E-05$
0.50	0.61109609	$-2.4\ E-05$	$-2.6\ E-05$

3 Advantages of 50-50 Construction in Practice

In application of the "rollback" methodology of binomial valuation of derivatives, 50-50 construction does not offer measurable practical advantage in terms of valuation superiority. The fact that such branching more closely models diffusion behavior in the quantum is, again, not a compelling reason to go against several decades of pedagogy and practice.

Nonetheless, there are circumstances when the simplified lattice branching can be exploited. For example, Monte Carlo applications sometimes employ "path running" algorithms rather than continuous replications; in that case, the method is simplified and accelerated when sample paths evolve with equal branching probabilities, and random *bits*, rather than random floating point numbers, can be used to define paths and the associated antithetics.

From the standpoint of implementing design, practical advantage may also attain from 50-50 branching. Interest rate dependent derivatives often use a lognormal rate movement model along the lines of the Black Derman and Toy model. In specifications without mean reversion, lattice models with equal branching probabilities can be specified in very general circumstances. Then, the same backward induction code can be employed for both interest rate derivatives and equity-like derivatives. This benefit is amplified in designs which numerically implement models for a joint distribution of rate and equity return factors.