Utility Maximization in Imperfected Markets

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Abstract. We analyze a problem of maximization of expected terminal wealth and consumption in markets with some "imperfection", such as constraints on the permitted portfolios, labor income, or/and nonlinearity of portfolio dynamics. By using general optional decomposition under constraints in multiplicative form, we develop a dual formulation. Then, under some conditions imposed on the model setting and the utility functions, we are able to prove an existence and uniqueness of an optimal solution to primal and to the corresponding dual problem by convex duality.

Key words: Consumption and Investment Optimization, Duality Theory, Convex and State Constraints, Utility Maximization, Optional Decomposition

JEL Classification: G11

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1 Introduction

Basic problems of mathematical finance are the problems of pricing, hedging or optimizing some portfolio choices, which could be formulated as the optimization problem of maximizing the expected value of some concave objective (eventually state-dependent) utility functions. The problems can be attacked by the stochastic optimal control methods as, for instance, in the papers of Merton (1971) [41], Duffie, Flemming, Soner and Zariphopoulou (1997) [17], or by a modern, more powerful and elegant method: the duality approaches. The difference is that, while the optimal control methods are wedded to the dynamic programming Hamilton-Jacobi-Bellman equation and based on the requirement of Markov state processes, the duality techniques, rather then rely on the Hamilton-Jacobi-Bellman equation, use the stochastic duality theory and permit us to deal with more general and non-markovian processes. The key point in this method is the duality relation of the set of self-financed wealth processes and the set of local supermartingale measures for the self-financed wealth process discounted by the numéraire.

Duality approaches have been used with success in treating portfolio optimization problems for incomplete financial markets in a continuous-time diffusion model such as in Karatzas, Lehoczky, Shreve and Xu (1991) [32], or in a more general framework, where the asset prices are semimartingales, as it is showed in series of papers of Kramkov and Schachermayer (1999 - 2001) [40]. The further extension to the case of constraints on the proportion of portfolio choice appears in Shreve and Xu (1992) [50], Cvitanic and Karatzas (1992) [7]. The extension to the case of constraints imposed on the amount addressed by Cuoco and Cvitanic (1998) [10], Cuoco (1997) [11].

Recently, Pham and Mnif (2002) [47] investigated the general structure of optimization financial problems with the presense of the so-called *liquidity* (or *American*) *constraints*. They considered the financial framework in a semimartingale setting, which is represented by the Föllmer-Kramkov optional decomposition under constraints in *additive form* (see, Föllmer and Kramkov (1997) [22]). Hence, it is general enough to incorporate many financial models, such as with constrained portfolios, labor income as well as large investor.

Motivated by the work of Pham and Mnif (2002), Long (2003) [43] considers the case, where the state processes have the Föllmer-Kramkov optional decomposition under constraints in *multiplicative form* (see, Föllmer and Kramkov (1997) [22]). Since the problem considered in Long (2003) is to optimize the expected utility of terminal wealth, so the problem is the simplest one in terms of objective. It is undoubtedly an important goal to generalize the study of optimal investment and consumption problems to the semimartingale setting used by Long (2003) [43]. This paper aims to solve the mentioned problem.

Like the model proposed by Pham and Mnif (2002), our formulation is sufficiently general to include as special cases the problems with constrained proportion portfolios, random endowment and large investor, as well as with the existence of labor income, which were considered in Cvitanic and Karatzas (1992, 1993) [7], El Karoui and Quenez (1996) [20], Cuoco and Cvitanic (1998) [10], El Karoui and Jeanblanc-Piqué (1998) [18], Rogers (2001) [45] and Klein and Rogers (2001) [37]. Using the general optional decomposition under constraints in a multiplicative form of Föllmer and Kramkov [22], we provide the duality characterization of the state processes in terms of a set of suitable probability measures and a term arising from the convex constraints; this set of probability measures is the dual set associated to the convex constraints on the family of state processes.

With this setting, under some conditions imposed on the model setting and on the utility functions, namely the asymptotic elasticity of the utility function is strictly less than 1, we are able then to prove an existence of an optimal solution to the original and dual problem.

The outline of the paper is organized as follows. Section 2 recalls the general framework of a financial model in Long (2003) [43]. After formulizing the problem in Section 3, in Section 4 we set up and analyze the properties of the dual set, which is the set of equivalent local martingale measures for state processes, associated with the term arising from the convex constrants. In Section 5, we provide the dual and primal sets in an abstract version and analyze the properties of the abstract setting. The existence and uniqueness of an optimal solution to the original and dual problem is given in Section 6.

2 The Model Setting

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ denote a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the "usual" conditions, here $T \in \mathbf{R}_+$ is a fixed time horizon and we assume that $\mathcal{F}_0$ is trivial. Except for processes which appear as integrand of stochastic integrals, all processes considered in the sequel are assumed to be real-valued, to have right-continuous paths with left limits (*càdlàg*), and to be adapted with respect to the given filtration; in particular they are all optional.

For the theory of stochastic integration we refer to [16], [49] and [28]. The stochastic integral of a predictable process π with respect to a semimartingale X will be denoted by $\int \pi dX$ or $\pi \bullet X$. We denote by $\mathbf{L}(X)$ the space of all predictable processes integrable with respect to X. The Émery distance between two semimartingales X and Y is defined as:

$$D(X,Y) = \sup_{|\pi|\leq 1} \left(\sum_{n\geq 1} 2^{-n}\mathbf{E}[\min(|(\pi \bullet (X-Y)_n)|, 1)] \right),$$

where the supremum is taken over the set of all predictable processes π bounded by 1. The corresponding topology is called the *semimartingale* topology.

In the sequel, for completeness, we recall some definitions and assumptions of the model setting from Long (2003) [43].

Let R be a $\mathbf{R}^d$-valued semimartingale in $(\Omega, \mathcal{F}, \mathbf{P})$. We prescribe a convex subset Π of $\mathbf{L}(R)$ containing the zero element and convex in the following sense: for any predictable process $\zeta \in [0,1]$ and for all $\pi^1, \pi^2 \in \Pi$ we have:

$$\zeta\pi^1 + (1-\zeta)\pi^2 \in \Pi.$$

We consider a family $\{\widetilde{G}^\pi : \pi \in \Pi\}$ of adapted processes with finite variation, null at 0 and satisfying the concavity property:

$$\widetilde{G}^{\zeta\pi^1+(1-\zeta)\pi^2} - \zeta \bullet \widetilde{G}^{\pi^1} - (1-\zeta) \bullet \widetilde{G}^{\pi^2} \in \mathcal{I}, \tag{1}$$

where $\mathcal{I}$ is the set of all (optional) nondecreasing adapted processes with initial value 0 and null at 0.

Now we consider the following family:

$$\widetilde{\mathbb{X}}_0 = \left\{\pi \bullet R + \widetilde{G}^\pi\right\}$$

We shall make the following standing assumption:

Standing Assumption 2.1 *Under the condition (1), the set $\widetilde{\mathbb{X}}_0$ is closed for semimartingale topology.*

Given $\widetilde{X}^0 \in \widetilde{\mathbb{X}}_0$, we define the set

$$\widetilde{\mathbb{X}}^b = \left\{\widetilde{X} - \widetilde{X}^0 : \widetilde{X} \in \widetilde{\mathbb{X}}_0 \text{ and } X - \widetilde{X}^0 \text{ is locally bounded from below}\right\}$$

so that $\widetilde{\mathbb{X}}^b$ is locally bounded from below, closed for the semimartingale topology null at 0, and containing the constant process 0.

Remark 2.1 *Under the relation (1), the family of semimartingales $\widetilde{\mathbb{X}}_0$ is a predictable convex set in the sense of Föllmer and Kramkov (1997)[22], i.e., for $\widetilde{X}^i \in \widetilde{\mathbb{X}}_0$ for $i = 1, 2$, and for any* *predictable process $\zeta \in [0,1]$ we have:*

$$\zeta \bullet \widetilde{X}^1 + (1-\zeta) \bullet \widetilde{X}^2 \in \widetilde{\mathbb{X}}_0 - \mathcal{I}$$

Now let us introduce the set $\mathcal{P}(\widetilde{\mathbb{X}}^b)$ of all nonnegative $\mathbf{P}$-local martingales Z with $Z_0 = 1$ such that there exists a process $A \in \mathcal{I}_p$ – the set of nondecreasing predictable processes, null at 0 – satisfying

$$Z(\widetilde{X}^b - A) \text{ is a } \mathbf{P}\text{-local supermartingale for any } \widetilde{X}^b \in \widetilde{\mathbb{X}}^b. \tag{2}$$

The next definition of the upper variation process is adopted from the one in Föllmer and Kramkov (1997) [22].

Definition 2.1 *The upper variation process of $\widetilde{\mathbb{X}}^b$ corresponding to $Z \in \mathcal{P}(\widetilde{\mathbb{X}}^b)$, defined as the element $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ in $\mathcal{I}_p$ satisfying (2) and is minimal with respect to this property, i.e. such that $(A - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)) \in \mathcal{I}_p$, for any $A \in \mathcal{I}_p$ satisfying (2).*

In the remainder of this paper, we shall make the following standing assumptions

Standing Assumption 2.2 *The upper variation process $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ exists.*

On the set $\mathcal{P}(\widetilde{\mathbb{X}}^b)$, we define the set

$$\mathcal{P}^*(\widetilde{\mathbb{X}}^b) = \left\{ \begin{array}{ll} Z \in \mathcal{P}(\widetilde{\mathbb{X}}^b): & Z \text{ is a } \mathbf{P}\text{-supermartingale and} \\ & \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) \text{ is a continuous process with finite variation} \end{array} \right\}$$

and its subset

$$\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b) = \Big\{ Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b) \colon Z \text{ is a positive and } \mathbf{P}\text{-martingale} \Big\}.$$

In the sequel, we identify a $Z \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ with a probability measure $Q \sim \mathbf{P}$ whose density process is

$$Z = (Z_t)_{0 \le t \le T} = \mathbf{E}\left[\frac{dQ}{d\mathbf{P}} \,\middle|\, \mathcal{F}_t \right]$$

and the upper variation process $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ with $Z \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ is frequently denoted by $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)$.

We assume that

Standing Assumption 2.3 $\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b) \neq \emptyset$

Let us introduce a strictly positive process S^0. In what follows, we assume that $\widetilde{X}^0$ can be chosen so as:

Standing Assumption 2.4 *$\widetilde{X}^0$ is a finite variation process with continuous paths and null at* 0.

We now define the family $\widetilde{\mathbb{W}}$ as follows:

$$\widetilde{\mathbb{W}} = \left\{\widetilde{W} = \mathcal{E}(\widetilde{X} - \widetilde{C});\ \widetilde{X} \in \widetilde{\mathbb{X}},\ \text{and}\ \widetilde{C} \in \mathcal{I}\right\} \tag{3}$$

$$\widetilde{\mathbb{W}}^b = \left\{\widetilde{W}_b = \mathcal{E}(\widetilde{X}_b - \widetilde{C});\ \widetilde{X}_b \in \widetilde{\mathbb{X}}^b,\ \text{and}\ \widetilde{C} \in \mathcal{I}\right\} \tag{4}$$

where $\mathcal{E}(\cdot)$ is the exponential semimartingale of Doléans-Dade.

Recall that for any semimartingale X null at 0 the Doléan-Dade exponential $\mathcal{E}(X)$ is a solution of the following stochastic differential equation:

$$Z = 1 + Z_{-} \bullet X, \quad Z_0 = 1. \tag{5}$$

Moreover any solution of this equation coincides with $\mathcal{E}(X)$ on the set $\{(\omega, t)\colon \mathcal{E}(X)_{-} \neq 0\}$.

For any $x > 0$, we define

$$\widetilde{\mathbb{W}}_x \triangleq x\widetilde{\mathbb{W}} = \{x\widetilde{W}\colon \widetilde{W} \in \widetilde{\mathbb{W}}\},$$

We are now interested on the family of state processes:

$$\begin{aligned}\mathbb{W} &\triangleq \{W = S^0\widetilde{W}\colon \widetilde{W} \in \widetilde{\mathbb{W}}\},\\ &= \left\{W = S^0\left(1 + \widetilde{W}_{-} \bullet \widetilde{X} - \widetilde{W}_{-} \bullet \widetilde{C}\right)\colon \widetilde{X} \in \widetilde{\mathbb{X}},\ \widetilde{C} \in \mathcal{I}\right\},\end{aligned} \tag{6}$$

with

$$\mathbb{W}_x \triangleq x\mathbb{W} = \{xW\colon W \in \mathbb{W}\}$$

We suppose that the process $\widetilde{W}_{-} \bullet \widetilde{C}$ can be represented by the formula:

$$\int_0^t \widetilde{W}_{s-} d\widetilde{C}_s = \int_0^t \widetilde{c}_s ds,\ \forall\ t \in [0, T]$$

We now define a consumption process:

Definition 2.2 *A consumption process $c(\cdot)$ is an $\mathcal{F}_t$-adapted nonnegative process, which is related to the accumulated proportion process by the formula*

$$\int_0^t c_s ds = \int_0^t S_s^0 \widetilde{W}_{s-} d\widetilde{C}_s, \quad 0 \leq t \leq T. \tag{7}$$

Put $\Lambda_t = t$, then in the standard notation of the stochastic calculus for semimartingales (7) can be written as follows:

$$c \bullet \Lambda = S^0 \widetilde{W}_- \bullet \widetilde{C},$$

or equivalently, we have $c_t = S_t^0 \widetilde{c}_t$, for any $t \in [0,T]$.

Given $x > 0$ and $\widetilde{X}^0 \in \widetilde{\mathbb{X}}_0$, we denote by $\mathcal{A}(x)$ the set of the pairs of processes (W, c), where $W \in \mathbb{W}_x$ and c satisfying (7).

One of the families of examples we have in mind for applications is described below.

Example (Cuoco and Liu (2000), Rogers (2001)).

This is an important example, generalizing a number of other papers in the subject: Cvitanic and Karatzas (1992, 1993) [7], El Karoui, Peng and Quenez (1997) [20], Cuoco and Cvitanic (1998) [10], Rogers (2001) [45], El Karoui and Jeanblanc-Piqué (1998) [18], for example. The numéraire S^0 and the wealth process W of the agent satisfies:

$$\begin{aligned}
\frac{dW_t}{W_t} &= \left[r_t dt + \pi_t \Big(\sigma_t dB_t + (b_t - r_t \mathbf{1}) dt \Big) + g(t, \pi_t) dt + e_t dt \right] - c_t dt \qquad (8) \\
W_0 &= x \\
dS_t^0 &= r_t S_t^0 dt, \quad S_0^0 = 1
\end{aligned}$$

where $\pi_t \in \Pi$, B is an n-dimensional Brownian motion, b, r, $V \equiv \sigma\sigma'$, V^{-1}, e are all bounded processes, and there is a uniform Lipschitz bound on g: for some $\theta < \infty$,

$$|g(t, x, \omega) - g(t, y, \omega)| \leq \theta |x - y|$$

for all x, y, t and ω. In our model the agent receives an income with a proportional (eventually stochastic) rate e_t per unit time.

The unconventional term in the dynamics (8) is the term involving g about which we assume:

- for $x \in \mathbf{R}^n$, $(t, w) \mapsto g(t, x, \omega)$ is an optional process;
- for each $t \in [0,T]$ and $\omega \in \Omega$, $x \mapsto g(t, x, \omega)$ is concave and upper semicontinuous.
- $g(t, 0, \omega) = 0$ for all $t \in [0,T]$ and $\omega \in \Omega$.

Suppose that $c_t = \frac{D_t}{W_t}$, and D_t is a nonnegative process. Now let $\widetilde{W} \triangleq \frac{W}{S^0}$, by Itô Lemma we have:

$$d\widetilde{W}_t = \widetilde{W}_t\left[\pi_t\left(\sigma_t dB_t + (b_t - r_t\mathbf{1})dt\right) + g(t,\pi_t)dt - D_t dt + e_t dt\right]$$

In this case, we choose

$$\widetilde{X}^0_t = \pi^0_t = \int_0^t e_s ds$$

so that $\widetilde{\mathbb{X}}^b = \{\pi \bullet R + \widetilde{G}^\pi\}$, with

$$\begin{aligned} dR_t &= \sigma_t dB_t + (b_t - r_t\mathbf{1})dt \\ d\widetilde{G}^\pi_t &= g(t,\pi_t)dt \end{aligned}$$

By the martingale representation theorem for Brownian motion (see, e.g. Karatzas and Shreve (1991)), any probability measure equivalent to $\mathbf{P}$ has a density process in the form:

$$Z^\nu \triangleq \frac{dB^\nu}{dB} = \mathcal{E}\left(-\int \sigma'_t V_t^{-1}(b_t - r_t\mathbf{1} + \nu)dB_t\right)$$

where $\nu \in \mathcal{M}$:

$$\mathcal{M} \triangleq \left\{\nu\colon \int_0^T |\sigma'_t V_t^{-1}\nu|^2 dt < \infty, \text{ and } \mathbf{E}[Z^\nu_T] = 1\right\}$$

Now by Girsanov's Theorem, the Doob-Meyer decomposition of $\widetilde{X}_b = \pi \bullet R + \widetilde{G}^\pi \in \widetilde{\mathbb{X}}^b$ under $P^\nu = Z^\nu_T\mathbf{P}$, $\nu \in \mathcal{M}$, is:

$$d\widetilde{X}_{b_t} = \pi_t\sigma_t dB^\nu_t + dA^{\nu,\pi}_t$$

where B^ν is a n-dimensional Brownian motion under P^ν and $A^{\nu,\pi}$ is the predictable compensator under P^ν:

$$dA^{\nu,\pi}_t = (g(t,\pi_t) - \pi_t\nu)dt$$

Now denote

$$\widetilde{g}(t,\nu) = \int_0^t \sup_{\pi\in\Pi}\left(g(s,\pi_s) - \pi_s\nu\right)ds$$

the convex conjugate of $-g(t,-\pi)$ and let $\widetilde{\mathcal{G}} = \{\nu \in \mathbf{R}^n : \widetilde{g}(t,\nu) < \infty\}$ its effective domain.

We deduce that $\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ consists of all probability measures $P^\nu, \nu \in \mathcal{M}(\widetilde{\mathcal{G}})$:

$$\mathcal{M}(\widetilde{\mathcal{G}}) \triangleq \{\nu \in \mathcal{M}\colon \nu \in \widetilde{\mathcal{G}} \text{ and } \widetilde{g}(t,\nu) \text{ is a continuous process with finite variation}\}$$

Moreover, the upper variation process is given by:

$$\widetilde{A}^{\widetilde{\mathbb{X}}^b}(P^\nu)_t = \int_0^t \widetilde{g}(s,\nu)ds$$

Since all coefficients are bounded, it is straightforward to verify that the model satisfies the Standing Assumption 2.1. Moreover, the closure property of $\widetilde{\mathbb{X}}$ may also be proved in this model under a Liptschitz condition on function g and the invariance of the Emery distance under translation, see Pham (2002) [48] for details.

Remark 2.2 *In the paper of Cuoco and Cvitanic (1998) [10], they preassummed that $\widetilde{g}$ is bounded on its effective domain.*

Remark 2.3 *Recall that in our framework, the labor income is restricted to be a continuous process with finite variation. Therefore our framework is not applicable to the general case considered by El Karoui and Jeanblanc-Picqué, where the income process e is of the general Markovian form $de_t = \mu(t, e_t)dt + \sigma(t, e_t)dB_t$.*

3 The Dual Set

We define the family $\mathbb{Y}$ of nonnegative semimartingales Y as follows

$$\mathbb{Y} = \left\{ Y = \frac{Z}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))\mathcal{E}(\widetilde{X}^0)S^0} \; : \; Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b) \right\}$$

and denote

$$\mathbb{Y}_y \triangleq y\mathbb{Y}, \quad y > 0$$

In the sequel, let us denote by $\mathbb{Y}_y^+ \subset \mathbb{Y}_y$ the subset containing all positive $Y \in \mathbb{Y}_y$. We also suppose that any $Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$ can be written as $Z = \mathcal{E}(N)$, where N is some $\mathbf{P}$-local martingale null at 0. Since $\widetilde{X}^0$ and $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ are continuous processes of finite variation, by using Proposition I.4.4.69 in Jacod and Shiryaev (1987) [28] we have $[\widetilde{X}^0, Y] = 0$ and $[\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) + \widetilde{X}^0, Y] = 0$ for any semimartingale Y with $Y_0 = 0$, therefore any $Y \in \mathbb{Y}_y$ can be rewritten as:

$$Y = y\frac{\mathcal{E}(N - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) - \widetilde{X}^0)}{S^0} \tag{9}$$

The following Lemma is taken from Long (2003) [43], we include it for completeness.

Lemma 3.1 *Given $x > 0$, for all $Y \in \mathbb{Y}$ and $(W,c) \in \mathcal{A}(x)$, the process $(YW + Yc \bullet \Lambda)$ is a $\mathbf{P}$-supermartingale.*

Proof. Without loss of generality, we may focus to the set $\mathcal{A}(1)$.

Since $(\widetilde{X}^0 + \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))$ is a continuous process with finite variation, by Itô's lemma and after straightforward calculations, from (9) we get:

$$YW + Yc \bullet \Lambda = 1 + Y_- W_- \bullet \left(\widetilde{X} - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) - \widetilde{X}^0 + N + [N, \widetilde{X}] \right) \tag{10}$$

By some algebras we also get

$$\begin{aligned} Z(\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)) &= 1 + Z_- \bullet (\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)) + [Z, \widetilde{X}] + \\ &\quad + (\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))_- \bullet Z \\ &= 1 + Z_- \bullet \left(\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) + [N, \widetilde{X}] \right) + \\ &\quad + (\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))_- \bullet Z, \end{aligned}$$

Since $Z(\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))$ is a $\mathbf{P}$-local supermartingale. The last term on the right-hand side of the above equality is a $\mathbf{P}$-local martingale, it follows then

$$Z_- \bullet \left(\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) + [N, \widetilde{X}] \right)$$

is also a $\mathbf{P}$-local supermartingale.

Moreover, since Z_- is positive and predictable, we deduce that

$$\left(\widetilde{X} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) + [N, \widetilde{X}] \right) \tag{11}$$

is a $\mathbf{P}$-local supermartingale. Since Y, W are nonnegative, by Remark VI.53.d in Dellacherie and Mayer (1982) [16], we deduce from (11) that the processes on the both sides of (10) is a $\mathbf{P}$-local supermartingales. Furthermore, since $Y \geq 0$, $W \geq 0$, $c \bullet \Lambda \in \mathcal{I}$ we have $YW + Yc \bullet \Lambda$ is bounded from below. We then deduce by Fatou's lemma that in fact, $YW + Yc \bullet \Lambda$ is a nonnegative $\mathbf{P}$-supermartingale. This completes the proof of the lemma. $\square$

Remark 3.1 *Since $Y \geq 0$, $c > 0$, then from the last lemma, we deduce that, for any $x > 0$, $W \in \mathbb{W}_x$, the product YW is a $\mathbf{P}$-supermartingale.*

From Lemma 3.1 we deduce that the process

$$\frac{ZW}{\mathcal{E}(\widetilde{X}^0)\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))S^0} + \frac{Zc}{\mathcal{E}(\widetilde{X}^0)\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))S^0} \bullet \Lambda$$

is a $\mathbf{P}$-supermartingale for any $Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$, and the budget constraint

$$v(W,c) \triangleq \mathbf{E}\Big[\frac{Z_T W_T}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_T)\mathcal{E}(\widetilde{X}^0_T)S^0_T} + \int_0^T \frac{Z_t c_t dt}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_t)\mathcal{E}(\widetilde{X}^0_t)S^0_t}\Big] \leq x, \tag{12}$$

is satisfied for any $(W,c) \in \mathcal{A}(x)$.

4 The Utility Maximization from Terminal Wealth and Consumption

In this paper, our goal is to generalize the study of optimal investment and consumption problems to the aforementioned semimartingale setting.

We first recall some classical definitions and properties of utility function.

Definition 4.1 *A utility function $U\colon (0,\infty)\times\Omega \to \mathbf{R}\cup\{-\infty\}$ is a strictly increasing, strictly concave, continuously differentiable function and satisfies the Inada conditions:*

$$U'(0) = \lim_{x\to 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x\to\infty} U'(x) = 0 \tag{13}$$

We now introduce the conjugate function of U:

$$\widetilde{U}(y) = \sup_{x>0}[U(x) - xy], \quad y > 0 \tag{14}$$

Recall that if $U(x)$ is defined as in Definition 4.1, then $\widetilde{U}(y)$ is a continuously differentiable, decreasing, strictly convex function satisfying:

$$\widetilde{U}'(0) = -\infty, \quad \widetilde{U}'(\infty) = 0, \quad \widetilde{U}(0) = U(\infty), \quad \widetilde{U}(\infty) = U(0) \tag{15}$$

and the following bidual relation:

$$U(x) = \inf_{y>0}[\widetilde{U}(y) + xy], \quad x > 0 \tag{16}$$

We also note that the derivative of $U(x)$ is the inverse function of the negative of the derivative of $\widetilde{U}(y)$, which we denote by I

$$I \triangleq -\widetilde{U}' = (U')^{-1} \tag{17}$$

It is well-known that

$$\widetilde{U}'(y) = -I(y), \quad y > 0, \text{ a.s.}, \tag{18}$$

and $I(y)$ attains the supremum in (14), i.e.

$$\widetilde{U}(y) = U(I(y)) - yI(y), \quad y > 0, \text{ a.s.} \tag{19}$$

The agent in our model has time-seperable utility structure as follows

Definition 4.2 *A (time-seperable, von Neumann-Morgenstern) preference structure is a pair of utility functions* $U_1 \colon \mathbf{R} \times [0,T] \to [-\infty, \infty]$ *and* $U_2 \colon \mathbf{R} \to [-\infty, \infty)$*, which measure the investor's utility from consumption and wealth, respectively.*

Definition 4.3 *Given an initial endowment* $x \in \mathbf{R}$*, the consumption plan* (W_T, c)*, here* W_T *is the terminal wealth, and* c *the consumption rate process throughout the liftetime investment, is called* x*-affordable if they are financeable from an initial wealth less or equal to* x*, i.e., the pair of a wealth and consumption process* (W, c) *belong to the set* $\mathcal{A}(x^*)$ *with* $0 < x^* \le x$*.*

Recall that a necessary condition for $(W, c) \in \mathcal{A}(x)$ is the budget constraint (12).

The agent's total expected uility from consumption over the period and expected utility of investment at the end of the period $[0, T]$ is defined as

$$J(x;\, W, c) \triangleq \mathbf{E}\left[\int_0^T U_1(c_t, t)dt + U_2(W_T)\right]. \tag{20}$$

The x-affordable consumption plan is said to be *x-feasible* if it satisfies:

$$J(x;\, W, c)^- = \mathbf{E}\left[\int_0^T U_1(c_t, t)^- dt + U_2(W_T)^-\right] < \infty$$

we denote the set of x-feasible consumption plans (W_T, c) by $\mathcal{A}^*(x)$. By misuse of notation, we shall write $(W, c) \in \mathcal{A}^*(x)$ instead of $(W_T, c) \in \mathcal{A}^*(x)$.

Given an initial endowment x and income stream X^0, an investor wishes to choose a consumption profile and investment policy so as to to maximize his total expected uility from consumption over the period and expected utility of investment at the end of the period $[0, T]$, with the value function:

$$u(x) = \sup_{(W,c)\in\mathcal{A}^*(x)} J(x;\, W, c), \quad x \in \mathbf{R}_+, \tag{21}$$

using feasible policies.

To ensure that (21) is meaningful, we impose the following assumption:

Assumption 4.1

$$u(x) < \infty, \quad \textit{for some } x > 0$$

Following Kramkov and Schachermayer (1999) [39], we require an asymptotic elasticity condition on U_1 and U_2:

$$AE(U_1(t,x)) \triangleq \limsup_{x\to\infty} \frac{xU_1'(t,x)}{U_1(t,x)} < 1, \ \forall\, t \in [0,T] \text{ and } AE(U_2(x)) \triangleq \limsup_{x\to\infty} \frac{xU_2'(x)}{U_2(x)} < 1 \tag{22}$$

5 The Abstract Setting

The main goal of this section is to provide a dual sets and their basic properties. With respect to the classical utility maximization from fixed terminal wealth, we have now to consider the whole path of the consumption process on the support of $\ell[0,T]$, here $\ell[0,T]$ stands for Lebesgue measure on $[0,T]$.

We now introduce some definitions and notations that will be useful in the rest of the paper. Define the finite measure space $(S, \mathcal{S}, \mu)$ as follows:

$$S = [0,T] \times \Omega, \quad \mathcal{S} = \mathcal{B}[0,T] \otimes \mathcal{F}, \quad \mu = (\ell[0,T] + \delta_T) \times \mathbf{P}$$

Let $\mathcal{L}^0_+$ denote the cone of non-negative functions on $\mathcal{L}^0(S, \mathcal{S}, \mu)$, a closed convex set usually abbreviated to $\mathcal{L}^0_+$.

Notice that, for Y^1, $Y^2 \in \mathcal{L}^0_+$, we have:

$$\int (Y^1, Y^2) d\mu = \mathbf{E}\Big[\int_0^T Y_t^1 Y_t^2 dt + Y_T^1 Y_T^2\Big] \tag{23}$$

Here and in what follows we denote

$$\langle Y^1, Y^2 \rangle_{s,t} = \int_s^t Y_u^1 Y_u^2 du + Y_t^1 Y_t^2 \mathbf{1}_{t=T}, \quad t \in [0,T] \tag{24}$$

and let

$$\langle Y^1, Y^2 \rangle_t \triangleq \langle Y^1, Y^2 \rangle_{0,t}, \quad \langle Y^1, Y^2 \rangle \triangleq \langle Y^1, Y^2 \rangle_{0,T} \tag{25}$$

For Y_1, $Y_2 \in \mathcal{L}^0_+$, we shall say that

$$Y_1 \equiv Y_2, \quad \text{if} \quad Y_1 = Y_2 \quad \mu - a.e.$$

On $\mathcal{L}^0_+$, we define a partial ordering by:

$$Y^1 \preceq Y^2 \Leftrightarrow Y^1 \leq Y^2, \quad \mu - \text{a.e.}$$

We say that a subset $\mathcal{C}$ of $\mathcal{L}^0_+$ is solid if

$$Y_2 \in \mathcal{C},\ Y_1 \preceq Y_2 \Rightarrow Y_1 \in \mathcal{C}$$

We define $\mathcal{L}^1$ as the Banach space of elements $Y = (Y)_t \in \mathcal{L}^0$, equipped with the norm

$$\|Y\|_1 = \mathbf{E}\left[\int_0^T |Y|_t dt + |Y|_T\right]$$

We also denote $\mathcal{L}^1_+ = \mathcal{L}^0_+ \cap \mathcal{L}^1$.

We define the abstract version of the primal and dual sets $\mathcal{A}^*(x)$ and $\mathbb{Y}_y$ as follows:

$$\mathbb{C}_x \quad = \quad \left\{ \begin{array}{c} g \in \mathcal{L}^0_+;\ g\colon S \to \mathbf{R}_+ \quad \text{such that} \quad g \preceq c, \\ \text{and} \quad g_T \leq W_T \quad \text{for some} \quad (W, c) \in \mathcal{A}^*(x) \end{array} \right\} \tag{26}$$

$$\overline{\mathbb{D}}_y \quad = \quad \Big\{ h \in \mathcal{L}^0_+ \colon h \preceq Y, \quad Y \in \mathbb{Y}_y \Big\} \tag{27}$$

We denote by $\mathbb{D}^+_y$ the subset of $\mathbb{D}_y$ consisting of all h such that $h > 0$ μ-a.e., and $\mathbb{D}_y$ the closure in $\mathcal{L}^0_+$ of $\overline{\mathbb{D}}_y$. From Standing Assumption 2.3 we deduce that $\mathbb{D}^+_y \neq \emptyset$ for any $y > 0$.

For later use, we summarize some of the basic properties of the set $\mathbb{C}$ and $\mathbb{D}$, as well as kind of "bipolar" relation between the these sets in the Lemma 5.1 below.

Lemma 5.1 *Let $g \in \mathcal{L}^0_+$, then $g \in \mathbb{C}$ if and only if*

$$v(g) \triangleq \sup_{h \in \mathbb{D}} \int \langle h, g \rangle d\mu \leq 1, \tag{28}$$

Proof. First of all, notice that by Fatou's lemma we have:

$$\sup_{h \in \mathbb{D}} \int \langle h, g \rangle d\mu = \sup_{h \in \overline{\mathbb{D}}} \int \langle h, g \rangle d\mu \tag{29}$$

so all we need to confirm (28) is to check the statement for $h \in \overline{\mathbb{D}}$.

The "if part" of the relation (29) is obvious, since YW is a $\mathbf{P}$-supermartingale (Remark 3.1) and the fact that h, g are nonnegative and dominated by Y, W in a sense of (26) and (27). What remains now is to prove the converse assertion.

Consider an adapted, nonnegative $\mathcal{F}_T$-measurable random variable A defined as

$$A = \int_0^T \frac{g_t}{\mathcal{E}(\widetilde{X}_t^0)S_t^0}dt + \frac{g_T}{\mathcal{E}(\widetilde{X}_T^0)S_T^0}$$

Since $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ is nondecreasing for any $Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$. Hence we have

$$\begin{aligned}
v(A) &\triangleq \sup_{Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)} \mathbf{E}^Q\left[\frac{A}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q))_T}\right] \\
&\leq \sup_{Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)} \mathbf{E}\left[\int_0^T \frac{Z_t g_t}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))_t \mathcal{E}(\widetilde{X}_t^0)S_t^0}dt + \frac{Z_T g_T}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))_T \mathcal{E}(\widetilde{X}_T^0)S_T^0}\right] \\
&= \sup_{Y \in \mathbb{Y}} \int \langle Y, g\rangle d\mu \ \leq \ 1 \qquad (30)
\end{aligned}$$

where the first inequality follows from the inclusion $\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b) \subset \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$. where the last equality by (28) and (29) and the definition of the set $\mathbb{Y}$. Then by the stochastic control lemma of Föllmer and Kramkov (1997) [22] (see proof in Long (2003) [43]), there exists a *càdlàg* version of the nonnegative process:

$$\widetilde{W}_t^b = \operatorname*{ess\,sup}_{Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)} \mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q))_t \mathbf{E}^Q\left[\frac{A}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_T)}\middle|\mathcal{F}_t\right], \quad 0 \leq t \leq T \qquad (31)$$

Moreover, for any $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$, the process $\widetilde{W}^b/\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q))$ is a Q-supermartingale. By the optional decomposition under constraints theorem in multiplicative form in Föllmer and Kramkov (1997) [22], the process $\widetilde{W}^b$ admits a decomposition:

$$\widetilde{W}^b = v(A)\mathcal{E}(\widetilde{X}_b - \widetilde{C}) = v(A) + \widetilde{W}_-^b \bullet \widetilde{X}_b - \widetilde{W}_-^b \bullet \widetilde{C} \qquad (32)$$

where $\widetilde{X}_b \in \widetilde{\mathbb{X}}^b$, $\widetilde{C} \in \mathcal{I}$. We now consider the process

$$W \triangleq S^0\mathcal{E}(X^0)\widetilde{W}^b = v(A)\mathcal{E}(X^0 + \widetilde{X}_b - \widetilde{C}) = v(A) + \widetilde{W}_- \bullet \widetilde{X} - \widetilde{W}_- \bullet \widetilde{C}$$

with $\widetilde{X} \triangleq X^0 + \widetilde{X}_b \in \widetilde{\mathbb{X}}_0$.

Let us define $W \triangleq S^0\left(\widetilde{W} - \frac{g}{S^0} \bullet \Lambda\right)$. Using the definition of $\widetilde{\mathbb{X}}_0$, (31) and (32) we get

$$\begin{aligned} W &= S^0\left(v(A)\mathcal{E}(\widetilde{X} - \widetilde{C}) - \frac{g}{S^0} \bullet \Lambda\right) \\ &= S^0\left(v(A) + \widetilde{W}_- \bullet \widetilde{X} - \widetilde{W}_- \bullet \widetilde{C} - \frac{g}{S^0} \bullet \Lambda\right) \\ &= S^0\left(1 + \widetilde{W}_- \bullet \widetilde{X} - \left(\widetilde{W}_- \bullet \widetilde{C} + 1 - v(A) + \frac{g}{S^0} \bullet \Lambda\right)\right) \end{aligned}$$

It is not hard to show that $W_T \geq g_T$, and W belongs to the set $\mathbb{W}$, with the cumulated consumption process defined as

$$\frac{c}{S^0} \bullet \Lambda \triangleq \widetilde{W}_- \bullet \widetilde{C} + 1 - v(A) + \frac{g}{S^0} \bullet \Lambda \succeq \frac{g}{S^0} \bullet \Lambda,$$

Hence, $(W, c) \in \mathcal{A}(1)$ is a pair of wealth process its corresponding consumption rate process that dominates g in a sense of (26). $\square$

As it can be seen, the value $v(A)$ is the least initial state value, which allows to dominate in the almost sure sense the $\mathcal{F}_T$ random variable A by a state process. In the financial context, $v(A)$ is usually called the superreplication cost of the European option A_T. Notice in particular that the expression of $v(A)$ does not depend on the choice of $\widetilde{X}^0$.

Lemma 5.2 *Given $x > 0$ The set $\mathbb{C}_x$ is convex, solid and closed under convergence in μ-measure.*

Proof. Note that the solidity of $\mathbb{C}_x$ is rather obvious. It remains to prove its convexity.

Let $(S^0\widetilde{W}^1, S^0\widetilde{c}^1)$ and $(S^0\widetilde{W}^2, S^0\widetilde{c}^2)$ are two pair of processes in $\mathcal{A}^*(x)$. Taking any $\epsilon^1 = 1 - \epsilon^2 \in (0,1)$ and defining the convex combinations

$$\begin{aligned} \widetilde{W}^* &= \epsilon^1\widetilde{W}^1 + \epsilon^2\widetilde{W}^2 \\ \widetilde{c}^* &= \epsilon^1\widetilde{c}^1 + \epsilon^2\widetilde{c}^2 \end{aligned}$$

By the predictable convexity property on the set $\widetilde{\mathbb{X}}_0$ and the associativity of the stochastic integral (see, e.g. Theorem 19 in Protter (1990) [49]), we find immediately that:

$$\begin{aligned} \widetilde{W}^* &= x + (\epsilon^1\widetilde{W}^1_- \bullet \widetilde{X}^1 + \epsilon^2\widetilde{W}^2_- \bullet \widetilde{X}^2) - \widetilde{c}^* \bullet \Lambda \\ &= x + \widetilde{W}^*_- \bullet \left(\frac{\epsilon^1\widetilde{W}^1_-}{\widetilde{W}^*_-} \bullet \widetilde{X}^1 + \frac{\epsilon^2\widetilde{W}^2_-}{\widetilde{W}^*_-} \bullet \widetilde{X}^2\right) - \widetilde{c}^* \bullet \Lambda \\ &= x + \widetilde{W}^*_- \bullet \overline{X} - (\widetilde{c}^* \bullet \Lambda + \widetilde{W}^*_- \bullet \overline{C}) \end{aligned}$$

where $\overline{X} \in \widetilde{\mathbb{X}}_0$, $\overline{D} \in \mathcal{I}$. We see that $\widetilde{c}^* \bullet \Lambda \preceq c^* \bullet \Lambda \triangleq \widetilde{c}^* \bullet \Lambda + \widetilde{W}^*_- \bullet \overline{D}$, and deduce that

$$\widetilde{c}^* \preceq c^*$$

Since the utility functions are nondereasing, then $(S^0\widetilde{W}^*, S^0c^*)$ is a pair of a wealth and a consumption process in $\mathcal{A}^*(x)$ (corresponding to $\overline{X}$). By the definition of $\mathbb{C}_x$, the convex combination $(S^0_T\widetilde{W}^*_T, S^0_t\widetilde{c}^*_t)$ is also in $\mathbb{C}_x$, hence $\mathbb{C}_x$ is convex.

Now let $(g^n)_{n\in\mathbf{N}} \in \mathcal{L}^0_+$ be a sequence in $\mathbb{C}_x$ converging in μ-measure; we may (and shall) by passing to a subsequence and suppose that the sequence converges μ-almost everywhere to limit g.

We will use lemma 5.1 to prove that f belong to the sets $\mathbb{C}_x$.

Since all processes under consideration are nonnegative, and all $h \in \overline{\mathbb{D}}$ are dominated by some $Y \in \mathbb{Y}$ μ-a.e., by Fatou's lemma and by (29), we have:

$$\begin{aligned}
\sup_{h\in\mathbb{D}} \int \langle h, g\rangle d\mu &\leq \sup_{Y\in\mathbb{Y}} \int \langle Y, f\rangle d\mu \\
&= \sup_{Z\in\mathcal{P}^*(\widetilde{\mathbb{X}}^b)} \int \langle \frac{Z}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))\mathcal{E}(\widetilde{X}^0)S^0}, g\rangle d\mu \\
&\leq \sup_{h\in\mathbb{D}} \liminf_{n\to\infty} \int \langle \frac{Z}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z))\mathcal{E}(\widetilde{X}^0)S^0}, g^n\rangle d\mu \\
&\leq x.
\end{aligned}$$

This proves the closeness property of $\mathbb{C}_x$. $\square$

The next lemma is taken from Long (2003) [43], we include it for completeness.

Lemma 5.3 *The set $\mathbb{D}$ is convex, solid and closed with respect to the topology of convergence in μ-measure.*

Proof. First note that the closeness of $\mathbb{D}$ follows immediately from its definition and the solidity of $\mathbb{D}$ is rather obvious. We now prove the remaining assertion.

Since 0 already belongs to the set $\mathbb{D}$, and the convexity is preserved under weak convergence, so all we need to verify the convexity for $h \in \overline{\mathbb{D}}^+$.

We first show the convexity of $\mathbb{Y}^+$, which then implies the the convexity of $\overline{\mathbb{D}}^+$ by the the solidity property of $\overline{\mathbb{D}}^+$.

First, let us recall the following properties of the exponential semimartingales of Doléans-Dade (see, e.g. Kallsen and Shirayev (2002) [29])

$$\mathcal{E}(X) = 1 + \mathcal{E}(X)_- \bullet X \tag{33}$$

$$X = X_0 + \frac{1}{\mathcal{E}(X)_-} \bullet \mathcal{E}(X) \tag{34}$$

Let Y^1 and Y^2 be processes in $\mathbb{Y}^+$, which have the following decompositions:

$$\begin{aligned} Y^1 &= \frac{Z^1}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1))\mathcal{E}(\widetilde{X}^0)S^0} = \frac{\mathcal{E}(N^1)}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1))\mathcal{E}(\widetilde{X}^0)S^0} \\ Y^2 &= \frac{Z^2}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2))\mathcal{E}(\widetilde{X}^0)S^0} = \frac{\mathcal{E}(N^2)}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2))\mathcal{E}(\widetilde{X}^0)S^0} \end{aligned}$$

where $Z^i = \mathcal{E}(N^i) \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$. Taking any $\epsilon^1 = 1 - \epsilon^2 \in [0,1]$ and defining the convex combinations

$$\widehat{Y} = \epsilon^1 Y^1 + \epsilon^2 Y^2$$

Define a process $A \in \mathcal{I}_p$ and a $\mathbf{P}$-local martingale N as follows:

$$A = \frac{\epsilon^1 Y^1_-}{\widehat{Y}_-} \bullet \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1) + \frac{\epsilon^2 Y^2_-}{\widehat{Y}_-} \bullet \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2) \tag{35}$$

$$N = \frac{\epsilon^1 Y^1_-}{\widehat{Y}_-} \bullet N^1 + \frac{\epsilon^2 Y^2_-}{\widehat{Y}_-} \bullet N^2 \tag{36}$$

We now check whether $\widehat{Z} \triangleq \mathcal{E}(N)$ belongs to the set $\mathcal{P}^*(\widetilde{\mathbb{X}}^b)$.

Fix any $\widetilde{X}_b \in \widetilde{\mathbb{X}}^b$, we need to show that $\widehat{Z}(\widetilde{X}_b - A)$ is a $\mathbf{P}$-local supermartingale. Since $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)$ with $i = 1, 2$ is a continuous process with finite variation, we deduce from I.4.34 c, and I.4.36 in Jacod and Shirayev (1987) [28] that A is also a continuous process with finite variation. We first prove that:

$$\frac{\epsilon^1 Z^1}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1))} + \frac{\epsilon^2 Z^2}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2))} = \frac{\widehat{Z}}{\mathcal{E}(A)} \tag{37}$$

For convienience, we denote $Y^i_0 \triangleq Y^i S^0 \mathcal{E}(\widetilde{X}^0)$. Without loss of generality assume that $S^0_0 = 1$. Hence, by (34) we have

$$Y^i_0 = \mathcal{E}(N^i - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)) = 1 + Y^i_0 \bullet (N^i - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)), \quad i = 1, 2 \tag{38}$$

Therefore,

$$
\begin{aligned}
\widehat{Y}_0 &= \mathcal{E}(N-A) \;=\; 1+\widehat{Y}_0 \bullet (N-A) \\
&= 1+\epsilon^1 Y_0^1 \bullet (N^1-\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1))+\epsilon^2 Y_0^2 \bullet (N^2-\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2)) \\
&= \epsilon^1 Y_0^1+\epsilon^2 Y_0^2
\end{aligned}
$$

where the third equality follows from (38), and we get (37).

From (37) and using the properties of the Doléans-Dade exponential semimartingales, we deduce that:

$$
\widehat{Z} = \frac{\epsilon^1 Y_-^1 \widehat{Z}_-}{\widehat{Y}_- Z_-^1} \bullet Z^1 + \frac{\epsilon^2 Y_-^2 \widehat{Z}_-}{\widehat{Y}_- Z_-^2} \bullet Z^2 \tag{39}
$$

For convinience, we denote

$$
\begin{aligned}
\overline{\epsilon}^1 &\triangleq \frac{\epsilon^1 Y^1 \widehat{Z}}{\widehat{Y} Z^1} \\
\overline{\epsilon}^2 &\triangleq \frac{\epsilon^2 Y^2 \widehat{Z}}{\widehat{Y} Z^2}
\end{aligned}
$$

From (39) we have

$$
\begin{aligned}
[\widehat{Z}, \widetilde{X}_b - A] &= \overline{\epsilon^1} \bullet [Z^1, \widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1)] + \overline{\epsilon^2} \bullet [Z^2, \widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2)] + \\
&\quad + \overline{\epsilon^1} \bullet [Z^1, \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1)] + \overline{\epsilon^2} \bullet [Z^2, \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2)] - [\widehat{Z}, A]
\end{aligned} \tag{40}
$$

Recall that we have:

$$
\begin{aligned}
Z^i(\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)) &= (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i))_- \bullet Z^i + Z_-^i \bullet (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)) + \\
&\quad + [Z^i, \widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)] \\
\widetilde{X}_b - A &= \frac{\epsilon^1 Y_-^1}{\widehat{Y}_-} \bullet (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1)) + \frac{\epsilon^2 Y_-^2}{\widehat{Y}_-} \bullet (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2))
\end{aligned}
$$

Using Ito's lemma and after some straightforward calculations we obtain:

$$\begin{aligned}
\widehat{Z}(\widetilde{X}_b - A) &= (\widetilde{X}_b - A)_- \bullet \widehat{Z} + \widehat{Z}_- \bullet (\widetilde{X}_b - A) + [\widehat{Z}, \widetilde{X}_b - A] \\
&= (\widetilde{X}_b - A)_- \bullet \widehat{Z} + \overline{\epsilon}^1_- Z^1_- \bullet (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1)) - \\
&\quad - \overline{\epsilon}^1_- (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1))_- \bullet Z^1 + \\
&\quad + \overline{\epsilon}^2_- Z^2_- \bullet (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2)) - \\
&\quad - \overline{\epsilon}^2_- (\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2))_- \bullet Z^2 + \overline{\epsilon}^1_- \bullet [Z^1, \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1)] + \\
&\quad + \overline{\epsilon}^2_- \bullet [Z^2, \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2)] - [\widehat{Z}, A] \qquad (41)
\end{aligned}$$

By the definition of $\mathcal{P}^*(\widetilde{\mathbb{X}}^b)$ then we have $Z^i(\widetilde{X}_b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)),\ i = 1,\, 2$ is a **P**-local supermartingale. Moreover, since $Z^i,\ \widehat{Z}$ are **P**-local supermartingale, $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)$ and A are predictable processes with finite variation, then Theorem VII.36 in Dellacherie and Mayer (1982) [16] implies that $[Z^i, \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^i)]$ and $[\widehat{Z}, A]$ are **P**-local martingale. Therefore (41) imples that $\widehat{Z}(\widetilde{X}_b - A)$ is a **P**-local supermartingale.

We conclude that $\widehat{Z}$ belongs to the set $\mathcal{P}^*(\widetilde{\mathbb{X}}^b)$ with the uppervariation process $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(\widehat{Z})$, which is continuous and satisfies

$$\widetilde{A}^{\widetilde{\mathbb{X}}^b}(\widehat{Z}) \preceq A$$

Since then, we have:

$$\begin{aligned}
\widehat{Y} &= \left(\frac{\epsilon^1 Z^1}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^1))\mathcal{E}(\widetilde{X}^0)S^0} + \frac{\epsilon^2 Z^2}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z^2))\mathcal{E}(\widetilde{X}^0)S^0} \right) \\
&= \frac{\widehat{Z}}{\mathcal{E}(A)\mathcal{E}(\widetilde{X}^0)S^0} \\
&\preceq \overline{Y} \triangleq \frac{\widehat{Z}}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(\widehat{Z}))\mathcal{E}(\widetilde{X}^0)S^0} \in \mathbb{Y} \subset \overline{\mathbb{D}}
\end{aligned}$$

As a result, we have proved the convexity property of $\mathbb{D}$. This completes the proof of the lemma.

□

6 Existence results and characterization of the optimal solution

In the newly established finite measure $(S, \mathcal{S}, \mu)$, we define a $\mathcal{S}$-measurable function $U\colon S \times \mathbf{R}^+ \to \mathbf{R} \cup \{-\infty\}$ such that:

$$U((t,\omega),x) = U_1(t,x), \quad t \in [0,T], \quad U((T,\omega),x) = U_2(x), \quad a.s. \tag{42}$$

and with the basic properties:

1. $s \mapsto U(s,x)$ is $\mathcal{S}$-measurable for all $x \geq 0$;
2. $x \mapsto U(s,x)$ is again a utility function in a sense of Definition 4.1 and satisfying (22) for every $s \in \mathcal{S}$.

We slightly abuse notation and omit the dependence in the state $s \in \mathcal{S}$ and write $U(x)$ in place of $U(s,x)$ henceforth.

We shall denote by $I\colon (0, U'(0)) \to (0,\infty)$ the continuous, strictly decreasing inverse of the marginal utility function U'. we set $I(y) = 0$ for $y > U'(0)$. Notice that:

$$\langle 1, I(g) \rangle = \int_0^T I_1(g_t, t)dt + I_2(g_T), \quad g \in \mathcal{L}^0_+$$

Following Pham and Mnif (2002), we formulate the next result.

Lemma 6.1

$$u(x) = \sup_{g \in \mathbb{C}_x} \int U(g)d\mu = \sup_{g \in \mathbb{C}_x} \mathbf{E}\left[\int_0^T U_1(g_t,t)dt + U_2(g_T)\right] \quad x > 0 \tag{43}$$

1. If $(W^*, c^*) \in \mathcal{A}^*(x)$ solves (21), then $g_t = c^*_t$ for $t \in [0,T]$ and $g_T = W^*_T$ solves (43),

2. Conversely, if $g^* \in \mathbb{C}_x$ solves (43), then $(W,c) \in \mathbb{C}_x$, such that $g^*_T \leq W_T$, $g^* \preceq c$, solves (21).

Proof. From (23) and the definition of U (42), then clearly we have the second equality in (43).

Moreover, since $\mathcal{A}^*(x) \subset \mathbb{C}_x$. Hence we already have:

$$u(x) \leq \sup_{g \in \mathbb{C}_x} \int U(g)d\mu \tag{44}$$

Now let $g \in \mathbb{C}_x$, there exists a pair of $(W, c) \in \mathcal{A}^*(x)$ dominating g in a sense of (26). Since U_1, U_2 are nondecreasing, we deduce that

$$\int U(g) d\mu \leq \mathbf{E}\left[\int_0^T U_1(c_t, t)dt + U_2(W_T)\right] \tag{45}$$

and so by (21):

$$\sup_{g \in \mathbb{C}_x} \int U(g) d\mu \leq u(x) \tag{46}$$

From (46) and (44) we have (43).

1. Suppose that $(W^*, c^*) \in \mathcal{A}^*(x)$ solves (21). Then $g \equiv c^* + W^* \delta_T \in \mathbb{C}_x$ and we have

   $$u(x) = \mathbf{E}\left[\int_0^T U_1(c_t^*, t)dt + U_2(W_T^*)\right] = \int U(g) d\mu$$

   which shows that g solves (43).

2. Suppose that $g^* \in \mathbb{C}_x$ solves (43), then there exists $(W, c) \in \mathcal{A}^*(x)$ dominating g^* in a sense of (26). Since U_1, U_2 are nondecreasing, then

   $$u(x) = \int U(g^*) d\mu \leq \mathbf{E}\left[\int_0^T U_1(c_t, t)dt + U_2(W_T)\right]$$

   which shows that (W, c) solves (21). $\Box$

We now define the conjugate function $\widetilde{U} \colon S \times \mathbf{R}_+ \to \mathbf{R} \cup \{\infty\}$:

$$\widetilde{U}(s, h) = \sup_{g > 0} [U(s, g) - \langle g, h \rangle], \quad h \in \mathcal{L}_+^0$$

To alleviate notations, we omit the dependence in the state $s \in S$.

Clearly, $\widetilde{U}$ is a continuously differentiable, decreasing, strictly convex function satisfying (15), (17), (18) and (19), and

$$\langle 1, \widetilde{U}(h) \rangle = \int_0^T \widetilde{U}_1(h_t, t)dt + \widetilde{U}_2(h_T)$$

where

$$\begin{aligned} \widetilde{U}_1(y, t) &= \sup_{x > 0} [U_1(x, t) - xy], \quad y > 0 \\ \widetilde{U}_2(y) &= \sup_{x > 0} [U_2(x) - xy], \quad y > 0 \end{aligned}$$

We now formulate dual problem:

$$\widetilde{u}(y) = \inf_{h\in\mathbb{D}_y} \widetilde{J}(y;h) \triangleq \inf_{h\in\mathbb{D}_y} \int \widetilde{U}(h)d\mu = \inf_{h\in\mathbb{D}_y} \mathbf{E}\left[\int_0^T \widetilde{U}_1(h_t,t)dt + \widetilde{U}_2(h_T)\right] \tag{47}$$

In order to proceed, we shall need the following assumption

Assumption 6.1

$$\widetilde{u}(y) < \infty, \quad \textit{for some } y > 0$$

Assumption 6.2 *$\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_T$ is bounded for any $Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$*

Assumption 6.3 *S^0 and X^0 are bounded from below.*

Clearly, the model described in Section 2.2 satisfying Assumptions 6.2 and 6.3.

We now state the main result of this paper.

Theorem 6.1 *Assume that Assumptions 4.1, 6.1, 6.2, 6.3 and (22) hold true. Then we have*

1. *Existence to the dual problem (47)*

   (a) *For all $y > 0$, $\widetilde{u}(y) < \infty$ and there exists a unique (in the sense of $\equiv$) optimal solution $h^y \in \mathbb{D}_y$ to problem (47). Moreover, $h^y \in \mathbb{D}_y^+$.*

   (b) *Given any $x > 0$, there exists a unique y^* solution of $\inf_{y>0}[\widetilde{u}(y) + xy]$ and characterized by*

   $$\mathbf{E}\left[\int_0^T I_1(h_t^*,t)dt + I_2(h_T^*)\right] = xy^*$$

   where h^* is the optimal solution of $\widetilde{u}(y^*)$.

2. *Existence to the primal problem: Given any $x > 0$. There exists a unique (in the sense of $\equiv$) optimal solution g^* to problem (43). This solution satisfies:*

   $$g_t^* \equiv I_1(h_t^*,t)\mathbf{1}_{t\leq T} + I_2(h_T^*)\mathbf{1}_T, \quad t \in [0,T]$$

   and the solution to problem (21) (W^*,c^*) satisfies:

   $$\mathbf{E}\left[h_T^* W_T^* + \int_0^T h_t^* c_t^* dt\right] = xy^*$$

3. We have the duality relation:

$$
\begin{aligned}
u(x) &= \inf_{y>0}[\widetilde{u}(y) + xy] \\
\widetilde{u}(y) &= \sup_{x>0}[u(x) - xy]
\end{aligned}
$$

The proof to Theorem 6.1 is broken into several lemmas.

Lemma 6.2 *Under Assumptions 6.2 and 6.3. Let $(y^n, h^n)_n$ be a sequence in $\mathbf{R}_+ \times \mathbb{D}_{y^n}$ such that y^n is bounded. Then, there exists a sequence $(y_1^n, h_1^n) \in conv\{(y^k, h^k),\, k \geq n\}$ that converges μ-almost everywhere, to some $(y^*, h^*) \in \mathbf{R}_+ \times \mathbb{D}_{y^*}$.*

Proof. First, notice that from the definition of $\mathbb{Y}_y$ and (27), it is not hard to show that the set

$$\{(y, h) \colon y \in \mathbf{R}_+,\, h \in \mathbb{D}_y\}$$

is convex.

The sequence of nonnegative y^n being bounded, it converges (up to a subsequence) to some nonnegative y^*. Moreover, $h \in \mathbb{D}_y$ is bounded in $\mathcal{L}^1$. Indeed, we have

$$
\begin{aligned}
\int h d\mu &\leq y\mathbf{E}\Big[\int_0^T \frac{Z_t dt}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_t)\mathcal{E}(X_t^0)S_t^0} + \frac{Z_T}{\mathcal{E}(\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_T)\mathcal{E}(X_T^0)S_T^0}\Big] \\
&< yC_1, \quad C_1 \in \mathbf{R}_+
\end{aligned} \tag{48}
$$

where by Assumptions 6.2, 6.3 and the fact that Z is a $\mathbf{P}$-supermartingale.

Using Lemma A1.1 of Delbaen and Schachermayer (1994) [13], we can find a sequence $h_1^n \in$ conv $\{(y^k, h^k),\, k \geq n\}$, which converges μ-a.e. to a function h^* taking values in $[0, \infty]$. Notice also that the limit h^* must be almost everywhere finite because (48). Moreover, $h^* \in \mathbb{D}_{y^*}$ by the convexity of the set

$$\{(y, h) \colon y \in \mathbf{R}_+,\, h \in \mathbb{D}_y\}$$

and Fatou's lemma. □

Lemma 6.3 *Under Assumptions 6.2, 6.3. Let $C > 0$, then the family*

$$\Big\{\widetilde{U}(h)^- \colon h \in \mathbb{D}_y,\ y \in [0, C]\Big\}$$

is uniformly integrable under μ.

Proof. Notice that

$$\int \widetilde{U}(h)^- d\mu = \mathbf{E}\Big[\int_0^T \widetilde{U}_1(h_t,t)^- dt + \widetilde{U}_2(h_T)^-\Big] \tag{49}$$

We closely follow the proof of Lemma 5.2 in Bouchard and Pham (2002) [4]. Assume that $\widetilde{U}_1(\infty,t) < 0$ for all $t \in [0,T]$ and $\widetilde{U}_2(\infty) < 0$ (otherwise there is nothing to prove).

First, we suppose that $\widetilde{U}_1(\infty,t) = -\infty$ for all $t \in [0,T]$, and $\widetilde{U}_1(\infty) = -\infty$. Let

$$\begin{aligned} \phi_1(.,t) &: \quad (-\widetilde{U}_1(0,t),\infty) \to [0,\infty), \quad \forall\, t \in [0,T] \\ \phi_2(.) &: \quad (-\widetilde{U}_2(0),\infty) \to [0,\infty), \end{aligned}$$

denote the inverse functions of $\widetilde{U}_i$. The function ϕ_i are convex and strictly increasing. Since $\widetilde{U}_1(0,t) = U_1(\infty,t) > 0$ for any $t \in [0,T]$ and $\widetilde{U}_2(0) = U_2(\infty) > 0$ hence $\phi_1(0,t)$ and $\phi_2(0)$ are well-defined and finite for any $t \in [0,T]$. It follows that for $y \geq 0$:

$$\phi_1(\widetilde{U}_1(h_t,t)^-,t) \leq \phi_1(0,t) + h_t, \quad t \in [0,T], \quad \phi_2(\widetilde{U}_2(h_T)^-) \leq \phi_2(0) + h_T, \tag{50}$$

Hence, because of (48) we obtain

$$\begin{aligned} \mathbf{E}\Big[\int_0^T \phi_1(\widetilde{U}_1(h_t,t)^-,t)dt + \phi_2(\widetilde{U}_2(h_T)^-)\Big] &\leq \mathbf{E}\Big[\int_0^T \phi_1(-\widetilde{U}_1(h_t,t),t)dt + \phi_2(-\widetilde{U}_2(h_T))\Big] + \\ &\quad + \int_0^T \phi_1(0,t)dt + \phi_2(0) \\ &= \mathbf{E}\Big[\int_0^T h_t dt + h_T\Big] + C_2 \\ &\leq yC_1 + C_2 \quad \forall\, y \geq 0,\ h \in \mathbb{D}_y \end{aligned} \tag{51}$$

with C_2 is some constant such that

$$\int_0^T \phi_1(0,t)dt + \phi_2(0) \leq C_2 < \infty \tag{52}$$

By (17) and the l'Hospital rule:

$$\lim_{x\to\infty} \frac{\phi_1(x,t)}{x} = \lim_{y\to\infty} \frac{y}{-\widetilde{U}_1(y,t)} = \lim_{y\to\infty} \frac{1}{-\widetilde{U}_1'(y,t)} = \infty, \quad t \in [0,T] \tag{53}$$

$$\lim_{x\to\infty} \frac{\phi_2(x)}{x} = \lim_{y\to\infty} \frac{y}{-\widetilde{U}_2(y)} = \lim_{y\to\infty} \frac{1}{-\widetilde{U}_2'(y)} = \infty \tag{54}$$

The uniformly integrability under $\mathbf{P}$ of the sequence $\widetilde{U}_1(h_t,t)^-$ and $\widetilde{U}_2(h_T)^-$ now follows from (51), (53), (54) by the de la Vallée-Poussin theorem. This proves the required results.

Now, we suppose that $\widetilde{U}_1(\infty,t) > -\infty$ for any $t \in [0,T]$ and $\widetilde{U}_2(\infty) > -\infty$. We may reduce the problem to the first case by defining the functions:

$$\phi_1(x,t) \triangleq \begin{cases} (-\widetilde{U}_1)^{-1}(x,t) & \text{for} \quad -\widetilde{U}_1(0,t) \le x \le -\widetilde{U}_1(\infty,t), \\ \psi_1(x,t) & \text{for} \quad x > -\widetilde{U}_1(\infty,t), \end{cases} \qquad t \in [0,T]$$

$$\phi_2(x) \triangleq \begin{cases} (-\widetilde{U}_2)^{-1}(x) & \text{for} \quad -\widetilde{U}_2(0) \le x \le -\widetilde{U}_2(\infty), \\ \psi_2(x) & \text{for} \quad x > -\widetilde{U}_2(\infty), \end{cases}$$

where ψ_i is chosen so that

$$\lim_{x\to\infty} \frac{\psi_i(x)}{x} = \infty$$

and $\phi_1(0,t) < \infty$ for any $t \in [0,T]$, $\phi_2(0) < \infty$. Finally, by the same arguments as in the first case, we obtain the required result. $\square$.

The next Corollary is a useful result from the last lemma. We denote the domain of any function U by $\mathrm{dom}(U) = \{x > 0 \colon U(x) < \infty\}$.

Corollary 6.1 *For each $y \in dom(\widetilde{u})$, there is some $h^y \in \mathbb{D}(y)$ for which the infimum defining in (47) is attained. Differentiability of U_1, U_2 implies strict convexity of $\widetilde{U}_1$, $\widetilde{U}_2$, which in turn implies uniqueness of the minimizing h^y. Moreover, $h^y \in \mathbb{D}_y^+$.*

Proof. We take a minimizing sequence $h^n \in \mathbb{D}_y$ such that:

$$\widetilde{u}(y) \le \int \widetilde{U}(h)d\mu = \mathbf{E}\left[\int_0^T \widetilde{U}_1(h_t^n,t)dt + \widetilde{U}_2(h_T^n)\right] \le \widetilde{u}(y) + n^{-1}. \tag{55}$$

By lemma A.1.1 of Delbaen and Schachermayer in [13], there exists a sequence (up to subsequence) $h_1^n \in \mathrm{conv}(h^n, h^{n+1}, \cdots)_{n\ge1}$, that are μ–almost everywhere convergent to limit h^y. We may suppose that h^y still satisfies the inequality (55). Since $\mathbb{D}_y$ is convex and closed in μ-measure, hence $h^y \in \mathbb{D}_y$. By lemma 6.3 and by applying Fatou's lemma to the sequence $(\widetilde{U}(h_1^n)^+)_{n\ge1}$, and to the right-hand side of the inequality (54) we obtain:

$$\widetilde{u}(y) \ge \liminf_{n\to\infty} \int \widetilde{U}(h_1^n)d\mu \ge \int \widetilde{U}(h^y)d\mu \ge \widetilde{u}(y)$$

The uniqueness assertion is immediated by general duality results (see Theorem V.26.3 in [44]).

We now prove that $h^y \in \mathbb{D}_y^+$. Fix any $h \in \mathbb{D}_y^+$, which is possible by Standing Assumption 2.3. Define the convex combination:

$$h^\delta = \delta h + (1-\delta)h^y, \quad \delta \in (0, 1/2)$$

Note that as δ tends to 0, we have $h^\delta \to h^y$.

Recall that for any convex function $\widetilde{U}$ we have

$$\widetilde{U}(x) \geq \widetilde{U}(y) + (x-y)\widetilde{U}'(y),$$

By the optimality of h^y and the convexity of $\widetilde{U}$ then we have:

$$\begin{aligned} 0 &\geq \frac{1}{\delta}\int \Big(\widetilde{U}(h^y) - \widetilde{U}(h^\delta)\Big)d\mu \\ &\geq \int \langle (h-h^y), I(h^\delta)\rangle d\mu \end{aligned} \tag{56}$$

First, we shall show that the family

$$\langle (h-h^y), I(h^\delta)\rangle^- \quad \text{is integrable under } \mu. \tag{57}$$

or equivalently,

$$\int_0^T \Big((h_t - h_t^y)I_1(h_t^\delta, t)\Big)^- dt + \Big((h_t - h_t^y)I_2(h_T^\delta)\Big)^-$$

is integrable under $\mathbf{P}$.

Indeed, since I_1, I_2 are decreasing on non-negative, we have:

$$\begin{aligned} \Big((h_t - h_t^y)I_1(h_t^\delta, t)\Big)^- &\leq h_t^y I_1((1-\delta)h_t^y, t), \\ \Big((h_T - h_T^y)I_2(h_T^\delta)\Big)^- &\leq h_T^y I_1((1-\delta)h_T^y) \end{aligned}$$

Applying Lemma 6.3 in Kramkov and Schachermayer (1999) [39], it follows that we can find some constants c_i, with $i = 1, 2..$, and some positive y_1, y_2 such that:

$$\begin{aligned} \Big((h_t - h_t^y)I_1(h_t^\delta, t)\Big)^- &\leq c_1\widetilde{U}_1(h_t^y, t)\mathbf{1}_{h_t^y \leq y_1} + h_t^y I_1((1-\delta)h_t^y, t)\mathbf{1}_{h_t^y > y_1}, \\ \Big((h_T - h_T^y)I_2(h_T^\delta)\Big)^- &\leq c_2\widetilde{U}_2(h_T^y)\mathbf{1}_{h_T^y \leq y_2} + h_T^y I_2((1-\delta)h_T^y)\mathbf{1}_{h_T^y > y_2}, \end{aligned}$$

Notice that since $\delta \in (0, 1/2)$ hence

$$(1-\delta)h_t^y \geq \frac{1}{2}y_1, \quad \forall\ h_t^y > y_1, \quad (1-\delta)h_T^y \geq \frac{1}{2}y_2, \quad \forall\ h_T^y > y_2,$$

From (15) it follows that I_1, I_2 are bounded on $\Big[\frac{1}{2}(y_1 \wedge y_2), \infty\Big[$. Hence we can find some $\widetilde{c}_i$ such that

$$\begin{aligned} \Big((h_t - h_t^y)I_1(h_t^\delta, t)\Big)^- &\leq c_1\widetilde{U}_1(h_t^y, t)\mathbf{1}_{h_t^y \leq y_1} + \widetilde{c}_1 h_t^y \\ \Big((h_T - h_T^y)I_2(h_T^\delta)\Big)^- &\leq c_2\widetilde{U}_2(h_T^y)\mathbf{1}_{h_T^y \leq y_2} + \widetilde{c}_2 h_T^y \end{aligned}$$

By (48) and since $\widetilde{u}(y) < \infty$, we get the desired result:

$$
\begin{aligned}
\int \langle (h - h^y), I(h^\delta)\rangle^- d\mu \quad &= \quad \mathbf{E}\Big[\int_0^T ((h_t - h_t^y)I_1(h_t^\delta, t))^- dt + ((h_t - h_t^y)I_2(h_T^\delta))^-\Big] \\
&\le \quad \mathbf{E}\Big[\int_0^T \Big(c_1\widetilde{U}_1(h_t^y, t)\mathbf{1}_{h_t^y \le y_1} + \widetilde{c}_1 h_t^y\Big)dt\Big] + \\
&\quad + \mathbf{E}\Big[c_2\widetilde{U}_2(h_T^y)\mathbf{1}_{h_T^y \le y_2} + \widetilde{c}_2 h_T^y\Big] \;<\; \infty
\end{aligned}
$$

Now, to prove that $h^y \in \mathbb{D}_y^+$ we assume the contrary. Notice that as δ tends to 0, we have $h \to 0$ everywhere. Moreover, we have $I_1(0,t) = \infty$ for any $t \in [0,T]$ and $I_2(0) = \infty$. Therefore, by sending δ to 0, (56) implies the contradition, since the right-hand side term goes to ∞ by Fatou's lemma. $\square$

Lemma 6.4 *Under Assumption 6.1, 6.2 and 6.3. Let the optimal solution to the problem (47) for some $y > 0$, is $h^y \in \mathbb{D}_y^+$. Then $\widetilde{u}(y)$ is differentiable in y and we have:*

$$
y\widetilde{u}'(y) = \mathbf{E}\Big[-\int_0^T h_t^y I_1(h_t^y, t)dt - h_T^y I_2(h_T^y)\Big]. \tag{58}
$$

Moreover, if in addition $h^y \in \overline{\mathbb{D}}_y^+$ then

$$
y\widetilde{u}'(y) = \mathbf{E}\Big[-\int_0^T Y_t^y I_1(h_t^y, t)dt - Y_T^y I_2(h_T^y)\Big] \tag{59}
$$

for any $Y^y \in \mathbb{Y}_y$ that dominates h^y in a sense of (27).

Proof. From Corollary 6.1 we know that the optimal solution to the problem (47) exists under Assumption 6.1. Now fix any $\delta > 0$ sufficiently small, we will show that

$$
\liminf_{\delta \downarrow 0} -\frac{\widetilde{u}(y(1+\delta)) - \widetilde{u}(y)}{\delta} \ge \mathbf{E}\Big[\int_0^T h_t^y I_1(h_t^y, t)dt + h_T^y I_2(h_T^y)\Big] \tag{60}
$$

and

$$
\limsup_{\delta \uparrow 0} -\frac{\widetilde{u}(y(1+\delta)) - \widetilde{u}(y)}{\delta} \le \mathbf{E}\Big[\int_0^T h_t^y I_1(h_t^y, t)dt + h_T^y I_2(h_T^y)\Big] \tag{61}
$$

Let $\delta > 0$. By using successively the definition of $\widetilde{u}(y)$, the convexity of $\widetilde{U}_1$, $\widetilde{U}_2$ and its properties we obtain:

$$
\begin{aligned}
-\frac{\widetilde{u}(y(1+\delta)) - \widetilde{u}(y)}{\delta} \quad &\ge \quad \mathbf{E}\Big[\int_0^T \frac{\widetilde{U}_1((1+\delta)h_t^y, t) - \widetilde{U}_1(h_t^y, t)}{-\delta}dt + \\
&\quad + \frac{\widetilde{U}_2((1+\delta)h_T^y) - \widetilde{U}_2(h_T^y)}{-\delta}\Big] \\
&\ge \quad \mathbf{E}\Big[\int_0^T h_t^y I_1((1+\delta)h_t^y, t)dt + h_T^y I_2((1+\delta)h_T^y)\Big]
\end{aligned}
$$

We then deduce by monotone convergence theorem:

$$\liminf_{\delta\downarrow 0} -\frac{\widetilde{u}(y(1+\delta))-\widetilde{u}(y)}{\delta} \geq \mathbf{E}\Big[\int_0^T h_t^y I_1(h_t^y,t)dt + h_T^y I_2(h_T^y)\Big] \tag{62}$$

Now, without loss of generality we assume that $\delta \in (-\frac{1}{2},0)$. By the same arguments in the case $\delta > 0$, we obtain:

$$-\frac{\widetilde{u}(y(1+\delta))-\widetilde{u}(y)}{\delta} \leq \mathbf{E}\Big[\int_0^T h_t^y I_1((1+\delta)h_t^y,t)dt + h_T^y I_2((1+\delta)h_T^y)\Big] \tag{63}$$

Same arguments as in the proof of Corollary 6.1 prove the the right-hand side of (63) is integrable under $\mathbf{P}$.

Since the right-hand side in (63) is integrable under $\mathbf{P}$. Therefore we can apply the dominated convergence theorem to (63) and obtain:

$$\limsup_{\delta\uparrow 0} -\frac{\widetilde{u}(y(1+\delta))-\widetilde{u}(y)}{\delta} \leq \mathbf{E}\Big[\int_0^T h_t^y I_1(h_t^y,t)dt + h_T^y I_2(h_T^y)\Big] \tag{64}$$

From (62), (64) and the convexity of $\widetilde{u}(y)$ we get (58).

Now suppose that $h^y \in \overline{\mathbb{D}}_y^+$. Since there exists a process $Y^y \in \mathbb{Y}_y$ such that $h^y \preceq Y^y$, we then have:

$$-\widetilde{u}'(y) \leq \mathbf{E}\Big[\int_0^T Y_t^y I_1(h_t^y,t)dt + Y_T^y I_2(h_T^y)\Big] \tag{65}$$

To prove the converse inequality, we take an arbitrary element $h \in \mathbb{D}_y$, $\delta \in (0,\frac{1}{2})$ and consider the process:

$$h^\delta = (1-\delta)h^y + \delta h$$

which also belongs to the set $\mathbb{D}_y$ by the convexity of $\mathbb{D}_y$. Notice also that $\lim_{\delta\to 0} h^\delta = h^y$. Since h^y solves $\widetilde{u}(y)$, then we have:

$$\mathbf{E}\Big[\int_0^T \widetilde{U}_1(h_t^y,t)dt + \widetilde{U}_2(h_T^y)\Big] \leq \mathbf{E}\Big[\int_0^T \widetilde{U}_1(h_t^\delta,t)dt + \widetilde{U}_2(h_T^\delta)\Big] \tag{66}$$

Then by the convexity of $\widetilde{U}_1$ and $\widetilde{U}_2$, we have:

$$\begin{aligned}
\int_0^T \widetilde{U}_1(h_t^y,t)dt + \widetilde{U}_2(h_T^y) \quad &\geq \quad \int_0^T \widetilde{U}_1(h_t^\delta,t)dt \\
&+ \int_0^T (h_t^y - h_t^\delta)\widetilde{U}_1'(h_t^\delta,t)dt + \\
&+ \widetilde{U}_2(h_T^\delta) + (h_T^y - h_T^\delta)\widetilde{U}_2'(h_T^\delta) \\
&\geq \quad \int_0^T \widetilde{U}_1(h_t^\delta,t)dt + \widetilde{U}_2(h_T^\delta) + \\
&+ \delta\Big(\int_0^T (h_t - h_t^y)I_1(h_t^\delta,t)dt + \\
&+ (h_T - h_T^y)I_2(h^y h_T^\delta)\Big)
\end{aligned} \tag{67}$$

Plugging (67) to (66) and dividing by δ, we obtain:

$$\mathbf{E}\Big[\int_0^T h_t^y I_1(h_t^\delta,t)dt + h_T^y I_2(h_T^\delta)\Big] \geq \mathbf{E}\Big[\int_0^T h_t I_1(h_t^\delta,t)dt + h_T I_2(h_T^\delta)\Big] \tag{68}$$

Since $h^\delta \geq (1-\delta)h^y$, by the decrease and nonnegative of I_1 and I_2 we have:

$$\begin{aligned}
0 \quad &\leq \quad \mathbf{E}\Big[\int_0^T h_t^y I_1(h_t^\delta,t)dt + h_T^y I_2(h_T^\delta)\Big] \\
&\leq \quad \mathbf{E}\Big[\int_0^T h_t^y I_1((1-\delta)h_t^y,t)dt + h_T^y I_2((1-\delta)h_T^y)\Big]
\end{aligned} \tag{69}$$

By the same arguments as in proof of Corollary 6.1 we deduce that the right-hand side of (69) is integrable under $\mathbf{P}$.

Therefore by applying the dominated convergence theorem to the left-hand side of (66), and Fatou's lemma to the right-hand side we get:

$$-\widetilde{u}'(y) \quad \geq \quad \mathbf{E}\Big[\int_0^T h_t I_1(h_t^y,t)dt + h_T I_2(h_T^y)\Big],\ \forall\ h \in \mathbb{D}_y \tag{70}$$

$$\geq \quad \mathbf{E}\Big[\int_0^T Y_t^y I_1(h_t^y,t)dt + Y_T^y I_2(h_T^y)\Big] \tag{71}$$

where the last inequality follows from the fact that Y^y belongs to $\mathbb{D}_y$. From (65) and (71) we get the desired result. $\square$

The following lemma is adopted from Lemma 5.3 in Bouchard and Pham (2002) [4].

Lemma 6.5 *Let Assumptions 6.1, 6.2 and 6.3 hold true. Given any $x \in dom(V)$ defined in (72), there exists a unique optimal solution $y^* > 0$ to the problem*

$$V(x) \triangleq \inf_{y>0}[\widetilde{u}(y) + xy] \tag{72}$$

Proof. Let $(y^n)_n \in \mathrm{dom}(\widetilde{u})$ be a minimizing sequence of the problem $\inf_{y>0}[\widetilde{u}(y) + xy]$. By Corollary 6.1 for any y^n there exists an optimal solution $h^{y^n} \in \mathbb{D}_{y^n}$ of $\widetilde{u}(y^n)$. Let us now fix any $\epsilon > 0$ and x_0 such that

$$\frac{\phi_1(x,t)}{x} \geq \frac{1}{\epsilon}, \quad t \in [0,T], \quad \frac{\phi_2(x)}{x} \geq \frac{1}{\epsilon} \tag{73}$$

for $x \geq x_0$, where $\phi_i,\ i = 1,\,2$ are defined as in Lemma 6.3.

From (73) it follows that:

$$\begin{aligned}
\widetilde{u}(y^n)^- &= \int \widetilde{U}(h^{y^n})^- d\mu \;=\; \mathbf{E}\Big[\int_0^T \widetilde{U}_1(h_t^{y^n}, t)^- dt + \widetilde{U}_2(h_T^{y^n})^-\Big] \\
&\leq \mathbf{E}\Big[\int_0^T x_0 dt + x_0\Big] + \epsilon\mathbf{E}\Big[\int_0^T \phi_1(\widetilde{U}_1(h_t^{y^n}, t)^-, t)dt + \phi_2(\widetilde{U}_2(h_T^{y^n})^-)\Big] \\
&\leq x_0 C_3 + \epsilon(C_2 + y^n C_1)
\end{aligned} \tag{74}$$

or equivalently, we have:

$$\widetilde{u}(y^n) \geq -x_0 C_3 - \epsilon(C_2 + y^n C_1) \tag{75}$$

Now we take n large enough, such that

$$\widetilde{u}(y^n) + xy^n \leq V(x) + 1$$

Hence by choosing $\epsilon = x/2$ it follows from (75) that $(y^n)_n$ is bounded.

By Lemma 6.2 there exists a sequence $(y_1^n, h_1^n) \in \mathrm{conv}\{(y^k, h^k),\, k \geq n\}$ that converges μ-a.e. to some $(y^*, h^*) \in \mathbf{R}_+ \times \mathbb{D}_{y^*}$. By the convexity of $\widetilde{U}_i,\ i = 1,\,2$ we have:

$$\widetilde{U}(h_1^n) \leq \sup_{m \geq n} \widetilde{U}(h^k)$$

so that

$$\lim_{n\to\infty} \int \widetilde{U}(h_1^n) d\mu + xy_1^n = V(x) \tag{76}$$

Hence we have:

$$\begin{aligned}
\widetilde{u}(y^*) + xy^* &= \int \widetilde{U}(h^x)^+ d\mu - \int \widetilde{U}(h^x)^- d\mu + xy^* \\
&\leq \liminf_{n\to\infty} \widetilde{U}(h_1^n)^+ d\mu - \lim_{n\to\infty} \widetilde{U}(h_1^n)^- d\mu + xy^* \\
&\leq \lim_{n\to\infty} \int \widetilde{U}(h_1^n) d\mu + xy_1^n \;=\; V(x)
\end{aligned}$$

where the first inequality follows from Lemma 6.3 and Fatou's lemma. The last inequality follows from (76).

To prove that $y^* > 0$, we assume the contrary, then:

$$\widetilde{u}(0) = \int \widetilde{U}(0)d\mu \leq \int \widetilde{U}(h)d\mu + xy = \mathbf{E}\Big[\int_0^T \widetilde{U}_1(h_t,t)dt + \widetilde{U}_2(h_T)\Big] + xy \tag{77}$$

for all $y > 0$ and $h \in \mathbb{D}_y$. Using the properties of utility functions, we have:

$$\mathbf{E}\Big[\int_0^T \Big(\widetilde{U}_1(h_t,t) + h_t I_1(h_t,t)\Big)dt + \widetilde{U}_2(h_T) + h_T I_2(h_T)\Big] \leq \widetilde{u}(0) \tag{78}$$

Plugging (78) into (77) and dividing by $y > 0$, we obtain:

$$\mathbf{E}\Big[\int_0^T h_t I_1(h_t,t)dt + h_T I_2(h_T)\Big] \leq x, \quad \forall\ y > 0,\ h \in \mathbb{D}_y \tag{79}$$

As $y \to 0$, h tends to 0 everywhere. Moreover, by the model setting we have $I_1(0,t) = \infty$ for any $t \in [0,T]$ and $I_2(0) = \infty$. Therefore, by sending y to 0 and using Fatou's lemma, (79) implies the contradiction since $x < \infty$. Finally, the uniqueness of y^* follows from the strict convexity of $\widetilde{u}$ on $\{\widetilde{u} < \infty\}$. $\square$

Lemma 6.6 *Given $x \in dom(V)$ and let y^* be an optimal solution of (70). Then for all $y > 0$ and $h \in \mathbb{D}_y$*

$$\int \langle h, I(h^*)\rangle d\mu - xy \leq \int \langle h^*, I(h^*)\rangle d\mu - xy^* = 0 \tag{80}$$

Proof. Fix $y > 0$, $h \in \mathbb{D}_y$ and define the convex combination:

$$(y^\epsilon, h^\epsilon) = \epsilon(y,h) + (1-\epsilon)(y^*,h^*), \quad \epsilon \in (0,1/2)$$

Note that as ϵ tends to 0, we have $(y^\epsilon, h^\epsilon) \to (y^*, h^*)$.

By the optimality of (y^*, h^*) and the convexity of $\widetilde{U}$ then we have:

$$\begin{aligned} 0 &\geq \frac{1}{\epsilon}\int \widetilde{U}\Big((h^*) - \widetilde{U}(h^\epsilon)\Big)d\mu + x(y^* - y) \\ &\geq \int \langle (h - h^*), I(h^\epsilon)\rangle d\mu + x(y^* - y) \end{aligned} \tag{81}$$

By the same arguments as in Corollary 6.1 it is not hard to show that the family

$$\langle (h - h^*), I(h^\epsilon)\rangle^- \quad \text{is integrable under } \mu. \tag{82}$$

Sending ϵ to 0 in (81), using (82) and Fatou's lemma again, we obtain:

$$0 \geq \int \langle (h - h^*), I(h^*) \rangle d\mu + x(y^* - y)$$

and (80) by choosing $(y, h) = \frac{1}{2}(y^*, h^*)$ and then $(y, h) = 2(y^*, h^*)$. $\square$

Proof of Theorem 6.1

1. By Lemma 6.3 in Kramkov and Schachermayer (1999) [39], we deduce that there exists some $y_0 > 0$ such that $\widetilde{u}(y) < \infty$ for all $y \geq y_0$. On the other hand, by using Lemma 6.3 in Kramkov and Schachermayer (1999) again, there exists $y_1 > 0$ such that for all $y \in (0, y_0)$ and $h \in \mathbb{D}$ we obtain

$$\begin{aligned} \widetilde{U}_1(yh_t, t) &\leq c(y)\widetilde{U}(y_0 h_t, t)\mathbf{1}_{y_0 h_t < y_1} + \widetilde{U}(yh_t, t)\mathbf{1}_{y_0 h_t \geq y_1} \\ &\leq c(y)\widetilde{U}(y_0 h_t, t)\mathbf{1}_{y_0 h_t < y_1} + \widetilde{U}(y\frac{y_1}{y_0}, t)\mathbf{1}_{y_0 h_t \geq y_1}, \quad c(y) < \infty \end{aligned}$$

where the last inequality follows from the decrease of $\widetilde{U}_1$. As regards $\widetilde{U}_2$, the same assertion follows. This proves that $\widetilde{u}(y) < \infty$ for $y < y_0$ and so $\widetilde{u}(y) < \infty$ for all $y \in (0, \infty)$.

The rest of the assertion 1(a) follows from Corollary 6.1, and by the same arguments as in the proof of Lemma 6.5.

1(b) Quite clearly that $\mathrm{dom}(V) = (0, \infty)$ whenever $\mathrm{dom}(\widetilde{u}) \neq \emptyset$. Then the assertion 1(b) follows from Lemma 6.5.

2. Moreover, as a result of the last lemma, $\widetilde{u}$ is differentiable at y^* and we shall have:

$$-y^*\widetilde{u}'(y^*) = \mathbf{E}\left[\int_0^T h_t^* I_1(h_t^*, t)dt + h_T^* I_2(h_T^*)\right] = xy^* \tag{83}$$

Let us define

$$g_t^* = I_1(h_t^*, t)\mathbf{1}_{t \leq T} + I_2(h_T^*)\mathbf{1}_T$$

We will show that g^* is a unique solution to the optimization problem (46).

Lemma 5.1, (71), 81 and (83) we deduce that g^* belong to the set $\mathbb{C}_{-\widetilde{u}'(y^*)} \equiv \mathbb{C}_x$.

Now, for an arbitrary $g \in \mathbb{C}_x$, by the convexity of $\widetilde{U}_i$, we have:

$$\begin{aligned}
\int_0^T U_1(g_t,t)dt + U_2(g_T) &\leq \int_0^T \widetilde{U}_1(h_t^*,t)dt + \widetilde{U}_2(h_T^*) + \\
&\quad + \int_0^T h_t^* g_t dt + h_T^* g_T \\
&\leq \int_0^T U_1(I_1(h_t^*,t),t)dt + U_2(I_2(h_T^*) + \\
&\quad + \int_0^T h_t^* g_t dt + h_T^* g_T - \\
&\quad - \int_0^T h_t^* I_1(h_t^*,t)dt - h_T^* I_2(h_T^*) \\
&\leq \int_0^T U_1(g_t^*,t)dt + U_2(g_T^*) + \\
&\quad + \left(\int_0^T h_t^* g_t dt + h_T^* g_T\right) - \\
&\quad - \left(\int_0^T h_t^* g_t^* dt + h_T^* g_T^*\right)
\end{aligned}$$

Taking expectation, we obtain:

$$\begin{aligned}
\int U(g)d\mu = \mathbf{E}\left[\int_0^T U_1(g_t,t)dt + U_2(g_T)\right] &\leq \mathbf{E}\left[\int_0^T U_1(g_t^*,t)dt + U_2(g_T^*)\right] + \\
&\quad + \mathbf{E}\left[\int_0^T g_t h_t^* dt + g_T h_T^* - xy^*\right] \\
&\leq \mathbf{E}\left[\int_0^T U_1(g_t^*,t)dt + U_2(g_T^*)\right] \\
&\leq \int U(g^*)d\mu
\end{aligned}$$

The second inequality follows from Lemma 5.1 and the fact that $g \in \mathbb{C}_x$, $h^* \in \mathbb{D}_{y^*}$. This proves the optimality of g^*.

Now let $(W^*, c^*) \in \mathcal{A}^*(x)$ be any element that dominates g^* in a sense of (26). By (83) and from Lemma 5.1, we have:

$$\begin{aligned}
xy^* = \int \langle h^*, I(h^*)\rangle d\mu = \int \langle h^*, g^*\rangle d\mu &\leq \mathbf{E}\left[\int_0^T h_t^* c_t^* dt + h_T^* W_T^*\right] \\
&\leq xy^*
\end{aligned}$$

and hence:

$$\mathbf{E}\left[\int_0^T h_t^* c_t^* dt + h_T^* W_T^*\right] = xy^*$$

3. For any fixed $x > 0$, by the definition of the dual set $\mathbb{D}_y$ we have:

$$u(x) \leq \inf_{y>0}[\widetilde{u}(y) + xy] \tag{84}$$

Moreover, we have:

$$\begin{aligned}
u(x) &= \int U(g^*)d\mu \;=\; \mathbf{E}\Big[\int_0^T U_1(g_t^*, t)dt + U_2(g_T^*)\Big] \\
&= \mathbf{E}\Big[\int_0^T \widetilde{U}_1(h_t^*, t)dt + \widetilde{U}_2(h_T^*)\Big] + xy^* \\
&= \int \widetilde{U}(h^*)d\mu \;=\; \widetilde{u}(y^*) + xy^*
\end{aligned}$$

This proves that:

$$u(x) = \inf_{y>0}[\widetilde{u}(y) + xy] \tag{85}$$

The second formula of assertion (3) follows from (85) and the general bidual property of the Legendre-transform (see, e.g. Theorem III.12.2 in Rockafellar (1970) [44]).□

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