Portfolio Selection with Two-Stage Preferences

Marco Taboga*

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Abstract

We propose a model of portfolio selection under ambiguity, based on a two-stage valuation procedure which disentangles ambiguity and ambiguity aversion. The model does not imply "extreme pessimism" from the part of the investor, as multiple priors models do. Furthermore, its analytical tractability allows to study complex problems thus far not analyzed, such as joint uncertainty about means and variances of returns.

*Bank of Italy and University of Turin. Address: Banca d'Italia, Servizio Studi, Via Nazionale 91, 00184 - Roma, Italy. e-mail: mtaboga@fastwebnet.it. Any views expressed in this article are mine and do not necessarily represent those of the Bank of Italy. I thank Erio Castagnoli, Peter Klibanoff, Elisa Luciano, Fabio Maccheroni, Massimo Marinacci, Sujoy Mukerji, two anonymous referees and seminar participants at the Bank of Italy for helpful comments and suggestions.

1 Introduction

Traditional portfolio selection models, such as Markowitz's (1952), assume that the distribution of asset returns is objectively known by the decision maker. Several models recently relaxed this assumption, taking into account the possibility that the decision maker is unable to form a unique probability distribution. Most of these models are firmly grounded in decision theory and make theoretical appeal to the literature about ambiguity. Ambiguity (or Knightian uncertainty) arises when a decision maker is unable to unambiguously assign probabilities to the events which are relevant to her decision.

We base portfolio selection on a model of decisions under ambiguity proposed and axiomatized by Klibanoff, Marinacci and Mukerji (2003 - henceforth KMM). According to their model a decision maker maximizes:

$$\mathrm{E}_{\mu}\left[\varphi\left(\mathrm{E}_{\pi}\left[u\left(W\right)\right]\right)\right]$$

where W is stochastic future wealth, φ and u are increasing and concave functions, μ and π are probability measures and E_{μ} and E_{π} denote expected values with respect to such measures. The inner expectation (with respect to the measure π) is akin to a von-Neumann Morgenstern expected utility. The investor realizes that to different distributions of asset returns correspond different values of the expected utility of wealth. The measure μ assigns second-order probabilities to the distributions which are deemed plausible by the investor and hence to expected utility values. Instead of simply averaging over expected utilities, the decision maker applies a concave transform before taking the outer expectation, because she dislikes mean-preserving spreads in expected utility values.

The model differs considerably from models previously used to study portfolio selection under ambiguity and carries new implications both from a positive and a normative perspective. Most existing models are based on Gilboa and Schmeidler's (1989 - henceforth GS) multiple priors preferences. Among them, Epstein and Wang (1994), Dow and Werlang (1992) and Chen and Epstein

(2002). Roughly speaking, an investor with multiple priors preferences has got several priors (probability distributions) and, after choosing a portfolio, she selects the prior which yields the lowest expected utility given her choice. Another class of models, based on robust control techniques, includes Maenhout (2004), Anderson, Hansen and Sargent (2003) and Uppal and Wang (2003). In the latter models agents have a reference prior, but recognize the possibility of misspecification and account for it in their decisions, by considering alternative priors and a penalty for deviating from the reference one.

The model we propose does not belong to either of the two classes mentioned above and we believe it has got some advantages over them. First, it allows to overcome a shortcoming of existing models: both those based on multiple priors and those based on robust control share a common feature, that is they require the solution of a maxmin problem; due to the limited analytical tractability of maxmin problems, these models restrict attention to very special sets of alternative priors, namely to sets of normal distributions wich have different means, but the same variances and covariances. Our model, instead, thanks to the smoothness of the preference functional we adopt and the simplicity of the first order condition for an optimal portfolio, allows to deal simultaneously with uncertainty about means, variances and covariances of returns. This is especially important in normative contexts, where uncertainty about variances and covariances has been proven to be as important as ambiguity about means (see e.g. Jagannathan and Ma - 2003). Another advantage of the model is that it allows for a distinction between ambiguity, coming from multiple probability distributions, and ambiguity aversion, parametrized by the concavity of the function φ. This distinction is not possible in the other models, where the set of priors over which minimization takes place determines both the degree of ambiguity and ambiguity aversion. Models of ambiguity aversion are attracting the attention of financial economists not only because they are a useful device to understand the behavioral implications of ambiguity aversion for financial decisions, but also because they are theoretically well-founded decision tools, suited to address real-world situations where the lack of sufficient information about the distribution of asset returns makes classical decision rules inadequate. In this perspective, the model makes it possible for the investor to separately quantify ambiguity (for example, stemming from objective imprecision of statistical estimates) and her subjective attitude towards ambiguity, parametrized by φ. Furthermore, as documented in the last section of the paper, there are situations in which preferences displaying "extreme pessimism", such as multiple priors, have paradoxical consequences, which our model has not. In Section 4 we will prove that in our framework it is optimal to take all priors into consideration, although giving less importance to the most optimistic ones: this is in sharp contrast with the multiple priors model, where the decision maker behaves as if she was considering only one of the priors.

The paper is organized as follows: Section 2 describes the salient features of the model. Section 3 conducts a preliminary analysis of the general properties of an optimal portfolio. Section 4 specializes to an instance of the model. Section 5 presents an application. Section 6 concludes the paper.

2 The model

We consider the one-period allocation problem of an agent who has to decide how to invest a unit of wealth at time 0, dividing it among $n+1$ assets. The gross return on the i-th asset after one period is a random variable denoted by R_i. The $(n \times 1)$ vector of the returns on the first n assets is denoted by R and the $(n \times 1)$ vector of portfolio weights, indicating the fraction of wealth invested in each of the first n assets is denoted by x.[1] The end-of-period wealth W_x

[1] We assume that there are no frictions of any kind: securities are perfectly divisible; there are no transaction costs or taxes; the agent is a price-taker, in that she believes that her choices do not affect the distribution of asset returns; there are no institutional restrictions, so that the agent is allowed to buy, sell or short sell any desired amount of any security (this

(depending on the portfolio choice x) is equal to:

$$W_x = R_{n+1} + x^{\intercal}\left(R - \overrightarrow{1} R_{n+1}\right)$$

where $\overrightarrow{1}$ is a column vector of 1s of dimension n. The above definition of W_x implicitly accomodates the requirement that the portfolio weights sum up to unity.

Let $(\Omega, \sigma(\Omega), \mu)$ be a measure space and assume that for each $\omega \in \Omega$ we are given a measure $\pi(\omega; \cdot)$ on the Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^{n+1}\right)$ of subsets of $\mathbb{R}^{n+1}$. Assume also that, for each $B \in \mathcal{B}\left(\mathbb{R}^{n+1}\right)$, $\pi(\omega; B)$ is $\sigma(\Omega)$-measurable. Both μ and $\pi(\omega; \cdot)$ are assumed to be probability measures. As a consequence, there exists a probability measure P defined on the product σ-algebra $\sigma(\Omega) \times \mathcal{B}\left(\mathbb{R}^{n+1}\right)$ such that:

$$P(A \times B) = \int_A \pi(\omega; B)\, d\mu(\omega) \qquad \forall A \in \sigma(\Omega), B \in \mathcal{B}\left(\mathbb{R}^{n+1}\right)$$

Furthermore, if $f \in L^1\left(\Omega \times \mathbb{R}^{n+1}\right)$ [2] , then the function

$$\omega \to \int_{\mathbb{R}^n} f(\omega, r)\, d\pi(\omega; r)$$

is μ-a.s. well-defined, belongs to $L^1(\Omega)$ and the conditional version of Fubini's theorem (see Ash and Doléans Dade (1999)) ensures that:

$$\int_{A \times B} f(\omega, r)\, dP(\omega, r) = \int_A \int_B f(\omega, r)\, d\pi(\omega; r)\, d\mu(\omega)$$

for any $A \in \sigma(\Omega)$ and $B \in \mathcal{B}\left(\mathbb{R}^{n+1}\right)$.

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assumption can be weakened, by simply requiring that at an optimum institutional restrictions are not binding).

[2] From now on, $L^p(\Omega)$, $p \in [1, \infty)$, denotes the space of all functions $f(\omega)$ defined on Ω such that $|f(\omega)|^p$ is integrable with respect to μ. Similarly, $L^p\left(\Omega \times \mathbb{R}^{n+1}\right)$, $p \in [1, \infty)$, denotes the space of all functions $f(\omega, r)$ defined on $\Omega \times \mathbb{R}^{n+1}$ such that $|f(\omega, r)|^p$ is integrable with respect to P, and, when a measure $\pi(\omega; \cdot)$ on $\mathcal{B}\left(\mathbb{R}^{n+1}\right)$ is given, $L^p\left(\mathbb{R}^{n+1}\right)$ denotes the space of functions $f(r)$ such that $|f(r)|^p$ is integrable with respect to $\pi(\omega; \cdot)$. Statements which hold almost surely with respect to μ, P or $\pi(\omega; \cdot)$ will be said to hold μ-a.s., P-a.s. or $\pi(\omega,)$-a.s. respectively.

We identify the return on the i-th asset R_i with the i-th component of the element extracted from the sample space $\left(\mathbb{R}^{n+1}, \mathcal{B}\left(\mathbb{R}^{n+1}\right)\right)$, so that each conditional probability measure $\pi(\omega; \cdot)$ on $\mathcal{B}\left(\mathbb{R}^{n+1}\right)$ can be interpreted as a distribution of asset returns. We assume that $R_i \in L^2\left(\Omega \times \mathbb{R}^{n+1}\right)$ and $R_i \in L^2\left(\mathbb{R}^{n+1}\right)$ for each i and each $\pi(\omega; \cdot)$ in a set of μ-measure 1.

The investor chooses the portfolio weights x in order to solve the following maximization problem:

$$\sup_{x \in \mathbb{R}^n} \mathrm{E}_\mu \left[\varphi\left(\mathrm{E}_\pi \left[u\left(W_x\right)\right]\right)\right] \tag{1}$$

which is a shorthand for:

$$\sup_{x \in \mathbb{R}^n} \int_\Omega \varphi\left(\int_{\mathbb{R}^{n+1}} u\left(W_x\right) d\pi(\omega; r)\right) d\mu(\omega)$$

As we have anticipated in the introduction, the above objective function, first axiomatized as a specification of preferences by Klibanoff, Marinacci and Mukerji (2003), is conceptually very simple. Uncertainty about future asset returns cannot be described by a unique probability distribution, but the investor is able to identify a set of probability measures which could be plausible descriptions of the randomness inherent in the asset allocation problem. Each probability measure $\pi(\omega;)$ yields a von-Neumann Morgenstern expected utility value $\mathrm{E}_{\pi(\omega)}\left[u\left(W_x\right)\right]$, where u is concave, strictly increasing and finite-valued. Since u is concave and strictly increasing and W_x is linear in the asset returns, then $u(W) \in L^1\left(\Omega \times \mathbb{R}^{n+1}\right)$, $u(W) \in L^1\left(\mathbb{R}^{n+1}\right)$ and $\mathrm{E}_{\pi(\omega)}[u(W)]$ is a $\sigma(\Omega)$-measurable function belonging to $L^1(\Omega)$. By considering all the probability measures $\pi(\omega; \cdot)$ for ω in Ω, we obtain a whole range of expected utility values; formally, we have a mapping $U : \Omega \rightarrow \mathbb{R}$ defined by:

$$U(\omega) = \mathrm{E}_{\pi(\omega)}\left[u\left(W_x\right)\right]$$

The investor subjectively assigns a degree of likelihood to the probability measures $\pi(\omega; \cdot)$, that is she forms a subjective measure μ, assigning a probability to a σ-algebra $\sigma(\Omega)$ of subsets of Ω. This is compatible, for example, with a Bayesian framework in which the distribution of asset returns is parametrized by a vector θ and the investor assigns a non-degenerate probability distribution to the parameter vector θ.

Since $U(\omega)$ is a measurable function and belongs to $L^1(\Omega)$, the investor is able to evaluate the integral:

$$\mathrm{E}_\mu[\varphi(U)]$$

where φ is again taken to be a concave, strictly increasing and finite-valued function. The above expectation is an "expected utility of expected utilities": if μ is non-degenerate, the expected utility U is a random variable; the investor, instead of simply maximizing the expected value $\mathrm{E}_\mu[U]$ of the possible values of U, maximizes the expectation $\mathrm{E}_\mu[\varphi(U)]$ of a concave transform of U. The concavity of φ reflects the fact that the investor dislikes mean-preserving spreads in expected utility values, as highlighted by Jensen's inequality:

$$\mathrm{E}_\mu[\varphi(U)] \leq \varphi(\mathrm{E}_\mu[U])$$

3 Preliminary analysis

If maximization problem (1) admits an interior solution x^* and if both u and φ are differentiable, the following first order necessary condition must be satisfied[3]:

$$\mathrm{E}_\mu\left[\varphi'(\mathrm{E}_\pi[u(W_{x^*})])\,\mathrm{E}_\pi\left[u'(W_{x^*})\left(R-\overrightarrow{1}R_{n+1}\right)\right]\right]=0 \tag{2}$$

When there is no ambiguity the above condition simplifies to:

$$\mathrm{E}_\pi\left[u'(W_{x^*})\left(R-\overrightarrow{1}R_{n+1}\right)\right]=0 \tag{3}$$

which has a simple economic interpretation: if the portfolio allocation is optimal, the marginal increase in utility obtained selling one dollar worth of asset

[3]We assume that differentiation under the integral sign is legitimate and the dominated convergence theorem applies: for example it suffices that $u'(W_{x^*})$ be bounded $\pi(\omega;\cdot)$-a.s. for any ω in a set of μ-measure 1 and $\varphi'\left(\mathrm{E}_{\pi(\omega)}[u(W_{x^*})]\right)$ be bounded μ-a.s.

$n+1$ and investing the proceeds into any one of the other assets, must have zero expected value. When ambiguity is present, i.e. μ is non-degenerate, condition (3) does not necessarily hold for every $\pi(\omega;\cdot)$; there might be some probability measures $\pi(\omega;\cdot)$ under which a reallocation of the portfolio increases the expected utility value $\mathrm{E}_\pi[u(W_x)]$; however, once one accounts for the effect of the reallocation on the whole range of expected utility values, the overall marginal benefit of the reallocation must be zero.

The first order condition is better understood in the light of the following proposition (proved in the Appendix):

Proposition 1 *Let φ and u be of class C^1. Assume at an optimum $u'(W_{x^*})$ is bounded $\pi(\omega;\cdot)$-a.s. for any ω in a set of μ-measure 1 and $\varphi'\left(\mathrm{E}_{\pi(\omega)}[u(W_{x^*})]\right)$ is bounded μ-a.s. Then, there exists a probability measure μ^*, equivalent to μ, such that condition (2) is equivalent to:*

$$\mathrm{E}_{\mu^*}\left[\mathrm{E}_\pi\left[u'(W_{x^*})\left(R-\overrightarrow{1}R_{n+1}\right)\right]\right]=0$$

or to:

$$\mathrm{E}_Q\left[u'(W_{x^*})\left(R-\overrightarrow{1}R_{n+1}\right)\right]=0 \tag{4}$$

where $Q=\pi\times\mu^*$ and

$$\frac{d\mu^*}{d\mu}=\frac{\varphi'(\mathrm{E}_\pi[u(W_{x^*})])}{\mathrm{E}_\mu[\varphi'(\mathrm{E}_\pi[u(W_{x^*})])]}$$

Condition (4) is easily compared to condition (3), arising in the case without ambiguity. The difference between the two optimality conditions lies in the probability measure used to evaluate expected marginal utility. Without ambiguity, the unique probability measure π is used, while in the presence of ambiguity expectations are formed under a different probability measure Q, which depends on the portfolio chosen by the investor (the Radon-Nykodym derivative $d\mu^*/d\mu$ is a function of x^*). This is consistent with the empirical evidence motivating models of ambiguity aversion, for example the classic Ellsberg's (1961) paradox, where the same prior cannot explain all of the decision maker's choices. Were the investor to choose a different portfolio, she would use another probability measure, different from Q, to evaluate her new choice.

Proposition 1 sheds some light on the implications of optimal choice in the KMM framework: while KMM (2003) axiomatize and provide a full characterization of individual preferences, their paper only presents some numerical examples to illustrate instances of individual choice and does not give a general characterization of the consequences of optimal choice within any class of decision problems. Equation (4) extends the classical result on equalization of expected marginal utilities in portfolio choice (see e.g. Cochrane - 2001) to the KMM framework. A qualitative analysis of the Radon-Nykodym derivative provides some insights on the optimal allocation rule and the dependence of Q on the portfolio chosen by the investor: when φ is concave, φ' is decreasing; this implies that the original measure μ is distorted by subtracting weight from those distributions which yield high expected utility values and adding more weight to those distributions which yield low expected utility values. Roughly speaking, the measure Q is a weighted average of the priors π, where the weights are not assigned according to the original hyper-prior μ, but according to a more cautios one (note that how probabilities are reallocated depends on x^* via the distributions of wealth it induces). This is a crucial difference with respect to max-min models à la Gilboa and Schmeidler (1989): in a max-min model the investor displays extreme pessimism and chooses the prior which yields the lowest expected utility, while all other priors are discarded; here all the priors are considered, but the investor gives more importance to the most pessimistic ones. Section 5 presents an example where this difference seems to imply a more reasonable behavior by the investor.

4 A tractable implementation

To implement the model described in the previous section, we need to specify the functions u and φ characterizing the investor's preferences, as well as the set of probability distributions π and a second-order probability distribution μ over them.

We assume u has CARA form:

$$u(W) = -e^{-\gamma W}$$

where γ is the coefficient of absolute risk aversion.

We further assume that, for each $\omega \in \Omega$ there exist an $(n \times 1)$ vector m_ω and an $(n \times n)$ symmetric and positive definite matrix Σ_ω such that, under the probability measure $\pi(\omega;)$, the vector of returns R has a multivariate normal distribution with expected value m_ω and variance-covariance matrix Σ_ω. Finally, we assume that R_{n+1} is equal to a constant R_f, i.e. the $(n+1)$-th asset is risk-free.

The joint hypothesis of normality and CARA utility implies that in the standard case when there is no ambiguity, i.e. $\Omega = \{\omega\}$ is a singleton, to solve maximization problem (1) is equivalent to solving Markowitz's portfolio selection problem

$$\sup_{x \in \mathbb{R}^n} \left(R_f + x^\intercal \left(m_\omega - \overrightarrow{1} R_f \right) - \frac{\gamma}{2} x^\intercal \Sigma_\omega x \right)$$

whose well-known solution is:

$$x^* = \frac{1}{\gamma} \Sigma_\omega^{-1} \left(m_\omega - \overrightarrow{1} R_f \right) \tag{5}$$

Although one will not generally be able to find a closed-form solution x^* to the first order condition

$$\mathrm{E}_\mu \left[\varphi' \left(\mathrm{E}_\pi \left[u \left(W_x \right) \right] \right) \mathrm{E}_\pi \left[u' \left(W_x \right) \left(R - \overrightarrow{1} R_f \right) \right] \right] = 0$$

it is possible to resort to numerical procedures to solve it. However, since iterated numerical integration is often a burdensome task, it is desirable to be able to calculate explicitly the two inner expected values, which is made possible by the above specification.

As to the function φ, we choose an investor with constant ambiguity attitude[4], as defined by KMM:

$$\varphi(U) = -e^{-\zeta U}$$

The following result characterizes the optimal portfolio allocation:

Proposition 2 *The first order condition for an optimal portfolio is solved by:*

$$x^* = \frac{1}{\gamma}\overline{\Sigma}^{-1}\left(\overline{m} - \overrightarrow{1}R_f\right) \tag{6}$$

where:

$$\begin{aligned} \overline{m} &= \mathrm{E}_\nu\left[m_\omega\right] \\ \overline{\Sigma} &= \mathrm{E}_\nu\left[\Sigma_\omega\right] \end{aligned}$$

and

$$\frac{d\nu}{d\mu} = \frac{\varphi'\left(\mathrm{E}_\pi\left[u\left(W_{x^*}\right)\right]\right)\mathrm{E}_\pi\left[u\left(W_{x^*}\right)\right]}{\mathrm{E}_\mu\left[\varphi'\left(\mathrm{E}_\pi\left[u\left(W_{x^*}\right)\right]\right)\mathrm{E}_\pi\left[u\left(W_{x^*}\right)\right]\right]} \tag{7}$$

The optimal portfolio (6) is easily compared to Markowitz's optimal portfolio (5): the vector of expected returns m_ω and the variance-covariance matrix Σ_ω, which are unique when Ω is a singleton, are replaced by an average of all the vectors m_ω and the matrices Σ_ω associated with the probability measures $\pi(\omega;)$. However, the averages are not computed using the original measure μ, which assigns second-order probabilities to the different priors, but another measure ν equivalent to μ: switching from μ to ν the investor attaches more weight to the most pessimistic priors and less weight to the most optimistic ones.

[4]As explained by KMM, the intuition behind constant ambiguity attitude is that translating distributions of expected utility values by a constant does not change the preference ordering.

Proposition 2 is of normative interest because it relates to a vast literature questioning the validity of Markowitz's portfolios when expected returns, variances and covariances are uncertain (see e.g.: Frankfurter, Phillips and Seagle - 1971, Barry - 1974, Bawa and Klein - 1976, and Jobson and Korkie - 1980). The proposition suggests that, according to the decision rule proposed in our model, Markowitz's portfolios are optimal even in the presence of parameter uncertainty, provided that appropriate averages of the parameter estimates made by the investor are used.

5 An example

In this section we apply the methodology explained in the previous section to a simple portfolio selection problem, with a single risky asset. The example is inspired by a recent strand of the asset pricing literature, which documents the existence of multiple regimes in stock returns (e.g.: Ang and Bekaert - 2001, Davis and Veronesi - 2001 and Whitelaw - 2001). It seems natural to interpret the existence of multiple regimes as ambiguity about asset returns: under each regime, returns have a different probability distribution and the investor is not able to identify which one correctly describes returns next period. We use the estimates reported by Guidolin and Timmerman (2004): analyzing a value-weighted portfolio of US stocks, they find that specification tests suggest the adoption of a Markov-switching model with two regimes, dubbed "bull" and "bear" respectively. Under both regimes monthly stock returns are normally distributed. A bull market is characterized by low volatility (3.3%) and high excess expected returns (1.11%), while in a bear market volatility is high (6.3%) and expected returns are low (-0.93%). The unconditional probability of being in a bull market is 79.1%, while that of being in a bear market is 20.9%. Table 1 summarizes their findings, adopting the notation of our model.

Table 1

Distributions of returns to the risky asset

	$m - R_f$	σ	$\mu(\omega_i)$
$\pi(\omega_1)$	1.11%	3.3%	79.1%
$\pi(\omega_2)$	-0.93%	6.3%	20.9%

Asset returns may follow either the probability distribution $\pi(\omega_1)$ or $\pi(\omega_2)$. Both distributions are normal, although they have different mean m and variance σ^2. $\mu(\omega_i)$ are the second order probabilities assigned to the two distributions.

We interpret the two regimes as two probability distributions ($\pi(\omega_1)$ and $\pi(\omega_2)$) and the unconditional probabilities of the two regimes as second-order probabilities ($\mu(\omega_1)$ and $\mu(\omega_2)$). Table 2 displays the optimal portfolio and the optimal reallocation of second-order probabilities as the coefficient of ambiguity aversion ζ increases (γ is set equal to 5).

Table 2

Optimal portfolios and re-allocations of second-order probabilities

ζ	$\nu(\omega_1)$	$\nu(\omega_2)$	$\overline{m} - R_f$	$\sqrt{\overline{\sigma^2}}$	x^*
1	77.45%	22.55%	0.651%	4.16%	74.78%
10	77.35%	22.65%	0.648%	4.17%	74.44%
100	76.46%	23.54%	0.630%	4.20%	71.27%
∞ (maxmin)	0	100%	-0.93%	6.30%	-46.87%

ζ is the coefficient of ambiguity aversion, $\nu(\omega_i)$ are the second-order probabilities, as re-allocated at an optimum, $\overline{m} - R_f$ and $\overline{\sigma^2}$ are the average excess return and variance of the risky asset under ν and x^* is the optimal share of wealth to be invested in the risky asset.

As predicted by Proposition 2 the optimal reallocation of probabilities puts more mass on the bear regime, where expected utility is lower, and less mass on the bull regime, where expected utility is higher. As ambiguity aversion increases, this behavior is more pronounced: therefore, the fraction of wealth invested in the risky asset diminishes. Table 2 also reports the optimal behavior of an agent who solves a maxmin problem (KMM prove that as the coefficient of ambiguity aversion goes to infinity, the optimal behavior coincides with that of a multiple priors agent à la Gilboa and Schmeidler). An agent considering only the worst case scenario would behave as if the probability distribution under the bear regime truly described asset returns. The outcome of applying a maxmin decision rule is clearly far from being reasonable: behaving as if the bear regime was the only possible regime and expected excess returns were negative, the agent would sell the risky asset short and ignore completely the fact that in the bull regime the expected excess return is positive. On the contrary, an investor with two-stage preferences has a more reasonable behavior and takes into consideration both regimes, although she prudentially overweighs the bear regime. We believe that this simple example, although admittedly stylized, illustrates why a decision rule à la KMM might be more sensible than a maxmin decision rule in some normative contexts. It is not uncommon for an agent to face an investment decision where no single prior seems adequate to describe the distribution of asset returns. In such a situation, it would be possible to follow a maxmin criterion, whereby the decision maker considers only the worst prior. Following the alternative criterion proposed by KMM, which is equally justifiable on formal grounds, the decision maker would average all the priors in a cautious way. The two criteria conform to sets of axioms giving a different definition of individual rationality. [5] Our example is meant to show how following the latter criterion one might avoid some odd consequences of maxmin; the argument would be strengthened by the simple observation that even a hundred-fold increase in the coefficient of ambiguity aversion produces only a

[5] Note that, although in GS's setting it is not possible to separate tastes and beliefs in the maxmin representation of individual preferences, the main equivalence theorem in GS guarantees that a decision maker applying a maxmin criterion when a set of probability measures is exogenously given or predetermined, conforms to GS's axioms.

small reallocation of weights among priors and that to obtain a reallocation close to that of the limiting maxmin case one would have to raise the ambiguity aversion to implausibly extreme levels.

6 Concluding remarks

We have addressed the problem of optimally selecting a portfolio of assets when the probabilistic distribution of asset returns is not known with precision. We adopt a specification of individual preferences, axiomatized by Marinacci, Klibanoff and Mukerji (2003), which posits that agents have many first-order priors, to which they assign probabilities by forming a second-order prior. Our model differs in many respects from previous models of portfolio selction under ambiguity (uncertainty about the true distribution of asset returns), based on Gilboa and Schmeidler's (1989) multiple priors preferences. The model is computationally very tractable, which makes it suitable to analyze complex problems not addressed by the previous literature, for example joint uncertainty about means and variances of returns. A novel feature of the model is that it allows to disentangle ambiguity and ambiguity aversion. Furthermore, the behavioral implications of the model differ from those of previous models based on multiple priors preferences: it turns out that at an optimum the investor behaves as if she took decisions using a weighted average of all the priors, but giving more weight to the more pessimistic ones; this is an important difference with respect to the multiple priors model, where the investor behaves as if she was considering only one prior, i.e. the worst one. We have also provided an example where this difference determines a very different optimal behavior by the investor.

7 Appendix

Proof of Proposition 1. The optimality condition is:

$$\mathrm{E}_{\mu}\left[\varphi'\left(\mathrm{E}_{\pi}\left[u\left(W\right)\right]\right)\mathrm{E}_{\pi}\left[u'\left(W\right)\left(R-\overrightarrow{1}R_{n+1}\right)\right]\right]=0$$

where $u'(W)$ is bounded $\pi(\omega;\cdot)$-a.s. for any ω in a set of μ-measure 1 and $\varphi'\left(\mathrm{E}_{\pi(\omega)}\left[u\left(W\right)\right]\right)$ is bounded μ-a.s.

Under the assumptions stated in Section 1, $\mathrm{E}_{\pi}\left[u\left(W\right)\right]$ is $\sigma(\Omega)$-measurable; furthermore, φ' is continuous since $\varphi\in C^1$, hence $\varphi'\left(\mathrm{E}_{\pi}\left[u\left(W\right)\right]\right)$ is $\sigma(\Omega)$-measurable; it is also strictly positive. Define:

$$\xi=\frac{\varphi'\left(\mathrm{E}_{\pi}\left[u\left(W\right)\right]\right)}{\mathrm{E}_{\mu}\left[\varphi'\left(\mathrm{E}_{\pi}\left[u\left(W\right)\right]\right)\right]}$$

where the denominator is finite given the boundedness assumption on $\varphi'\left(\mathrm{E}_{\pi(\omega)}\left[u\left(W\right)\right]\right)$.

The optimality condition can be rewritten as:

$$\mathrm{E}_{\mu}\left[\xi\mathrm{E}_{\pi}\left[u'\left(W\right)\left(R-\overrightarrow{1}R_{n+1}\right)\right]\right]=0$$

Since $\xi>0$ and $\mathrm{E}_{\mu}\left[\xi\right]=1$, ξ can be used to define a change of measure and write the optimality condition as:

$$\mathrm{E}_{\mu^*}\left[\mathrm{E}_{\pi}\left[u'\left(\widetilde{W}\right)\left(R-\overrightarrow{1}R_{n+1}\right)\right]\right]=0$$

where μ^* is another probability measure, absolutely continuous with respect to μ, with Radon-Nikodym derivative

$$\frac{d\mu^*}{d\mu}=\xi$$

The above double expectation is just a double integral:

$$\int_{\Omega}\int_{\mathbb{R}^n}u'\left(W\right)\left(R-\overrightarrow{1}R_{n+1}\right)d\pi\left(\omega;r\right)d\mu^*\left(\omega\right)$$

Since $u'(W)$ is bounded $\pi(\omega;\cdot)$-a.s. for any ω in a set of μ-measure 1 and $R_i\in L^2\left(\mathbb{R}^n\right)$ for any i, the inner integral belongs to $L^1(\Omega)$, Tonelli's theorem guarantees that the integral

$$\int_{\Omega\times\mathbb{R}^n}u'\left(W\right)\left(R-\overrightarrow{1}R_{n+1}\right)d\pi\times\mu^*$$

is well-defined and equals the above double integral.

Thus, the optimality condition becomes:

$$\mathrm{E}_Q\left[u'(W)\left(R-\overrightarrow{1}R_{n+1}\right)\right]=0$$

where $Q=\pi\times\mu^*$. ■

Proof of Propositon 2. Recall that

$$\mathrm{E}_\pi\left[u'(W)\left(R-\overrightarrow{1}R_f\right)\right]=-\gamma\mathrm{E}_\pi\left[u(W)\right]\left[m_\omega-\overrightarrow{1}R_f-\gamma\Sigma_\omega x\right]$$

The first order condition

$$\mathrm{E}_\mu\left[\varphi'\left(\mathrm{E}_\pi\left[u(W)\right]\right)\mathrm{E}_\pi\left[u'(W)\left(R-\overrightarrow{1}R_f\right)\right]\right]=0$$

becomes:

$$\mathrm{E}_\mu\left[-\gamma\varphi'\left(\mathrm{E}_\pi\left[u(W)\right]\right)\mathrm{E}_\pi\left[u(W)\right]\left(m_\omega-\overrightarrow{1}R_f-\gamma\Sigma_\omega x\right)\right]=0 \tag{8}$$

Since

$$-\gamma\varphi'\left(\mathrm{E}_\pi\left[u(W)\right]\right)\mathrm{E}_\pi\left[u(W)\right]$$

is strictly positive we can divide both sides of (8) by its expectation, obtaining:

$$\mathrm{E}_\mu\left[\xi\left(m_\omega-\overrightarrow{1}R_f-\gamma\Sigma_\omega x\right)\right]=0$$
$$\xi=\frac{\varphi'\left(\mathrm{E}_\pi\left[u(W)\right]\right)\mathrm{E}_\pi\left[u(W)\right]}{\mathrm{E}_\mu\left[\varphi'\left(\mathrm{E}_\pi\left[u(W)\right]\right)\mathrm{E}_\pi\left[u(W)\right]\right]}$$

or

$$\mathrm{E}_\nu\left[m_\omega\right]-\overrightarrow{1}R_f-\gamma\mathrm{E}_\nu\left[\Sigma_\omega\right]x=0$$

where the measure ν is defined by $\frac{d\nu}{d\mu}=\xi$. Invertibility of $\overline{\Sigma}=\mathrm{E}_\nu\left[\Sigma_\omega\right]$ is proved as follows: for each ω, Σ_ω is positive definite, i.e. for any $x\neq 0$

$$x^\top\Sigma_\omega x>0$$

but this implies that also $\overline{\Sigma}$ is positive definite (hence invertible), because:

$$x^\top\overline{\Sigma}x=x^\top\mathrm{E}_\nu\left[\Sigma_\omega\right]x=\mathrm{E}_\nu\left[x^\top\Sigma_\omega x\right]>0$$

This proves the proposition in the text. ■

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