# Using Hermite Expansions for Fast and Arbitrarily Accurate Computation of the Expected Loss of a Loan Portfolio Tranche in the Gaussian Factor Model.

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**Abstract**

We propose a fast algorithm for computing the expected tranche loss in the Gaussian factor model with arbitrary accuracy using Hermite expansions. No assumptions about homogeneity of the portfolio are made. The algorithm is a generalization of the algorithm proposed in [4]. The advantage of the new algorithm is that it allows us to achieve higher accuracy in almost the same computational time. It is intended as an alternative to the much slower Fourier transform based methods [2].

## 1 The Gaussian Factor Model

Let us consider a portfolio of $N$ loans. Let the notional of loan $i$ be equal to the fraction $f_i$ of the notional of the whole portfolio. This means that if loan

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$i$ defaults and the entire notional of the loan is lost the portfolio loses fraction $f_i$ or $100f_i\%$ of its value. In practice when a loan $i$ defaults a fraction $r_i$ of its notional will be recovered by the creditors. Thus the actual loss given default (LGD) of loan $i$ is

$$LGD_i = f_i(1 - r_i) \tag{1}$$

fraction or

$$LGD_i = 100f_i(1 - r_i)\% \tag{2}$$

of the notional of the entire portfolio.

We now describe the Gaussian m-factor model of portfolio losses from default. The model requires a number of input parameters. For each loan $i$ we are give a probability $p_i$ of its default. Also for each $i$ and each $k = 1, \ldots, m$ we are given a number $w_{i,k}$ such that $\sum_{k=1}^{m} w_{i,k}^2 < 1$. The number $w_{i,k}$ is the loading factor of the loan $i$ with respect to factor $k$. Let $\phi_1, \ldots, \phi_m$ and $\phi^i, i = 1, \ldots, N$ be independent standard normal random variables. Let $\Phi(x)$ be the cdf of the standard normal distribution. In our model loan $i$ defaults if

$$\sum_{k=1}^{m} w_{i,k}\phi_k + \sqrt{1 - \sum_{k=1}^{m} w_{i,k}^2}\phi^i < \Phi^{-1}(p_i) \tag{3}$$

This indeed happens with probability $p_i$. The factors $\phi_1, \ldots, \phi_m$ are usually interpreted as the state of the global economy, the state of the regional economy, the state of a particular industry and so on. Thus they are the factors that affect the default behavior of all or at least a large group of loans in the portfolio. The factors $\phi^1, \ldots, \phi^N$ are interpreted as the idiosyncratic risks of the loans in the portfolio.

Let $I_i$ be defined by

$$I_i = I_{\{loan\ i\ defaulted\}} \tag{4}$$

We define the random loss caused by the default of loan $i$ as

$$L_i = f_i(1 - r_i)I_i, \tag{5}$$

where $r_i$ is the recovery rate of loan $i$. The total loss of the portfolio is

$$L = \sum_i L_i \tag{6}$$

An important property of the Gaussian factor model is that the $L_i$'s are not independent of each other. Their mutual dependence is induced by the dependence of each $L_i$ on the common factors $\phi_1, \ldots, \phi_m$. Historical data supports the conclusion that losses due to defaults on different loans are correlated with each other. Historical data can also be used to calibrate the loadings $w_{i,k}$.the $L_i$'s are not independent of each other. Their mutual dependence is induced by the dependence of each $L_i$ on the common factors $\phi_1, \ldots, \phi_m$. Historical data supports the conclusion that losses due to defaults on different loans are correlated with each other. Historical data can also be used to calibrate the loadings $w_{i,k}$.

# 2 Conditional Portfolio Loss $L$

When the values of the factors $\phi_1, \ldots, \phi_m$ are fixed, the probability of the default of loan $i$ becomes

$$p^i = \Phi^{-1}\left(\frac{p_i - \sum_k w_{i,k}\phi_k}{\sqrt{1 - \sum_k w_{i,k}^2}}\right) \tag{7}$$

The random losses $L_i$ become conditionally independent Bernoulli variables with the mean given by

$$E_{cond}(L_i) = f_i(1 - r_i)p^i \tag{8}$$

and the variance given by

$$VAR_{cond}(L_i) = f_i^2(1 - r_i)^2 p^i(1 - p^i) \tag{9}$$

By the Central Limit Theorem the conditional distribution of the portfolio loss $L$, given the values of the factors $\phi_1, \ldots, \phi_m$, can be approximated by the normal distribution with the mean

$$E_{cond}(L) = \sum_i E_{cond}(L_i) \tag{10}$$

and the variance

$$VAR_{cond}(L) = \sum_i VAR_{cond}(L_i) \tag{11}$$

In [4] it was shown that for portfolios of 125 names this approximation leads to accurate results.

When the approximation to the conditional distribution of $L$ given by the Central Limit Theorem is deemed insufficiently accurate, an arbitrarily accurate representation of the conditional distribution of the portfolio loss $L$ can be obtained from its Hermite series expansion. For historical reasons this expansion is also known as the Charlier series expansion [3], [1].

# 3 The Hermite Expansion of the Conditional Distribution of the Portfolio Loss $L$

Let $F(x)$ be the c.d.f. of the conditional distribution of the portfolio loss $L$. So that

$$P(L \leq x) = F(x) \tag{12}$$

For each fixed value of the factors $\phi_1, \ldots, \phi_m$ we define the normalized conditional loss $\tilde{L}$ by

$$\tilde{L} = \frac{L - E_{cond}(L)}{\sqrt{VAR_{cond}(L)}} \tag{13}$$

Let $\tilde{F}(x)$ be the c.d.f. of the distribution of the normalized conditional portfolio loss $\tilde{L}$. So that

$$P(\tilde{L} \leq x) = \tilde{F}(x) \tag{14}$$

We define the Hermite polynomial $H_n(x)$ of degree $n$ by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{\frac{-x^2}{2}} \tag{15}$$

Let $c_n$ be defined by

$$c_n = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} H_n(x) d\tilde{F}(x) \tag{16}$$

Then we have

$$\tilde{F}(x) = \sum_{i=0}^{\infty} \int_{-\infty}^{x} c_i H_i(t) \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} dt \tag{17}$$

The series above converges in the sense of distributions (generalized functions) [5]. A good reference on the theory of distributions (generalized functions) is [5]. Let us pick a finite $N$. Then we have

$$\tilde{F}(x) \approx \sum_{i=0}^{N} c_i \int_{-\infty}^{x} H_i(t) \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} dt \tag{18}$$

As before the approximation is in the sense of generalized functions. Equation (18) implies that the distribution of the normalized conditional portfolio loss $\tilde{L}$ can be approximated by a distribution with the density

$$\tilde{\rho}(x) = \sum_{i=0}^{N} c_i H_i(x) \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{19}$$

The function $\tilde{\rho}(x)$ is not necessarily nonnegative and therefore may not be a probability density in the strict sense. However, as is explained in [5], this does not affect the validity of our final result (24). Therefore we may treat $\tilde{\rho}(x)$ as a real probability density.

The distribution of the unnormalized loss $L$ can be approximated by a distribution with density

$$\rho(x) = \sum_{i=0}^{N} \frac{c_i}{\sqrt{VAR_{cond}(L)}} H_i\left(\frac{x - E_{cond}(L)}{\sqrt{VAR_{cond}(L)}}\right) \frac{e^{\frac{-\left(\frac{x-E_{cond}(L)}{\sqrt{VAR_{cond}(L)}}\right)^2}{2}}}{\sqrt{2\pi}} \tag{20}$$

The joint distribution of the factors $\phi_1, \ldots, \phi_m$ and the portfolio loss $L$ can be approximated by a distribution with density

$$\rho_{joint}(\phi_1, \ldots, \phi_m, L) = \rho(L) \prod_{k=1}^{m} \rho_{G,0,1}(\phi_k), \tag{21}$$

where $\rho_{G,0,1}(x)$ stands for the Gaussian density with mean 0 and variance 1.

Observe that the coefficient $c_n$ depends only on the moments of the distribution $\tilde{F}(x)$. Since $L_i$'s are independent Bernoulli random variables these moments are known analytically. Thus in the case under consideration all the $c_n$'s are known analytically.

If in equation (20) we set $N = 1$ we obtain the standard approximation by the normal density proposed in [4]. Thus the algorithm proposed here is a generalization of the algorithm in [4]. We show later that it gives good numerical results even when the portfolio size is too small for the normal approximation to be accurate.

# 4 Expected Loss of a Tranche of Loan Portfolio

Let $0 \leq a < b \leq 1$. We define a tranche loss profile $Tl_{a,b}(x)$ by

$$Tl_{a,b}(x) = \frac{min(b-a, max(x-a,0))}{b-a} \tag{22}$$

Number $a$ is called the attachment point of a tranche, while $b$ is called the detachment point of a tranche. The expected loss of a tranche is then

$$TLoss(a,b) = \int Tl_{a,b}(L)\rho_{joint}(\phi_1,\ldots,\phi_m,L)d\phi_1\ldots\phi_m L \tag{23}$$

This can be rewritten as a double integral

$$TLoss(a,b) = \int\int Tl_{a,b}(L)\rho(L)dL\prod_{k=1}^{m}\rho_{G,0,1}(\phi_k)d\phi_1\ldots\phi_m \tag{24}$$

The inside integral with respect to $L$ can be done analytically for fixed values of the factors $\phi_1,\ldots,\phi_m$. The outside integral has to be computed numerically. However, since it is an integral of a bounded smooth function with respect to m-dimensional Gaussian density, it is one of the simpler integrals to compute numerically.

# 5 Numerical Example

In this section we test the proposed algorithm on several portfolios of smaller size. For these portfolios the approximation to the conditional distribution of the portfolio loss $L$ given by the Central Limit Theorem is not very accurate, because of their small size. However, the Hermite expansion produces very good results. We apply the proposed algorithm to the single factor Gaussian model of a portfolio with $n$ names. We take $n$ to be 25 (size of DJ iTraxx Australia), 30 (size of DJ iTraxx ex Japan), 50 (size of DJ iTraxx CJ) and 100 (size of DJCDX.NA.HY). We choose a single factor model because it is the one most frequently used in practice. For each $n$ we compute the loss of the equity tranche with the attachment point $a = 0$ or $a = 0\%$ and the detachment point 3%. The parameters of the porfolio are

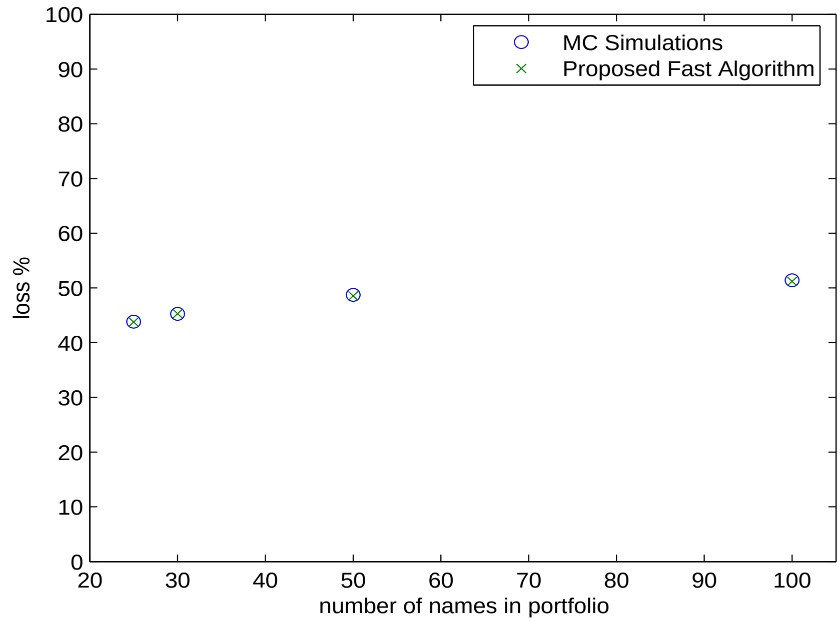
$$f_i = \frac{1}{n}$$

$$p_i = 0.015 + \frac{0.05(i-1)}{n-1}$$
$$r_i = 0.5 - \frac{0.1(i-1)}{n-1}$$
$$w_{i1} = 0.5 - \frac{0.1(i-1)}{n-1}, \tag{25}$$

where $i = 1, \ldots, n$. Finally, we choose $N = 5$ in (18).

In Figure 1 we compare the expected loss computed using $10^6$ Monte Carlo samples with the expected loss computed using formula (24).[1] The agreement between the two is good.

Figure 1: Equity Tranche Loss in the Gaussian Single Factor Model




[1]The author has the code implementing the algorithm described here in MATLAB, VBA for Excel and C.

# 6 Conclusions.

To obtain the results in Figure 1 we only needed to perform a single one dimensional numerical integration for each tranche. This is an improvement over the Fourier transform based methods [2] which require computing a large number of Fourier transforms for each tranche. Each individual Fourier transform is as computationally expensive as (24).

The Hermite expansion (18) can be used to achieve arbitrary accuracy when the normal approximation is insufficiently accurate. The proposed algorithm is as fast as the algorithm proposed in [4] because the inside integral in (24) can be done analytically.

We also comment that the algorithm can be extended trivially to the case of non-constant recovery rates and recovery rates correlated with the state of the factor variables.

# 7 Acknowledgments

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