Exponentially damped Lévy flights, multiscaling and slow convergence in stock markets.

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Abstract

We have previously examined the role of autocorrelations in the sum of stochastic variables together with the existence of scaling power laws [1]. Here we employ such an approach to analyze sluggish convergence [2] in data coming from the S& P500 index. We also employ our suggested exponentially damped Lévy flight [3] to assess the multiscaling properties in the data.

Key words: Lévy distributions; Econophysics; Multiscaling. 05.40.+j;02.50.-r

1 Introduction

We have previously put forward a connection between autocorrelation and the presence of power laws [1]. We show that the typical scaling properties of abruptly truncated Lévy flights (TLFs) [2] are linked to autocorrelation in data. Here we move up and show how the property of sluggish convergence to a Gaussian [2] can be explained by particular features of nonlinear autocorrelations.

Owing to its sharp truncation, the characteristic function (CF) of the TLF is not infinitely divisible [4–6]. To make the CF infinitely divisible again, an

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exponentially damped Lévy flight (EDLF) has been suggested by some of us [3] and used to analyze foreign exchange rate data [7]. This paper blends such novel theoretical results with an analysis of actual data coming from the S& P500 index. We consider daily closing values for the period from January/1980 to December/2002. The structure of the paper is as follows: section 2 examines the problem of convergence to the Gaussian regime. Section 3 shows the EDLF and its multiscaling properties, and section 4 concludes.

2 Slow convergence and nonlinear correlations

Consider the sum of random variables x_i of zero mean given by $\sum_{i=1}^{n} x_i$. For reduced variables, the CF $\psi(z)$ of a process with finite second moments can be written as $\psi(z) = e^{-z^2(1+w(z))/2}$ [8]. If $w(z) = 0, \forall z$ one gets the CF of the Gaussian distribution. In [1] we showed how a scaling power law in the probability of return to the origin ($P(0) = 0$) emerges if the standard deviation is also governed by a power law and $w_n(z)$ has some special properties. We employ such an approach to exchange rate data [1] and get results that are similar to the originally suggested by Mantegna and Stanley [2]. Here we move up to extend the methodology and examine sluggish convergence to a Gaussian. The statistical moments of order p of x_i and S_n are μ_{ip} and ν_{ip} respectively. A nonlinear autocorrelation can be tracked by

$$< p_1 p_2 \cdots p_k >_n = \sum_{i_1 \cdots i_k}^{n} (< x_{i_1}^{p_1} \cdots x_{i_k}^{p_k} > - < x_{i_1}^{p_1} > \cdots < x_{i_k}^{p_k} >) \qquad (1)$$

where $p_1 p_2 \cdots p_k$ are positive integers, and $i_1 \neq i_2 \neq \cdots \neq i_k$. Our task is to analyze the contribution of such nonlinear terms in the process of convergence. If we take $w(z) = w_R + \imath w_I$, and further carry out an expansion in series, it can be shown that

$$w_R = -\frac{1}{12}(\mu^4 - 3)z^2 + O(z^4) = -\frac{K}{12}z^2 + O(z^4) \qquad (2)$$

$$w_I = \frac{1}{3}\mu^3 z + O(z^3) = \frac{Sk}{3}z + O(z^3) \qquad (3)$$

where the kurtosis K and skewness Sk of the process are made explicit; actually they are the leading terms in the series expansion of $w(z)$. For independent and identically distributed (IID) variables, K and Sk approach zero at a pace governed by ratios $K \to 1/n$ and $Sk \to 1/\sqrt{n}$. So according to the central limit theorem (CLT) [8],$w(z) \to 0$, and the Gaussian regime is reached. Now let us show how the nonlinear correlations in Eq. (1) can prevent the Gaussian regime to be reached, which may also explain the slow convergence observed in real data [2]. If $\Psi(z) = e^{-z^2(1+\Omega(z))/2}$ is the CF of the reduced sum variable, then from Eqs. (2) and (3) it can be shown that

$$\Omega_R = -\frac{1}{12}K_n z^2 + O(z^4), \frac{1}{12}K_n = \sum_{i=1}^{n}\frac{\mu_{i2}^2\mu_{i4}}{12\nu_{n2}^2} + \frac{1}{4}(1-\frac{\sigma_{n2}^2}{\nu_{n2}^2}) - \tag{4}$$

$$\frac{1}{12}\frac{< 1111 >_n +6 < 112 >_n +4 < 13 >_n +3 < 22 >_n}{\nu_{n2}^2} \tag{5}$$

$$\equiv \frac{1}{12}K_n^0 + K_n^1 + K_n^2 \tag{6}$$

and

$$\Omega_I = \frac{1}{3}Sk_n z + O(z^3), \tag{7}$$

$$\frac{1}{3}Sk_n = \sum_{i=1}^{n}\frac{\mu_{i2}^{3/2}\mu_{i3}}{3\nu_{n2}^{3/2}} + \frac{< 111 >_n +3 < 12 >_n}{\nu_{n2}^{3/2}} \equiv \frac{1}{3}Sk_n^0 + Sk_n^1 \tag{8}$$

where $\sigma_{np} = \mu_{1p} + \cdots \mu_{np}$. Terms K_n^0 and Sk_n^0 are standard in the sum of IID variables; they move toward zero according the CLT. But if at least one of the terms K_n^1,K_n^2 or Sk_n^1, which involve the nonlinear autocorrelations, is greater than nil, then $\Omega(z) = \Omega_R + \imath\Omega_I$ falls greater than nil. Here the process cannot reach the Gaussian regime. It is worth to note that slow convergence takes place even for short-range correlated processes, provided that either K_n^2 or Sk_n^1 departures from zero (K_n^1 has only the usual autocorrelation function $< 11 >$).

The sluggish convergence can be illustrated with data coming from daily variations of the S& P500 index. Fig. 1(a) displays the behavior of the kurtosis for daily returns of the index. The process remains bounded at nonzero values for several days and goes to Gaussian regime in an ultra slow pace. Compare with previous works dealing with the S& P 500 [2] and the Ibovespa index (Brazilian stock market) [9], where it is claimed that Gaussian regime is reached after 20-30 days. From our results, the crossover time seems to be greater.

The kurtosis and skewness does not decay to zero (at least in the time window considered). Fig. 1(b) shows K_n^2, which involves nonlinear correlations of fourth order. Thus splitting up the kurtosis makes it clear how K_n^1 (not shown) and K_n^2 contribute to the nonzero kurtosis, thereby preventing the Gaussian regime to be reached. And that happens thanks to the presence of the nonlinear autocorrelations. The same behavior is observed for the skewness (not shown).

3 Exponentially Damped Lévy Flights and Multiscaling

Although our previous approach allows one to study a number of statistical properties, it does not provide a model on its own for a stochastic process. The nonstable, finite-variance TLF itself is such a model. Other modified versions of the TLF are the smoothly truncated Lévy flight and the gradually truncated Lévy flight [4–6]. The sharp cutoff of the TLF renders its CF not infinitely divisible. Now we present our EDLF [3], for which the CF is infinitely divisible. Consider a symmetric Lévy distribution of index α, that is $L(S_{\Delta t}) = \frac{1}{\pi}\int_0^\infty e^{-\gamma \Delta t q^\alpha} \cos(q S_{\Delta t})$. Let $z_{\Delta t} = S_t - S_{t-\Delta t}$ (here we employ Δt instead of n). The EDLF is defined as $P(\Delta t) = \eta L(z_{\Delta t}) f(z_{\Delta t})$, where η is a normalizing constant, and $f(z_{\Delta t})$ is the change carried out on the distribution:

$$f(\|z_{\Delta t}\|) = \begin{cases} 1, \|z_{\Delta t}\| < l_c \\ (\Delta t^{-1/\alpha}\|z_{\Delta t}\| + \vartheta)^{\beta_1} e^{H(z_{\Delta t})}, l_c < \|z_{\Delta t}\| < l_{max} \\ 0, \|z_{\Delta t}\| > l_{max} \end{cases} \tag{9}$$

where $H(z_{\Delta t}) = \lambda_1 + \lambda_2[1 - \|z_{\Delta t}\|/l_{max}]^{\beta_2} + \lambda_3(\|z_{\Delta t}\| - l_c)^{\beta_3}$, and $\vartheta, \lambda_1, \lambda_2 \leq 0, \beta_1, \beta_2$ and β_3 are parameters describing the deviations from the Lévy stable distribution, l_c is the size step at which the distribution begins to deviate from the Lévy, and l_{max} is the size step at which an abrupt truncation is carried out. The TLF obtains if $f(z_{\Delta t}) = 0, (\|z_{\Delta t}\| > l_c)$, and $f(z_{\Delta t}) = 1, (\|z_{\Delta t}\| \leq l_c)$.

One striking difference between the TLF and the EDLF is the presence of multiscaling in the latter, which can also emerge in other modified Lévy flights [4–6].

By scaling $z_{\Delta t}$ together with the truncation parameters, a distribution can be collapsed onto the $\Delta t = 1$ distribution. Power laws for both the K^{th} absolute moment and norm of the CF of the EDLF can then be found [7].

To examine multiscaling, let $< \|z_{\Delta t}\| >^K = \frac{1}{n}\sum_{t=1}^n \|S_t - S_{t-\Delta t}\|^K$, and $\|\tilde{z}\|^K = \frac{1}{n}\sum_{t=1}^n \|S_t - S_{t-1}\|^K$ be the K^{th} sample mean of lagged absolute values of S_t at time intervals Δt and 1 respectively. For moments that are low enough (such as $0 < K < a$), $P(z_{\Delta t})$ is expected to be approximated by $L(z_{\Delta t})$, which in turn does not depend on the truncation parameters. Thus we expect $< \|z_{\Delta t}\|^K > \approx \Delta t^{K/\alpha} < \|\tilde{z}\|^K >$ to hold for lower moments. This means that ratio $R(K, \Delta t) =< \|z_{\Delta t}\|^K > / < \|\tilde{z}\|^K >$ scales with Δt as $R(K, \Delta t) = \Delta t^{K/\alpha}$. Here multiscaling is related to nonlinear behavior in the partition function $R(K, \Delta t)$.

The norm of the CF ($\phi(K)$)can also be used to assess parameter γ by taking into account the same assumption that $P(z_{\Delta t}) \approx L(z_{\Delta t})$ for low values of $\|K\|$.

Since $\phi(K) \equiv E[e^{iKz}] = E[\cos(Kz) + \imath \sin(Kz)] = E[\cos(Kz)] + E[\imath \sin(Kz)]$ then the squared norm of the CF gives $\|\phi\|^2 = E^2[\cos(Kz)] + E^2[\sin(Kz)]$. For some K and Δt, $\|\phi(K)\|$ can be estimated by $\|\tilde{\phi}(K)\|^2 =< \cos(Kz) >^2 + < \sin(Kz) >^2$. By assuming that $\ln[\phi(K)] \approx -\gamma \Delta t \|K\|^\alpha$ for $0 < K < \alpha$, the estimated norm in logs of the CF is $\ln \|\phi_L(K)\|$ and then we can expect that $\ln[\tilde{\phi}_L(K)] \approx -\gamma \Delta t \|K\|^\alpha$

Fig. 2(a) displays sample ratios $R(K, \Delta t)$ for K ranging from 0.0 (bottom lines) to $K = 4.0$ (top lines) at intervals of 0.1 in log-log plots of data coming from daily returns of S& P 500. They exhibit power law dependence on Δt. By fitting $\ln R(K, \Delta t) = \xi \ln \Delta t$ for every K, we get the corresponding scaling exponents shown in Fig. 2(b). The curves show linear dependence on K, for $0 < K < \alpha$. However scaling breaks down after $K > \alpha$, and a nonlinear behavior (multiscaling) steps in. Fig. 3(a) displays the sample logarithm of the absolute CF versus Δt for K ranging from 0.00 (upper lines) to 0.025 (bottom lines) at intervals of 0.0005. A power law dependence on Δt is clearcut for the S&P500. By fitting $\ln \|\tilde{\phi}_L(K)\| = \zeta \Delta t$ for every K, the estimated values of ζ versus $\|K\|^\alpha$ are plotted in Fig. 3(b). An approximate linear behavior for all $\kappa(\alpha) = \|K\|^\alpha$ tracks mere single scaling, and a linear behavior for initial values of $\kappa(\alpha) < \alpha_0$ followed by a nonlinear pattern after $\kappa(\alpha) > \alpha_0$ captures multiscaling.

4 Conclusions

This paper shows that nonlinear autocorrelations may prevent convergence to the Gaussian regime. It enlarges previous results obtained by some of the authors [1]. The slow convergence is illustrated with data from the S&P500 index.

We further employ a modified version of the truncated Lévy flight whose characteristic function is infinitely divisible, namely our exponentially damped Lévy flight. Then we examine the multiscaling property of our distribution, and illustrate it with the S&P500.

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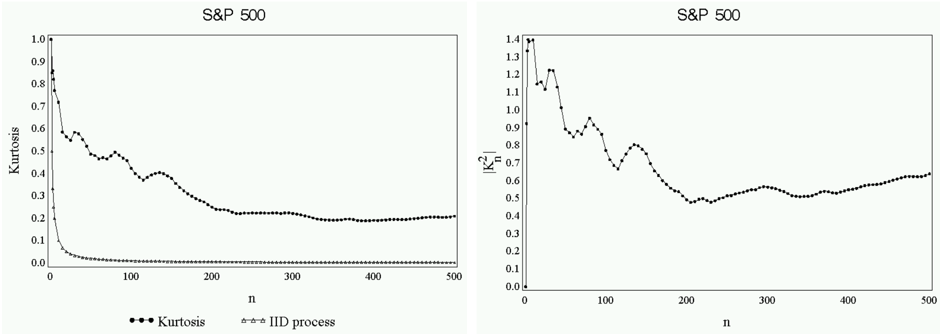

Fig. 1. (a) Behavior of the kurtosis for daily returns of the S&P 500 index (black circles), and an IID process (triangles) (n =days). (b) Contribution of K_n^2.

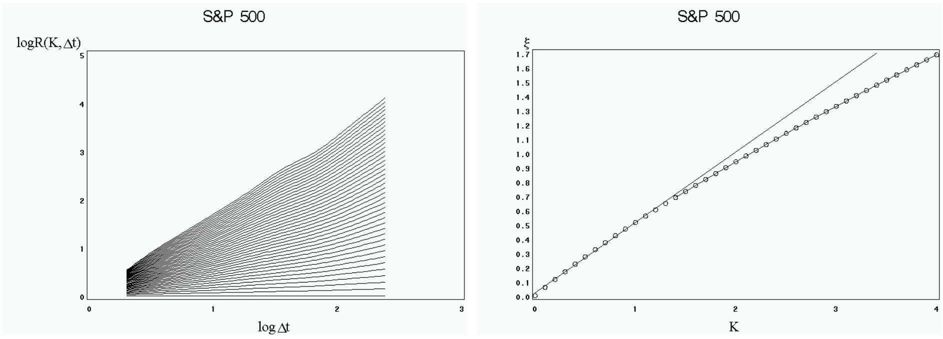

Fig. 2. (a) Estimated $R(K, \Delta t)$ for several K. (b) Estimated multiscaling exponents ξ.

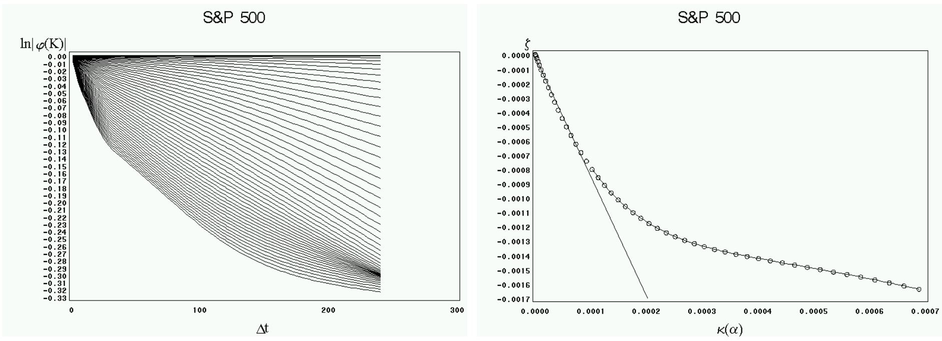

Fig. 3. (a) Estimated ratios $\ln \|\phi_L(K)\|$ for several K. (b) Estimated multiscaling exponents ζ.

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