August 2, 2005

The Degree of Stability of Price Diffusion

Abstract

The distributional form of financial asset returns has important implications for the theoretical and empirical analyses in economics and finance. It is now a well-established fact that financial return distributions are empirically nonstationary, both in the weak and the strong sense. One first step to model such nonstationarity is to assume that these return distributions retain their shape, but not their localization (mean μ) or size (volatility σ) as the classical Gaussian distributions do. In that case, one needs also to pay attention to skewedness and kurtosis, in addition to localization and size. This modeling requires special Zolotarev parametrizations of financial distributions, with a four parameters, one for each relevant distributional moment. Recently popular stable financial distributions are the Paretian scaling distributions, which scale both in time T and frequency ω. For example, the volatility of the lognormal financial price distribution, derived from the geometric Brownian asset return motion and used to model Black-Scholes (1973) option pricing, scales according to $T^{0.5}$. More generally, the volatility of the price return distributions of Calvet and Fisher's (2002) Multifractal Model for Asset Returns (MMAR) scales according to $T^{\frac{1}{\alpha_Z}}$, where the Zolotarev stability exponent α_Z measures the degree of the scaling, and thus of the nonstationarity of the financial returns.

Keywords: Stable distributions, price diffusion, stability exponent, Zolotarev parametrization, fractional Brownian motion, financial markets

Acknowledgement 1 *This article is a corrected, updated and extended adaptation of Chapter 3 of Cornelis A. Los, FINANCIAL MARKET RISK: MEASUREMENT & ANALYSIS, Routledge/Taylor & Francis Group, 2003.*

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1 Introduction

As we discussed in Los (2005a), the distributional form of financial asset returns has important implications for the theoretical and empirical analyses in economics and finance. For example, asset, portfolio and option pricing theories are typically based on the shape of these distributions, which some researchers have tried to recover from financial market prices. For example, Jackwerth and Rubinstein (1996) and Melick and Thomas (1997) did this for the options markets.

Stable distributions, which are distributions which retain their shape over time, but not necessarily their size - are currently *en vogue* for risk valuation, asset and option pricing, and portfolio management, long after having been in fashion for a short - lived period in the 1960s. They provide much more realistic financial risk profiles, in particular in the high frequency FX markets, where, for example, excess kurtosis is found, but also in the persistent stock markets (Hsu, Miller and Wichern, 1974; Mittnik and Rachev, 1993a and b; Chobanov *et al.*, 1996; McCulloch, 1996; Cont, Potters and Bouchaud, 1997; Gopikrishnan *et al.*, 1998; Müller *et al.*, 1998; Los, 2000).

The scientific debate - about what kind of distributions best represent financial time series - is not yet settled, and maybe never will. Some authors claim that the financial market return distributions are close to being Paretian stable (Mandelbrot, 1962, 1963a and b and c, 1966; Fama, 1963, 1965a and b; McFarland, Petit and Sung, 1982; Rachev and Mittnik, 2000); others that they are close to Student $-t$ distributions (Boothe and Glasserman, 1987). Still others reject any single distribution, and claim that they can best be modeled by portfolios of distributions (Caldéron - Rossel and Ben - Horim, 1982). However, everybody can agrees on a few empirical observations like: FX return rates are fat - tailed and show high kurtosis. In other words, in the FX markets extreme values are more prevalent than the conventional Gaussian distribution suggests, *i.e.,* extreme risks in the FX markets are *abnormally frequent*.

In addition, a new controversy has arisen in the financial research community as to whether the second moment of the distribution of rates of returns actually exists, *i.e.,* whether it converges to a (time - normalized) constant, or not. As emphasized by Müller, Dacorogna and Pictet (1998), this question is central to computational finance, since financial models heavily rely on the existence of the volatility of returns, σ (Los, 2001). Some empirical financial distributions, such as the rates of return of the S&P500 Index exhibit such non - existent, *i.e.,* non - convergent volatilities. Their variances are not only nonstationary, they are essentially unpredictable!

As we observed in Los (2005a), financial market risk has been associated with this volatility of returns σ, ever since in the 1950s Markowitz attempted to put portfolio theory on a scientific footing (Markowitz, 1952; 1991, original 1959), using only the first two distributional moments - the mean μ and the variance σ^2. From the Sharpe ratio for measuring the portfolio performance of mutual funds (Sharpe, 1966) to dynamic fundamental asset and derivative pricing models, the volatility or risk constant σ is always present. Of course, for full - scale global multi - currency, multi - asset investment portfolio valuation, one investigates the whole covariance matrix $\mathbf{\Sigma}$, instead of only independent variances σ^2.

Thus the main motivation for studying stable distributions is the need to evaluate *extreme risks* in the financial markets, *i.e.*, the fat tails of the financial return distributions. Regrettably, most of the current models for assessing such risks are still based on the assumption that financial market returns are distributed according to the Gaussian distribution, which has only two parameters, the two first moments of Markowitz. With the Gaussian distribution the evaluation of the frequency of occurrence of extreme risks is directly related to the measurement of the volatility size σ, but in the case of fat - tailed distributions that is no longer the case.

Thus, to broaden the set of our distributional benchmarks for financial returns, in this paper we focus on the statistical theory of stable marginal distributions of investment returns, in particular, of their Paretian time-and-frequency scaling distributions, *irrespective of the structure of their temporal dependence*. In the case of scaling distributions, we want to have a theoretical concept of statistical frequency distributions that exhibits the property of self - similarity and to show how that property is related to certain time intervals via stable scaling laws of time aggregation. Later on we will establish a (not yet specified) connection between the frequency of occurrence and the timing of occurrence of certain financially risky events.

In this paper, we explain, first, the difference between linear and affine relations and time series. Then, in Sections 2 and 3, some invariant properties of stable distributions are defined, like of weighted mixtures, choice maximization and aggregation, closely following Mandelbrot (1962, 1963b). In Section 4 the scaling properties of Pareto-Lévy distributions are analyzed. Next, in Section 5, we focus on the particular parametrizations of stable distributions of Zolotarev, following the very clear explanation by Nolan (1999a and b) and Rachev and Mittnik (2000). In Section 6, we also provide some examples of empirical financial research, which use this new theory of stable distributions. In Section 7 we connect this discussion of stable distributions to current research of the efficiency and stability of financial markets, via Calvet and Fisher's (2002) Multifractal Model of Asset Returns, and in Section 8 we discuss some of the essential weaknesses of current statistical approaches to identify these distributions from inexact and irregular data. We recommend to use the engineering signal processing modeling technologies to identify the crucial financial market stability exponents, as we explained in Los (2005b).

2 Affine Traces of Speculative Prices

Although, in Los (2005a) we stated that correlatedness was a form of linear dependence, we did not yet define what that concept represents. In this section, we will define linearity, affinity, time - invariance, and time - dependence, all within the context of a financial system by using simple operator algebra.[1]

[1] An early user of such operator algebra was the famous Polish economist Oskar Lange, 1904 - 1965 (Lange, 1970). As a graduate student at Columbia University, I used such operator algebra in 1978 to solve the complex nonlinear growth system of Michael Kalecki (1945). I had picked up Lange's use of operator algebra at the London School of Economics in 1975-76. My solution of Kalecki's dynamic economic development system was to the delight of my economics lecturer Duncan Foley of Barnard College. Kalecki's dynamic system was more realistic, because it could model more complex nonstationary behavior, than Samuelson's more familiar, but much simpler accelerator - multiplier economic growth system, which can only model stationary behavior (i.e., trends, infinite sinusoidal waves, etc.) (Samuelson, 1947). It was an early indication to me that engineering concepts of signal processing could have relevant use in the study of the nonstyationary financial market processes

2.1 Linearity Versus Affinity

2.1.1 System Transformations

Let's first define what is meant by such a crucial concept as a *system*.

Definition 2 *A **system** is a mathematical model of a physical process that relates the input function (or source) to the output function (or response). Thus, a system can be considered a mapping of an input $X_i(t)$ into an output $X_o(t)$. Using the symbol f to symbolize this mapping, we have*

$$X_o(t) = f\{X_i(t)\} \tag{1}$$

*and f is the **system operator**, which transforms the inputs $X_i(t)$ into outputs $X_o(t)$.*

f may be a linear or a nonlinear system operator. We will shortly define system linearity.

Definition 3 *A system is **invertible** when*

$$X_i(t) = f^{-1}X_o(t) \tag{2}$$

Thus, in an invertible system the output can just as well be the input, and *vice versa*.

Definition 4 *A system is **time - invariant** when*

$$X_o(t+\tau) = f\{X_i(t+\tau)\} \tag{3}$$

where τ is an arbitrary constant, representing a time interval.

Time intervals have no influence on the output of a time - invariant system, since the system does not change within such time intervals.

Definition 5 *L is called the **linear operator** and the system represented by L is called a **linear system**, if the operator L satisfies the following two conditions of additivity and homogeneity:*

$$\begin{aligned} L\{X_{i1}(t) + X_{i2}(t)\} &= L\{X_{i1}(t)\} + L\{X_{i2}(t)\} \\ &= X_{o1}(t) + X_{2o}(t) \ \textit{(additivity)} \end{aligned} \tag{4}$$

$$\begin{aligned} L\{cX_i(t)\} &= cL\{X_i(t)\} \\ &= cX_o(t) \ \textit{(homogeneity)} \end{aligned} \tag{5}$$

For example, the familiar time lag - operator L, which delays the input by one time period, is linear, as can be easily checked, since it satisfies the two defining properties of linearity. Notice that

$$X_{t-\tau} = L^{\tau}X_t \tag{6}$$

Thus, multiple period lags consist of a geometric series of linear one - period lag operators.

Remark 6 *Note that the* ***first difference operator*** Δ *can be derived from the time lag - operator, since*

$$\Delta = 1 - L \tag{7}$$

This easy to check, since

$$\begin{aligned} \Delta X(t) &= X(t) - X(t-1) \\ &= X(t) - LX(t) \\ &= (1-L)X(t) \end{aligned} \tag{8}$$

Now we also see why the Geometric Brownian Motion (GBM) can be written, in discrete time fashion, as

$$\begin{aligned} \Delta x(t) &= (1-L)x(t) \\ &= \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d. \end{aligned} \tag{9}$$

with the rate of return $x(t) = \Delta \ln X(t)$*, when* $X(t)$ *is the market price.*

Since the lag operator is linear, the first difference operator is also linear. *Higher - order difference operators* can easily be expressed as products of the first difference operator:

$$\Delta^d = (1-L)^d \tag{10}$$

for any real (integer or fractional) constant order $d \in \mathbb{R}$. These higher - order difference operators play an increasingly important role in empirical financial research.

Example 7 *Bachelier's (1900) simpler Random Walk (RW) can also be viewed as a linear system, when we focus on the first price differences* $\Delta X(t)$*, since we can write*

$$\Delta X(t) = \varepsilon(t), \text{ with } \varepsilon(t) \sim i.i.d. \tag{11}$$

In this model conception, the series of time - dependent prices $\{X(t)\}$ is linearly transformed, or *filtered*, into innovations, which are assumed to be independently and identically distributed (= strongly stationary) or i.i.d.. Consequently, to empirically test this RW model, we compute the first differences of such price series and then test if the resulting series of innovations are, indeed, independent and strongly stationary. If not, the price series cannot be described by the RW model. Recently, we executed non - parametric independence and stationarity tests on high - frequency, minute - by - minute Asian FX series in Los (1999), which are both not independent and nonstationary, and on weekly Asian stock market returns in Los (2000), which show a fair amount of stationarity, but do not show independence.

However, the real order d of financial system differentiation is often empirically measured to be a fraction and not an integer.

Definition 8 *A* ***fractional difference operator*** *is* $\Delta^d = (1 - L)^d$ *for* $d = non\,\text{integer} \in \mathbb{R}$.

We met these empirically important fractional difference operators again in Los (2005b), where we discussed Fractional Brownian Motion (FBM). The FBM and the related Multifractal Model of Asset returns (MMAR) can better explain the observed simultaneous phenomena of nonstationarity and long - term dependence or Long Memory.

In Los (2005b), we discussed the two major types of their time dependence: serial (or short-term) dependence and global (or long - term) dependence. This global time dependence can only be modelled by fractional difference operators, since it requires that the power of volatility or financial market risk never dies off. In other words, the arbitraging financial market trading processes never cease to operate due to the long memory of historical news events.

Now, there is a difference between linear and affine system operations.

2.1.2 Affine Transformations

Definition 9 *M is called the* ***affine operator*** *when*

$$\begin{aligned} X_o(t) &= M\left\{X_i(t)\right\} \\ &= cX_i(t) + d \end{aligned} \tag{12}$$

where c and d are ***amplifying*** *and vertical* ***frame shifting*** *constants, respectively.*

The affine operator is clearly nonlinear, since, first, it is not additive:

$$\begin{aligned} M\left\{X_1(t)\right\} + M\left\{X_2(t)\right\} &= c\left[X_1(t) + X_2(t)\right] + 2d \\ &\neq M\left\{X_1(t) + X_2(t)\right\} \\ &= c\left[X_1(t) + X_2(t)\right] + d \end{aligned} \tag{13}$$

and, second, it is not homogeneous, since

$$
\begin{aligned}
M\{cX(t)\} &= cX(t) + d \\
&\neq cM\{X(t)\} \\
&= cX(t) + cd
\end{aligned}
\tag{14}
$$

However, we can always transform an affine data series into a linear data series by taking *deviations from the mean*, since

$$
\begin{aligned}
x_o(t) &= \left[X_o(t) - \frac{1}{T}\sum_{t=1}^{T} X_o(t)\right] \\
&= [cX_i(t) + d] - \left[\frac{c}{T}\sum_{t=1}^{T} X_i(t) + \frac{1}{T}\sum_{t=1}^{T} d\right] \\
&= c\left[X_i(t) - \frac{1}{T}\sum_{t=1}^{T} X_i(t)\right] + d - d \\
&= c\left[X_i(t) - \frac{1}{T}\sum_{t=1}^{T} X_i(t)\right] \\
&= cx_i(t)
\end{aligned}
\tag{15}
$$

which is clearly additive and homogeneous, and thus linear. Thus, we've found a second reason to compute deviations from the mean, or first differences, before we analyze a financial time series: to derive linearity.

3 Invariant Properties: Stationarity Versus Scaling

We learned in Los (2005a) that stationarity in the wide sense (weak stationarity) is defined by constant, invariant risk:

$$\sigma_t = \sigma_s, \text{ with } t, s \in T \tag{16}$$

We learned also that Bachelier's RW process has invariant normalized risk. As long as the scaling factor remains invariant over time, we can transform any horizon risk linearly into normalized risk by proper scaling. By scaling we normalize the horizon risk of an asset to its own invariant asset risk class. In the case of the RW process, we use the so-called Fickian scaling. The RW process risk scales *self - similarly* according to the number of periods n, where the total time of observation is $T = n\tau$, since we can express the self - similarity of the horizon risk of the RW process within the constant investment or trading horizon τ as follows:

$$\begin{aligned}\sigma_\tau &= \frac{1}{n^{0.5}}\sigma_T \\ &= \left(\frac{\tau}{T}\right)^{0.5}\sigma_T \end{aligned} \tag{17}$$

or, in inverse form,

$$\sigma_T = \sigma_\tau n^{0.5} \tag{18}$$

But there are distributions which have different scaling exponents than the *Fickian scaling exponent* $\lambda = 0.5$ of an RW (or Arithmetic Brownian Motion on the basis of the market prices $X(t)$), or of a GBM (on the basis of the investment return $x(t) = \ln(X(t)/X(t-1))$). It appears now that these *non - Fickian scaling exponents* are most prevalent in empirical finance and not the usually presumed Fickian exponent. A subgroup of such statistical scaling distributions are the Pareto - Lévy power laws.[2]

Definition 10 *A **(Pareto - Lévy) scaling distribution** (or **power law**) is a frequency distribution $P(X(\tau) > x)$ of independent random variables $X(\tau)$ with a scaling factor σ_T, which is dependent on the frequency of (observed) occurrence, such that*

$$P(X(t) > x) \sim \sigma_T = \sigma_\tau n^\lambda \tag{19}$$

where λ is the scaling exponent, the total time of observation is $T = n\tau$ and τ is the minimal trading horizon, e.g., a minute, an hour, a day, a month, or a year, etc.

The essence of power laws is the inherent *self- similarity* over the n trading periods: no matter what the size of n, the power law will have the same shape. The shape of the power law is

[2] Vilfredo Pareto (1848 - 1923) was an Italian sociologist and professor of political economy at the University of Lausanne, Switzerland. In his book *Mind and Society* (1916; English translation, 1935), Pareto states that individuals act irrationally, but that mass action becomes more logical the greater the number of individuals involved, because their desires and illusions cancel out. He thought that society, like physics, is a system of forces in equilibrium. Mathematics can therefore be applied to explain why the equilibrium holds, making a science of society possible. Unfortunately, Pareto's theory did not recognize that irrational behavior can also occur on a mass scale, *e.g.*, like bubbles and catastrophes in the financial markets, and therefore his theory cannot account for crowd behavior. In 1897 Pareto found that the distribution of incomes for individuals was approximately log - normally distributed for 97% of the population. But for the last 3% of the population incomes increased more sharply. We now know from Finance theory why that is, because the more wealth one has, the more one can risk. The wealthy can *leverage* their wealth in ways the average, middle income, individual cannot.

determined by the exponent λ. Thus, the size of the shape of a financial return distribution determined by the number of trading periods n and the *fundamental market volatility* σ_τ, which is a measure of the fundamental energy or "noise" of a market.

Remark 11 *Notice that we make a distinction between the trading time τ and the actual time t. Thus, only when the trading time unit is the same as the actual time unit $\tau = 1$, $T = n$, and the power law can be expressed in terms of the total time of observation:*

$$P(X(t) > x) \sim \sigma_T = \sigma_\tau T^\lambda \tag{20}$$

For some financial time series, we must distinguish between trading time and observation time, like for FX series, where the tick - by - tick trading is often more frequent than the recording of transaction prices by commercial bank quotations. Researchers have often only access to the regularly spaced price quotations and not to the more frequent and irregularly spaced tick - by - tick transaction prices.

A Pareto-Lévy power law can be written in logarithmic form as an affine relation:

$$\ln \sigma_T = \lambda \ln n + \ln \sigma_\tau \tag{21}$$

so that, in principle, the exponent λ can be found from the expression:

$$\lambda = \frac{\ln \sigma_T - \ln \sigma_\tau}{\ln n} \tag{22}$$

In terms of financial risk theory, Peters (1994, pp.27 - 37) appropriately calls this relationship the *term structure of volatility.* It depends on the *horizon* (or maturity) volatility σ_T and the fundamental market volatility σ_τ - which depends on the uniform trading horizon τ - as well as the number n of uniform, equally spaced, trading times.

How easy is it to compute the invariant scaling exponent λ from the observations? Not as easy as it appears, since, *a priori*, we do not know the fundamental market volatility σ_τ, the basic standard deviation (risk) of the unit of observation, *i.e.,* the observation "noise". This has to be measured first, somehow, or at least simultaneously, with the horizon volatility. We discussed this epistemological issue in greater detail already in Los (2005b), where we showed that this problem is solved by the latest advances in nonstationary (engineering) signal processing. The relevance if this term structure of volatility for portfolio risk management and Value - at - Risk issues is discussed in Los (2005c).

4 Invariances of Pareto - Lévy Scaling Distributions

Many objects that come in different sizes have self - similar power law distributions of their relative abundance over large size ranges, of the form:

$$f(x) \sim x^{\lambda} \tag{23}$$

A recent example of the application of such scaling laws of financial volatility to the analysis of financial long - term dependence is Batten, Ellis and Mellor (1999). The only prerequisite for such a self - similar law to prevail in a given size range is the absence of an inherent size scale. Thus, invariance of scaling results from the fact that homogeneous power laws lack natural scales: they do not harbor a characteristic unit (such as a unit length, a unit time, or a unit mass).

Remark 12 *Real - world data are never completely scale - invariant because of "end effects." For example, no living village has fewer than one inhabitant or more than* 100 *million inhabitants - except the proverbial "global village," which is more of a simile, than a reality.*

Mandelbrot (1962, 1963b) discusses three invariances of scaling, or self - similarities, of stable Pareto - Lévy power law distributions:

(1) invariance of scaling under weighted mixture (= weighted linear combination);

(2) invariance of scaling under choice maximization (minimization); and

(3) invariance of scaling under aggregation.

More invariances are possible, as Fig. 1 shows, but they are all related to the three invariances defined by Mandelbrot.

Let's discuss Mandelbrot's three scale invariances in some sufficient detail to understand the concept of "scaling distributions," and as an example of "distributional stability."

Scheme	Stability Property[a]
Summation	$X_1 \stackrel{d}{=} a_n(X_1 + \cdots + X_n) + b_n$
Maximum	$X_1 \stackrel{d}{=} a_n \max_{1 \le i \le n} X_i + b_n$
Minimum	$X_1 \stackrel{d}{=} a_n \min_{1 \le i \le n} X_i + b_n$
Multiplication	$X_1 \stackrel{d}{=} A_n(X_1 X_2 \cdots X_n)^{C_n}$
Geometric Summation	$X_1 \stackrel{d}{=} a(p)(X_1 + \cdots + X_{T(p)}) + b(p)$
Geometric Maximum	$X_1 \stackrel{d}{=} a(p) \max_{1 \le i \le T(p)} X_i + b(p)$
Geometric Minimum	$X_1 \stackrel{d}{=} a(p) \min_{1 \le i \le T(p)} X_i + b(p)$
Geometric Multiplication	$X_1 \stackrel{d}{=} A(p)(X_1 X_2 \cdots X_{T(p)})^{C(p)}$

[a] Notation "$X_1 \stackrel{d}{=}$" stands for "equality in distribution."

Figure 1: Stable probabilistic schemes

4.1 Weighted Mixtures

Suppose that the random variable X_W is a weighted mixture of the independent random variables $X(\tau)$, and denote by p_τ the probability that X_W is identical to $X(\tau)$. Since

$$\begin{aligned} P(X_W > x) &= \sum_\tau p_\tau P(X(\tau) > x) \\ &\sim \sum_\tau p_\tau \sigma_\tau n^\lambda \\ &= \sigma_W n^\lambda \end{aligned} \tag{24}$$

we see that the weighted mixture X_W is also scaling and the scale parameter $\sigma_W = \sum p_\tau \sigma_\tau$ is a *weighted average* of the separate scale parameters σ_τ. (The sign $\sim$ means "is proportional to"). Thus, scaling is invariant under weighted mixture (= weighted linear combination) of random variables.

4.2 Choice Maximization

Ex post, when the values of $X(\tau)$ are known, let X_M be the maximum value. This X_M is also scaling with the scale parameter $\sigma_M = \sum \sigma_\tau$, since, in order that X_M is the maximum, *i.e.,* $X_M \leq x$,where x is a value, it is both necessary and sufficient that $X_\tau \leq x$ for every τ. Hence we have the product

$$P(X_M \leq x) = \prod_\tau P(X_\tau \leq x) \tag{25}$$

Consequently,

$$\begin{aligned} P(X_M > x) &= 1 - P(X_M \leq x) \\ &= 1 - \prod_\tau P(X_\tau \leq x) \\ &= 1 - \prod_\tau (1 - P(X_\tau > x)) \\ &\sim 1 - \prod_\tau (1 - \sigma_\tau n^\lambda) \\ &\sim \sum \sigma_\tau n^\lambda = \sigma_M n^\lambda \end{aligned} \tag{26}$$

for sufficiently small σ_τ, where $\sigma_M = \sum \sigma_\tau$.

4.3 Aggregation

Let X_A be the sum of the random variables X_τ. The *aggregate* X_A is also scaling, with a scale parameter that is again the sum of the separate weights $\sigma_A = \sum \sigma_\tau$. Using a similar argument as for the weighted mixtures

$$\begin{aligned} P(X_A > x) &= \sum_\tau P(X(\tau) > x) \\ &\sim \sum_\tau \sigma_\tau n^\lambda \\ &= \sigma_A n^\lambda \end{aligned} \tag{27}$$

where $\sigma_A = \sum \sigma_\tau$. Mixtures combined with aggregation leave the scaling distribution invariant - up - to - scale.

5 Zolotarev Parametrization of Stable Distributions

We will now discuss stable distributions in general, by following closely Nolan's (1999a and b) admirably clear theoretical presentation, and we will see where the Pareto - Lévy scaling laws of Mandelbrot, which exhibit infinite variance in the limit, fit in as a subsection of stable distributions. Interestingly, the study of general stable distributions was begun by Paul Lévy in 1924 in his study of normalized sums of i.i.d. variables. Stable distributions are a class of distributions, that includes the Gaussian and Cauchy distributions in a family that allows skewness and heavy tails (= excess kurtosis). Distributions with heavy tails are empirically observed in economics, finance, insurance, telecommunications and physics.

Remark 13 *In finance, the interest in the skewness of return distributions has primarily emerged in the context of the discussion about the empirical truthfulness of the Capital Asset Pricing Model (CAPM), which is based on Markowitz' Nobel Memorial Prize - winning Mean - Variance Analysis. That model assumes normal distributions and/or quadratic wealth-utility preference functions, which don't include preferences for skewness and kurtosis. However, the moment a certain degree of skewness is preferred by the investors, the conventional CAPM is no longer a model of market efficiency (Kraus and Litzenberger, 1976; Friend and Westerfield, 1980). In other words, the empirically observed skewness implies that the CAPM cannot represent an efficient market model for the empirical markets..*

Some people have objected against the use of stable distributions with infinite variance, because empirical data exhibit bounded ranges. However, that is not what it means, since the rates of return of the S&P500 Index have indeterminate (= "infinite") variance! Moreover, bounded data sets are routinely modeled by Gaussian distributions which have infinite support. Thus the epistemological question is, why would distributions with theoretical infinite support with empirically bounded ranges be methodologically acceptable, while distributions with theoretical finite support and empirically unbounded ranges would not be? After all, we're primarily interested in the *shape* of the distributions, not in their size.

It is now an established empirical fact that the shape characteristics of stable distributions, other than the Gaussian, are more conform those of the frequency distributions we empirically observe, in particular in finance (Rachev and Mittnik, 2000). In addition, stable distributions provide a realistic fit with very parsimonious parametrizations. Furthermore, infinite variances are not restricted to stable distributions. If a distribution has asymptotic power decay on its tails, then the number of its moments is limited. If the exponent of such power decay is less than 2, then the distribution will have infinite variance, as we already learned in Los (2005b).

We turn now to Zolotarev's definition and parametrization of stable distributions, since that is currently the most popular theoretical representation (Zolotarev, 1986; Adler, Feldman and Taqqu, 1998)

5.1 Definitions of Stable Distributions

Definition 14 *(**Original definition of stable distribution**): A random variable X is **stable**, or **stable in the wide sense**, if for X_1 and X_2 independent copies of X and for any positive constants a and b, we have*

$$aX_1 + bX_2 \stackrel{d}{=} cX + d \tag{28}$$

for all choices of a and b and for some nonnegative $c \geq 0$ and some $d \in \mathbb{R}$. Thus if the weighted sum of X_1 and X_2 equals in distribution an affine relationship.

The symbol $\stackrel{d}{=}$ means equality in distribution, *i.e.,* both expressions have the same probability law, although the size of the distribution is indeterminate.

Definition 15 *The random variable X is **strictly stable** or **stable in the narrow sense** if this relationship holds with the "intercept" $d = 0$, thus if their weighted sum equals in distribution a **linear** relationship.*

Definition 16 *A random variable is **symmetrically stable** if it is stable and symmetrically distributed around 0, e.g.,*

$$X \stackrel{d}{=} -X \tag{29}$$

In other words, the equation

$$aX_1 + bX_2 \stackrel{d}{=} cX + d \tag{30}$$

states that the shape of the distribution of X is preserved affinely, *i.e.*, up to scale c and shift d under addition. For *scaling distributions*, which are a subset of stable distributions, this is, of course, equivalent to the invariances under weight mixture and aggregation of Mandelbrot's (1963a) Pareto - Lévy distributions. The word stable is used because the shape of the distribution is stable or unchanged under sums of this additive type. As already mentioned, there are not only additive stable, but also max - stable, min - stable and geometrically stable distributions, that preserve stability under choice maximization, choice minimization, etc.

There are other equivalent definitions of stable random variables. Here is a variation of the original definition of an (additive) stable distribution:

Definition 17 ***(Variation of definition of stable distribution)*** *X is stable (in the wide sense) if and only if for all $n > 1$ there exist constants c_n and $d_n \in \mathbb{R}$ such that*

$$X_1 + X_2 + ... + X_n \stackrel{d}{=} c_n X + d_n \tag{31}$$

where $X_1, ..., X_n$ are independent, identical copies of X.

It appears that the only possible choice for c_n is that it is an exponential function of n: $c_n = n^{\lambda} = n^{\frac{1}{\alpha_Z}}$. X is again strictly stable if and only if $d_n = 0$ for all n. Thus a defining invariance property of stable distributions is that linear combinations of stable random variables are also stable.

The most concrete way to describe all possible stable distributions is through their characteristic functions, or Fourier transforms (*Cf.* Los, 2005a), which is what we will do next. All stable distributions are scale and location shifts of standardized stable distributions, just like any Gaussian $X \sim N(\mu, \sigma^2)$ is the scale and location shift affine transform $X = \sigma Z + \mu$ of the standardized Gaussian $Z \sim N(0, 1)$, for which standardized probability tables exist.

Following Nolan, we will present the popular *standardized* or *reduced* parametrization of stable distributions of Zolotarev.[3] This standardized parametrization of stable distributions uses the *sign* (or modified Heaviside) function, which is defined as:

$$sign(\omega) \left\{ \begin{array}{l} = -1 \text{ for } \omega < 0 \\ = 0 \text{ for } \omega = 0 \\ = +1 \text{ for } \omega > 0 \end{array} \right\} \tag{32}$$

Theorem 18 ***(Zolotarev, 1986, Standardized Parametrization of Stable Distribution)*** *A random variable X is stable if and only if $X \stackrel{d}{=} cZ + d$, with $c \geq 0, d \in \mathbb{R}$, and $Z = (\alpha_Z, \beta)$ is a*

[3] Other parametrizations are possible, but currently not as popular (*Cf.* Rachev and Mittnik, 2000). Since in the cuent paper I emphasize concepts, definitions and empirical measurements of risk, all Theorems, Lemma's and Propositions will be given without proof. Such mathematical proofs can be found in the references.

random variable with the following characteristic function, where $0 < \alpha_Z \leq 2, -1 \leq \beta \leq 1$,

$$E\left\{e^{j\omega z}\right\} = \int_{-\infty}^{+\infty} e^{j\omega z} dG(z)$$
$$\left\{ \begin{array}{l} = e^{\left(-|\omega|^{\alpha_Z}\left[1+j\beta \tan\frac{\pi\alpha_Z}{2} sign(\omega)(|\omega|^{1-\alpha_Z}-1)\right]\right)} \; if\; \alpha_Z \neq 1 \\ = e^{\left(-|\omega|\left[1+j\beta\frac{2}{\pi}(sign(\omega)\ln|\omega|\right]\right)} \; if\; \alpha_Z = 1 \end{array} \right\} \tag{33}$$

where G is the stable distribution function corresponding to the stable density function of Z.

The key idea of Zolotarev's fundamental Theorem is that the parameters α_Z and β determine the shape of the stable distribution, while c is a scale parameter and d is a shift parameter. It shows that the standardized stable distribution has only two parameters: (1) an *index of stability, or stability (shape) exponent* $\alpha_Z \in (0, 2]$ and (2) a *skewness parameter* $\beta \in [-1, 1]$. For the $\alpha_Z = 1$ case, $0. \ln 0$ is always interpreted as 0.

Remark 19 *The* ***Non - Standardized Stable Distribution*** *of the random variable* $X \sim \mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 0)$ *(e.g., in Mittnik et al., 1998, and in Rachev and Mittnik, 2000) has the characteristic function*

$$E\left\{e^{j\omega x}\right\} = \int_{-\infty}^{+\infty} e^{j\omega x} dH(x)$$
$$\left\{ \begin{array}{l} = e^{\left(-\gamma^{\alpha_Z}|\omega|^{\alpha_Z}\left[1+j\beta \tan\frac{\pi\alpha_Z}{2} sign(\omega)(\gamma|\omega|^{1-\alpha_Z}-1)\right]+j\delta\omega\right)} \; if\; \alpha_Z \neq 1 \\ = e^{\left(-\gamma|\omega|\left[1+j\beta\frac{2}{\pi}(sign(\omega)(\ln|\omega|+\ln\gamma\right]+j\delta\omega\right)} \; if\; \alpha_Z = 1 \end{array} \right\} \tag{34}$$

where H is the stable distribution function corresponding to the stable density function of X. As we already discussed in Los (2005a), this non - standardized stable distribution has four parameters (1) a stability exponent $\alpha_Z \in (0, 2]$*, (2) a skewness parameter* $\beta \in [-1, +1]$*, (3) a scale parameter* $\gamma > 0$*, and (4) a location parameter* $\delta \in \mathbb{R}$*.*

Remark 20 *This is the theoretical expression, of course The actual computation of all stable densities is always approximate in the sense that the density function* $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; k), k = 0, 1$ *is approximated by the Fast Fourier Transformation of these stable characteristic functions.*

5.2 General Properties of Stable Distributions

Although explicit formulas exist for stable characteristic functions, in general no explicit formulas exist for the corresponding stable distribution densities. However, the theoretical properties of such distribution densities are well known. The basic property of stable distribution densities is given by the following so - called idealization theorem.

Theorem 21 *All (non - degenerate) stable distributions are continuous distributions with an infinitely differentiable density.*

The probability density function (pdf) of a standardized $Z(\alpha_Z, \beta)$ stable distribution will be denoted by $f(z|\alpha_Z, \beta)$ and the cumulative distribution function (c.d.f.) will be denoted by $F(z|\alpha_Z, \beta)$. All stable densities are *unimodal*, *i.e.,* they have each one "peak."[4] The mode $m(\alpha_Z, \beta)$ of a $Z(\alpha_Z, \beta)$ distribution can be *numerically* computed, even though no explicit algebraic formula for it exists. By the symmetry property, the densities have uni-modes such that:

$$m(\alpha_Z, -\beta) = -m(\alpha_Z, \beta) \tag{35}$$

Furthermore, stable densities are positive on the whole real line, unless $\alpha_Z < 1$ and ($\beta = +1$ or $\beta = -1$), in which case the support is half a line. In more precise terms:

Lemma 22 *The* ***support*** *of a stable* $X(\alpha_Z, \beta, \gamma, \delta)$ *distribution is*

$$\text{support } f(z|\alpha_Z, \beta) = \left\{ \begin{array}{l} \left[\delta - \tan(\frac{\pi\alpha_Z}{2}), \infty\right) \text{ if } \alpha_Z < 1 \text{ and } \beta = 1 \text{ (positively skewed)} \\ \left(-\infty, \delta + \gamma\tan(\frac{\pi\alpha_Z}{2})\right] \text{ if } \alpha_Z < 1 \text{ and } \beta = -1 \text{ (negatively skewed)} \\ (-\infty, +\infty) \text{ otherwise} \end{array} \right\} \tag{36}$$

Remark 23 *Notice that the constant* $\tan(\frac{\pi\alpha_Z}{2})$ *is an important ingredient of stable distributions. It shows an essential discontinuity at* $\alpha_Z = 1$*, since as* $\alpha_Z \uparrow 1$, $\tan(\frac{\pi\alpha_Z}{2}) \uparrow +\infty$ *and* $\alpha_Z \downarrow 1$, $\tan(\frac{\pi\alpha_Z}{2}) \downarrow -\infty$*, while* $\tan(\frac{\pi\alpha_Z}{2})$ *is undefined at* $\alpha_Z = 1$.

Another basic property of stable distributions is their symmetry.

Proposition 24 *(**Symmetry Property**) For any* α_Z *and* β,

$$Z(\alpha_Z, -\beta) \stackrel{d}{=} Z(\alpha_Z, \beta) \tag{37}$$

Therefore, the density and distribution function of a $Z(\alpha_Z, \beta)$ random variable satisfy $f(z|\alpha_Z, \beta) = f(-z|\alpha_Z, -\beta)$ and $F(z|\alpha_Z, \beta) = 1 - F(-z|\alpha_Z, \beta)$.

It's important to consider now a few special cases to understand these distributions and their densities:

(1) When $\beta = 0$, the symmetry property says $f(z|\alpha_Z, \beta) = f(-z|\alpha_Z, \beta)$, so the pdf and c.d.f. are symmetric around 0.

[4] This unimodality, or "one - peakedness" of stable distributions is a potential shortcoming for research into empirical financial distributions, since some of such distributions have been observed to be multi - modal. Multi - modality occurs, for example, in chaotic distributions. Such multiple equilibria, chaotic distributions can be generated by price diffusion equations, which contain a combination of linear and parabolic or higher-order components, like the price diffusion equations of some options. This is an extremely interesting area of both theoretical and empirical research into financial turbulence (Los, 2005d).

(2) When $\beta > 0$, the distribution is skewed to the right with the right tail of the distribution heavier than the left tail: $P(Z > z) > P(Z < -z)$ for large $z > 0$. When $\beta = 1$, the stable distribution is *totally skewed to the right*.

(3) By the symmetry property, the behavior of the $\beta < 0$ cases is reflecting the behavior of the $\beta > 0$ cases, with a heavier left tail. Thus when $\beta < 0$, the distribution is skewed to the left with the left tail of the distribution heavier than the right tail: $P(Z > z) > P(Z < -z)$ for large $z > 0$. When $\beta = -1$, the distribution is *totally skewed to the left*.

(4) The stability exponent $\alpha_Z \in (0, 2]$ determines the kurtosis of the distribution: the peakedness at δ and the fatness of the tails. As the stability exponent α_Z decreases, three things occur to the distribution density: its peak gets higher, the region flanking the peak gets lower, and the tails get heavier, or, in summary: the kurtosis of the distribution increases. *Vice versa*, when the stability exponent α_Z increases, the kurtosis of the distribution decreases. For example, when $\alpha_Z = 2$, the distribution is normal with its variance equal to $\sigma^2 = 2\gamma^2$. In that case

$$\tan(\frac{\pi\alpha_Z}{2}) = \tan(\pi) = 0 \tag{38}$$

so the characteristic function is real and hence the distribution is always symmetric, no matter what the value of β. The next characteristic of stable distributions is the most interesting.

(5) When the stability exponent $\alpha_Z < 2$, the second moment, or variance, becomes infinite, or, more precisely, undefined. Its computation no longer converges to a unique value. When $1 < \alpha_Z < 2$, the first moment still exists, but when $\alpha_Z \leq 1$, the first moment or theoretical (population) average also becomes infinite or undefined and its computation no longer converges to a unique value (Samorodnitsky and Taqqu, 1994). Thus, there is only a very limited range of the stability exponent α_Z for which both the first and second moments of stable distributions exist. By *existence of moments* we mean that they have a well - defined value that can be computed and identified within a prespecified error range, no matter how small.

Of course, we can always compute a (sample) average or a variance of a finite data set. Non - existent or undefined theoretical (population) averages and variances just mean that there is no convergence to well - defined values, even when we substantially enlarge the data set. The computed mean and variance of that data set will never converge to a specific mean and variance, but will continue to "wander." It will never settle on a specific value.

This is not a theoretical abstraction, as one can observe from the computation of the variance of the rates of return of the S&P500 Index. These Index rates have a well-defined, convergent finite mean, but no defined, convergent variance. Peters, 1994, pp. 200 - 205 provides many additional theoretical and empirical examples. These cases are seldom mentioned in the classical statistical literature, thereby creating the erroneous impression that these cases are pathological and special. But they are actually regularly occurring empirical cases in the financial markets!

5.3 Different Zolotarev Parametrizations

Historically, several different Zolotarev parametrizations have been used for stable distributions, for which, in general, no closed form parametrization exists (because of the discontinuity at $\alpha_Z = 1$). We give the three most often used Zolotarev parametrizations. Here is the first one.

Definition 25 *A random variable X is the* ***parametrized stable distribution*** $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 0)$ *if*

$$X \stackrel{d}{=} \gamma Z + \delta \tag{39}$$

where $Z = Z(\alpha_Z, \beta)$ is implicitly given by its characteristic function in Theorem 1.

This is the Zolotarev parametrization used for current numerical work on stable distributions. It has the simplest form for the characteristic function that is continuous in all parameters.

Remark 26 *Notice that γ is the scale parameter and δ the location parameter, in a rather natural fashion. For the standardized version $\gamma = 1$ and $\delta = 0$, so that $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 0) = \mathbf{S}(\alpha_Z, \beta; 0)$.*

Let's show some numerical examples of stable distributions to demonstrate their properties mentioned in the preceding subsection. Fig. 2 provides a graphical representation of stable densities in the $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 0) = \mathbf{S}(\alpha_Z, 0.8, 1, 0; 0)$ parametrization, with the stability exponent α_Z (alpha) as indicated.

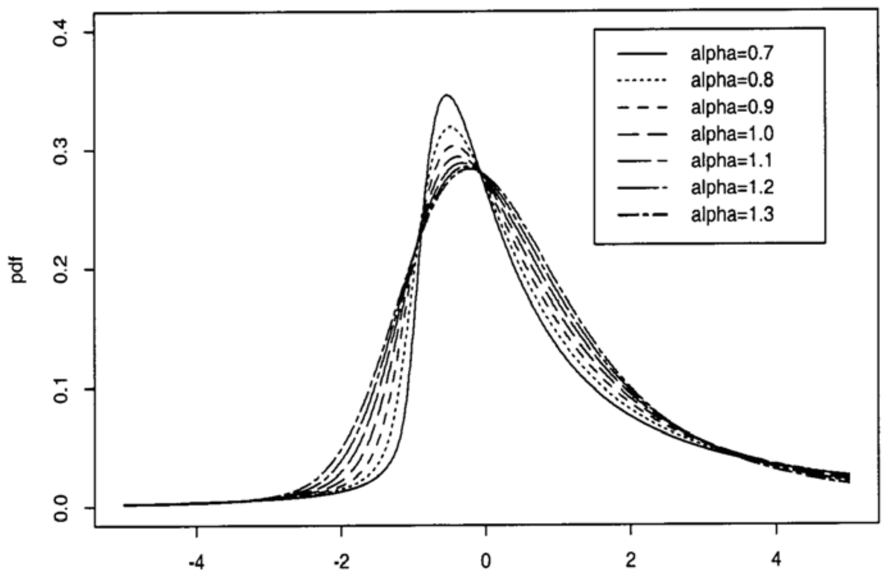

Figure 2: Stable density in the Zolotarev $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 0) = \mathbf{S}(\alpha_Z, 0.8, 1, 0; 0)$ parametrization

Here is the second Zolotarev parametrization:

Definition 27 *A random variable X is characterized by the parametrized stable distribution $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 1)$ if*

$$X \left\{ \begin{array}{l} \stackrel{d}{=} \gamma Z + (\delta + \beta\gamma \tan \frac{\pi \alpha_Z}{2}), \text{ if } \alpha_Z \neq 1 \\ \stackrel{d}{=} \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \ln \gamma), \text{ if } \alpha_Z = 1 \end{array} \right\} \tag{40}$$

where $Z = Z(\alpha_Z, \beta)$ is implicitly given by its characteristic function in Zolotarev's 1986 Theorem.

This $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 1)$ parametrization is the most common one currently in use, since it produces the simplest characteristic function, which is jointly continuous in all four parameters, and has therefore preferable algebraic properties. But it's practical disadvantage is that the location of the mode is unbounded in any neighborhood of $\alpha_Z = 1$.

Fig. 3 provides a graphical representation of stable densities in the $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 1) = \mathbf{S}(\alpha_Z, 0.8, 1, 0; 1)$ parametrization, with the stability exponent α_Z (alpha) being varied similarly as in Fig. 2. Notice in Fig. 3, that the mode is near 0 for α_Z near 0 or 2, or $\alpha_Z = 1$, but diverges to $+\infty$ as $\alpha_Z \uparrow 1$ and diverges to $-\infty$ as $\alpha_Z \downarrow 1$. When $\beta = 0$, both these parametrizations are identical.

Remark 28 *As $\alpha_Z \uparrow 2$ both parametrized distributions converge in distribution to a distribution with standard deviation $\sqrt{2}\gamma$ and not γ, as maybe would have been expected! In fact, when $\alpha_Z < 2$,*

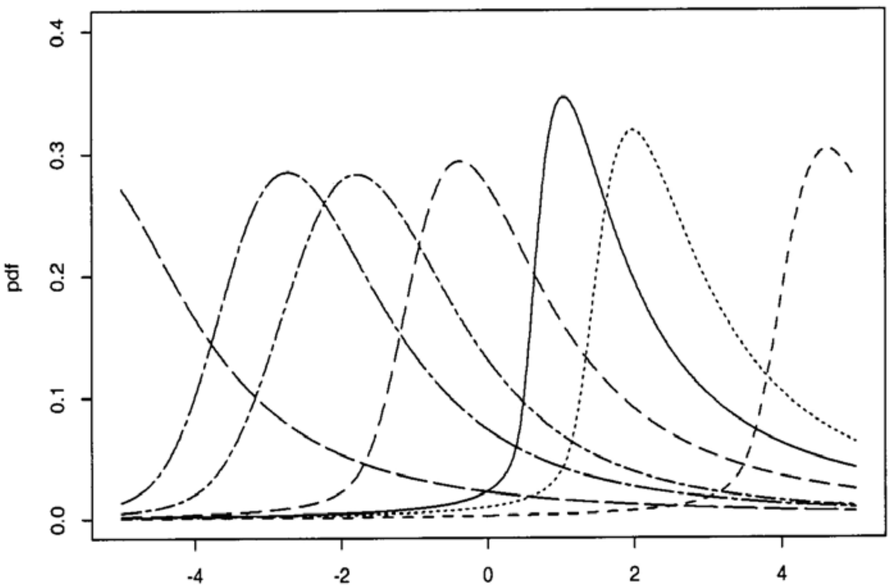

Figure 3: Stable density in the Zolotarev $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 1) = \mathbf{S}(\alpha_Z, 0.8, 1, 0; 1)$ parametrization

no standard deviation exists. Thus, for comparison purposes, one should multiply γ by $\sqrt{2}$ to make the scale parameter of the stable distribution comparable with that of the standard Gaussian distribution, i.e., the deviation $\sigma = \sqrt{2}\gamma$, or, equivalently, $\gamma = \frac{1}{\sqrt{2}}\sigma$.

The third Zolotarev parametrization focuses on the mode as a location parameter, since, as we saw, every stable distribution has a mode.

Definition 29 *A random variable X is characterized by the parametrized stable distribution $\mathbf{S}(\alpha_Z, \beta, \gamma, \delta; 2)$ if*

$$X \stackrel{d}{=} \alpha_Z^{-1/\alpha_Z} \gamma [Z - m(\alpha_Z, \beta)] + \delta \tag{41}$$

where $Z = Z(\alpha_Z, \beta)$ is implicitly given by its characteristic function in Theorem 1 and $m(\alpha_Z, \beta)$ is the mode of Z.

5.4 Tail Properties and Stable Paretian laws

When the stability exponent $\alpha_Z = 2$, the resulting Gaussian distribution has well understood asymptotic tail properties. For the purpose of comparison, we'll briefly discuss in this section the crucial tail properties of non - Gaussian ($\alpha_Z < 2$) stable distributions. In risk theory, it is the tails of such stable distributions, representing the less likely, outlying and sometimes catastrophic events, that are most important for financial analysts, hedgers, and insurance and re-insurance companies.

Theorem 30 ***(Tail Approximation)****. Let $X \sim \mathbf{S}(\alpha_Z, \beta; 0)$ with $0 < \alpha_Z < 2$, $-1 < \beta \leq 1$. Then, as $x \to \infty$,*

$$P(X > x) \sim c_{\alpha_Z}(1+\beta)x^{-\alpha_Z} \tag{42}$$

$$f(x|\alpha_Z, \beta; 0) \sim \alpha_Z c_{\alpha_Z}(1+\beta)x^{-(\alpha_Z+1)} \tag{43}$$

where $c_{\alpha_Z} = \Gamma(\alpha_Z)(\sin \frac{\pi\alpha_Z}{2})/\pi$.

Remark 31 *Notice the gamma function Γ, which is such that $\Gamma(\alpha_Z + 1) = \alpha_Z \Gamma(\alpha_Z) = \alpha_Z!$, with $\Gamma(1) = 1$.*

Remark 32 *Using the symmetry property, the lower tail properties are similar. For all $\alpha_Z < 2$ and $-1 < \beta$, the upper tail probabilities and densities are asymptotic power laws (i.e., scaling distributions).*

Having developed this arsenal of concepts and definitions of stable distributions, we'll can now define more specific non - Gaussian distributions, in particular the Pareto and heavy tailed distributions, which figure prominently in the recent financial research literature (Müller *et al.*, 1990; Janicki and Weron, 1994; Mantegna and Stanley, 1995; Samorodnitsky and Taqqu, 1994) .

Definition 33 ***Pareto distributions*** *are probability laws with upper tail probabilities given exactly by the right hand side of the Tail Approximation Theorem.*

Remark 34 *The term stable Paretian laws is used to distinguish between the fast decay of the Gaussian distributions and the Pareto - like tail behavior in the $\alpha_Z < 2$ case.*

Definition 35 *A distribution is said to be* ***heavy - tailed*** *if it's tails are heavier than exponential.*

Remark 36 *For $\alpha_Z < 2$, stable distributions have one tail (when $\alpha_Z < 1$ and $\beta = \pm 1$), or both tails (in all other cases) that are asymptotically power laws with heavy tails.*

One important consequence of heavy tails is that not all moments exist, or, when they exist, they may be fractional. In other words, the literature on frequency distributions has considerably expanded our arsenal of moments discussed in Los (2005a): from integer moments to fractional moments! This provides the direct connection to Los (2005b).

Definition 37 ***Fractional absolute moments:***

$$E\{|X|^p\} = \int_{-\infty}^{\infty} |x|^p f(x)dx \tag{44}$$

where p is any - integer or fractional - real number.

The Tail Approximation Theorem implies that for $0 < \alpha_Z < 2$, the moments $E\{|X|^p\}$ are finite for $0 < p < \alpha_Z$, and that $E\{|X|^p\} = +\infty$ for all $p \geq \alpha_Z$. Thus ,when $\alpha_Z < 2$, $E\left\{|X|^2\right\} = E\left\{X^2\right\} = +\infty$ and stable distributions do not have finite second moments or variances. This is the worrisome theoretical case to which Mandelbrot (1963, 1966) referred in the 1960s and which was then dismissed by most mathematicians as pathological. But empirical observations in the financial markets since the 1960s have demonstrated that this case is empirically more prevalent than was presumed by the theoreticians.

In fact, this is an important case for anybody studying financial risk, since it implies that particular investment return series may have measurable stable distributions, but still exhibit infinite or undefined risk! The empirical scientific question is, do such strange financial distributions exist in empirical reality? The unfortunate answer is: yes, since this are the distributions of variables moving in the range of the so - called persistent or pink noise, *i.e.,* noise that lies in the range between white and red noise (Los, 2005b & c).

Example 38 *The logarithmic plot of Fig. 4 (which we borrowed from Mantegna and Stanley, 2000, p. 69) shows that the high - frequency pdf for* $\Delta t = 1$ *minute price changes of the S&P500 Index with an empirically measured* $\alpha_Z = 1.67$ *lies between the Gaussian pdf with* $\alpha_Z = 2.00$ *and the pdf of a Lévy stable distribution with* $\alpha_Z = 1.40$ *and a scaling factor of* $\gamma = 0.00375$.

Example 39 *Fig. 5 shows that the daily observations on the rates of return of the S&P500 Index in 1998 exhibit considerable persistence, unlike Gaussian rates of return. The variance or volatility of these daily rates of return, computed over longer and longer horizons dissipates. But this dissipation of the S&P500's volatility is not gradual and smooth. It shows sudden and completely unpredictable discontinuities and the volatility never converges to a uniquely defined value. Peters (1994, pp. 141 - 146) observed similar phenomena and found that this volatility dissipation process is* ***antipersistent*** *(Los, 2005b).*

Let's analyze the specific case of the first moment, or mean, of stable distributions in somewhat greater detail.

Proposition 40 *When* $1 < \alpha_Z \leq 2$, $E\{|X|\} < \infty$ *and the mean of* $X \sim \mathbf{S}(\alpha_Z, \beta, \gamma_k, \delta_k; k)$ *exists, for* $k = 0, 1, 2$, *respectively, the mean is*

$$\mu \left\{ \begin{array}{l} = E\{X\} = \delta_1 \\ = \delta_0 - \beta\gamma_0 \tan\frac{\pi\alpha_Z}{2} \\ = \delta_2 - \alpha_Z^{-1/\alpha}\gamma_2\left(\beta \tan\frac{\pi\alpha_Z}{2} + m(\alpha_Z, \beta)\right) \end{array} \right\} \tag{45}$$

In other words, there is a clear relationships between the various location parameters, δ_1, δ_2, and δ_2 of these three parametrizations. On the other hand, when $\alpha_Z \leq 1$, the first absolute

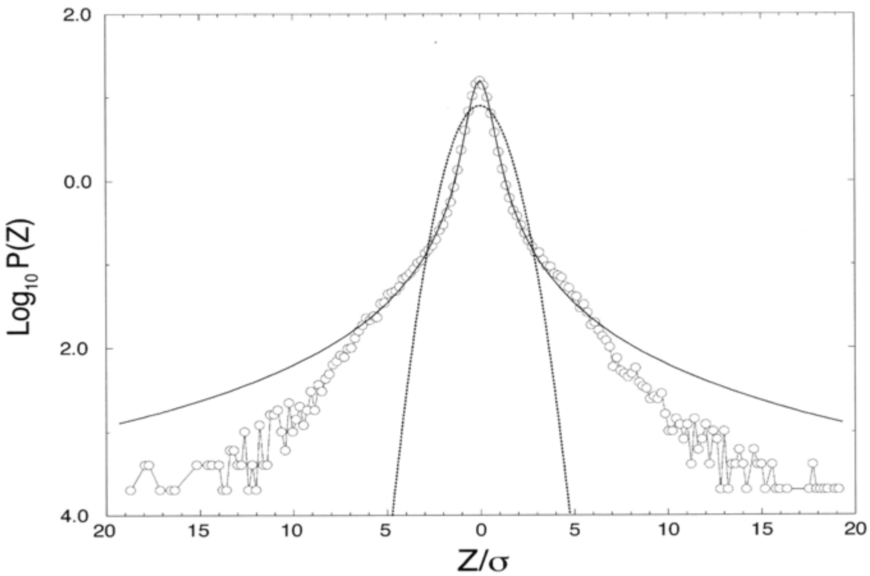

Figure 4: Comparison of the $\Delta t = 1$ minute p.d.f. for high - frequency S&P500 price changes (white circles) with the Gaussian p.d.f (dotted line, smallest p.d.f. in middle) with $\alpha_Z = 2.00$ and with a Lévy stable p.d.f. (solid line, largest p.d.f.) of $\alpha_Z = 1.40$ and scale factor $\gamma = 0.00375$ (same as that of the S&P500).

moment is infinite, $E\{|X|\} = +\infty$, and these means are undefined. What happens geometrically with a stable distribution when its absolute mean does not exist?

Consider what happens to the mean of $X \sim \mathbf{S}(\alpha_Z, \beta; 0)$ as $\alpha_Z \downarrow 1$. Even though the mode of the distribution stays close to 0, it has a mean $\mu = \beta \tan \frac{\pi \alpha_Z}{2}$. When $\beta = 0$, the distribution is symmetric and the mean is always 0. When $\beta > 0$, the mean $\mu \uparrow +\infty$, because both tails are getting heavier, but the right tail is heavier than the left. By symmetry, the $\beta < 0$ case has the mean $\mu \downarrow -\infty$. Finally, when α_Z reaches 1, the tails are too heavy for the integral

$$E\{X\} = \int_{-\infty}^{\infty} x f(x) dx \tag{46}$$

to converge and the mean becomes undefined or infinite: $E\{X\} \to \infty$.

However, this geometric description depends on the particular Zolotarev parametrization chosen. For example, the second parametrization, a $\mathbf{S}(\alpha_Z, \beta; 1)$ distribution, keeps the mean at 0 by shifting the whole distribution by an increasing amount as $\alpha_Z \downarrow 1$. For the third Zolotarev

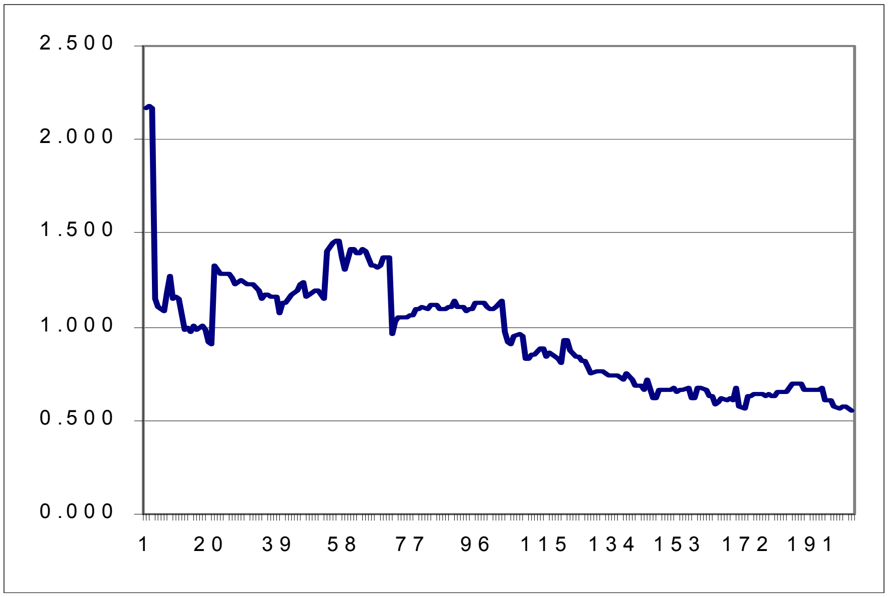

Figure 5: Non - convergent moving variance of 253 daily rates of return (in 100%) of the S&P500 stock market index in 1998, computed with a moving (horizon) window of $\tau = 50$ observations. Notice that none of the window variances is the same and that they wander aimlessly.

parametrization, a $\mathbf{S}(\alpha_Z, \beta; 2)$ distribution keeps the mode exactly at 0, and the mean behaves like the mean of a $\mathbf{S}(\alpha_Z, \beta; 0)$ distribution. Thus a stable empirical distribution with a non - existent mean can best be parametrized by the first Zolotarev parametrization, when a parametrization is required (usually for computational purposes).

5.5 Generalized Central Limit Theorem (GCLT)

The classical Central Limit Theorem states that the normalized sums of i.i.d. variables with finite variance converges to a Gaussian distribution (Gnedenko and Kolmogorov, 1954). But the Generalized Central Limit Theorem shows that if the finite variance (= finite risk) assumption is dropped, the only possible resulting limits are stable distributions.

Theorem 41 ***(Generalized Central Limit Theorem, or GCLT)****. Let $X_1, X_2, ..., X_n$ be an i.i.d. sequence of random variables. There exist constants $c_n > 0, d_n \in \mathbb{R}$ and a non - degenerate random variable Z with*

$$c_n(X_1 + ...X_n) - d_n \xrightarrow{d} Z \tag{47}$$

if and only if Z is stable, in which case $c_n = n^{-1/\alpha_Z}$ for some $0 < \alpha_Z \leq 2$.

Remark 42 *Recall from Chapter 1 that for the normalized i.i.d. Random Walk volatility we have the stability exponent $\alpha_Z = 2$ and thus the normalizing constant $c_n = n^{-0.5}$.*

This GCLT implies that the only possible distributions with unique *domains of attraction* are stable distributions!

Definition 43 *A random variable X is in the* ***domain of attraction (DOA)*** *of Z if and only if there exist constants $c_n > 0, d_n \in \mathbb{R}$ with*

$$c_n(X_1 + ...X_n) - d_n \xrightarrow{d} Z \tag{48}$$

where $X_1, X_2, ...$are i.i.d. distributed copies of X.

By $DOA(Z)$ we will indicate the set of all random variables that are in the domain of attraction of Z. As Mittnik, Rachev and Paolella (1998) and Rachev and Mittnik (2000) properly emphasize, a DOA is an important and, perhaps, even desirable property. Loosely speaking, any distribution in the DOA of a specified stable distribution has properties which are close to the properties of the stable distribution. These authors reason that, therefore, decisions will, in principle, not be affected by adopting an "idealized" stable distribution instead of using the true empirical distribution. Furthermore, they claim that it is it is possible to check whether or not a distribution is in the DOA of a stable distribution by examining only the tails of the distribution, since only these parts specify the DOA properties of the distribution. The stability, or continuity, of the adopted distribution is valid for any distribution with the appropriate tail.[5]

6 Examples of Closed Form Stable Distributions

Although there are closed forms for the characteristic functions of all stable distributions, there are no closed formulas for the distribution densities and unctions for all but a few stable distributions, like for the Gaussian, Cauchy and Lévy distributions we encountered in Chapter 1. Here are their respective special closed form densities.

[5] Of course, this reasoning only applies when the distribution has one unique DOA around its mode. But as we commented earlier, empirical return distributions may have more than one mode, and thus more than one DOA, when multiple price equilibria coexist in a turbulent market.

Definition 44 ***Gaussian distributions****: $X \sim N(\mu, \sigma^2)$ if it has density*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})}, -\infty < x < +\infty \tag{49}$$

The normal distribution has an *infinite support* (infinite domain) on the whole real line from $-\infty$ to $+\infty$. In terms of Zolotarev's formula, $Z(2, \beta) \stackrel{d}{=} Z(2, 0) = N(0, 2)$.

Definition 45 ***Cauchy distributions****: $X \sim Cauchy(\gamma, \delta)$ if it has density*

$$f(x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2}, -\infty < x < +\infty \tag{50}$$

The Cauchy distribution has also an *infinite support* (infinite domain) on the whole real line from $-\infty$ to $+\infty$. In terms of Zolotarev's formula, $Z(1, 0) = Cauchy(1, 0)$.

Remark 46 *It can be easily shown that a Cauchy variable X, which has a stable distribution and is almost certainly finite, has an infinite variance and an infinite mean! (Cf. Los, 2005c)*

Definition 47 ***Lévy distributions****: $X \sim Lévy(\gamma, \delta)$ if it has density*

$$f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\delta)^{3/2}} e^{(-\frac{\gamma}{2(x-\delta)})}, \delta < x < \infty \tag{51}$$

The Lévy distribution has only *support in the positive domain* on the half line from δ to ∞. In terms of Zolotarev's formula, $Z(0.5, 0) = Lévy(1, 0)$.

Both Gaussian and Cauchy distributions are symmetric, bell-shaped curves, but the Cauchy distribution has much heavier tails than the Gaussian distribution, *i.e.,* the pricing events further away from the mean are more likely to occur than under a Gaussian distribution. This is the reason why stable return distributions other than the Gaussian are called *heavy tailed.* In contrast to both the Gaussian and Cauchy distributions, the Lévy distribution is highly skewed, with all the probability concentrated on $x > 0$, and it has even a heavier tail than the Cauchy distribution. General stable distributions allow for varying degrees of tail heaviness and varying degrees of skewness.

Table 1 demonstrates clearly the heavier tail probabilities of the Cauchy and Lévy distributions, compared to the tail probabilities of the Gaussian distribution.

Other than the Gaussian distribution, the Cauchy distribution, the Lévy distribution, and the reflection of the Lévy distribution, there are no known closed form expressions for general stable

c	**Normal**	**Cauchy**	**Lévy**
0	0.5000	0.5000	1.0000
1	0.1587	0.2500	0.6827
2	0.0228	0.1476	0.5205
3	0.001347	0.1024	0.4363
4	0.00003167	0.0780	0.3829
5	0.0000002866	0.0628	0.3453

Table 1: Comparison of Tail P(X>c) Probabilities

densities and it is even unlikely that any other stable distributions, than the ones mentioned, have closed forms for their densities. Although there is no closed formula for the normal distribution function, there are numerical tables and accurate numerical computer algorithms for the standard distribution function (*e.g.,* Mantegna, 1994). Financial analysts use such computed numerical values in normal models, *e.g.,* for the valuation of Black - Scholes options. Similarly, we have now also computer programs (like Nolan's STABLE.EXE software, available from his web site:

http://www.cas.american.edu/~jpnolan/stable.html)

to compute quantities of interest for stable distributions. So, it is possible to use such programs to solve empirical problems, like the valuation of the risk in various assets and derivatives. Precise tabulations of the skewed Stable distributions can be found in McCulloch and Panton (1997, 1998)

7 Stable Parameter Estimation and Diagnostics

Nolan (1999b) discusses in detail the methods for estimating stable parameters from empirical data and the methods for model verification, *i.e.,* how to assess whether the estimated stable parameters actually do a good job describing the empirical data.

7.1 Parameter Computation

There are basically four methods of distributional parameter identification:

(1) The computation of α_Z, β, γ, and δ is usually performed by minimizing a distance function. (Mittnik, Rachev and Polella, 1998)), like the Kolmogorov Distance.

Definition 48 ***Kolmogorov Distance (KD):***

$$\rho = \sup_{x \in \mathbb{R}} \left| F(x) - \widehat{F_{\mathbf{S}}}(x) \right| \tag{52}$$

where $F(x)$ is the empirical distribution and $\widehat{F}(x)$ the estimated distribution function for a particular parametrization $\mathbf{S}$.

This method is used mostly when one is concerned about kurtosis.

(2) Alternatively, one maximizes numerically the so-called likelihood function of stable distributions.

Definition 49 ***Likelihood Function (ML)***

$$L(\alpha_Z, \beta, \gamma, \delta) = \prod_{t-1}^{T} \mathbf{S}(\alpha_Z, \beta; 0) \left(\frac{x - \delta}{\gamma} \right) \frac{1}{\gamma} \tag{53}$$

which is maximized with respect to the four parameters $\alpha_Z, \beta, \gamma, \delta$.

Under the i.i.d. assumptions the resulting estimates are consistent and asymptotically normal with the asymptotic covariance matrix given by the inverse of the usual Fisher information matrix, *i.e.,* the matrix of second derivatives of the Likelihood Function evaluated at the ML point values (Mittnik et al., 1996).

(3) The oldest method is the *quantile/fractile method* of Fama and Roll (1971) for the symmetric case and McCulloch (1986) for the general case. This method tries to match certain data quantiles with those of stable distributions.

(4) But the scientifically most convincing method is to compute the moments directly from the *empirical characteristic function*, as is recommended by Nolan (1999a and b).

Nolan (1999b) provides many valuable applications of simulated data, exchange rate data, CRSP stock prices, Abbey National share prices, radar noise, ocean wave energy, and simulated unstable data. Here, we reproduce Nolan's example of fitting stable distributions to exchange rate data.

Example 50 *Daily exchange rate data for* 15 *different currencies were recorded (in U.K. pounds) over a 16 year period (2 January 1980 to 21 May 1996). The data was logarithmically transformed by*

$$\begin{aligned} y(t) &= \Delta \ln X(t+1) \\ &= \ln X(t+1 - \ln X(t) \end{aligned} \tag{54}$$

giving $T = 4,274$ transformed data observations. The transformed data were fit with a stable distribution, using the maximum Likelihood Function method. The results, with 95% confidence intervals, are given in Fig 6. These empirical data are clearly not Gaussian: the heavy tails in

country	α	β	γ	δ
Australia	1.479 ± 0.047	0.033 ± 0.080	0.00413 ± 0.00013	-0.00015 ± 0.00022
Austria	1.559 ± 0.047	-0.119 ± 0.092	0.00285 ± 0.00009	0.00014 ± 0.00015
Belgium	1.473 ± 0.047	-0.061 ± 0.080	0.00306 ± 0.00010	0.00009 ± 0.00016
Canada	1.574 ± 0.047	-0.051 ± 0.093	0.00379 ± 0.00012	0.00004 ± 0.00020
Denmark	1.545 ± 0.047	-0.119 ± 0.090	0.00272 ± 0.00008	0.00022 ± 0.00014
France	1.438 ± 0.047	-0.146 ± 0.078	0.00245 ± 0.00008	0.00028 ± 0.00013
Germany	1.495 ± 0.047	-0.182 ± 0.085	0.00244 ± 0.00008	0.00019 ± 0.00013
Italy	1.441 ± 0.046	-0.043 ± 0.076	0.00266 ± 0.00009	0.00017 ± 0.00014
Japan	1.511 ± 0.047	-0.148 ± 0.086	0.00368 ± 0.00012	0.00013 ± 0.00019
Netherlands	1.467 ± 0.047	-0.167 ± 0.081	0.00244 ± 0.00008	0.00016 ± 0.00013
Norway	1.533 ± 0.047	-0.070 ± 0.088	0.00253 ± 0.00008	0.00005 ± 0.00013
Spain	1.512 ± 0.047	-0.007 ± 0.083	0.00268 ± 0.00008	0.00012 ± 0.00014
Sweden	1.517 ± 0.047	-0.081 ± 0.085	0.00256 ± 0.00008	0.00006 ± 0.00013
Switzerland	1.599 ± 0.047	-0.179 ± 0.100	0.00295 ± 0.00009	0.00014 ± 0.00016
United States	1.530 ± 0.047	-0.088 ± 0.088	0.00376 ± 0.00012	0.00009 ± 0.00020

Figure 6: Identification of the four parameter of the Zolotarev parametrization of stable FX distributions.

the data causes the sample variance to be large, and the Gaussian fit poorly describes both the center and the tails of the distribution. Although the stable distribution fit does a reasonable job of describing the FX rate data, it never captures the extreme "peakedness " of FX rate data. With the stability or tail exponent $1 < \alpha_Z < 2$, we must conclude that although the mean of these daily FX returns exists, the variance is undefined and thus also the fourth moment. In other words, the values of the variance and of the kurtosis of each of the FX series do not converge, but they "wander" aimlessly when more data are aggregated.[6] *In other words, the volatilities of these FX data are undefined and, therefore, cannot be priced or hedged by the usual option pricing or hedging formulas! The currency with the heaviest tails ($\alpha = \alpha_Z = 1.441$) and thus most extreme outlying values was the Italian lire, while the one with the lightest tails ($\alpha = \alpha_Z = 1.530$) was the Swiss Franc. Notice also that the Australian distribution was the only one in this period with a slight positive skewness ($\beta > 0$), indicating the depreciation of the Australian dollar versus the U.K. Pound. All other currencies showed negative skewness ($\beta < 0$) and thus appreciated versus the U.K. Pound over the length of this 16 year period. For a similar, but earlier, set of daily foreign exchange data and their statistical properties, see Hsieh (1988).*

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[6] *Interestingly, Nolan's (1999b) and Mittnik et al.'s (1999) measurements using the ML method and the implied conclusion regarding the nonconvergence of the variance of FX returns appears to conflict with the measurements by Müller, Dacorogna and Pictet (1998). The Nolan - Mittnik measurements of α_Z are between 1.44 and 1.78. Müller, Dacorogna and Pictet use so - called bootstrap and jackknife methods and find values for the tail exponent α_Z between 3 and 5 for various US Dollar exchange rates for various time intervals, suggesting that the second moment does converge. This inconsistency of the respective empirical measurements is not easily resolved. But my own α_Z measurements are compatible with the Nolan - Mittnik measurements (Cf. Chapter 8, Section 8.42). Moreover the nonconvergence of the variance has been observed by myself and several other researchers. Perhaps, Müller, Dacorogna and Pictet inverted the exponent and actually measured the homogeneous Lipschitz $\alpha_L = H$ (Los, 2005b). In that case their measured tail exponent is $1/3 = 1.33 \leq \alpha_Z \leq 2.00 = 1/5$ and, thus, much more in agreement with the (somewhat tighter) Nolan - Mittnik measurements of $1.44 \leq \alpha_Z \leq 1.78$. Both the physics and the financial literature is full of confusion between the homogeneous or uniform Lipschitz α_L (= Hurst - Hölder exponent) and Zolotarev's tail or stability exponent $\alpha_Z = \frac{1}{\alpha_L}$, since most authors don't bother to index the particular α!*

8 The Degree of Stability of Price Diffusion

Let's return now, for a moment to our original concern: the stable distributions of financial market rates of return, as generated by a general price diffusion equation. For example, we know that the volatility of the lognormal financial price distribution, derived from the geometric Brownian asset return motion and used to model Black-Scholes (1973) option pricing, scales according to $T^{0.5}$, since $\sigma_T = \sigma_\tau T^{0.5}$ As we have seen in Section 3, this implies that the Black-Scholes model assumes that there is no difference between actual clock time and trading time and that the unit of actual clock time is the same as the unit of trading time, or $\tau = 1$.

But Calvet, Fisher and Mandelbrot (1997) propose a more general configuration, where a distinction can be made between actual clock time and trading time, where the unit of clock time is not necessarily equal to the unit of trading time, $\tau \neq 1$, and where there may not even be a uniform trading time. The trading time units may be of unequal, fractional length and not uniformly distributed: $0 < \tau < 1$. In their Multifractal Model for Asset Returns (MMAR) the volatility of the price return distributions scales according to $T^{\frac{1}{\alpha_Z}}$ (Calvet and Fisher, 2002). Thus the Zolotarev stability exponent α_Z measures not only the degree of the time-scaling of the financial market return distributions produced by a price diffusion, but also the degree of their kurtosis, stability, or lack of stationarity. Since the MMAR is now considered the best theoretical and empirical model of efficient financial market price diffusion - it captures both the empirically observable Long Memory phenomenon and it is arbitrage-free (it produces a martingale pricing time series) - it is essential that Zolotarev's uniform stability exponent α_Z is accurately measured.

9 Conclusion: Diagnostics of a Skeptic

In principle, it should be no surprise that one can fit the financial market return data better with the four parameter (Zolotarev) stable distribution model than with the two parameter Gaussian model, since there are two more degrees of freedom available. But the relevant scientific question is whether or not the fitted stable distribution actually describes the empirical financial pricing data well. In models of financial data, like rates of investment, stock prices or foreign exchange rates, we're interested in the whole distribution, and not only in the tails, even though risk sensitive financial managers may want to focus on the extreme values in these tails (Hols and DeVries, 1991).

An important caveat is that non - Gaussian stable distributions are heavy - tailed distributions, but most heavy - tailed distributions are not stable. In fact, *it is not possible to directly prove that a given empirical data set is or is not stable!* (Pincus and Kalman, 1997) Therefore, the elegance of the stable distributions may turn out to be irrelevant for empirical financial research, Gaussian or not, because of changes in the financial and economic situations over time that produce nonstationary, unstable time series, for which no definite stable distributions exist (Los, 2005c).

Even testing for normality or "Gausianity" is still an active field of research and not as "cut and dried" as standard statistics and, in particular, econometrics textbooks (even in specialized textbooks such as Gourieroux and Jasiak, 2001) make it out to be! The best we can do at this point is to determine whether the financial market time series are consistent with the hypothesis of distributional stability. But all these tests will fail if the departure from stability is small or occurs in an unobserved part of the range of observations. For example, it is found that because of the curvature (reflecting the degree of kurtosis) in the distribution functions, it is very difficult to compare the fitted and he empirical density functions visually, especially with respect to the (important) tails, where observations are, per definition, scarce.

10 APPENDIX: Software

For more detailed information on stable distributions, papers and software, see John Nolan's expert web site at the American University:

http://academic2.american.edu/~jpnolan/stable/stable.html

where you can find STABLE.EXE (900 KB) which calculates stable densities, cumulative distribution functions and quantiles. It also includes stable random number generation and maximum likelihood estimation of stable parameters using a fast 3−dimensional cubic spline interpolation of stable densities. STABLE.TXT (16 KB) provides the description of the STABLE.EXE program.

Huston McCulloch of Ohio State University provides a stable distribution random number generator in the form of MATLAB® M - Files: STABRND.M:

http://www.econ.ohio-state.edu/jhm/jhm.html

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