Investment Optimization under Constraints

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Abstract. We analyze general stochastic optimization financial problems under constraints in a general framework, which includes financial models with some "imperfection", such as constrained portfolios, labor income, random endowment and large investor models. By using general optional decomposition under constraints in a multiplicative form, we first develop a dual formulation under minimal assumption modeled as in Pham and Mnif (2002) [Ph-M], Long (2002) [L02a]. We then are able to prove an existence and uniqueness of an optimal solution to primal and to the corresponding dual problem. An optimal investment to the original problem then can be found by convex duality, similarly to the case considered by Kramkov and Schachermayer (1999) [K-Sch].

Key words: Stochastic Optimization, Investment Optimization, Duality Theory, Convex and State Constraints, Optional Decomposition

JEL Classification: G11

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1 Introduction

Basic problems of mathematical finance are the problems of pricing, hedging or optimizing some portfolio choices, which could be formulated as the optimization problem of maximizing the expected value of some concave objective (eventually state-dependent) utility functions. The problems can be attacked by the stochastic optimal control methods as, for instance, in the papers of Merton (1971) [M], Duffie, Flemming, Soner and Zariphopoulou (1997) [D-F-W-Z], or by a modern, more powerful and elegant method: the duality approaches. The difference is that, while the optimal control methods are wedded to the dynamic programming Hamilton-Jacobi-Bellman equation and based on the requirement of Markov state processes, the duality techniques, rather then rely on the Hamilton-Jacobi-Bellman equation, use the stochastic duality theory and permit us to deal with more general and non-markovian processes. The key point in this method is the duality characterizations of the set of wealth processes provided by the set of martingale measures for state processes.

Duality approaches have been used with success in treating portfolio optimization problems for incomplete financial markets in a continuous-time diffusion model such as in Karatzas, Lehoczky, Shreve and Xu (1991) [K-L-S-X], or in a more general framework, where the asset prices are semimartingales, as it is showed in series of papers of Kramkov and Schachermayer (1999 - 2001) [K-Sch]. The further extension to the case of constraints on the proportion of portfolio choice appears in Shreve and Xu (1992) [Sh-X], Cvitanic and Karatzas (1992) [Cv-Ka]. The extension to the case of constraints imposed on the amount addressed by Cuoco and Cvitanic (1998) [Cv-Cu], Cuoco (1997) [Cu]. However, all the mentioned papers above dealt with the Itô processes model.

Recently, Pham and Mnif (2002) [Ph-M] generalized the financial framework and developed it into a general structure that has an optional decomposition in an *additive form*. This framework is universal enough to incorporate many financial models, such as with constrained portfolios, random endowment and large investor, as well as reinsurance models.

In this paper, we study the general structure of optimization financial problems of an agent, whose wealth process admits an optional decomposition in a *multiplicative form*. Like the model proposed by Pham and Mnif (2002), our formulation is also sufficiently general to include as special cases the problems with constrained proportion portfolios, random endowment and large investor, as well as with the existence of labor income, which were considered in Cvitanic and Karatzas (1992, 1993) [Cv-Ka], El Karoui and Quenez (1996) [E-Q], Cuoco and Cvitanic (1998) [Cv-Cu], El Karoui and Jeanblanc-Piqué (1998) [E-J], Rogers (2001) [RG] and Klein and Rogers (2001) [K-RG]. Using the general optional decomposition under constraints in a multiplicative form of Föllmer and Kramkov [F-K], we provide the duality characterization of the state process in terms of a set of suitable probability measures, which are associated to the convex constraints on the family of wealth processes. We are then able to prove an existence and uniqueness of an optimal solution to our primal problem. However, to ensure an existence to the dual problem, we need to enlarge the set of abstract setting by considering the closure of this set with respect to the norm of $\mathbf{L}^0(\mathbf{P})$. With this setting, under minimal assumptions similar to those Pham and Mnif (2002) [Ph-M], we are also able to prove an existence of an optimal solution to the dual problem. The solution to the original problem then can be found by convex duality, similarly to the case considered by Kramkov and Schachermayer (1999) [K-Sch], but with careful attention to the behavior of the additional term arising from the convex and state space constraints.

The outline of the paper is organized as follows. Section 2 describes the general framework of a financial model. In Section 3 we set up and analyze the properties of the dual set, which is the set of equivalent local supermartingale measures for state processes. In Section 4, we formulate the optimization problem and the so-called budget constraint. The existence and uniqueness of an optimal solution to the original problem is given in Section 6, after analyzing the properties of our abstract setting. In Section 7 we provide and analyze the duality theorem. Section 8 devotes concluding remarks. For completeness, in Appendix we prove a stochastic control lemma needed in the context of this paper, which was given without proof in Föllmer and Kramkov (1997) [F-K].

2 The Market Setting

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ denote a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the "usual" conditions, here $T \in \mathbf{R}_+$ is a fixed time horizon and we assume that $\mathcal{F}_0$ is trivial. Except for processes which appear as integrand of stochastic integrals, all processes considered in the sequel are assumed to be real-valued, to have right-continuous paths with left limits (*càdlàg*), and to be adapted with respect to the given filtration; in particular they are all optional.

For the theory of stochastic integration we refer to Dellacherie and Mayer (1982) [De-Ma], Protter (1990) [P] and Jacod and Shiryaev (1987) [J-S]. The stochastic integral of a predictable process π with respect to a semimartingale X will be denoted by $\int \pi dX$ or $\pi \bullet X$. We denote by $\mathbf{L}(X)$ the space of all predictable processes integrable with respect to X. The Émery distance between two semimartingales X and Y is defined as:

$$D(X,Y) = \sup_{|\pi|\le 1}\left(\sum_{n\ge 1} 2^{-n}\mathbf{E}[\min(|(\pi \bullet (X-Y)_n)|, 1)]\right),$$

where the supremum is taken over the set of all predictable processes π bounded by 1. The corresponding topology is called the *semimartingale* topology.

Let R be a $\mathbf{R}^n$-valued semimartingale in $(\Omega, \mathcal{F}, \mathbf{P})$. We prescribe a convex subset Π of $\mathbf{L}(R)$ containing the zero element and convex in the following sense: for any predictable process $\zeta \in [0,1]$

and for all $\pi^1, \pi^2 \in \Pi$ we have:

$$\zeta\pi^1 + (1-\zeta)\pi^2 \in \Pi.$$

We consider a family $\{H^\pi : \pi \in \Pi\}$ of adapted processes with finite variation, with initial value 0. In the following, we shall denote by $\mathcal{O}$ the set of all nondecreasing adapted processes with initial value 0 and introduce an ordering $\preceq$ on $\mathcal{O}$ indicating that A is *not greater* than B ($A \preceq B$) if $(B - A)$ is a nondecreasing process. We shall assume the following concavity property:

$$H^{\zeta\pi^1+(1-\zeta)\pi^2} \preceq \zeta \bullet H^{\pi^1} + (1-\zeta) \bullet H^{\pi^2}, \tag{1}$$

We then consider the following family:

$$\widetilde{\mathcal{X}} = \left\{ \pi \bullet R + H^\pi \right\}$$

We shall make the following closure property assumption:

Standing Assumption 2.1 *Under the condition (1), the set $\widetilde{\mathcal{X}}$ is closed for semimartingale topology.*

Given $\widetilde{X}^0 \in \widetilde{\mathcal{X}}$, we define the set

$$\widetilde{\mathcal{X}}_b = \left\{ \widetilde{X} - \widetilde{X}^0 : \widetilde{X} \in \widetilde{\mathcal{X}} \text{ and } \widetilde{X} - \widetilde{X}^0 \text{ is locally bounded from below} \right\} \tag{2}$$

so that $\widetilde{\mathcal{X}}_b$ is locally bounded from below, closed for the semimartingale topology, null at 0 and containing the constant process 0.

Remark 2.1 *Under the relation (1), the family of semimartingales $\widetilde{\mathcal{X}}$ (or $\widetilde{\mathcal{X}}_b$) clearly is a predictable convex set in the sense of [F-K], i.e. for $X^i \in \widetilde{\mathcal{X}}($ or $\widetilde{\mathcal{X}}_b)$ $(i = 1, 2)$, and for any predictable process ζ such that $0 \leq \zeta \leq 1$ we have:*

$$\zeta \bullet X^1 + (1-\zeta) \bullet X^2 \in \widetilde{\mathcal{X}} - \mathcal{O} \ (\text{ or } \widetilde{\mathcal{X}}_b - \mathcal{O}) \tag{3}$$

For $x > 0$, we denote by $\widetilde{\mathcal{W}}(x)$ the family of nonnegative semimartingales defined as follows:

$$\widetilde{\mathcal{W}}(x) \quad = \quad \left\{ \widetilde{W}(x) = x\mathcal{E}(\widetilde{X} - \widetilde{D}); \, \widetilde{X} \in \widetilde{\mathcal{X}}, \text{ and } \widetilde{D} \in \mathcal{O} \right\} \tag{4}$$

$$\widetilde{\mathcal{W}}_b(x) \quad = \quad \left\{ \widetilde{W}_b(x) = x\mathcal{E}(\widetilde{X}_b - \widetilde{D}); \, \widetilde{X}_b \in \widetilde{\mathcal{X}}_b, \text{ and } \widetilde{D} \in \mathcal{O} \right\} \tag{5}$$

where $\mathcal{E}(\cdot)$ is the exponential semimartingale of Doléans-Dade. Clearly,

$$\widetilde{\mathcal{W}}(x) = x\widetilde{\mathcal{W}}(1) = \{x\widetilde{W} : \widetilde{W} \in \widetilde{\mathcal{W}}(1)\},$$

Now let us introduce the set $\mathcal{P}(\widetilde{\mathcal{X}}_b)$ of all nonnegative local-martingales Z with $Z_0 = 1$ such that for any $X_b \in \widetilde{\mathcal{X}}_b$, there exists a process $A \in \mathcal{O}_p$ – the set of nondecreasing predictable processes with initial values $A_0 = 0$ – satisfying

$$Z(\widetilde{X}_b - A) \ \text{ is a } \mathbf{P}\text{-local supermartingale for any } \widetilde{X}_b \in \widetilde{\mathcal{X}}_b. \tag{6}$$

The next definition of the upper variation process is adopted from the one in Föllmer and Kramkov (1997) [F-K].

Definition 2.1 *The upper variation process of $\widetilde{\mathcal{X}}_b$ corresponding to $Z \in \mathcal{P}(\widetilde{\mathcal{X}}_b)$, defined as the element $A^{\widetilde{\mathcal{X}}_b}(Z)$ in $\mathcal{O}_p$ satisfying (6) and is minimal with respect to this property, i.e. such that $A^{\widetilde{\mathcal{X}}_b}(Z) \preceq A$ for any $A \in \mathcal{O}_p$ satisfying (6).*

In the context of this paper, we assume throughout that

Standing Assumption 2.2 *The upper variation process $A^{\widetilde{\mathcal{X}}_b}(Z)$ exists.*

On the set $\mathcal{P}(\widetilde{\mathcal{X}}_b)$, we define the set

$$\mathcal{P}^*(\widetilde{\mathcal{X}}_b) = \Big\{ Z \in \mathcal{P}(\widetilde{\mathcal{X}}_b) \colon A^{\widetilde{\mathcal{X}}_b}(Z) \text{ is a continuous process with finite variation} \Big\},$$

and its subset

$$\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b) = \Big\{ Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b) \colon Z \text{ is a positive and } \mathbf{P}\text{-martingale} \Big\}.$$

We can identify any $Z \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$ with a probability $Q \sim \mathbf{P}$, whose density process is $Z = (Z_t)_{t\in[0,T]}$, $Z_t = \mathbf{E}[dQ/d\mathbf{P}|\mathcal{F}_t]$.

In the remainder of this paper, we shall make the following standing assumptions

Standing Assumption 2.3 $\mathcal{P}^*(\widetilde{\mathcal{X}}_b) \neq \emptyset$

Let us introduce a positive process S^0. In what follows, we assume that $\widetilde{X}^0$ can be chosen so as:

Assumption 2.1 *$\widetilde{X}^0$ is a finite variation process with continuous paths and null at 0.*

Assumption 2.2

$$\mathbf{E}\Big[\frac{Z_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)\mathcal{E}(\widetilde{X}^0_T)S^0_T} \Big] < \infty \quad \forall\, Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b)$$

Remark 2.2

1. *In general, we shall choose $\widetilde{X}^0 \equiv H^0$ corresponding to the element of $\widetilde{\mathcal{X}}$ for $\pi = 0$.*
2. *A sufficient condition for Assumption 2.2 is that $\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)\mathcal{E}(\widetilde{X}^0_T)S^0_T$ is bounded away from zero.*

We are now interested on the family of state processes:

$$\begin{aligned} \mathcal{W}(x) &\triangleq \{W = S^0\widetilde{W} \colon \widetilde{W} \in \widetilde{\mathcal{W}}(x)\}, &(7)\\ &= \Big\{ W = S^0\Big(x + \widetilde{W}_- \bullet \widetilde{X} - \widetilde{W}_- \bullet \widetilde{D}\Big) \colon \widetilde{X} \in \widetilde{\mathcal{X}},\ \widetilde{D} \in \mathcal{O} \Big\}, &(8) \end{aligned}$$

It is clear that

$$\mathcal{W}(x) = x\mathcal{W}(1)$$

We suppose that the process $\widetilde{W}_- \bullet \widetilde{D}$ can be represented by the formula:

$$\int_0^t \widetilde{W}_{s-} d\widetilde{D}_s = \int_0^t \widetilde{c}_s ds,\ \forall\, t \in [0,T]$$

In a financial context, $W(x) \in \mathcal{W}(x)$ can be interpreted as a wealth process with an initial nonrandom endowment x of an economic agent, who is taking part in a financial market consisting of $n+1$ assets: one bond and n stocks, and with finite horizon time T. We suppose that the bond with positive price process S^0 is chosen as a numéraire. In a sense of Föllmer and Kramkov (1997) [F-K], $R = (R^i)_{1\le i\le n}$ is stated as the $\mathbf{R}^n$-valued *return* process of n stocks, and the increasing process $\widetilde{D}$ appears in (8) as the accumulated proportion consumption process. In this framework, an agent can decide at any time $t \in [0,T]$, what proportion $\pi_t = (\pi^i_t)_{1\le i\le n}$ of his wealth to invest in the i^{th}-stock. Of course these decisions can only be based on the current information $\mathcal{F}_t$ without anticipation of the future. The set Π models constraints on proportion portfolios π. Process H^π

allows to take into account the term arising from labor income and large investor. Process $\widetilde{X}^0$ describes the wage income (rate) throughout investment life-time. With $\widetilde{X}^0$ chosen in advance, we state that the proportion income process $\widetilde{X}^0$ is spanned by the market assets and therefore is not a source of new uncertainty.

We now define a consumption process:

Definition 2.2 *A consumption process $c(\cdot)$ is an $\mathcal{F}_t$-adapted nonnegative process, which is related to the accumulated proportion process by the formula*

$$\int_0^t c_s ds = \int_0^t S_s^0 \widetilde{W}_{s-} d\widetilde{D}_s, \quad 0 \leq t \leq T. \tag{9}$$

Put $\Lambda_t = t$, then in the standard notation of the stochastic calculus for semimartingales (9) can be written as follows:

$$c \bullet \Lambda = S^0 \widetilde{W}_- \bullet \widetilde{D},$$

and we have $c \equiv S^0 \widetilde{c}$.

One of the families of examples we have in mind for applications is described below. We refer the reader to Pham and Mnif (2002) and Pham (2002) for more explicitly examples.

Example (Cuoco and Liu (2000), Rogers (2001)).

This is an important example, generalizing a number of other papers in the subject: Cvitanic and Karatzas (1992, 1993) [Cv-Ka], El Karoui, Peng and Quenez (1997) [E-Q], Cuoco and Cvitanic (1998) [Cv-Cu], Rogers (2001) [RG], El Karoui and Jeanblanc-Piqué (1998) [E-J], for example. The numéraire S^0 and the wealth process W of the agent satisfies:

$$\begin{aligned} dW_t &= W_t\left[r_t dt + \pi_t\left(\sigma_t dB_t + (b_t - r_t\mathbf{1})dt\right) + g(t,\pi_t)dt + e_t dt\right] - c_t dt, \quad (10)\\ W_0 &= x \\ dS_t^0 &= r_t S_t^0 dt, \quad S_0^0 = 1 \quad (11)\end{aligned}$$

where $\pi_t \in \Pi$, B is an n-dimensional Brownian motion, b, r, $V \equiv \sigma\sigma^T$ (the superscript T stands for "transpose"), V^{-1}, e are all bounded processes, and there is a uniform Lipschitz bound on g: for some $\theta < \infty$,

$$|g(t,x,\omega) - g(t,y,\omega)| \leq \theta |x - y|$$

for all x, y, t and ω. In our model the agent receives an income with a proportional (eventually stochastic) rate e_t per unit time.

The unconventional term in the dynamics (10) is the term involving g about which we assume:

- for $x \in \mathbf{R}^n$, $(t,w) \mapsto g(t,x,\omega)$ is an optional process;
- for each $t \in [0,T]$ and $\omega \in \Omega$, $x \mapsto g(t,x,\omega)$ is concave and upper semicontinuous.
- $g(t,0,\omega) = 0$ for all $t \in [0,T]$ and $\omega \in \Omega$.

Suppose that $c_t = \frac{D_t}{W_t}$, and D_t is a nonnegative process. Now let $\widetilde{W} \triangleq \frac{W}{S^0}$, by Itô Lemma we have:

$$d\widetilde{W}_t = \widetilde{W}_t\left[\pi_t\left(\sigma_t dB_t + (b_t - r_t\mathbf{1})dt\right) + g(t,\pi_t)dt - D_t dt + e_t dt\right]$$

In this case, we choose

$$\widetilde{X}_t^0 = H_t^0 = \int_0^t e_s ds$$

so that $\widetilde{X}_b = \pi \bullet R + H^\pi$, with

$$\begin{aligned} dR_t &= \sigma_t dB_t + (b_t - r_t\mathbf{1})dt \\ dH_t^\pi &= g(t,\pi_t)dt \end{aligned}$$

By the martingale representation theorem for Brownian motion (see, e.g. Karatzas and Shreve (1991)), any probability measure equivalent to $\mathbf{P}$ has a density process in the form:

$$Z^{\upsilon} \triangleq \frac{dB^{\upsilon}}{dB} = \mathcal{E}\Big(-\int \sigma_t^T V_t^{-1}(b_t - r_t\mathbf{1} + \upsilon)dB_t\Big)$$

where $\upsilon \in \mathcal{M}$:

$$\mathcal{M} \triangleq \Big\{\upsilon\colon \int_0^T |\sigma_t^T V_t^{-1}\upsilon|^2 dt < \infty, \text{ and } \mathbf{E}[Z_T^{\upsilon}] = 1\Big\}$$

Now by Girsanov's Theorem, the Doob-Meyer decomposition of $\widetilde{X}_b = \pi \bullet R + H^{\pi} \in \widetilde{\mathcal{X}}_b$ under $P^{\upsilon} = Z_T^{\upsilon}\mathbf{P}$, $\upsilon \in \mathcal{M}$, is:

$$d\widetilde{X}_{b_t} = \pi_t\sigma_t dB_t^{\upsilon} + dA_t^{\upsilon,\pi}$$

where B^{υ} is a n-dimensional Brownian motion under P^{υ} and $A^{\upsilon,\pi}$ is the predictable compensator under P^{υ}:

$$dA_t^{\upsilon,\pi} = (g(t,\pi_t) - \pi_t\upsilon)dt$$

Now denote

$$\widetilde{g}(t,\upsilon) = \int_0^t \sup_{\pi\in\Pi}\Big(g(s,\pi_s) - \pi_s\upsilon\Big)ds$$

the convex conjugate of $-g(t,-\pi)$ and let $\widetilde{\mathcal{H}} = \{\upsilon \in \mathbf{R}^n\colon \widetilde{g}(t,\upsilon) < \infty\}$ its effective domain.

We deduce that $\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$ consists of all probability measures $P^{\upsilon}, \upsilon \in \mathcal{M}(\widetilde{\mathcal{H}})$:

$$\mathcal{M}(\widetilde{\mathcal{H}}) \triangleq \{\upsilon \in \mathcal{M}\colon \upsilon \in \widetilde{\mathcal{H}} \text{ and } \widetilde{g}(t,\upsilon) \text{ is a continuous process with finite variation}\}$$

Moreover, the upper variation process is given by:

$$A^{\widetilde{\mathcal{X}}_b}(P^{\upsilon})_t = \int_0^t \widetilde{g}(s,\upsilon)ds$$

Since all coefficients are bounded, it is straightforward to verify that the model satisfies the Standing Assumptions 2.2, 2.3, and Assumptions 2.1, 2.2. Moreover, the closure property of $\widetilde{\mathcal{X}}$ may also be proved in this model under a Liptschitz condition on function g and the invariance of the Emery distance under translation, see Pham (2002).

Remark 2.3

1. *In the paper of Cuoco and Cvitanic (1998) [Cv-Cu], they preassummed that $\widetilde{g}$ is bounded on its effective domain.*
2. *Recall that in our framework, the labor income is restricted to be a continuous process with finite variation. Therefore our framework is not applicable to the general case considered by El Karoui and Jeanblanc-Picqué, where the income process e is of the general Markovian form $de_t = \mu(t,e_t)dt + \sigma(t,e_t)dB_t$.*

3 The Dual Setting

We define the family $\mathcal{Y}(y)$ of nonnegative semimartingales Y with $Y_0 = y$ and such that

$$\mathcal{Y}(y) = \Big\{Y = y\frac{Z}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z))\mathcal{E}(\widetilde{X}^0)S^0}\ :\ Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b)\Big\}$$

In the sequel, let us denote by $\mathcal{Y}_+(y) \subset \mathcal{Y}(y)$ the subset containing all positive $Y \in \mathcal{Y}(y)$. We also suppose that any $Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b)$ can be written as $Z = \mathcal{E}(N)$, where N is some $\mathbf{P}$-local martingale null at 0. Since $\widetilde{X}^0$ and $A^{\widetilde{\mathcal{X}}_b}(Z)$ are continuous processes of finite variation, by using Proposition

I.4.4.69 in Jacod and Shiryaev (1987) [J-S] we have $[\widetilde{X}^0, Y] = 0$ and $[A^{\widetilde{\mathcal{X}}_b}(Z) + \widetilde{X}^0, Y] = 0$ for any semimartingale Y with $Y_0 = 0$, therefore any $Y \in \mathcal{Y}(y)$ can be rewritten as:

$$Y = y\frac{\mathcal{E}(N - A^{\widetilde{\mathcal{X}}_b}(Z) - \widetilde{X}^0)}{S^0} \tag{12}$$

Lemma 3.1 *For all $x > 0$, $y > 0$, $Y \in \mathcal{Y}(y)$ and $W \in \mathcal{A}(x)$, the process $(YW + Yc \bullet \Lambda)$ is a* $\mathbf{P}$*-supermartingale.*

Proof. Since

$$\begin{array}{rcl} \mathcal{Y}(y) & = & y\mathcal{Y}(1) \\ \mathcal{A}(x) & = & x\mathcal{A}(1) \end{array}$$

for all $x > 0$, $y > 0$, then we may focus without loss of generality to the sets $\mathcal{Y}(1)$ and $\mathcal{A}(1)$.

Recall that for any semimartingale X null at 0 the Doléan-Dade exponential $\mathcal{E}(X)$ is a solution of the following stochastic differential equation:

$$Z = 1 + Z_- \bullet X, \quad Z_0 = 1.$$

Moreover any solution of this equation coincides with $\mathcal{E}(X)$ on the set $\{(\omega, t) \colon \mathcal{E}(X)_- \neq 0\}$.

Since $(\widetilde{X}^0 + A^{\widetilde{\mathcal{X}}_b}(Z))$ is a continuous process with finite variation, by Itô's lemma and after straightforward calculations, from (12) we get:

$$YW + Yc \bullet \Lambda = 1 + Y_- W_- \bullet \left(\widetilde{X} - A^{\widetilde{\mathcal{X}}_b}(Z) - \widetilde{X}^0 + N + [N, \widetilde{X}]\right) \tag{13}$$

By some algebras we also get

$$\begin{array}{rcl} Z(\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z)) & = & 1 + Z_- \bullet (\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z)) + [Z, \widetilde{X}] + \\ & & + \, (\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z))_- \bullet Z \\ & = & 1 + Z_- \bullet \left(\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z) + [N, \widetilde{X}]\right) + \\ & & + \, (\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z))_- \bullet Z, \end{array}$$

Since $Z(\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z))$ is a $\mathbf{P}$-local supermartingale. The last term on the right-hand side of the above equality is a $\mathbf{P}$-local martingale, it follows then

$$Z_- \bullet \left(\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z) + [N, \widetilde{X}]\right)$$

is also a $\mathbf{P}$-local supermartingale.

Moreover, since Z_- is positive and predictable, we deduce that

$$\left(\widetilde{X} - \widetilde{X}^0 - A^{\widetilde{\mathcal{X}}_b}(Z) + [N, \widetilde{X}]\right) \tag{14}$$

is a $\mathbf{P}$-local supermartingale. Since Y, W are nonnegative, by Remark VI.53.d in Dellacherie and Mayer (1982) [De-Ma], we deduce from (14) that the processes on the both sides of (13) is a $\mathbf{P}$-local supermartingales. Furthermore, since $Y \geq 0$, $W \geq 0$, $c \bullet \Lambda \in \mathcal{O}$ we have $YW + Yc \bullet \Lambda$ is bounded from below. We then deduce by Fatou's lemma that in fact, $YW + Yc \bullet \Lambda$ is a nonnegative $\mathbf{P}$-supermartingale. This completes the proof of the lemma. $\square$

Remark 3.1 *Assumption 2.2 implies that in fact $\mathcal{Y}(y) \subset \mathbf{L}^1(\mathbf{P})$.*

Remark 3.2 *From the last lemma, we deduce that, for any $x > 0$, the product YW is a* $\mathbf{P}$*-supermartingale for all $W \in \mathcal{W}(x)$.*

4 The Utility Function and the Optimization Problem

We now consider an economic agent, which has a utility function $U\colon (0,\infty) \to \mathbf{R}$ for wealth. At first, we recall some classical definitions and properties of utility function.

Definition 4.1 *A utility function $U\colon (0,\infty)\times\Omega \to \mathbf{R}\cup\{-\infty\}$ is a uppersemicontinuous, concave, continuously differentiable and nondecreasing on its (convex) domain. Moreover, it satisfies the Inada-type conditions:*

$$U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty \quad \textit{and} \quad U'(\overline{x}) \triangleq \lim_{x\uparrow\overline{x}} U'(\overline{x}) = 0 \quad \textit{a.s.} \tag{15}$$

To alleviate notations, we omit the dependence in the state $\omega \in \Omega$ and we write $U(x)$. We assume that $\inf\{x > 0\colon U(x) > -\infty\} = 0$ a.s. and $\sup\{x > 0\colon U(x) > -\infty\} = \infty$.

We set:

$$\overline{x} = \sup\{x > 0\colon U(x) > -\infty\}, \tag{16}$$

so that the convex domain of U, dom $U \triangleq \{x > 0\colon U(x) > -\infty\}$ satisfies $\mathrm{int}(\mathrm{dom}U) = (0,\overline{x})$, a.s. Notice that by the uppersemicontinuity of U, we have $U(\overline{x}) < \infty$ if $\overline{x} < \infty$.

We shall denote by $I(.)$ the (continuous, strictly decreasing) inverse of the marginal utility function $U'(.)$; this function maps $(0, U'(0))$ onto $(0,\overline{x})$, extended by continuity on $(0,\infty)$ by setting $I(y) = 0$ for $y > U'(0)$.

We also introduce the (state-dependent) conjugate function of U

$$\widetilde{U}(y) \triangleq \sup_{x>0}[U(x) - xy], \quad y > 0, \tag{17}$$

It is well-known (see e.g. [R]) that this function is nonincreasing, convex differentiable on $(0,\infty)$ with $\widetilde{U}(y) = U(\overline{x})$, and satisfies

$$\widetilde{U}'(y) = -I(y), \quad y > 0, \text{ a.s.}, \tag{18}$$

We also know that $I(y)$ attains the supremum in (17), i.e.

$$\widetilde{U}(y) = U(I(y)) - yI(y), \quad y > 0, \text{ a.s.} \tag{19}$$

From Lemma 3.1 we deduce that the process

$$\frac{ZW}{\mathcal{E}(\widetilde{X}^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z))S^0} + \frac{Zc}{\mathcal{E}(\widetilde{X}^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z))S^0} \bullet \Lambda$$

is a $\mathbf{P}$-supermartingale for any $Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b)$, since then

$$v(W,c) \triangleq \mathbf{E}\left[\frac{Z_T W_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)\mathcal{E}(\widetilde{X}^0_T)S^0_T} + \int_0^T \frac{Z_t c_t dt}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_t)\mathcal{E}(\widetilde{X}^0_t)S^0_t}\right] \le x, \tag{20}$$

for all $Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b)$.

In a financial context, the formula (20) is stated as a constraint on the consumption plans.

For a given initial capital $x > 0$, the goal of the agent is to *maximize the expected value* $\mathbf{E}[U(W_T)]$. The value function of this problem is denoted by:

$$\begin{aligned} u(x) &= \sup_{W\in\mathcal{W}(x)} \mathbf{E}[U(W_T)], \quad x > 0 & (21)\\ &= \sup_{W\in\mathcal{W}(x)} \mathbf{E}[U(W_T \wedge \overline{W})] & (22) \end{aligned}$$

where

$$\overline{W} = \sup\{W > 0\colon U(W) > -\infty\},$$

Remark 4.1 *When U is a deterministic function with $\overline{W} = \infty$, the problem (21) is an utility maximization problem from terminal state. However, U could be on the form $U(x+B)$, where B is an $\mathcal{F}_T$-measurable nonnegative random variable, problem (21) becomes an utility-based pricing problem, see Karatzas and Kou (1996). U also can be on the form:*

$$U(x) = \begin{cases} -l(B-x) & 0 \le x \le B, \\ -\infty & x > B \end{cases}$$

where l is a convex nondecreasing function on $\mathbf{R}_+$. This is a shortfall risk minimization problem in finance and insurance (cf. Pham and Mnif (2002)).

5 The Abstract Setting

Following the line of research of Kramkov and Schachermayer (1999) [K-Sch], we dualize the optimization problem (21). First, we need to pass from the sets of processes $\mathcal{W}(x)$, $\mathcal{Y}(y)$ to the sets $\mathcal{C}(x)$ and $\mathcal{D}^0(y)$ of random variables dominated by the elements of $\mathcal{W}(x)$ and $\mathcal{Y}(y)$ respectively. The sets $\mathcal{C}(x)$ and $\mathcal{D}^0(y)$ are defined as follows:

$$\mathcal{C}(x) = \{F \in \mathbf{L}^0_+(\mathcal{F}_T) : F \le W_T, \text{ for some } W \in \mathcal{W}(x)\} \tag{23}$$

$$\mathcal{D}^0(y) = \{g \in \mathbf{L}^0_+(\mathcal{F}_T) : g \le Y_T, \text{ for some } Y \in \mathcal{Y}(y)\} \tag{24}$$

for any $x > 0$, $y > 0$.

We denote by $\mathcal{D}^0_+(y)$ the subset of $\mathcal{D}^0(y)$ consisting of all positive g.

Now let us denote the closure of $\mathcal{D}^0(y)$ in $\mathbf{L}^0(\mathbf{P})$ by $\mathcal{D}(y)$.

Remark 5.1 *Since the set $\mathcal{Y}(y) \subset \mathbf{L}^1(\mathbf{P})$, then it is clear that $\mathcal{D}^0(y) \subset \mathbf{L}^1(\mathbf{P})$, we then deduce from Vitali's Convergence Theorem [1] that $\mathcal{D}(y) \subset \mathbf{L}^1(\mathbf{P})$.*

We observe that

$$\mathcal{C}(x) = x\mathcal{C}(1) \quad \forall\ x > 0 \tag{25}$$

and the analogous relations for $\mathcal{D}(y)$, $\mathcal{D}^0(y)$, $\mathcal{W}(x)$, $\widetilde{\mathcal{W}}(x)$ with any $x > 0,\ y > 0$.

The duality relation between $\mathcal{C}(x)$ and $\mathcal{D}(y)$ (or equivalently between $\mathcal{A}^*_T(x)$ and $\mathcal{Y}(y)$ is a basic theme in mathematical finance (see, e.g. Delbaen and Scharchermayer (1994), Kramkov and Schachermayer (1999) [D-Sch, K-Sch]... and the references therein)).

The next lemma presents the key polarity properties of "bipolar"-type, which the primal and dual above quantities are related by.

Lemma 5.1 *Suppose that $F \in \mathbf{L}^0_+(\mathcal{F}_T)$ and $g \in \mathcal{D}(1)$, then*

$$F \in \mathcal{C}(1) \text{ iff } \upsilon(F) \triangleq \sup_{g \in \mathcal{D}(1)} \mathbf{E}[gF] \le 1, \tag{26}$$

Proof. First of all, notice that by Fatou's lemma we have:

$$\sup_{g \in \mathcal{D}(1)} \mathbf{E}[gF] = \sup_{g \in \mathcal{D}^0(1)} \mathbf{E}[gF] \tag{27}$$

so all we need to confirm (26) is to check the statement for $g \in \mathcal{D}^0(1)$.

The "if part" of the relation (26) is obvious, since YW is a $\mathbf{P}$-supermartingale. What remains now is to prove the converse assertion.

Consider an adapted, nonnegative $\mathcal{F}_T$-measurable random variable A_T defined as

$$A_T = \frac{F}{\mathcal{E}(\widetilde{X}^0)_T S^0_T}$$

[1] Vitali's Convergence Theorem states that if f_n is a sequence of uniformly $\mathbf{P}$-integrable functions, $f_n \to f$ a.e., and $|f| < \infty$ a.e., then $f \in \mathbf{L}^1(\mathbf{P})$ and $f_n \to f$ in $\mathbf{L}^1(\mathbf{P})$.

Since $\mathcal{Y}(1) \subset \mathcal{D}^0(1)$ then for any $Y \in \mathcal{Y}(1)$, by (27) we should have:

$$\sup_{Y\in\mathcal{Y}(1)} \mathbf{E}[Y_T F] \leq \sup_{g\in\mathcal{D}(1)} \mathbf{E}[gF] \leq 1 \tag{28}$$

Notice that all random variables under consideration are nonnegative. Plugging (12) into (28), we obtain

$$\begin{aligned}\sup_{Q\in\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)} \mathbf{E}^Q\left[\frac{A_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_T}\right] \quad &\leq \quad \sup_{Z\in\mathcal{P}^*(\widetilde{\mathcal{X}}_b)} \mathbf{E}\left[\frac{Z_T F}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z))_T \mathcal{E}(\widetilde{X}^0)_T S_T^0}\right] \\ &= \quad \sup_{Y\in\mathcal{Y}(1)} \mathbf{E}[Y_T F] \ \leq \ 1 \ < \ \infty \end{aligned}\tag{29}$$

where the first inequality follows from the inclusion $\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b) \subset \mathcal{P}^*(\widetilde{\mathcal{X}}_b)$. Then by the stochastic control lemma A.1 there exists a *càdlàg* version of the nonnegative process:

$$\widetilde{W}_t^b = \operatorname*{ess\,sup}_{Q\in\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b), T\in\mathcal{T}_t} \mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_t \mathbf{E}^Q\left[\frac{A_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_T}\middle|\mathcal{F}_t\right], \quad 0 \leq t \leq T \tag{30}$$

Moreover, for any $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$, the process $\widetilde{W}^b/\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))$ is a Q-supermartingale. By the optional decomposition under constraints (see Corollary 3.1 in Föllmer and Kramkov (1997) [F-K]), the process $\widetilde{W}^b$ admits a decomposition:

$$\widetilde{W}^b = \widetilde{W}_0^b \mathcal{E}(\widetilde{X}_b - \widetilde{D}) = \widetilde{W}_0^b + \widetilde{W}_-^b \bullet \widetilde{X}_b - \widetilde{W}_-^b \bullet \widetilde{D} \tag{31}$$

where $\widetilde{X}_b \in \widetilde{\mathcal{X}}_b$ defined as in (2), $\widetilde{D} \in \mathcal{O}$, and

$$v(A) \triangleq \widetilde{W}_0 = \widetilde{W}_0^b = \sup_{Q\in\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)} \mathbf{E}^Q\left[\frac{A_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_T}\right] \leq 1, \tag{32}$$

We now consider process

$$\widetilde{W} = \mathcal{E}(X^0)\widetilde{W}^b = \widetilde{W}_0 + \widetilde{W}_- \bullet \widetilde{X} - \widetilde{W}_- \bullet \widetilde{D}$$

where $\widetilde{X} \in \widetilde{\mathcal{X}}$.

Let us consider process $W = S^0\widetilde{W}$. Using the definition of $\widetilde{\mathcal{X}}$, (31) we get

$$\begin{aligned} W \quad &= \quad S^0\Big(\widetilde{W}_0\mathcal{E}(\widetilde{X} - \widetilde{D})\Big) \\ &= \quad S^0\Big(\widetilde{W}_0 + \widetilde{W}_- \bullet \widetilde{X} - \widetilde{W}_- \bullet \widetilde{D}\Big) \\ &= \quad S^0\Big(1 + \widetilde{W}_- \bullet \widetilde{X} - (\widetilde{W}_- \bullet \widetilde{D} + 1 - \widetilde{W}_0)\Big) \end{aligned}$$

where $\widetilde{X} \in \widetilde{\mathcal{X}}$. It is obvious that $W \in \mathcal{W}(x)$, $W_T \geq F$. Moreover, since $\widetilde{W}_0 \leq 1$, and $\widetilde{W}$ is a nonnegative process, $\widetilde{D}$ is an increasing process, then $\widetilde{W}_- \bullet \widetilde{D}$ is a nonnegative increasing process. As a result, we see that W belongs to the set $\mathcal{W}(1)$. Hence, $W \in \mathcal{W}(1)$ is a wealth process that dominates F in a sense of (23). □

Characterization (32) in the last lemma means that $v(A)$ is the least initial state value, which allows to dominate in the almost sure sense the $\mathcal{F}_T$ random variable A by a state process. In the financial context, $v(A)$ is usually called the superreplication cost of the European option A_T. Notice in particular that the expression of $v(A)$ does not depend on the choice of $\widetilde{X}^0$.

Lemma 5.2 *For any $x > 0$, the set $\mathcal{C}(x)$ is convex, solid* [2] *and closed for the topology of convergence in measure.*

[2] A subset $\mathcal{C} \in \mathbf{L}_+^0(\Omega, \mathcal{F}, \mathbf{P})$ is called solid, if $0 \preceq h \preceq f$ and $f \in \mathcal{C}$ implies that $h \in \mathcal{C}$.

Proof. Note that the solidity of $\mathcal{C}(x)$ is rather obvious. It remains to prove its convexity.

Let $S^0\widetilde{W}^1$ and $S^0\widetilde{W}^2$ are two pair of processes in $\mathcal{W}(x)$. Taking any $\zeta^1 = 1-\zeta^2 \in [0,1]$ and defining the convex combinations

$$\begin{aligned} \widetilde{W}^* &= \zeta^1\widetilde{W}^1 + \zeta^2\widetilde{W}^2 \\ \widetilde{c}^* &= \zeta^1\widetilde{c}^1 + \zeta^2\widetilde{c}^2 \end{aligned}$$

By the predictable convexity property on the set $\widetilde{\mathcal{X}}$ and the associativity of the stochastic integral (see, e.g. Theorem 19 in Protter (1990) [P]), we find immediately that:

$$\begin{aligned} \widetilde{W}^* &= x + (\zeta^1\widetilde{W}^1_- \bullet \widetilde{X}^1 + \zeta^2\widetilde{W}^2_- \bullet \widetilde{X}^2) - \widetilde{c}^* \bullet \Lambda \\ &= x + \widetilde{W}^*_- \bullet \left(\frac{\zeta^1\widetilde{W}^1_-}{\widetilde{W}^*_-} \bullet \widetilde{X}^1 + \frac{\zeta^2\widetilde{W}^2_-}{\widetilde{W}^*_-} \bullet \widetilde{X}^2 \right) - \widetilde{c}^* \bullet \Lambda \\ &= x + \widetilde{W}^*_- \bullet \overline{X} - (\widetilde{c}^* \bullet \Lambda + \widetilde{W}^*_- \bullet \overline{D}) \\ &\preceq \widehat{W} \triangleq x + \widetilde{W}^*_- \bullet \overline{X} - \widetilde{W}^*_- \bullet \overline{D} \end{aligned}$$

with $\overline{X} \in \widetilde{\mathcal{X}}$, $\overline{D} \in \mathcal{O}$. Clearly $\widehat{W}$ belongs to the set $\widetilde{\mathcal{W}}(x)$ (corresponding to $\overline{X}$). By the definition of $\mathcal{C}(x)$, the convex combination $S^0\widetilde{W}^*$ is also in $\mathcal{C}(x)$, hence $\mathcal{C}(x)$ is convex.

Now let $(F^n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{C}(x)$ converging to $F \in \mathbf{L}^0_+(\mathcal{F}_T)$ a.s. We will use lemma 5.1 to prove that F belong to the sets $\mathcal{C}(x)$.

Since all $g \in \mathcal{D}^0(y)$ are dominated by $Y \in \mathcal{Y}(y)$ and all processes under consideration are nonnegative, by Fatou's lemma and by (27), we have:

$$\begin{aligned} \sup_{g\in\mathcal{D}(1)} \mathbf{E}[gF] &\leq \sup_{Y\in\mathcal{Y}(1)} \mathbf{E}[Y_TF] \\ &= \sup_{Z\in\mathcal{P}^*(\widetilde{\mathcal{X}}_b)} \mathbf{E}\left[\frac{Z_TF}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)\mathcal{E}(\widetilde{X}^0_T)S^0_T}\right] \\ &\leq \sup_{g\in\mathcal{D}(1)} \liminf_{n\to\infty} \mathbf{E}\left[\frac{Z_TF^n}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)\mathcal{E}(\widetilde{X}^0_T)S^0_T}\right] \\ &\leq xy. \end{aligned}$$

This proves the closeness property of $\mathcal{C}(x)$ and completes the proof of the lemma. □

Remark 5.2 *If we let the initial capital value in the market setting also change, then we shall have:*

$$\langle \zeta, \mathcal{C}(x_1)\rangle \oplus \langle (1-\zeta), \mathcal{C}(x_2)\rangle \subset \mathcal{C}(\zeta x_1 + (1-\zeta)x_2)$$

i.e.,

$$\langle \zeta, (F_{x_1}, f_{x_1})\rangle + \langle (1-\zeta), (F_{x_2}, f_{x_2})\rangle \in \mathcal{C}(\zeta x_1 + (1-\zeta)x_2)$$

where $(F_{x_i}, f_{x_i}) \in \mathcal{C}(x_i),\ i = 1,2.$

Lemma 5.3 *For any $y > 0$, the set $\mathcal{D}(y)$ is convex, solid and closed with respect to the topology of convergence in measure.*

Proof. First note that the closeness of $\mathcal{D}(y)$ follows immediately from its definition and the solidity of $\mathcal{D}(y)$ is rather obvious. We now prove the remaining assertion.

Since 0 already belongs to the set $\mathcal{D}(y)$ and the convexity is preserved under weak convergence so all we need to verify the convexity for $g \in \mathcal{D}^0_+(y)$.

We first show that $\mathcal{D}^0_+(y)$ is a convex set for all elements $Y \in \mathcal{Y}_+(y)$, then the convexity of the set $\mathcal{D}^0_+(y)$ follows from the the solidity property of $\mathcal{D}^0_+(y)$.

Let Y^1 and Y^2 are processes in $\mathcal{Y}_+(y)$. By the definition of $\mathcal{Y}_+(y)$ and from (12), we shall have:

$$\begin{aligned} Y^1 &= y\frac{Z^1}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^1))\mathcal{E}(\widetilde{X}^0)S^0} = y\frac{\mathcal{E}(N^1)}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^1))\mathcal{E}(\widetilde{X}^0)S^0} \\ Y^2 &= y\frac{Z^2}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^2))\mathcal{E}(\widetilde{X}^0)S^0} = y\frac{\mathcal{E}(N^2)}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^2))\mathcal{E}(\widetilde{X}^0)S^0} \end{aligned}$$

where $Z^i = \mathcal{E}(N^i) \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b)$. Taking any $\zeta^1 = 1-\zeta^2 \in (0,1)$ and defining the convex combinations

$$\widehat{Y} = \zeta^1 Y^1 + \zeta^2 Y^2 \tag{33}$$

Notice that $\widehat{Y}$ is strictly positive. Let us fix some $\widetilde{X}_b \in \widetilde{\mathcal{X}}_b$, and define a process $A \in \mathcal{O}_p$ and a **P**-local martingale N as follows:

$$A = \frac{\zeta^1 Y^1_-}{\widehat{Y}_-} \bullet A^{\widetilde{\mathcal{X}}_b}(Z^1) + \frac{\zeta^2 Y^2_-}{\widehat{Y}_-} \bullet A^{\widetilde{\mathcal{X}}_b}(Z^2) \tag{34}$$

$$N = \frac{\zeta^1 Y^1_-}{\widehat{Y}_-} \bullet N^1 + \frac{\zeta^2 Y^2_-}{\widehat{Y}_-} \bullet N^2 \tag{35}$$

We now check whether $\widehat{Z} \triangleq \mathcal{E}(N)$ belongs to the set $\mathcal{P}^*(\widetilde{\mathcal{X}}_b)$.

We shall show that $\widehat{Z}(\widetilde{X}_b - A)$ is a **P**-local supermartingale. Since $A^{\widetilde{\mathcal{X}}_b}(Z^i)$ are continuous processes with finite variation, we deduce from I.4.34 c, and I.4.36 in Jacod and Shirayev (1987) [J-S] that A is also a continuous process with finite variation. First, we prove that

$$\frac{\zeta^1 Z^1}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^1))} + \frac{\zeta^2 Z^2}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^2))} = \frac{\widehat{Z}}{\mathcal{E}(A)} \triangleq \widehat{Y}_0 \tag{36}$$

For convenience, denote $Y^i_0 \triangleq Y^i S^0 \mathcal{E}(\widetilde{X}^0)$. Hence we have:

$$Y^i_0 = \mathcal{E}(N^i - A^{\widetilde{\mathcal{X}}_b}(Z^i)) = 1 + Y^i_0 \bullet (N^i - A^{\widetilde{\mathcal{X}}_b}(Z^i)), \quad i = 1,\, 2$$

Therefore,

$$\begin{aligned} \widehat{Y}_0 &= \mathcal{E}(\widehat{N} - A^{\widetilde{\mathcal{X}}_b}(\widehat{Z})) = 1 + \widehat{Y}_0 \bullet (\widehat{N} - A^{\widetilde{\mathcal{X}}_b}(\widehat{Z})) \\ &= 1 + \zeta^1 Y^1_0 \bullet (N^1 - A^{\widetilde{\mathcal{X}}_b}(Z^1)) + \zeta^2 Y^2_0 \bullet (N^1 - A^{\widetilde{\mathcal{X}}_b}(Z^1)) \\ &= \zeta^1 Y^1_0 + \zeta^2 Y^2_0 \end{aligned}$$

and we get (36).

Recall the exponential semimartingales of Doléans-Dade have the following properties (see, e.g. Kallsen and Shirayev (2002) [K-Sh])

$$\begin{aligned} \mathcal{E}(X) &= 1 + \mathcal{E}(X)_- \bullet X \\ X &= X_0 + \frac{1}{\mathcal{E}(X)_-} \bullet \mathcal{E}(X) \end{aligned}$$

From (35) and using the properties of the Doléans-Dade exponential semimartingales, we deduce that:

$$\widehat{Z} = \frac{\zeta^1 Y^1_- \widehat{Z}_-}{\widehat{Y}_- Z^1_-} \bullet Z^1 + \frac{\zeta^2 Y^2_- \widehat{Z}_-}{\widehat{Y}_- Z^2_-} \bullet Z^2 \tag{37}$$

For convenience, we denote

$$\begin{aligned} \overline{\zeta}^1 &\triangleq \frac{\zeta^1 Y^1 \widehat{Z}}{\widehat{Y} Z^1} \\ \overline{\zeta}^2 &\triangleq \frac{\zeta^2 Y^2 \widehat{Z}}{\widehat{Y} Z^2} \end{aligned}$$

From (37) we have

$$\begin{aligned}[\widehat{Z}, \widetilde{X}_b - A] &= \overline{\zeta^1} \bullet [Z^1, \widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^1)] + \overline{\zeta^2} \bullet [Z^2, \widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^2)] + \\ &+ \overline{\zeta^1} \bullet [Z^1, A^{\widetilde{\mathcal{X}}_b}(Z^1)] + \overline{\zeta^2} \bullet [Z^2, A^{\widetilde{\mathcal{X}}_b}(Z^2)] - [\widehat{Z}, A] \end{aligned} \tag{38}$$

Recall that we have:

$$\begin{aligned} Z^i(\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^i)) &= (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^i))_- \bullet Z^i + Z^i_- \bullet (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^i)) + \\ &+ [Z^i, \widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^i)] \\ \widetilde{X}_b - A &= \frac{\zeta^1 Y^1_-}{\widehat{Y}_-} \bullet (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^1)) + \frac{\zeta^2 Y^2_-}{\widehat{Y}_-} \bullet (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^2)) \end{aligned}$$

Using Ito's lemma and after some straightforward calculations we obtain:

$$\begin{aligned} \widehat{Z}(\widetilde{X}_b - A) &= (\widetilde{X}_b - A)_- \bullet \widehat{Z} + \widehat{Z}_- \bullet (\widetilde{X}_b - A) + [\widehat{Z}, \widetilde{X}_b - A] \\ &= (\widetilde{X}_b - A)_- \bullet \widehat{Z} + \overline{\zeta}^1_- Z^1_- \bullet (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^1)) - \\ &- \overline{\zeta}^1_- (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^1))_- \bullet Z^1 + \\ &+ \overline{\zeta}^2_- Z^2_- \bullet (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^2)) - \\ &- \overline{\zeta}^2_- (\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^2))_- \bullet Z^2 + \overline{\zeta}^1_- \bullet [Z^1, A^{\widetilde{\mathcal{X}}_b}(Z^1)] + \\ &+ \overline{\zeta}^2_- \bullet [Z^2, A^{\widetilde{\mathcal{X}}_b}(Z^2)] - [\widehat{Z}, A] \end{aligned} \tag{39}$$

By the definition of $\mathcal{P}^*(\widetilde{\mathcal{X}}_b)$ then we have $Z^i(\widetilde{X}_b - A^{\widetilde{\mathcal{X}}_b}(Z^i))$, $i = 1, 2$ is a **P**-local supermartingale. Moreover, since Z^i, $\widehat{Z}$ are **P**-local supermartingale, $A^{\widetilde{\mathcal{X}}_b}(Z^i)$ and A are predictable processes with finite variation, then Theorem VII.36 in Dellacherie and Mayer (1982) [De-Ma] implies that $[Z^i, A^{\widetilde{\mathcal{X}}_b}(Z^i)]$ and $[\widehat{Z}, A]$ are **P**-local martingale. Therefore (39) imples that $\widehat{Z}(\widetilde{X}_b - A)$ is a **P**-local supermartingale.

We conclude that $\widehat{Z}$ belongs to the set $\mathcal{P}^*(\widetilde{\mathcal{X}}_b)$ with the uppervariation process $A^{\widetilde{\mathcal{X}}_b}(\widehat{Z})$, which is continuous and satisfies

$$A^{\widetilde{\mathcal{X}}_b}(\widehat{Z}) \preceq A$$

Since then, we have:

$$\begin{aligned} \widehat{Y} &= y\left(\frac{\zeta^1 Z^1}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^1))\mathcal{E}(\widetilde{X}^0)S^0} + \frac{\zeta^2 Z^2}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^2))\mathcal{E}(\widetilde{X}^0)S^0}\right) \\ &= y\frac{\widehat{Z}}{\mathcal{E}(A)\mathcal{E}(\widetilde{X}^0)S^0} \\ &\preceq \ \overline{Y} \triangleq y\frac{\widehat{Z}}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\widehat{Z}))\mathcal{E}(\widetilde{X}^0)S^0} \in \mathcal{Y}(y) \subset \mathcal{D}^0(y) \end{aligned}$$

As a result, we have proved the convexity property of $\mathcal{D}(y)$. This completes the proof of the lemma.
□

6 The Primal Problem - Existence and Uniqueness

We next formulate a lemma in the line of the work of Pham and Mnif (2002) [Ph-M]. We refer the reader to their work for the proof.

Lemma 6.1

$$u(x) = \sup_{F \in \mathcal{C}(x)} \mathbf{E}[U(F)] = \sup_{F \in \mathcal{C}(x)} \mathbf{E}[U(F \wedge \overline{W})] \quad x > 0 \tag{40}$$

(i) If $W^* \in \mathcal{W}(x)$ solves (21), then $F \equiv W^*_T \in \mathcal{C}(x)$ solves (40),
(ii) Conversely, if $F^* \in \mathcal{C}(x)$ solves (40), then $W \in \mathcal{C}(x)$, such that $F^* \leq W_T$, solves (21).

In this section, we focus on the existence and uniqueness of a solution to the primal optimization problem (21) and (40).

To exclude the trivial case, we shall assume throughout the paper

Standing Assumption 6.1

$$u(x) < \infty, \quad \textit{for some } x > 0. \tag{41}$$

We make the following assumption for utility functions.

Assumption 6.1 *We have either*
(i) U is bounded, or
(ii) there exist $\lambda \in (0,1)$, $\overline{Y} \in \mathcal{Y}(1)$ such that

$$\frac{1}{\overline{Y}_T} \in \mathbf{L}^{\overline{p}}(\mathbf{P}), \quad \textit{for some } \overline{p} > \frac{\lambda}{1-\lambda}, \tag{42}$$

$x_0 \in dom\,(U)$ satisfying $U(x_0) \in \mathbf{L}^p(\mathbf{P})$, where $p = \frac{\overline{p}}{\lambda(1+\overline{p})}$ and $\Upsilon \in \mathbf{L}^p(\mathbf{P})$, $k \in \mathbf{L}^\infty(\mathbf{P})$ such that

$$U^+(x) \le kx^\lambda + \Upsilon \quad \forall\, x \in dom\,(U) \cap [x_0, \infty) \tag{43}$$

Remark 6.1 *The above assumptions are very similar to Assumption 4.1 in Pham and Mnif (2002). However, there is a minor error in Pham and Mnif (2002). In the context of their paper, these authors should make the assumption $\frac{S_T^0}{Z_T} \in \mathbf{L}^{\overline{p}}(\mathbf{P})$ in order to get the desired result.*

Lemma 6.2 *Under Assumption 6.1, the family $\Big\{U^+(F),\, F \in \mathcal{C}(x)\Big\}$ is uniformly $\mathbf{P}$-integrable.*

Proof. The arguments are similar to the proof in Lemma 4.3 in Pham and Mnif (2002), we include it for completeness.

If the function U is bounded, then this assertion is trivial.

Otherwise, for any $F \in \mathcal{C}(x)$, there exists a wealth process $W \in \mathcal{W}(x)$ dominating F in a sense of (23).

$$\begin{aligned}
\mathbf{E}[F^{\lambda p}] &\le (\mathbf{E}[\overline{Y}_T F])^{\lambda p} \left(\mathbf{E}\left[\overline{Y}_T^{-\frac{\lambda p}{1-\lambda p}}\right]\right)^{1-\lambda p} \\
&\le \left(\mathbf{E}[\overline{Y}_T F]\right)^{\lambda p} \left(\mathbf{E}[\overline{Y}_T^{-\overline{p}}]\right)^{1-\lambda p} \\
&\le x^{\lambda p} \left(\mathbf{E}[\overline{Y}_T^{-\overline{p}}]\right)^{1-\lambda p} \\
&< \infty
\end{aligned} \tag{44}$$

where the first inequality is followed by applying Hölder's inequality, the second and third inequality follow from Lemma 5.1 and (42).

Since U is increasing on its domains, there exists a positive constant such that for all $F \in \mathcal{C}(x)$:

$$\begin{aligned}
\mathbf{E}[U^+(F)^p] &= \mathbf{E}[U^+(F)^p \mathbf{1}_{F \le x_0}] + \mathbf{E}[U^+(F)^+ \mathbf{1}_{F > x_0}] \\
&\le \mathbf{E}[U^+(x_0)^p] + (const)\,(\mathbf{E}[(kH^\lambda)^p] + \mathbf{E}[\Upsilon^p]\Big)
\end{aligned}$$

By (44) and assumptions on k and Υ, this proves the $\mathbf{L}^p(\mathbf{P})$-boundedness of the family $\Big\{U^+(F)^p,\, F \in \mathcal{C}(x)\Big\}$ and therefore its uniform integrability under $\mathbf{P}$. □

Theorem 6.1 *Under Assumption 6.1,*

(i) the optimal solution $F^* \in \mathcal{C}(x)$ to problem (40) exits for all $x > 0$, and $\overline{W}$ is a solution to (40) for $x \geq \upsilon(\overline{W})$. Moreover, it is unique if U is strictly concave on its domains,

(ii) the function u is nondecreasing and concave, on $(0, \infty)$ and equal to $\mathbf{E}[U(\overline{W})$ on $(\upsilon(\overline{W})$. Moreover if $U(W) < U(\overline{W})$, then u is strictly increasing on $[0, \upsilon(\overline{W}]$ and $\upsilon(F^*) = x$ for any $x \in [0, \upsilon(\overline{W}]$, and if U is strictly concave on its domains, then u is strictly concave on $[0, \upsilon(\overline{W})]$.

Since the proof is straightforward, let us omit it and refer the reader to paper of Pham and Mnif (2002) for discussion.

7 The Dual Problem

The main result below proves that under certain conditions, the value of the primal problem, expressed as a supremum over some set, is equal to the value of the dual problem, expressed as an infimum over some other set. The result presented here adopts the argument of Kramkov and Schachermayer (1999) [K-Sch].

We now consider the following optimization problem:

$$\begin{aligned} \widetilde{u}(y) &= \inf_{g \in \mathcal{D}(y)} \widetilde{J}(y; g) \triangleq \inf_{g \in \mathcal{D}(y)} \mathbf{E}[\widetilde{U}(g)] && (45) \\ &= \inf_{g \in \mathcal{D}(1)} \mathbf{E}[\widetilde{U}(yg)] && (46) \end{aligned}$$

where

$$\widetilde{U}(y) = \sup_{x>0} [U(x) - xy], \quad y > 0$$

In order to proceed, we shall need the following standing assumption

Standing Assumption 7.1

$$\widetilde{u}(y) < \infty, \quad \textit{for some } y > 0 \tag{47}$$

Assumption 7.1 *(i) There exists an $x^0 \in dom\,(U)$ with $\frac{x_T^0}{S_T^0} \in \mathbf{L}^\infty(\mathbf{P})$ and $U(x_T^0) \in \mathbf{L}^1(\mathbf{P})$, $\lambda \in (0,1)$, $\Upsilon \in \mathbf{L}^1(\mathbf{P})$, such that:*

$$xU'(x) \leq \lambda U(x) + \Upsilon,$$

$\mathbf{P}$*–a.s., for all $x \in dom\,(U)) \cap [x^0, \infty)$,*

(ii) $\forall\, \zeta > 0, \exists\, \delta_\zeta$ real-valued in $[0, \zeta]$, such that

$$\begin{aligned} \delta_\zeta \mathcal{E}(\widetilde{X}_T^0) \mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T) S_T^0 &\in dom\,(U) \\ U(\delta_\zeta \mathcal{E}(\widetilde{X}_T^0) \mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T) S_T^0) &\in \mathbf{L}^1(\mathbf{P}), \quad \forall\, Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b) \end{aligned}$$

Assumption 7.1 (i) is equivalent to the asymptotic elasticity condition of Kramkov and Scharchermayer (1999) [K-Sch], we refer the reader to Remark 5.1 in Pham and Mnif [Ph-M] for discussion. Since then it is a sufficient condition for Standing Assumption 7.1 (see Note 2 in Kramkov and Scharchermayer (2001) [K-Sch]).

Theorem 7.1 *If Assumption 6.1 and Assumption 7.1 hold true, then in addition to the assertions in Theorem 6.1 we have:*

(i) For any $y > 0$, the optimal solution $yg^* \in \mathcal{D}(y)$ to the problem (45) exists. Moreover, the differentiability of U implies the strict convexity of $\widetilde{U}$, which in turn implies uniqueness of this solution.

(ii) The value functions u and $\widetilde{u}$ are conjugate,

$$\begin{aligned} \widetilde{u}(y) &= \sup_{x>0} [u(x) - xy], \quad y > 0, && (48) \\ u(x) &= \inf_{y>0} [\widetilde{u}(y) + xy], \quad x > 0, && (49) \end{aligned}$$

and there exists an optimal solution y^* to the problem (49) for all $x \in (0, \upsilon(\overline{W}))$. The function u is continuously differentiable on $(0,\infty)$ and the function $\widetilde{u}$ is strictly convex on $(\widetilde{u} < \infty)$.

(iii) If $y^* g^* \in \mathcal{D}(y^*)$, is the optimal solution to the problem $\widetilde{u}(y^*)$ in (45), then the solution to the optimization problem (40) is given by:

$$F^* = I(y^* g^*) \tag{50}$$

and the solution to (21) W^* satisfies:

$$\mathbf{E}[g^* W_T^*] = x \tag{51}$$

for all $t \in [0,T]$. Moreover, if g^* belongs to $\mathcal{D}^0(1)$ then we also have:

$$\mathbf{E}[Y_T^* W_T^*] = x \tag{52}$$

for any $Y^* = \frac{Z^*}{\mathcal{E}(\widetilde{X}^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z^*))S^0} \in \mathcal{Y}(1)$ that dominates g^* in a sense of (24). If in addition, $Z^* \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$ then the wealth process W^* can be determined as follows:

$$W_t^* = S_t^0 \left(\mathcal{E}(\widetilde{X}_t^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q^*)_t)\mathbf{E}^{Q^*}\left[\frac{A_T^*}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q^*)_T)} \middle| \mathcal{F}_t \right] \right) \tag{53}$$

where A_T^* defined as

$$A_T^* = \frac{F^*}{\mathcal{E}(\widetilde{X}^0)_T S_T^0}$$

The proof of Theorems 7.1 is broken into several lemmas. Some lemmas are slightly modified results of that of Kramkov and Schachermayer (1999) [K-Sch], which we include for completeness.

Lemma 7.1 *Under Assumption 7.1, the family $\left\{ \widetilde{U}^-(g);\, g \in \mathcal{D}(y) \right\}$ is uniformly integrable.*

Proof. We suppose that $\widetilde{U}(\infty) < 0$, otherwise there is nothing to prove, and let $\phi\colon (-\widetilde{U}(0), -\widetilde{U}(\infty)) \to (0,\infty)$ denote the convex increasing inverse of $-\widetilde{U}$, and and by the Remark 5.1 we shall have

$$\begin{aligned} \mathbf{E}[\phi(\widetilde{U}^-(g))] &\leq \mathbf{E}[\phi(-\widetilde{U}(g))] + \phi(0) \\ &= \mathbf{E}[g] + \phi(0) \\ &< \infty, \end{aligned}$$

By (18) and l'Hospital rule,

$$\lim_{x \to -\widetilde{U}(\infty)} \frac{\phi(x)}{x} = \lim_{g \to \infty} \frac{g}{-\widetilde{U}(g)} = \lim_{g \to \infty} \frac{1}{I(g)} = \infty$$

where $I(g) \triangleq -\widetilde{U}(g)$. By applying the de la Vallée-Poussin theorem, the sequence $(\widetilde{U}^-(g^n))_{n \geq 1}$ is uniformly integrable follows from noting that $(g^n)_{n\geq 1}$ remains finite in $\mathbf{L}^1(\mathbf{P})$ (see Remark 5.1). $\square$

The next corollary is a useful result from the last lemma.

Corollary 7.1 *For any $y > 0$, there is some $g^* \in \mathcal{D}(y)$ for which the infimum defining in (45) is attained. Differentiability of U implies strict convexity of $\widetilde{U}$, which in turn implies uniqueness of the minimizing g^*.*

Proof. We take a minimizing sequence $g^n \in \mathcal{D}(y)$ such that:

$$\widetilde{u}(y) \leq \mathbf{E}\left[\widetilde{U}(g^n) \right] \leq \widetilde{u}(y) + n^{-1}. \tag{54}$$

By lemma A.1.1 of Delbaen and Schachermayer [D-Sch], there exists a sequence $g_1^n \in \text{conv}(g^n, g^{n+1}, \cdots)_{n\geq 1}$, that are $\mathbf{P}$–almost surely convergent to limit g^*. We may suppose that g^* still satisfies the inequality (54). Since $\mathcal{D}(y)$ is convex and closed, then $(g^n)_{n\geq 1} \in \mathcal{D}(y)$ converges almost surely to $g^* \in \mathcal{D}(y)$. By lemma 7.1 and by applying Fatou's lemma to the sequence $(\widetilde{U}^+(g_1^n))_{n\geq 1}$, and to the right-hand side of the inequality (54) we obtain:

$$\widetilde{u}(y) \geq \liminf_{n\to\infty} \mathbf{E}[\widetilde{U}(g_1^n] \geq \mathbf{E}[\widetilde{U}(g^*)] \geq \widetilde{u}(y) \tag{55}$$

The uniqueness assertion is immediated by general duality results (see Theorem V.26.3 in Rockafellar (1970) [R]). □

Let us fix $n \in \mathbf{N}$ and introduce the sets:

$$\mathcal{B}_n(x) \triangleq \{\mathcal{C}(x) \in \mathbf{L}_+^\infty(\mathbf{P})\colon 0 \leq F \leq n \ \ \forall\, t \in [0,T]\} \tag{56}$$

then $\mathcal{B}_n(x)$ is convex and compact in the topology $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$.

Lemma 7.2 *Under the conditions stated in Theorem 7.1, the functions u and $\widetilde{u}$ are dual:*

$$\begin{aligned} \widetilde{u}(y) &= \sup_{x>0}[u(x) - xy], \quad y > 0, \\ u(x) &= \inf_{y>0}[\widetilde{u}(y) + xy], \quad x > 0, \end{aligned}$$

The function u is continuously differentiable on $(0, \infty)$ and the function $\widetilde{u}$ is strictly convex on $(\widetilde{u} < \infty)$.

Proof. Since $\mathcal{D}(y)$ is a closed convex subset of $\mathbf{L}^1(\mathbf{P})$. By the Minimax Theorem (see, e.g., Aubin and Ekeland (1984) [A-E], or Rogers (2001) [RG]) we then have:

$$\sup_{F\in\mathcal{B}_n(x)} \inf_{g\in\mathcal{D}(y)} \mathbf{E}[U(F) - gF] = \inf_{g\in\mathcal{D}(y)} \sup_{F\in\mathcal{B}_n(x)} \mathbf{E}[U(F) - gF\Big]$$

It follows that:

$$\begin{aligned} \lim_{n\to\infty} \sup_{F\in\mathcal{B}_n(x)} \inf_{g\in\mathcal{D}(y)} \mathbf{E}[U(F) - gF] &= \sup_{x>0} \sup_{F\in\mathcal{C}(x)} \inf_{g\in\mathcal{D}(y)} \Big(\mathbf{E}[U(F)] - \mathbf{E}[gF]\Big) \\ &= \sup_{x>0}[u(x) - xy] \\ &= \widetilde{u}(y) \end{aligned} \tag{57}$$

where the last equality follows from Lemma 5.1.
Now let

$$\widetilde{U}^n(g) \triangleq \sup_{0\leq F\leq n} [U(F) - Fg]$$

and $\widetilde{u}^n(y) \triangleq \inf_{g\in\mathcal{D}(y)} \mathbf{E}[\widetilde{U}^n(g)]$. From (57), it is clear that the proof will be complete provided we can prove that

$$\lim_{n\to\infty} \widetilde{u}_n(y) = \widetilde{u}(y), \quad y > 0.$$

Evidently, we have $\widetilde{u}^n \leq \widetilde{u}$. We now proceed to prove the inverse inequality.
Suppose that $g^n \in \mathcal{D}(y)$ are such that

$$\widetilde{u}^n(y) \leq \mathbf{E}[\widetilde{U}^n(g^n)] \leq \widetilde{u}^n(y) + n^{-1}.$$

Since $\mathcal{D}(y)$ is a closed and convex set, then by using again Lemma A.1.1 of Delbaen and Schachermayer in [D-Sch], we can find a sequence $g_1^n \in \text{conv}(g^n, g^{n+1}, \cdots)_{n\geq 1}$ in $\mathcal{D}(y)$ which converges almost surely to a variable $g \in \mathcal{D}(y)$. From the properties of utility functions, it is immediate that $\widetilde{U}^n(g) = \widetilde{U}(g)$ for $U'(n) \leq g$. Notice that U_1' and U' are decreasing functions, we deduce from

Lemma 7.1 that the sequence $(\widetilde{U}^n)^-(g_1^n)_{n\geq 1}$ is uniformly integrable. Similarly as in the proof of the previous lemma, the convexity of $\widetilde{U}^n$ and Fatou's lemma now imply:

$$\widetilde{u}^n(y) \geq \lim_{n\to\infty} \mathbf{E}[\widetilde{U}^n(g^n)] \geq \liminf_{n\to\infty} \mathbf{E}[\widetilde{U}^n(g_1^n] \geq \mathbf{E}[\widetilde{U}(g)] \geq \widetilde{u}(y),$$

as required.

Since function U is differentiable, the conjugate function $\widetilde{U}$ are strictly convex by general duality results (see, Theorem V.26.3 in Rockafellar (1970) [R]). Using the strict convexity of $\widetilde{U}$ we can easily deduce that $\widetilde{u}$ is also a strict convex function on $(\widetilde{u} < \infty)$. Therefore, the continuously differentiability of the dual function u of $\widetilde{u}$ follows by using again Theorem V.26.3 in Rockafellar (1970) [R]. $\square$

We prove the following results for later use.

Lemma 7.3 *Under Assumption 7.1 (ii), then for all $x \in (0, v(\overline{W}))$, there exists an optimal solution $y^* > 0$ to the optimization problem (49).*

Proof. Fix any $x > 0$. Under assumption 7.1 (ii), there exists δ_x real-valued in $[0, x]$ such that:

$$\begin{aligned} \delta_x \mathcal{E}(\widetilde{X}_T^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)S_T^0 &\in dom\,(U) \quad \forall\, Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b) \\ U(\delta_x \mathcal{E}(\widetilde{X}_T^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)S_T^0) &\in \mathbf{L}^1(\mathbf{P}), \quad \forall\, Z \in \mathcal{P}^*(\widetilde{\mathcal{X}}_b) \end{aligned}$$

By definitions of $\widetilde{U}_1$, $\widetilde{U}$, and from (12), we have

$$\widetilde{U}(yg) \geq U(\delta_x \mathcal{E}(\widetilde{X}_T^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)S_T^0) - y\delta_x Z_T, \quad \forall\, y > 0, \forall\, g \in \mathcal{D}(1)$$

By definition of $\widetilde{J}(y, g)$ in (45) and since $\mathbf{E}[Z_T] \leq 1$ we get

$$\widetilde{J}(y; g) \geq U(\delta_x \mathcal{E}(\widetilde{X}_T^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)S_T^0) - y\delta_x,$$

for all $y > 0$ and $g \in \mathcal{D}(1)$.
Taking infimum in this last inequality over $g \in \mathcal{D}(1)$ implies:

$$\widetilde{u}(y) + xy \geq U(\delta_x \mathcal{E}(\widetilde{X}_T^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)S_T^0) + y(x - \delta_x),$$

for all $y > 0$ and $g \in \mathcal{D}(1)$.
Since

$$U(\delta_x \mathcal{E}(\widetilde{X}_T^0)\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Z)_T)S_T^0) > -\infty$$

and $(x - \delta_x) \geq 0$, we deduce that $y \to \widetilde{u}(y) + xy$ is a proper convex function. Moreover, $\widetilde{u}(y) + xy \to \infty$ as $y \to \infty$, this shows that the infimum $\widetilde{u}(y) + xy$ is attained in $y^* \in \mathbf{R}_+$.

To prove that $y^* > 0$, we assume the contrary, then:

$$\widetilde{u}(0) = \inf_{g\in\mathcal{D}(1)} \widetilde{J}(0, g) \leq \mathbf{E}[\widetilde{U}(yg)] + xy, \quad \forall\, y > 0. \tag{58}$$

By the properties of utility functions, we have:

$$\widetilde{U}(yg) + ygI(yg) \leq \widetilde{u}(0) \tag{59}$$

Plugging (59) into (58) and dividing by $y > 0$, we obtain for all $y > 0$, $g \in \mathcal{D}(1)$:

$$\mathbf{E}[gI(yg)] \leq x$$

By the properties of utility functions (see Section 4) and our model setting, as $y \to 0$ and $I(yg) \to \overline{W}$. We get by Fatou's lemma

$$\mathbf{E}[g\overline{W}] \leq x, \quad \forall\, g \in \mathcal{D}(1)$$

This implies $v(\overline{W}) \leq x$ the contradiction since $x < v(\overline{W})$ and this completes the proof of the lemma. $\square$

The following Lemma is taken from Pham and Mnif (2002).

Lemma 7.4 *Under Assumption 7.1 (i), there exist $\alpha \in (0,1)$, $c > 0$ and $\Gamma \in \mathbf{L}^1(\mathbf{P})$, such that:*

$$gI(\alpha g) \leq c\Big[\widetilde{U}(g)\mathbf{1}_{g\leq U'(x^0)} + U(x^0)\mathbf{1}_{g>U'(x^0)}\Big] + \Gamma$$

$\mathbf{P}$–a.s. for all $g > 0$.

Proof. Take $\alpha \in (\lambda, 1)$ and consider the case $g \leq U'(x^0)$. By assumption 7.1 (i), the definitions and properties of $\widetilde{U}$ (see Section 4 (17), (18) and (19)), we have:

$$\begin{aligned} \alpha I(\alpha g) &= I(\alpha g)U'(I(\alpha g)) \\ &\leq \lambda U(I(\alpha g)) + \Upsilon \\ &\leq \lambda(\widetilde{U}(g) + gI(\alpha g)) + \Upsilon \end{aligned}$$

Therefore we obtain:

$$gI(\alpha g) \leq \frac{\lambda}{\alpha - \lambda}\widetilde{U}(g) + \frac{\Upsilon}{\alpha - \lambda} \tag{60}$$

for all $0 < g \leq U(x^0_t, .)$.

In the case $U(x^0) < g$, since I are strictly decreasing, by using (60) we get

$$\begin{aligned} gI(\alpha g) &\leq gI(\alpha U'(x^0)) \\ &\leq c\widetilde{U}(U'(x^0))) + \Gamma \\ &\leq cU(x^0) + \Gamma \end{aligned} \tag{61}$$

with $c = \frac{\lambda}{\alpha-\lambda}$ and $\Gamma = \frac{\Upsilon}{\alpha-\lambda}$, where the last inequality comes from the following properties of utility function:

$$\begin{aligned} \widetilde{U}(U'(x)) &= U(I(U'(x))) - U'(x)I(U'(x)) \\ &\leq U(I(U'(x))) \\ &= U(x) \end{aligned}$$

From (60) and (61) we get the desired result. □

Lemma 7.5 *Under Assumption 7.1 (i), suppose that the optimal solution to the problem (45) for some $y > 0$ be $yg^* \in \mathcal{D}(y)$ then $\widetilde{u}(y)$ is differentiable in y and we have:*

$$\widetilde{u}'(y) = \mathbf{E}[-g^* I(yg^*)]. \tag{62}$$

Moreover, if in addition $g^* \in \mathcal{D}^0(1)$ then

$$\widetilde{u}'(y) = \mathbf{E}[-Y^*_T I(yg^*)] \tag{63}$$

for any $Y^* \in \mathcal{Y}(1)$ that dominates g^* as in (24).

Proof. From Corollary 7.1 we know that, under Assumption 7.1 the optimal solution to the problem (45) exists. Now fix any δ sufficiently small, we will show that

$$\liminf_{\delta\downarrow 0} -\frac{\widetilde{u}(y+\delta) - \widetilde{u}(y)}{\delta} \geq \mathbf{E}[g^* I(yg^*)] \tag{64}$$

and

$$\limsup_{\delta\uparrow 0} -\frac{\widetilde{u}(y+\delta) - \widetilde{u}(y)}{\delta} \leq \mathbf{E}[g^* I(yg^*)] \tag{65}$$

Let $\delta > 0$. By using successively the definition of $\widetilde{u}(y)$ in (40), the convexity of $\widetilde{U}_1$, $\widetilde{U}$ and its properties (see Section 4: equation (18)) we obtain:

$$\begin{aligned} -\frac{\widetilde{u}(y+\delta) - \widetilde{u}(y)}{\delta} &\geq \mathbf{E}\Big[\frac{\widetilde{U}((y+\delta)g^*) - \widetilde{U}(yg^*)}{-\delta}\Big] \\ &\geq \mathbf{E}[g^* I((y+\delta)g^*)] \end{aligned}$$

We then deduce by monotone convergence theorem:

$$\liminf_{\delta\downarrow 0} -\frac{\widetilde{u}(y+\delta)-\widetilde{u}(y)}{\delta} \geq \mathbf{E}[g^* I(yg^*)] \tag{66}$$

By the same arguments in the case $\delta < 0$, with $y+\delta > 0$, we obtain:

$$-\frac{\widetilde{u}(y+\delta)-\widetilde{u}(y)}{\delta} \leq \mathbf{E}[g^* I((y+\delta)g^*)] \tag{67}$$

Under Assumption 7.1 (i) and by lemma 7.4, we have for $\delta < 0$ sufficiently small:

$$0 \leq g^* I((y+\delta)g^*) \leq \Gamma + c\Big[\widetilde{U}(yg^*)\mathbf{1}_{yg^* \leq U'(x^0)} + U(x^0)\mathbf{1}_{yg^* > U'(x^0)}\Big] \tag{68}$$

We first show that the right-hand side in (68) is integrable.

Since $\widetilde{u}(y) < \infty$, we already have:

$$\mathbf{E}[\widetilde{U}(yg^*)] < \infty \tag{69}$$

By definition of $\widetilde{U}$ we have:

$$\widetilde{U}(yg^*) \geq U(x^0) - yx^0 g^* \tag{70}$$

The right-hand side of the (70) is integrable with respect to $\mathbf{P}$ by Assumption 7.1 (i), and Remark 5.1(ii), hence

$$\mathbf{E}[\widetilde{U}(yg^*)] > -\infty \tag{71}$$

By (69), (71) and Assumption 7.1 (i) we deduce that the right-hand side in (68) is integrable. Therefore we can apply the dominated convergence theorem to (67) and obtain:

$$\limsup_{\delta\uparrow 0} -\frac{\widetilde{u}(y+\delta)-\widetilde{u}(y)}{\delta} \leq \mathbf{E}[g^* I(yg^*)] \tag{72}$$

From (66), (72) and the convexity of $\widetilde{u}(y)$ we get:

$$-\widetilde{u}'(y) = \mathbf{E}[g^* I(yg^*)] \tag{73}$$

Now suppose that $g^* \in \mathcal{D}^0(1)$. Since there exists a process $Y^* \in \mathcal{Y}(1)$ such that $g^* \leq Y^*_T$, we then have:

$$-\widetilde{u}'(y) \leq \mathbf{E}[Y^*_T I(yg^*)] \tag{74}$$

To prove the converse inequality, we take an arbitrary element $g \in \mathcal{D}(1)$, let $\delta \in (0,1)$ and consider the process:

$$g^\delta = (1-\delta)g^* + \delta g$$

which also belongs to the set $\mathcal{D}(1)$. Notice also that $\lim_{\delta\to 0} g^\delta = g^*$. Since yg^* solves $\widetilde{u}(y)$, then we have:

$$\mathbf{E}[\widetilde{U}(yg^*)] \leq \mathbf{E}[\widetilde{U}(yg^\delta)] \tag{75}$$

Recall that for any convex function $\widetilde{U}$

$$\widetilde{U}(x) \geq \widetilde{U}(y) + (x-y)\widetilde{U}'(y),$$

Then by the convexity of $\widetilde{U}$, we have:

$$\begin{aligned}\widetilde{U}(yg^*) &\geq \widetilde{U}(yg^\delta) + (yg^* - yg^\delta)\widetilde{U}'(yg^\delta)\\ &\geq \widetilde{U}(yg^\delta) + \delta y(g-g^*)I(yg^\delta)\end{aligned} \tag{76}$$

Plugging (76) to (75) and dividing by δ, we obtain:

$$\mathbf{E}[yg^* I(yg^\delta_T)] \geq \mathbf{E}[ygI(yg^\delta_T)] \tag{77}$$

Since $g^\delta \geq (1-\delta)g^*$, then by Lemma 7.4 we have:

$$0 \leq \mathbf{E}[yg^* I(yg^\delta_T)\Big] \leq c\Big[\widetilde{U}(yg^*)\mathbf{1}_{yg^* \leq U'(x^0)} + U(x^0)\mathbf{1}_{yg^* > U'(x^0)}\Big] + \Gamma$$

for some $c > 0$ and $\Gamma \in \mathbf{L}^1(\mathbf{P})$. By the same arguments as in (68), we apply the dominated convergence theorem to the left-hand side of (77), Fatou's lemma to the right-hand side and using (73) we get:

$$\begin{aligned} -\widetilde{u}'(y) &\geq \mathbf{E}[gI(yg^*)], \quad \forall\, g \in \mathcal{D}(1) && (78)\\ &\geq \mathbf{E}[Y^*_T I(yg^*)] && (79) \end{aligned}$$

where the last inequality follows from the fact that Y^*_T belongs to $\mathcal{D}(1)$.

From (74) and (79) we get the desired result. □

We now turn to proof of Theorem 7.1.

Proof of Theorem 7.1

Lemma 7.1, 7.2, 7.3 and corollary 7.1 already verified assertion (i) and (ii) of Theorem 7.1. Now we will show the validity of the remaining assertion.

Suppose that y^* is an optimal solution to problem (49), its existence follows from Lemma 7.3. We deduce from Corollary 7.1 that there exists an optimal solution y^*g^*, with $g^* \in \mathcal{D}(1)$, to a problem $\widetilde{u}(y^*)$ in (45). Moreover, as a result of the last lemma, $\widetilde{u}$ is differentiable at y^* and we shall have:

$$-\widetilde{u}'(y^*) = \mathbf{E}[g^* I(y^*g^*)] = x \tag{80}$$

Let us define $F^* \triangleq I(y^*g^*)$. We will show that F^* is a unique solution to the optimization problem (40).

From (78), Lemma 5.1 and (80) we deduce that F^* belong to the set $\mathcal{C}(-\widetilde{u}'(y^*)) \equiv \mathcal{C}(x)$.

Now, for an arbitrary pair $F \in \mathcal{C}(x)$ we have:

$$\begin{aligned} U(F) &\leq \widetilde{U}(y^*g^*) + y^*g^*F \\ &\leq U(I(y^*g^*) + y^*g^*F - y^*g^*I(y^*g^*) \\ &\leq U(F^*) + (y^*g^*F - y^*g^*F^*\Big) \end{aligned}$$

where the second inequality follows from the definition of $\widetilde{U}$.

Taking expectation, we obtain:

$$\begin{aligned} \mathbf{E}[U(F)] &\leq \mathbf{E}[U(F^*)] + y^*\mathbf{E}[g^*F - x] \\ &\leq \mathbf{E}[U(F^*)] \end{aligned}$$

The last inequality follows from the property (26) of $F \in \mathcal{C}(x)$.
This proves the optimality of F^*, since F is an arbitrary element of $\mathcal{C}(x)$.

Now let $W^* \in \mathcal{W}(x)$ be a element that dominates F^* in a sense of (23). By (80) and (26) we have:

$$x = \mathbf{E}[g^*F^*] \leq \mathbf{E}[g^*W_T] \leq x$$

and this proves (51).

Now suppose that $g^* \in \mathcal{D}^0(1)$, and let $Y^* \in \mathcal{D}(1)$ be a element that dominates g^* in a sense of (24). By (80) and (26) we have:

$$x = \mathbf{E}[g^*F^*] \leq \mathbf{E}[g^*W^*_T] \leq \mathbf{E}[Y^*_T W^*_T] \leq x$$

and this proves (52).

By the same arguments as in the proof to the sufficient condition of the Lemma 5.1 we get the formula for the optimal solution W^* (53) to the primal optimization problem (21) and this completes the proof of the Theorem 7.1. □

8 Concluding Remarks

This paper has examined the individual's investment problem for an agent, whose wealth processes belong to some predictably convex sets of semimartingales. The framework incorporates many financial models such as with labor income, non-linearity of wealth processes (large investor) and constrained proportion of portfolio choice. The above analysis makes sense since the Standing Assumption 2.1 is only a mild condition, which almost every models analyzed in the financial literature satisfy. We omit this issue in this paper, since it has been extensively carried out in Pham and Mnif (2002) [Ph-M] and Pham (2002) [Ph], we then refer the reader to their papers for examples and discussion.

Appendix: A Stochastic Control Lemma

The result presented here is a slight modification of Lemma A.1 [F-K], which we need in the context of this paper.

Let $\widetilde{\mathcal{X}}_b$ be a family of semimartingales, which are locally bounded from below. Let us introduce the class $\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$ of all probability $Q \sim P$, such that any $X_b \in \widetilde{\mathcal{X}}_b$ is a special semimartingale under Q with the following property: There exists a process $A \in \mathcal{O}_p$ - the set of nondecreasing predictable processes with $A_0 = 0$, such that $X_b - A$ is a local supermartingale under Q for any $X_b \in \widetilde{\mathcal{X}}_b$. Assume that $\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b) \neq \emptyset$ and we denote by $A^{\widetilde{\mathcal{X}}_b}(Q)$ the upper variation process of $\widetilde{\mathcal{X}}_b$ with respect to $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$. $\mathcal{T}_t$ denotes the set of stopping times T with values in $[t, T]$, and $\mathcal{T} \equiv \mathcal{T}_0$. Notice that the below result still holds true with T replaced by ∞. As before, all processes are assumed to be real-valued, *càdlàg* and to be adapted with respect to the given filtration $(\mathcal{F}_t)_{t\geq 0}$. For simplicity we assume hereafter that the initial σ-field $\mathcal{F}_0$ is trivial.

Lemma A.1 *Let $(f_t)_{t\geq 0}$ be a nonnegative process such that*

$$\sup_{Q\in\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)} \sup_{T\in\mathcal{T}} \mathbf{E}^Q \left[\frac{f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_T} \right] < +\infty, \tag{81}$$

There exists a nonnegative process $(V_t)_{t\geq 0}$ such that for $t \geq 0$

$$V_t = \operatorname*{ess\,sup}_{Q\in\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b), T\in\mathcal{T}_t} \mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_t \mathbf{E}^Q \left[\frac{f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))_T} \middle| \mathcal{F}_t \right]$$

almost surely. Moreover, for any $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$ the process $\frac{V}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))}$ is a Q-supermartingale.

Proof. We have to show that $\frac{V}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(Q))}$ is a Q-supermartingale and that V admits an *càdlàg* modification. We will mimic an argument by Föllmer and Kramkov (1997) [F-K].

Let the probability measure $\mathbf{P}$ be an element of $\overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$. In the sequel, we identify a probability $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$ with its density process $z = (z_t)_{t\leq T}, z_t = \mathbf{E}[dQ/d\mathbf{P}|\mathcal{F}_t]$. For $t \geq 0$ we denote by $\mathcal{Z}_t$ the set of density processes z which are equal to 1 on the interval $[0, t]$. Throughout we will use the notation

$$A^{\widetilde{\mathcal{X}}_b}(z) = A^{\widetilde{\mathcal{X}}_b}(Q),$$

for any $z \in \mathcal{Z}_t$, corresponds to $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathcal{X}}_b)$. Since $A^{\widetilde{\mathcal{X}}_b}(z) = A^{\widetilde{\mathcal{X}}_b}(\mathbf{P})$ on $[0, t]$, we get

$$V_t = \operatorname*{ess\,sup}_{z\in\mathcal{Z}_t, T\in\mathcal{T}_t} \mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P})_t) \mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T} \middle| \mathcal{F}_t \right]$$

Let z_1, z belong to $\mathcal{Z}_t$ and T_1, T be in $\mathcal{T}_t$, and set

$$\begin{aligned} K &= \left\{ \omega\colon \mathbf{E}\left[\frac{z_{1_{T_1}} f_{T_1}}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_{T_1}} \middle| \mathcal{F}_t \right] \geq \mathbf{E}\left[\frac{z_{2_T} f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T} \middle| \mathcal{F}_t \right] \right\}, \\ T &= T_1 \mathbf{1}_K + T \mathbf{1}_{K^c} \\ z &= z_1 \mathbf{1}_K + z \mathbf{1}_{K^c}. \end{aligned}$$

Because $A \in \mathcal{F}_t$, the process z belongs to $\mathcal{Z}_t$ and the random time T is a stopping time. In particular, we have

$$\begin{aligned} A^{\widetilde{\mathcal{X}}_b}(z) &= A^{\widetilde{\mathcal{X}}_b}(z_1)\mathbf{1}_A + A^{\widetilde{\mathcal{X}}_b}(z)\mathbf{1}_{A^c} \\ \mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T} \middle| \mathcal{F}_t \right] &= \mathbf{E}\left[\frac{z_{1_{T_1}} f_{T_1}}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_{T_1}} \middle| \mathcal{F}_t \right] \bigvee \mathbf{E}\left[\frac{z_{2_T} f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T} \middle| \mathcal{F}_t \right] \end{aligned}$$

i.e., the family $\left\{\mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T}\middle|\mathcal{F}_t\right]\right\}_{T\in\mathcal{T}_t}$ is closed under pairwise maximization. Note that we also have:

$$\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P})_t)\mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T}\middle|\mathcal{F}_t\right] \geq 0$$

Proposition D.2 in Appendix D from Karatzas and Shreve (1998) [Ka-S] imply that for all $t \in \mathcal{T}$ and any $s \leq t$, the assumption (81) guarantees

$$\mathbf{E}[V_t|\mathcal{F}_s] < \infty,$$

moreover

$$\mathbf{E}[V_t|\mathcal{F}_s] = \operatorname*{ess\,sup}_{z\in\mathcal{Z}_t, T\in\mathcal{T}_t} \mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P})_t)\mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T}\middle|\mathcal{F}_s\right] \tag{82}$$

Since $\mathcal{T}_t \subset \mathcal{T}_s$ and $\mathcal{Z}_t \subset \mathcal{Z}_s$, it follows that:

$$\begin{aligned}\mathbf{E}\left[\frac{V_t}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P})_t)}\middle|\mathcal{F}_s\right] &\leq \operatorname*{ess\,sup}_{z\in\mathcal{Z}_s, T\in\mathcal{T}_s} \mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T}\middle|\mathcal{F}_s\right] \\ &= \frac{V_s}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P})_s)}\end{aligned}$$

Hence $V/\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))$ is a supermartingale.

To finish the proof we must show that the process V admits an *càdlàg* modification, or equivalently, there must exist a modification for a supermartingale $\widetilde{V} = V/\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))$. This is the case if and only if the function $(\mathbf{E}[\widetilde{V}_t])_{t\geq 0}$ is right-continuous.

When $s = 0$, the equality (82) takes the form

$$\mathbf{E}[\widetilde{V}_t] = \sup_{z\in\mathcal{Z}_t, T\in\mathcal{T}_t} \mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T}\right] \tag{83}$$

Let t, $(t_n)_{n\geq 1}$ be positive numbers such that $t_n \downarrow t$, $n \to +\infty$, and $t_n < t+1$, $n \geq 1$. Since $\widetilde{V}$ is a supermartingale, we have

$$\mathbf{E}[\widetilde{V}_t] \geq \lim_{n\to+\infty} \mathbf{E}[\widetilde{V}_{t_n}]. \tag{84}$$

To prove the reverse inequality, we fix $\epsilon > 0$ choose a process $z = z(\epsilon) \in \mathcal{Z}_t$ and a stopping time $T(\epsilon) \in \mathcal{T}_t$ such that

$$\mathbf{E}[\widetilde{V}_t] < \mathbf{E}\left[\frac{z_T f_T}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_T}\right] + \epsilon, \quad P(T > t) = 1 \tag{85}$$

This is possible by (83) and the right-continuity of the processes under consideration. For $n \geq 1$ we define the stopping time T^n and the process z^n as follows:

$$T^n = \begin{cases} T, & \text{if } T \geq t_n, \\ t+1, & \text{if } T < t_n, \end{cases} \qquad z^n_u = \begin{cases} z_u/z_{t_n}, & \text{if } T \geq t_n \text{ and } u \geq t_n \\ 1, & \text{if } T < t_n \text{ or } u < t_n \end{cases}$$

We have $z^n \in \mathcal{Z}_{t_n}$, $T^n \in \mathcal{T}_{t_n}$. Moreover $T^n \to T$ and $z^n_{T^n} \to z_T$ almost surely as n tends to ∞. Now we deduce from Fatou's lemma and (83) and (85) that

$$\mathbf{E}[\widetilde{V}_t] \leq \liminf_{n\to+\infty} \mathbf{E}\left[\frac{z^n_{T^n} f_{T^n}}{\mathcal{E}(A^{\widetilde{\mathcal{X}}_b}(\mathbf{P}))_{T^n}}\right] + \epsilon \leq \lim_{n\to+\infty} \mathbf{E}[\widetilde{V}_{T^n}] + \epsilon$$

Since ϵ is an arbitrary positive number and by (84) that the function $(\mathbf{E}[\widetilde{V}_t])_{t\geq 0}$ is a right-continuous functions. This completes the proof of the lemma. □

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