GENERAL OPTION EXERCISE RULES, WITH APPLICATIONS TO EMBEDDED OPTIONS AND MONOPOLISTIC EXPANSION

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This paper provides a general framework for pricing of real options in continuous time for wide classes of payoff streams that are functions of Lévy processes. As applications, we calculate the option values of multi-stage investment/disinvestment problems (sequences of embedded options, which we call *Russian dolls*), and study two models of expansion of a monopoly. In the first model, each time when the stochastic demand reaches the boundary of the inaction region or crosses it, the monopoly increases capital stock but uses the same production technology. We assume that above a certain level, the stochastic demand factor increases slower than in the standard geometric Lévy models, and demonstrate that then the investment threshold is lower than in the standard models. Moreover, in the intermediate range between the regimes of the fast and slower growth, the monopoly may find it optimal to simultaneously increase the capital stock and decrease the output price. The second model is driven by two factors: one factor follows a process with upward jumps and describes the dynamics of the frontier technology, the other - demand uncertainty. The impact of these factors on new technology adoption is analyzed. It is shown that depending on the situation and type of uncertainty, the diffusion uncertainty and jump uncertainty can produce opposite effects.

Keywords: embedded options, technology adoption, capital expansion.

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1. Introduction

The paper presents the general method for valuation of and optimal exercise strategies for contingent claims of American type. In all dynamic models in economics under uncertainty one needs to calculate the expected present values (EPV) of streams of payoffs that are acquired or lost at some point in time. In many instances, such streams are acquired or lost at random time. In some situations, this random time is exogenous, in other cases it is chosen by an optimizing agent. In the latter case, one needs not only to calculate the EPV, but also to find when it is optimal to acquire or abandon the stream of payoffs. For example, one may consider the following debt covenants - a firm becomes bankrupt if its operating profit drops below zero. Here the random time is a hitting time of the interval $U = (-\infty, h]$, and h is given exogenously, it is the value of the underlying stochastic variable for which the profit is zero. On the other hand, if a firm files for Chapter 11 - the U is the same as above, but h is the choice variable of the firm that maximizes the value of the equity.

The standard methods are not involved only when the underlying uncertainty is modelled as a (geometric) Brownian motion, a decision has to be made only once, and the payoff function is of a simple form. In particular, in the now-classical theory of real options, the price of an underlying asset is modelled as a geometric Brownian motion, and optimal exercise strategies are described by simple explicit formulas (see Dixit and Pindyck (1996) and the bibliography therein). The primary goal of this paper is to explain an alternative method that gives explicit analytical answers for fairly general classes of stochastic processes and payoff functions, and for embedded options with arbitrary number of embedded options. The method is more efficient even in the case of Gaussian processes and provides solutions in a more meaningful form. The form of the solution for the option value that we obtain suggests the following description of the optimal exercise strategy. It is optimal to exercise the right for the stream of stochastic payoffs, $g_t = g(X_t)$, the first time the EPV of the infimum stream $\underline{g}_t = \inf_{0\le s\le t} g_s$ becomes non-negative. We call the last statement a *universal record-setting bad news principle*. This principle, stated and applied here under less restrictive conditions than in Boyarchenko and Levendorskiĭ (2005), naturally generalizes and extends Bernanke's (1983) bad news principle and record-setting news principles spelled out in Boyarchenko (2004).

There are real options models that allow for jumps of a fixed size, with exponentially distributed time of arrival (see Dixit and Pindyck (1996)), and more general models use geometric Lévy processes. However, in many real life situations, commodity price processes exhibit mean reverting features (see Dixit and Pindyck (1996), Metcalf and Hasset (1995), Schwartz (1997)). Also, the dynamics of prices of many commodities is bimodal of a sort: a long period of moderate fluctuations in the region of high prices may be followed by a period of moderate fluctuations in the region of low prices and vice versa, and the transition periods are typically short. The standard device for situations of this sort are regime switching models. Unfortunately, the standard mean-reverting and regime switching models lead to fairly complicated formulas (see, e.g., Asmussen et al. (2004), François and Morellec (2004), Guo and Zhang (2004)), and mean-reverting models are analytically tractable in the Gaussian case only. In addition, regime switching models lead to systems of unknown functions whereas the method of the paper needs only one unknown function. In Section 3, we extend the classical theory to allow for a fairly general functions of the Brownian motion with embedded jumps or more general Lévy processes to model prices, and, to some extent, bridges the gap between analytically tractable (geometric) Lévy models and less tractable models, such as mean-reverting processes or switching models with non-Gaussian uncertainty. In Section 4, we demonstrate that it is possible to obtain analytical solutions for sequences of embedded options of an arbitrary length. We assume that each embedded option entitles its owner for a monotone stream, and the differences between the streams are also monotone. Monotone means that streams and differences are all either non-decreasing or non-increasing. We call these options *Russian dolls*: expanding and contracting, respectively. Using the results obtained in Section 3, we explicitly solve the optimal stopping problem for the most distant option, express its value as the EPV of a certain stream, and prove that this procedure can be iterated. Natural examples of Russian dolls are multi-stage projects, an investment program in a growing industry, and a disinvestment program for a firm in a declining industry, when the number of investment/disinvestment actions is finite, and the sizes of investment/disinvestments are fixed in advance.

In Sections 5 and 6, we consider two models of monopolistic expansion, when the number of investment stages may be infinite, and the sizes of investments/disinvestments are not specified in advance. In these two models, we study how the decisions in the geometric Brownian motion framework may change if more general types of uncertainty are introduced.

Problems of timing investment or disinvestment, capital expansion/contraction program, timing new technology adoption and other problems in the real options theory are simplified if a competitive firm is considered, and the price of output is the primitive of the model. Optimal investment/disinvestment rules change (and may change significantly) if the strategic interactions are introduced (see Dixit and Pindyck (1996), Grenadier (2000, 2002), Smit and Trigeorgis (2004), Murto (2004) and the bibliography therein). Indeed, if the dynamics of the inverse demand curve is taken as the primitive, then the presence of competitors influences the price dynamics and/or investment decisions. To separate the dependence of investment/disinvestment decisions on a chosen model of uncertainty from the influence of strategic interactions, we consider only the case of a monopoly which takes the inverse demand curve as given; the underlying uncertainty is modelled as the demand uncertainty (and uncertainty in the new technology factor). We leave for the future the study of strategic interactions under non-standard specifications of uncertainty.

In the model constructed in Section 5, the monopoly increases capital stock when the stochastic demand becomes sufficiently high, but uses the same production technology. The primitives of the model are the Cobb-Douglas production function, and the inverse demand curve. For simplicity, we assume that the investment is completely irreversible; an extension to the case of partially reversible investment will be published elsewhere. The optimal capital expansion program can be obtained as the limit of a sequence of optimal stopping problems (investment of chunks of capital). We make the standard assumption that the inverse demand is a monotone function of the stochastic factor; then each optimization problem in the sequence is equivalent to the optimal exercise of a call-like American option with a monotone payoff stream. Hence, we can use general results for simple options obtained in Section 3. We consider an example when at high levels of the underlying stochastic factor the inverse demand grows slower than in the standard exponential models, and demonstrate that the investment threshold in terms of the price process may become much lower than in the standard models. The real options approach recognizes the value of waiting when an irreversible decision has to be made in an uncertain environment and recommends higher (respectively, lower) exercise threshold for a real call (respectively, put) option than the naive net present value (NPV) rule does. The higher the uncertainty, the more does the exercise threshold of a real option differ from the one prescribed by the NPV rule. However, practitioners are known (see, for example, Lander and Pinches

(1998)) to be uncomfortable with too high trigger prices of investment, which the classical real options theory provides. The literature on strategic interactions demonstrates that the investment threshold may decrease significantly due to the competition. This conclusion is generally valid if the investment threshold is expressed in terms of the inverse demand curve; in terms of the price process, the answer may be the same as in the model of price-taking firms (optimality of myopic behavior: see Leahy (1993) and Dixit and Pindyck (1996)). The results obtained in the paper demonstrate that large differences between the NPV and real options exercise prices can be naturally explained as artifacts of modelling the underlying price as a geometric Brownian motion or more generally, as an exponential of a Lévy process. The differences decrease if we use more general dependence on the stochastic factor. We also demonstrate that if above a certain level, the rate of growth of the stochastic demand factor decreases, then in the intermediate range, a monopoly may find it optimal to *increase* the capital stock and simultaneously *decrease* the output price. Similar results can be obtained in a model of a firm in a declining industry (we do not include this model in the paper in order to save space): if the expected rate of decline is small but accelerates below a certain barrier, then during a transition period between a graceful decline and rapid fall, the firm may find it optimal to disinvest part of capital and decrease the production but increase prices.

Note that the use of an appropriate (not exponential) monotone function makes it unnecessary to impose exogenous restrictions on the capital stock available for investment, or on the returns to capital, which typically arise in the geometric Brownian motion model (see Dixit and Pindyck (1996)).

In the second model (Section 6), the monopoly has an option to increase the output by adoption of a new technology. Here we have the second course of uncertainty: the evolution of the technology frontier. We generalize the model in Abel and Eberly (2002), who modelled both factors as Brownian motions. For a different model of technology innovations, also for the diffusion uncertainty, see e.g. Grenadier and Weiss (1997), Alvarez and Steinbacka (2001) and the bibliography therein. We confront the implications of Gaussian models of uncertainty vs. jump models of uncertainty, and demonstrate that the word "uncertainty" does not mean too much in itself: depending on the situation and type of uncertainty, the diffusion uncertainty and jump uncertainty can produce opposite effects. Although we have two sources of uncertainty, we manage to reduce the problem of timing adoption to an optimal stopping problem on the line.

The rest of the paper is organized as follows. In Section 2, we formulate the problem of valuation of streams of payoffs which accumulate during a random time interval, give a short overview of basic facts of the theory of Lévy processes, introduce the class of jump-diffusion processes which will be used as model examples, define the EPV operators, and specify their properties. In Section 3, we state and prove theorems for basic types of simple real options assuming that the optimal stopping time is the hitting time of a semi-bounded interval.

In Section 4, we present sufficient conditions of optimality in the class of all stopping times, and derive general theorems about embedded options with the arbitrary number of sequentially embedded options. In Section 5, the underlying inverse demand curve is modelled as an arbitrary monotone function of a Lévy process, and a problem of timing investment of a marginal unit of capital is solved. In Section 6, a model of new technology adoption is examined. Section 7 concludes. Technical details are presented in the Appendix.

2. Preliminaries

Let $q > 0$ be a constant discount rate, $X = \{X_t\}_{t\geq 0}$ be a process with i.i.d. increments (i.e. a Lévy process), and $\tau = \tau_U$ be the hitting time of a subset $U \subset \mathbb{R}$. In many situations in economics and finance, the following problems need to be solved.

Problem I. How to calculate stochastic expressions

$$V_1(x;\tau) = E^x\left[\int_0^\tau e^{-qt}g(X_t)dt\right];$$

$$V_2(x;\tau) = E^x\left[\int_\tau^\infty e^{-qt}g(X_t)dt\right];$$

$$V_3(x;\tau) = E^x\left[e^{-q\tau}G(X_\tau)\right].$$

Here and below $E^x[f(X_t)] := E[f(X_t)|\, X_0 = x]$. By definition, $V_1(x;\tau)$ is the present value of the stream g that accumulates only until the random date τ; $V_2(x;\tau)$ is the present value of the stream g that starts to accrue at the random date τ; and $V_3(x;\tau)$ is the present value of the instantaneous payoff G which is received at the random date τ. All the three values are conditioned on the current realization, x, of the underlying stochastic variable.

Problem II. Let V be one of the functions V_j or

$$V(x;\tau) = E^x\left[\int_0^\tau e^{-qt}g(X_t)dt\right] + E^x\left[e^{-r\tau}G(X_\tau)\right].$$

Find the optimal stopping time τ which maximizes $V(x;\tau)$.

For example, $g(X_t)$ is the profit stream of the firm, $G(X_t)$ is the scrap value, and then τ is the optimal time to exit the declining industry.

Under fairly weak regularity conditions, we reduce the problem of calculation of the stochastic expressions listed above to a boundary value problem in the inaction region U^c, with the boundary condition specified on U. In order to solve explicitly the Bellman equation in the inaction region, we use the Wiener-Hopf factorization technique in the operator form, as in analysis, but we interpret the operators in the formulas for the solutions as the EPV-operators under supremum and infimum processes, which greatly simplifies the proofs of optimality. We use the form of the solution for the value function to guess a natural candidate for the optimal action region. If the latter is a semi-infinite interval, then we give a very short and easy proof of optimality in the class of hitting times of semi-infinite intervals. For the proof of optimality in the class of all stopping times, one has to verify additional conditions for the value function.

2.1. **Process specification.** We need several basic facts of the theory of Lévy processes. The moment generating function of a Lévy process can be represented in the form $E\left[e^{zX_t}\right] = e^{t\Psi(z)}$; the function Ψ is called the Lévy exponent. The latter naturally appears when we calculate the action of the infinitesimal generator of X_t, denoted L, on exponential functions: $Le^{zx} = \Psi(z)e^{zx}$. In the paper, we restrict ourselves to the class of jump-diffusion processes introduced in Duffie et al. (2000), with the infinitesimal generator of the form

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + bu'(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x))\,F(dy). \tag{2.1}$$

Here the density of jumps, $F(dy)$, or Lévy density, is given by

$$F(dy) = c^+\lambda^+e^{-\lambda^+y}\mathbf{1}_{(0,+\infty)}(y)dy + c^-(-\lambda^-)e^{-\lambda^-y}\mathbf{1}_{(-\infty,0)}(y)dy, \tag{2.2}$$

$\mathbf{1}_{(a,\,b)}(\cdot)$ denotes the indicator function of the interval $(a,\,b)$, $c^{\pm} > 0$, and $\lambda^- < 0 < \lambda^+$. The coefficient c^+ (respectively, c^-) characterizes the intensity of upward jumps (respectively, downward jumps). The parameter λ^+ describes the relative intensity of large jumps: the smaller the λ^+, the larger is the probability of large upward jumps as opposed to small ones. Conversely, the smaller the λ^-, the larger is the probability of large downward jumps. If one of the $c^{\pm}$ is zero, there are no jumps in the corresponding direction. The method of the paper can be applied to much more general Lévy processes – see Boyarchenko and Levendorskiĭ (2002a, b, 2005). As we we will show, the choice (2.2) leads to simple formulas, and the calculations are not much more difficult than in the Gaussian case. At the same time, different terms in (2.1) can represent different stochastic factors. For instance, the Gaussian component, represented by the first two terms, can be used to account for the industry specific uncertainty, and the jump part – for the idiosyncratic one. Should we use a one-factor Gaussian model, and study, for example, how the investment threshold changes due to the change of the variance, we could not separate the impact of the industry specific and idiosyncratic shocks. Also, we can independently change the size and intensity of downward and upward jumps by changing the parameters $c^{\pm}$ and $\lambda^{\pm}$.

Computing the action of the infinitesimal generator (2.1) on e^{zx}, we obtain the exponent $\Psi(z)$ corresponding to the Lévy density (2.2) (for the calculation, see the Appendix):

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z}. \tag{2.3}$$

2.2. **EPV operators.** Let $T \in \mathbb{R}_+$ be an exponentially distributed random variable with the mean q^{-1} independent of the process $X = \{X_t\}_{t\ge 0}$. The $\mathrm{Prob}(T = t) = qe^{-qt}$, therefore

$$E^x[g(X_T)] := qE^x\left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right].$$

Introduce the *normalized EPV-operator* of a stochastic process X:

$$\mathcal{E}g(x) = qE^x\left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right].$$

This operator calculates the EPV of a stream $qg(X_t)$. The normalization is convenient because

$$(\mathcal{E}\mathbf{1})(x) = qE^x\left[\int_0^{+\infty} e^{-qt}\mathbf{1}(X_t)dt\right] = q\int_0^{+\infty} e^{-qt}dt = 1.$$

Applying $\mathcal{E}$ to $g(x) = e^{zx}$ and using the equality $E\left[e^{zX_t}\right] = e^{t\Psi(z)}$, we obtain that $\mathcal{E}$ acts on exponential functions as the multiplication operator by the number $q(q-\Psi(z))^{-1}$:

$$\mathcal{E}e^{zx} = q\int_0^{+\infty} e^{-(q-\Psi(z))t+zx}dt = q(q-\Psi(z))^{-1}e^{zx}. \tag{2.4}$$

To ensure that the expectation were finite, it is necessary and sufficient that the real part of $q - \Psi(z)$ were positive. Since $(q-L)e^{zx} = (q-\Psi(z))e^{zx}$, we conclude that $q^{-1}(q-L)$ and $\mathcal{E}$ are mutual inverses. To make this statement precise, we need to specify function spaces between which $q^{-1}(q-L)$ and $\mathcal{E}$ act. The simplest choice is $C_c(\mathbb{R})$ and $C_c^2(\mathbb{R})$, the Banach space of continuous functions vanishing at infinity, and the Banach space of twice continuously differentiable functions, whose derivatives up to the second order vanish at infinity. One of the basic facts of the theory of Lévy processes is that $q^{-1}(q-L) : C_c^2(\mathbb{R}) \to C_c(\mathbb{R})$ is invertible, and its inverse is $\mathcal{E}$ (see, e.g., Sato (1999)). However, in many applications in economics and finance, it is necessary to allow for the exponential growth as $x \to \pm\infty$. The rate of growth can be (and, typically, is) different in the positive and negative directions. Suppose, we need to consider functions, which grow not faster than $e^{\sigma_+ x}$ (respectively, $e^{\sigma_- x}$) as $x \to +\infty$ (respectively, as $x \to -\infty$). Here $\sigma_- \le 0 \le \sigma_+$. Then it is natural to consider spaces $C_0^s([\sigma_-, \sigma_+]; \mathbb{R})$, of functions, which are continuously differentiable up to the order s, and satisfy estimates: for $j \le s$, $g^{(j)}(x)e^{-\sigma_+ x} \to 0$, as $x \to +\infty$, and $g^{(j)}(x)e^{-\sigma_- x} \to 0$, as $x \to -\infty$. These functions form the Banach space with the norm

$$||g|| = \sum_{j=0}^{s} \sup_{\mathbb{R}} (e^{\sigma_+ x} + e^{\sigma_- x})^{-1} |g^{(j)}(x)|.$$

For instance, if the price of the firm's output is modelled as $P_t = P(X_t) = e^{X_t}$, then we may take any $\sigma_- < 0$ and $\sigma_+ > 1$ to include constant functions and $P = P(x)$ in the space $C_0^2([\sigma_-, \sigma_+]; \mathbb{R})$. To ensure that $q^{-1}(q - L)$ acts from $C_0^2([\sigma_-, \sigma_+]; \mathbb{R})$ to $C_0([\sigma_-, \sigma_+]; \mathbb{R}) := C_0^0([\sigma_-, \sigma_+]; \mathbb{R})$, the moment generating function must be defined at $z = \sigma_\pm$, and if we want $q^{-1}(q - L)$ to be invertible, then even more stringent condition must be imposed.

Lemma 2.1. *a) If $\Psi(\sigma_\pm)$ are well-defined, then the operator $q^{-1}(q - L) : C_0^2([\sigma_-, \sigma_+]; \mathbb{R}) \to C_0([\sigma_-, \sigma_+]; \mathbb{R})$ is bounded.*

b) If, in addition, $q - \Psi(\sigma_\pm) > 0$, then this operator is invertible with the inverse $\mathcal{E}$:

$$q^{-1}(q - L)\mathcal{E} = \mathcal{E} q^{-1}(q - L) = I. \tag{2.5}$$

For the proof, see, e.g., Boyarchenko and Levendorskiĭ (2002a) and (2002b, Chapter 15). Note that a similar definition and lemma are valid if we assume that the functions in question are measurable, locally bounded, and grow not too fast at infinity:

$$|g(x)| \le c e^{\sigma_- x}, \quad \text{for } x < 0, \tag{2.6}$$

and

$$|g(x)| \le c e^{\sigma_+ x}, \quad \text{for } x > 0, \tag{2.7}$$

where $\sigma_- < 0 < \sigma_+$ and c are independent of x. (The same conditions should hold for all derivatives up to the order s).

We will also need the normalized EPV-operators of the supremum process $\bar{X}_t = \sup_{0 \le s \le t} X_s$ and the infimum process $\underline{X}_t = \inf_{0 \le s \le t} X_s$. These EPV-operators act as follows:

$$\mathcal{E}^+ g(x) := qE^x \left[\int_0^\infty e^{-qt} g(\bar{X}_t) dt \right] := qE \left[\int_0^\infty e^{-qt} g(\bar{X}_t) dt \mid X_0 = x \right]$$

and

$$\mathcal{E}^- g(x) := qE^x \left[\int_0^\infty e^{-qt} g(\underline{X}_t) dt \right] := qE \left[\int_0^\infty e^{-qt} g(\underline{X}_t) dt \mid X_0 = x \right].$$

Evidently, $\mathcal{E}^+ g(x) = E^x[g(\bar{X}_T)]$ and $\mathcal{E}^- g(x) = E^x[g(\underline{X}_T)]$, where T is the exponential random variable introduced at the beginning of this subsection. It is straightforward to check that $\mathcal{E}^+$ and $\mathcal{E}^-$ also act on an exponential function e^{zx} as multiplication operators by numbers, which we denote $\kappa_q^+(z)$ and $\kappa_q^-(z)$, respectively:

$$\mathcal{E}^+ e^{zx} = \kappa_q^+(z) e^{zx}, \quad \mathcal{E}^- e^{zx} = \kappa_q^-(z) e^{zx}. \tag{2.8}$$

These numbers are

$$\kappa_q^+(z) = qE\left[\int_0^\infty e^{-qt}e^{z\bar{X}_t}dt\right] = E\left[e^{z\bar{X}_T}\right], \tag{2.9}$$

$$\kappa_q^-(z) = qE\left[\int_0^\infty e^{-qt}e^{z\underline{X}_t}dt\right] = E\left[e^{z\underline{X}_T}\right]. \tag{2.10}$$

2.3. **Wiener-Hopf factorization.** The Wiener-Hopf factorization formula reads: for $z \in i\mathbb{R}$

$$E[e^{zX_T}] = E[e^{z\bar{X}_T}]E[e^{z\underline{X}_T}].$$

For the reader convenience, we recall the outline of the proof. The formula above is based on a trivial observation that $X_T = \bar{X}_T + X_T - \bar{X}_T$, on an evident fact that $\underline{X}_T$ and $X_T - \bar{X}_T$ are the same in law (to see this, it suffices to draw a picture of the sample path and turn it upside down), and less obvious fact that $\underline{X}_T$ and $X_T - \bar{X}_T$ are independent. For more details and further references, see Rogers and Williams (2000), Section I.29.

Using (2.4), (2.9) and (2.10), we obtain an equivalent version of the Wiener-Hopf factorization formula: for $z \in i\mathbb{R}$

$$\frac{q}{q - \Psi(z)} = \kappa_q^+(z)\kappa_q^-(z) \tag{2.11}$$

(see, e.g., Sato (1999), Section 45). It is evident from (2.9) and (2.10) that $\kappa_q^+(z)$ (respectively, $\kappa_q^-(z)$) admits the analytic continuation into the half-plane $\Re z < 0$ (respectively, $\Re z > 0$), and it is continuous up to the boundary. Also, $\kappa_q^\pm(0) = 1$. The next important property - each of the factors has no zeroes on its domain - follows from explicit analytical formulas for $\kappa_q^-(z)$ and $\kappa_q^+(z)$ (see, e.g., Sato (1999), Section 45, equations (45.2) and (45.3)). If $(\kappa_q^\pm)(z)$ and $(\kappa_q^\pm)^{-1}(z)$ grow not faster than a polynomial in the half-plane $\mp\Re z \ge 0$, then the factorization (2.11) satisfying the above properties is unique (see e.g. Boyarchenko and Levendorskiǐ (2002a, b)), which allows one to guess the factors in many cases, in particular, for jump-diffusions with the Lévy exponent (2.2). Let $\beta_{1,2}^-$ and $\beta_{1,2}^+$ be the negative and positive solutions of the characteristic equation

$$q - \Psi(z) = 0. \tag{2.12}$$

(They are separated by $\lambda^-, 0$, and λ^+: $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+$.) Then

$$\kappa_q^\pm(z) = \frac{\beta_1^\pm}{\beta_1^\pm - z} \cdot \frac{\beta_2^\pm}{\beta_2^\pm - z} \cdot \frac{\lambda^\pm - z}{\lambda^\pm}. \tag{2.13}$$

Decomposing $\kappa_q^\pm(z)$ into a sum of simple fractions:

$$\kappa_q^\pm(z) = a_1^\pm \frac{\beta_1^\pm}{\beta_1^\pm - z} + a_2^\pm \frac{\beta_2^\pm}{\beta_2^\pm - z}, \tag{2.14}$$

where $a_{1,2}^\pm > 0$, we derive

$$\left(\mathcal{E}^+ g\right)(x) = \sum_{j=1,2} a_j^+ \int_0^{+\infty} \beta_j^+ e^{-\beta_j^+ y} g(x+y)dy, \tag{2.15}$$

$$\left(\mathcal{E}^- g\right)(x) = \sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-) e^{-\beta_j^- y} g(x+y)dy \tag{2.16}$$

(for the proof and explicit expressions for $a_j^\pm$, see the Appendix).

Applying $\mathcal{E}$, $\mathcal{E}^+$ and $\mathcal{E}^-$ to $g(x) = e^{zx}$ and using (2.4) and (2.8)–(2.11), we obtain the third version of the Wiener-Hopf factorization formula:

$$\mathcal{E}g(x) = \mathcal{E}^+\mathcal{E}^-g(x) = \mathcal{E}^-\mathcal{E}^+g(x). \tag{2.17}$$

By linearity, (2.17) holds for linear combinations of exponents and integrals of exponents, hence for wide classes of functions. Equation (2.17) means that the normalized EPV-operator of a Lévy process admits a factorization into a product of the normalized EPV-operators of the supremum and infimum processes.

Introduce Y^+, a random variable on $\mathbb{R}_+$, defined as $\bar{X}_T$ for X_t started at 0, and Y^-, a random variable on $\mathbb{R}_-$, defined as $\underline{X}_T$ for X_t started at 0. Then we can write

$$\mathcal{E}^+g(x) = E[g(x + Y^+)], \quad \mathcal{E}^-g(x) = E[g(x + Y^-)]. \tag{2.18}$$

It follows from (2.15) and (2.16), that the probability densities of Y^+ and Y^- are

$$\sum_{j=1,2} a_j^+\beta_j^+e^{-\beta_j^+y}dy \quad \text{and} \quad \sum_{j=1,2} a_j^-(-\beta_j^-)e^{-\beta_j^-y}dy,$$

respectively.

2.4. Properties of EPV operators.

Proposition 2.2. *(a) If $g(x) = 0\ \forall\ x \ge h$, then $\forall\ x \ge h$*

$$(\mathcal{E}^+g)(x) = 0, \quad ((\mathcal{E}^+)^{-1}g)(x) = 0. \tag{2.19}$$

(b) If $g(x) = 0\ \forall\ x \le h$, then $\forall\ x \le h$

$$(\mathcal{E}^-g)(x) = 0, \quad ((\mathcal{E}^-)^{-1}g)(x) = 0. \tag{2.20}$$

Proof. Statements concerning $\mathcal{E}^\pm$ follow immediately from (2.18). For the model example of jump-diffusion processes, properties of $(\mathcal{E}^\pm)^{-1}$ become evident upon the notice that $(\mathcal{E}^+)^{-1}$ and $(\mathcal{E}^-)^{-1}$ admit the following analytical representation:

$$\left((\mathcal{E}^+)^{-1}g\right)(x) \;=\; c_+^1g'(x) + c_+^0g(x) + b_+\int_0^{+\infty}\lambda^+e^{-\lambda^+y}g(x+y)dy, \tag{2.21}$$

$$\left((\mathcal{E}^-)^{-1}g\right)(x) \;=\; c_-^1g'(x) + c_-^0g(x) + b_-\int_{-\infty}^{0}(-\lambda^-)e^{-\lambda^-y}g(x+y)dy, \tag{2.22}$$

where $b_\pm$, $c_\pm^0$, $c_\pm^1$ are constants (see the Appendix for the proof and explicit expressions for $b_\pm$, $c_\pm^0$, $c_\pm^1$). The proof of the statements about the inverse operators $(\mathcal{E}^\pm)^{-1}$ for wide classes of Lévy processes is presented in Boyarchenko and Levendorskiĭ (2002, Chapter 15). The proof is based on the fact that $(\kappa_q^\pm)^{-1}(z)$ grows not faster than a polynomial in the half-plane $\mp\Re z \ge 0$. □

Corollary 2.3. *a) If $g(x) \ge 0\ \forall x$, then $(\mathcal{E}^+g)(x) \ge 0,\ \forall x$. If, in addition, there exists x_0 such that $g(x) > 0\ \forall x > x_0$, then $(\mathcal{E}^+g)(x) > 0\ \forall x$.*

b) If $g(x) \ge 0\ \forall x$, then $(\mathcal{E}^-g)(x) \ge 0,\ \forall x$. If, in addition, there exists x_0 such that $g(x) > 0$ $\forall x < x_0$, then $(\mathcal{E}^-g)(x) > 0\ \forall x$.

c) If g is monotone, then $\mathcal{E}^+g$ and $\mathcal{E}^-g$ are also monotone.

Proof. Follows immediately from (2.18). □

3. Simple options

We start with obtaining the values of several basic types of options on payoff streams g; in the end of this Section, we also consider options with instantaneous payoffs. We formulate and prove sufficient optimality conditions in the class of hitting times of semi-infinite intervals. For simplicity of presentation, we assume that $\{X_t\}$ is a jump-diffusion process with the Lévy exponent given by (2.2) [1]. Depending on the situation, payoff streams satisfy one of the conditions (2.7) and (2.6) or both, where $\sigma_\pm$ satisfy $q - \Psi(\sigma\pm) > 0$. If Ψ is given by (2.2), an equivalent condition is $\beta_1^- < \sigma_- < 0 < \sigma_+ < \beta_1^+$, where β_1^+ and β_1^- are the positive and negative roots of the characteristic equation (2.12), closest to 0. We will formulate the main results in two forms: using the EPV operators $\mathcal{E}^\pm$, and independent random variables $Y^+ = \bar{X}_T$ and $Y^- = \underline{X}_T$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, where $T \sim \mathrm{Exp}(q)$ is the exponential random variable independent of $\{X_t\}$, and X_t starts at 0. We would like to emphasize the fact that even though in the statements of main theorems the values of options are given by stochastic expressions, analytical formulas for those values are available as well. For general Lévy processes, the analytical expressions are rather involved, but for diffusion processes with embedded jumps, option values are given by relatively simple analytical formulas. We present an example of the analytical solution for the case of the call-like American option later in this Section.

3.1. **Valuation of an option to abandon a stream which is an increasing function of the stochastic factor.** Let g be an increasing function, the model example being $g(x) = Ge^{zx} - C$, where Ge^{zx} is the operating profit of an active firm, and C is the coupon payment on the debt. If the operating profit becomes too low (X_t becomes too low), it may become optimal to default on the debt. Suppose that the firm defaults when X_t reaches or falls below h. For $h \in \mathbb{R}$, we denote by $\tau = \tau_h^- = \inf\{t > 0 \,|\, X_t \le h\}$ the hitting time of $(-\infty, h]$, and by $\mathbf{1}_{(h,\infty)}$, the indicator function of (h,∞), and the multiplication-by-$\mathbf{1}_{(h,\infty)}$ operator. We want to find

$$V_-(x) = \sup_h E^x\left[\int_0^{\tau_h^-} e^{-qt} g(X_t)dt\right].$$

We will obtain the solution to the latter problem in three steps. First, we assume that h is given and write down the boundary value problem for the value function. Second, using the Wiener-Hopf factorization method, we obtain an explicit solution of the boundary value problem. Finally, using the explicit analytical expression for the value function, we guess the natural candidate for the optimal exercise boundary, h, and verify its optimality.

Theorem 3.1. *Let g be a measurable locally bounded function satisfying (2.7). Then*

$$V(x;h) := E^x\left[\int_0^{\tau_h^-} e^{-qt} g(X_t)dt\right] = q^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g(x). \tag{3.1}$$

Proof. Fix h, and consider the normalized value function

$$\mathcal{V}(x;h) = E^x\left[\int_0^{\tau_h^-} qe^{-qt} g(X_t)dt\right]. \tag{3.2}$$

[1] All results are valid for wide classes of Lévy processes satisfying the (ACP)-condition; for the definition, see e.g. Sato (1999), p.288. This is a fairly weak regularity condition. For example, it is satisfied if, for every $t > 0$, there exist a measurable function p_t such that $E^x[u(X_t)] = \int_{\mathbb{R}} p_t(x+y)u(y)dy$

It was shown in Boyarchenko and Levendorskiĭ (2002b, Theorem 2.12, p. 63) that the stochastic expression (3.2) is a bounded solution to the following boundary value problem:

$$(q-L)\mathcal{V}(x;h) = qg(x), \quad x>h, \tag{3.3}$$

$$\mathcal{V}(x;h) = 0, \quad x\le h. \tag{3.4}$$

Evidently, (3.3) is the Bellman equation in the inaction region. Using (2.5), we may rewrite (3.3) as

$$\mathcal{E}^{-1}\mathcal{V}(x;h) = g(x), \quad x>h. \tag{3.5}$$

Had the last equation been valid for any $x\in\mathbb{R}$, we would have applied the operator $\mathcal{E}$ to both sides of the equation and obtained the normalized value function as $\mathcal{V}(x;h)=\mathcal{E}g(x)$. We want to consider (3.5) on the whole axis but the values of the LHS for $x\le h$ are unknown. Hence, problem (3.5), (3.4) is equivalent to the following problem: find a pair of functions $\mathcal{V}(\cdot;h)$ and $g^-\in L^\infty(\mathbb{R})$ such that

(i) $\mathcal{V}(\cdot;h)$ vanishes on $(-\infty,h]$;
(ii) g^- vanishes on $(h,+\infty)$: $g^-(x)=0$ for $x>h$;
(iii) functions $\mathcal{V}(\cdot;h)$ and g^- satisfy

$$\mathcal{E}^{-1}\mathcal{V}(x;h) = g(x)+g^-(x), \quad \forall\, x. \tag{3.6}$$

By (2.17), $\mathcal{E}^{-1}=(\mathcal{E}^+)^{-1}(\mathcal{E}^-)^{-1}$. Apply the operator $\mathcal{E}^+$ to both sides of (3.6):

$$(\mathcal{E}^-)^{-1}\mathcal{V}(x;h) = \mathcal{E}^+g(x)+\mathcal{E}^+g^-(x), \quad \forall\, x. \tag{3.7}$$

On the strength of (2.19), $\mathcal{E}^+g^-(x)=0$ for $x>h$. We may use this property in order to get rid of $g^-(x)$. The multiplication of a function $f(x)$ by $\mathbf{1}_{(h,+\infty)}$ replaces values $f(x), x\le h$, with zeroes. Therefore $\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g^-(x)=0$ for all x. Also notice, that by (2.20), $(\mathcal{E}^-)^{-1}\mathcal{V}(x;h)=0$ for $x\le h$, therefore the multiplication by $\mathbf{1}_{(h,+\infty)}$ will not change the LHS in (3.7). Thus, multiplying both sides of (3.7) by $\mathbf{1}_{(h,+\infty)}$, we obtain an equivalent problem

$$(\mathcal{E}^-)^{-1}\mathcal{V}(x;h) = \mathbf{1}_{(h,+\infty)}\mathcal{E}^+g(x), \quad \forall\, x. \tag{3.8}$$

To finish the proof, it remains to apply the operator $\mathcal{E}^-$ to both sides of (3.8) and multiply them by q^{-1}. □

Using the independent random variables Y^+ and Y^- on $\mathbb{R}_+$ and $\mathbb{R}_-$, we may write (3.1) as

$$V(x;h) = q^{-1}E[\mathbf{1}_{(h,+\infty)}(x+Y^-)g(x+Y^++Y^-)]. \tag{3.9}$$

Theorem 3.2. *Assume that g is a measurable locally bounded function satisfying (2.7), and there exists h_* such that*

$$\mathcal{E}^+g(x)>0,\ x>h_*, \quad \text{and} \quad \mathcal{E}^+g(x)<0,\ x<h_*. \tag{3.10}$$

Then it is optimal to abandon the stream g the first time X_t reaches or drops below h_*, and $V_-(x)$, the rational price of the option, is given by (3.9) with $h=h_*$.

Proof. On the strength of (2.18), we may write (3.1) as

$$V(x;h) = q^{-1}E\left[\left(\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g\right)(x+Y^-)\right].$$

Consider the RHS of the last equation: multiplication by $\mathbf{1}_{(h,+\infty)}$ replaces values of $\mathcal{E}^+g(x)$ on $(-\infty,h]$ with zeroes. If $h>h_*$, then some positive values will be replaced with zeroes, and this will reduce the option value. If $h<h_*$, then some negative values will contribute to the

expected value, and the option value will be reduced again. Evidently, in order to maximize the option value, one has to replace all negative values (and only them) of $\mathcal{E}^+g(x)$ with zeroes. This proves optimality in the class of hitting times of intervals of the form $(-\infty, h]$. In Subsection 4.1, we show that if g is monotone then $\tau^-_{h_*}$ is optimal in the class of all stopping times. □

3.2. **Valuation of an option to abandon a stream which is a decreasing function of the stochastic factor.** Let $g(X_t)$ be a decreasing function of X_t, the model example being $g(X_t) = R - Ce^{zX_t}$, where R is the revenue of an active firm (assumed constant for simplicity), and Ce^{zX_t} is the stochastic cost of production driven by supply shocks. If the profit $g(X_t)$ becomes too low (X_t becomes too high), it may become optimal to stop the production. Suppose that the firm stops producing when X_t reaches or overshoots h. For $h \in \mathbb{R}$, we denote by $\tau = \tau^+_h = \inf\{t > 0 \,|\, X_t \ge h\}$ the hitting time of $[h, \infty)$, and by $\mathbf{1}_{(-\infty,h)}$, the indicator function of $(-\infty, h)$, and the multiplication-by-$\mathbf{1}_{(-\infty,h)}$ operator. We want to find

$$V_+(x) = \sup_h E^x \left[\int_0^{\tau^+_h} e^{-qt} g(X_t)dt \right].$$

Theorem 3.3. *Let g be a measurable locally bounded function satisfying (2.6). Then*

$$V(x;h) := E^x \left[\int_0^{\tau^+_h} e^{-qt} g(X_t)dt \right] = q^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^- g(x), \tag{3.11}$$

or equivalently,

$$V(x;h) = q^{-1}E[\mathbf{1}_{(-\infty,h)}(x + Y^+)g(x + Y^- + Y^+)]. \tag{3.12}$$

Proof. For the proof, repeat all the steps in the proof of Theorem 3.2 with $(-\infty, h)$, $\mathcal{E}^-$ and $\mathcal{E}^+$ in place of $(h, +\infty)$, $\mathcal{E}^+$ and $\mathcal{E}^-$, respectively. □

The proof of the next theorem is a straightforward modification of the proof of Theorem 3.2.

Theorem 3.4. *Assume that g is a measurable locally bounded function satisfying (2.6), and there exists h^* such that*

$$\mathcal{E}^+g(x) > 0,\ x < h^*, \quad \text{and} \quad \mathcal{E}^+g(x) < 0,\ x > h^*. \tag{3.13}$$

Then it is optimal to abandon the stream g the first time X_t reaches or overshoots h^*, and $V_+(x)$, the rational price of the option, is given by (3.12) with $h = h^*$.

3.3. **Valuation of an option to acquire a stream which is a decreasing function of the stochastic factor (put-like option).** In this subsection, we consider a stochastic expression

$$V^-(x) = \sup_h E^x \left[\int_{\tau^-_h}^{+\infty} e^{-qt} g(X_t)dt \right],$$

which is the value of the right for the stream of payoffs that starts to accrue when the boundary h is reached or crossed from above.

Theorem 3.5. *Let g be a measurable locally bounded function satisfying conditions (2.6)-(2.7). Then*

$$V(x;h) := E^x \left[\int_{\tau^-_h}^{+\infty} e^{-qt} g(X_t)dt \right] = q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+ g(x), \tag{3.14}$$

or equivalently,

$$V(x;h) = q^{-1}E[\mathbf{1}_{(-\infty,h]}(x+Y^-)g(x+Y^+ + Y^-)]. \tag{3.15}$$

Proof. Observe that the normalized value function

$$\begin{aligned}\mathcal{V}(x;h) &= E^x\left[\int_{\tau_h^-}^{+\infty} qe^{-qt}g(X_t)dt\right]\\ &= E^x\left[\int_0^{+\infty} qe^{-qt}g(X_t)dt\right] - E^x\left[\int_0^{\tau_h^-} qe^{-qt}g(X_t)dt\right].\end{aligned}$$

We may use (3.1) to obtain the value of the last stochastic expression:

$$\mathcal{V}(x;h) = \mathcal{E}g(x) - \mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g(x).$$

Using the Wiener-Hopf factorization $\mathcal{E} = \mathcal{E}^+\mathcal{E}^-$, we proceed as follows:

$$\mathcal{V}(x;h) = \mathcal{E}^-\left(\mathbf{1} - \mathbf{1}_{(h,+\infty)}\right)\mathcal{E}^+g(x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+g(x).$$

□

Theorem 3.6. *Assume that g is a measurable locally bounded function satisfying conditions (2.7)-(2.6), and there exists h_* such that*

$$\mathcal{E}^+g(x) < 0,\ x > h_*, \quad \text{and} \quad \mathcal{E}^+g(x) > 0,\ x < h_*. \tag{3.16}$$

Then h_* is the optimal exercise threshold, and $V^-(x)$, the rational price of the option, is given by (3.15) with $h = h_*$.

Proof. The proof is the straightforward modification of the proof of Theorem 3.2. □

3.4. **Valuation of an option to acquire a stream which is an increasing function of the stochastic factor (call-like option).** In this subsection, we obtain the value of the stochastic expression

$$V^+(x) = \sup_h E^x\left[\int_{\tau_h^+}^{+\infty} e^{-qt}g(X_t)dt\right],$$

which is the value of the right for the stream of payoffs that starts to accrue at a stopping time τ_h^+. The following two theorems are the mirror reflections of Theorem 3.5 and Theorem 3.6: the direction on the real axis changes, the supremum process and $\mathcal{E}^+$ are interchanged with the infimum process and $\mathcal{E}^-$, respectively.

Theorem 3.7. *Let g be a measurable locally bounded function satisfying conditions (2.6)-(2.7). Then*

$$V(x;h) := E^x\left[\int_{\tau_h^+}^{+\infty} e^{-qt}g(X_t)dt\right] = q^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-g(x), \tag{3.17}$$

or equivalently,

$$V(x;h) = q^{-1}E[\mathbf{1}_{[h,+\infty)}(x+Y^+)g(x+Y^- + Y^+)]. \tag{3.18}$$

Theorem 3.8. *Assume that g is a measurable locally bounded function satisfying conditions (2.7)-(2.6), and there exists h^* such that*

$$\mathcal{E}^- g(x) > 0,\ x > h^*, \quad \text{and} \quad \mathcal{E}^- g(x) < 0, x < h^*. \tag{3.19}$$

Then h^* is the optimal exercise threshold, and $V^+(x)$, the rational option price, is given by (3.18) with $h = h^*$.

As the first application of Theorem 3.8, consider an investor who chooses time τ to invest capital I into a technology that produces a commodity at rate G ever after. The output is sold on the spot at the market price e^{X_t}, where X_t follows a Lévy process. We view I as the present value of a stream qI of future expenditures. Let $g(x) = Ge^x - qI$. It may become optimal to acquire this stream if x becomes sufficiently large, i.e., at random time τ_h^+. The investor's problem is equivalent to choosing the investment threshold h so as to maximize the option value of investment, which is the problem just solved above. Equation (3.19) is equivalent to $G\kappa_q^-(1)e^{h^*} - qI = 0$, therefore the trigger price of investment is

$$e^{h^*} = \frac{qI}{G\kappa_q^-(1)}. \tag{3.20}$$

It remains to compute the option value of investment when the investment threshold is chosen optimally:

$$V^+(x) = q^{-1}(\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} \mathcal{E}^- g)(x) = q^{-1}(\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}(\cdot)\left(G\kappa_q^-(1)e^{\cdot} - qI\right))(x).$$

Using (3.20), we write the option value as

$$V^+(x) = I(\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}(\cdot)(e^{\cdot - h^*} - 1))(x).$$

Next, we use (2.15) to obtain, for $x < h^*$

$$\begin{aligned} V^+(x) &= I \sum_{j=1,2} a_j^+ \left[e^{x-h^*} \int_{h^*-x}^{+\infty} \beta_j^+ e^{(1-\beta_j^+)y} dy - \int_{h^*-x}^{+\infty} \beta_j^+ e^{-\beta_j^+ y} dy \right] \\ &= I \sum_{j=1,2} a_j^+ e^{\beta_j^+(x-h^*)} \left(\frac{\beta_j^+}{\beta_j^+ - 1} - 1 \right) = I \sum_{j=1,2} \frac{a_j^+ e^{\beta_j^+(x-h^*)}}{\beta_j^+ - 1}. \end{aligned} \tag{3.21}$$

As the next example, consider a simplest regime switching model, when the market price of output assumes only two values: $P_d < P_u$. Transitions from one state to another are determined by an underlying jump-diffusion process X: if $X_t < 0$, then $P(X_t) = P_d$, and if $X_t \geq 0$, then $P(X_t) = P_u$. Assume that $P_d < qI < P_d$. We may regard the investment opportunity as the option to acquire a stream $g(X_t)$ given by $g(X_t) = g_- := P_u - qI < 0$ for $X_t < 0$, and $g(X_t) = g_+ := P_d - qI > 0$ otherwise. In order to find the trigger value of the stochastic factor X_t, we need to solve $\mathcal{E}^- g(h) = 0$. For $x \leq 0$, $\mathcal{E}^- g(x) = \mathcal{E}^- g_- = g_- < 0$, and for $x > 0$,

$$\begin{aligned} \mathcal{E}^- g(x) &= \sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-) e^{-\beta_j^- y} g(x+y) dx \\ &= \sum_{j=1,2} a_j^- \left[\int_{-\infty}^{-x} (-\beta_j^-) e^{-\beta_j^- y} g_- dy + \int_{-x}^0 (-\beta_j^-) e^{-\beta_j^- y} g_+ dy \right] \\ &= \sum_{j=1,2} a_j^- \left[g_- e^{\beta_j^- x} + g_+(1 - e^{\beta_j^- x}) \right]. \end{aligned}$$

Therefore, h^* is a unique positive solution of the equation

$$\sum_{j=1,2} a_j^- g_+ = (g_+ - g_-) \sum_{j=1,2} a_j^- e^{\beta_j^- h}.$$

3.5. **Good and bad news principles.** Both Theorems 3.2 and 3.4 deal with options to abandon a decreasing stream $g_t = g(X_t)$; only the directions on X-axis are different, which explains the difference in the statements of the optimal exercise rules. In terms of g_t, both rules can be stated as

Good news principle: exercise the option to abandon a stream g_t the first time the EPV of the supremum stream $\bar{g}_t = \sup_{0\le s\le t} g_s$ becomes non-positive.

Theorems 3.6 and 3.8 deal with options to acquire an increasing stream g_t; once again, the difference in the statements are due to the difference of directions on the X-axis. In terms of g_t, both rules can be stated as

Bad news principle: exercise the option to acquire a stream g_t the first time the EPV of the infimum stream $\underline{g}_t = \inf_{0\le s\le t} g_s$ becomes non-negative.

Remark 3.1. It is interesting that in view of (2.18), optimal exercise rules (3.10), (3.13), (3.16) and (3.19) can be formulated in essentially the same form as in many models in economics, where uncertainty is modeled as draws from a given distribution: exercise the option when the expectation of a certain function of a random variable becomes positive (or negative).

3.6. **Options with instantaneous payoffs.** To consider options with an instantaneous payoff G, one should express G as the EPV of a stream g: $G = q^{-1}\mathcal{E}g$, where $g = (q - L)G = q\mathcal{E}^{-1}G$. Then

$$E^x\left[e^{-q\tau}G(X_\tau)\right] = E^x\left[e^{-q\tau}E_\tau\left[\int_\tau^{+\infty} e^{-q(t-\tau)}g(X_t)dt\right]\right] = E^x\left[\int_\tau^{+\infty} e^{-qt}g(X_t)dt\right].$$

Function g is well-defined if G and its first and second derivatives satisfy (2.6) and (2.7). We assume that G satisfies this condition, and, using equalities $\mathcal{E}^-g = \mathcal{E}^-q\mathcal{E}^{-1}G = q(\mathcal{E}^+)^{-1}G$ and $\mathcal{E}^+g = \mathcal{E}^+q\mathcal{E}^{-1}G = (\mathcal{E}^-)^{-1}G$, obtain analogs for Theorems 3.8 and 3.6 for options with instantaneous payoffs:

Theorem 3.9. *Assume that there exists $h^* \in \mathbb{R}$ such that*

$$(\mathcal{E}^+)^{-1}G(x) < 0,\ x < h^*, \quad \text{and } (\mathcal{E}^+)^{-1}G(x) > 0,\ x > h^*. \tag{3.22}$$

Then the option with the payoff G must be exercised the first time the threshold h^* is reached or crossed from below, and the option value is

$$V^* = \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G. \tag{3.23}$$

Theorem 3.10. *Assume that there exists $h_* \in \mathbb{R}$ such that*

$$(\mathcal{E}^-)^{-1}G(x) < 0,\ x > h_*, \quad \text{and } (\mathcal{E}^+)^{-1}G(x) > 0,\ x < h_*. \tag{3.24}$$

Then the option with the payoff G must be exercised the first time the threshold h_* is reached or crossed from above, and the option value is

$$V_* = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G. \tag{3.25}$$

4. Extensions and ramifications

As in Section 3, we assume that X_t is a jump-diffusion with the Lévy exponent (2.2); all results admit generalizations for a Lévy process satisfying the (ACP)-condition.

4.1. **Optimality conditions in the class of all stopping times, and options to swap a stream for another one.** Consider a firm, which contemplates the switch from one type activity to another one, addition of a new production unit, adoption of a new technology, closure of one of the operating units, etc. In all these cases, the firm has an option to swap the old stream of profits, $g_o(X_t)$, for the new one, $g_n(X_t)$. (The cost of the switch can be included into the new stream as the stream of coupon payments.) The option will be realized at the optimal random time τ. We assume that $\tau \in \mathcal{M}$, where $\mathcal{M}$ is the class of stopping times satisfying $\tau < +\infty$, a.s. Formally, we need to find the stopping time which maximizes the normalized value function

$$V(g_1, g_2; x) = \sup_{\tau\in\mathcal{M}} E^x\left[\int_0^\tau e^{-qt}g_o(X_t)dt + \int_\tau^{+\infty} e^{-qt}g_n(X_t)dt\right]. \tag{4.1}$$

This class of options contains all the classes studied in Section 3: Theorems 3.6 and 3.8 consider the case $g_o = 0$ and $g_n = g$, and Theorems 3.2 and 3.4 - the case $g_o = g$ and $g_n = 0$. To ensure that the stochastic expression in (4.1) was well-defined, we require that both g_o and g_n satisfy (2.7) and (2.6). Assume that at high positive levels of the stochastic factor X_t, stream $g_o(X_t)$ dominates stream $g_n(X_t)$, and the latter dominates the former at low negative levels. To be more specific,

$$\text{function } g(x) := g_n(x) - g_o(x) \quad \text{is non} - \text{increasing on } \mathbb{R}, \tag{4.2}$$

and

$$\lim_{x\to-\infty} g(x) > 0, \qquad \lim_{x\to+\infty} g(x) < 0. \tag{4.3}$$

Then it is natural to presume that it is optimal to swap stream $g_o(X_t)$ for $g_n(X_t)$ when the stochastic factor X_t reaches a certain threshold h_* from above or crosses it.

Theorem 4.1. *Suppose that functions g_o and g_n satisfy conditions (2.7), (2.6), (4.2) and (4.3). Then: a) equation*

$$\mathcal{E}^+g_o(h) = \mathcal{E}^+g_n(h) \tag{4.4}$$

has a unique solution, denote it h_*;

b) the optimal stopping time is $\tau^-_{h_*}$;

c) the option value is given by

$$V(g_o, g_n; x) = q^{-1}\mathcal{E}^-\left\{\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g_o + \mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g_n\right\}(x), \tag{4.5}$$

or equivalently,

$$V(g_o, g_n; x) = q^{-1}E[\mathbf{1}_{(h_*,+\infty)}(x+Y^-)g_o(x+Y^++Y^-)+\mathbf{1}_{(-\infty,h_*]}(x+Y^-)g_n(x+Y^++Y^-)]. \tag{4.6}$$

Proof. a) Since g is monotone, it is measurable, and since it satisfies (2.7) and (2.6), function $\mathcal{E}^+g$ is continuous. Function g satisfies (4.3), therefore $\mathcal{E}^+g$ satisfies (4.3) as well. Since g is monotone, $\mathcal{E}^+g$ is also monotone. We conclude that equation $\mathcal{E}^+g(x) = 0$ has a zero. To prove that a zero is unique, assume that there exists $x^0 \in \mathbb{R}$ such that for all $x \geq x^0$, $g(x) = c\,(< 0)$. Then $\mathcal{E}^+g(x) = c$ on the same interval but it follows from (4.2) and (2.9) that $\mathcal{E}^+g$ is a strictly decreasing continuous function on $(-\infty, x^0)$; hence, the zero is unique. If such x^0 does not exist, then $\mathcal{E}^+g$ is a strictly decreasing continuous function on $\mathbb{R}$, and the zero is unique.

c) Assuming b) has been proved, equality (4.5) follows from (3.1) and (3.14).

b) will be deduced from the following lemma, which is of independent interest. In particular, this lemma can be applied to optimal stopping rules of a more general form then in Theorem 4.1.

Lemma 4.2. *Let an open set $U \subset \mathbb{R}$ and a measurable function W_* satisfy the following conditions:*

$$W_*(x) = g_o(x), \qquad x \in U; \tag{4.7}$$

$$W_*(x) \ge g_o(x), \qquad x \notin U, \text{ a.e.}; \tag{4.8}$$

$$\mathcal{E}W_*(x) = \mathcal{E}g_n(x), \qquad x \notin U; \tag{4.9}$$

$$\mathcal{E}W_*(x) \ge \mathcal{E}g_n(x), \qquad x \in U. \tag{4.10}$$

Then τ_*, the hitting time of U^c, is the optimal stopping time in the class $\mathcal{M}$, and $V_* := q^{-1}\mathcal{E}W_*$ is the rational option price.

Remark 4.1. a) We can reformulate Lemma 4.2 as follows: the option price is generated by a measurable stream W_*. Conditions (4.7)-(4.8) state that in the inaction region, this stream coincides with the stream which the option generates prior to exercise, and in the action region, the former equals or exceeds the latter. In the action region, after the action is undertaken, the EPV of the stream matters, and evidently, the option value is generated by stream $g_n(X_t)$ (condition (4.9)). On the other hand, in the inaction region, the option value must be at least as big as the EPV of the stream $g_n(X_t)$ (condition (4.10)).

b) The difference in the formulation between pairs (4.7)-(4.8) and (4.9)-(4.10) is due to the irreversibility of the swap. In the completely reversible case, the option value is generated by the stream $W(x) = \max\{g_o(x), g_n(x)\}$, and conditions (4.9)-(4.10) hold without the EPV operators $\mathcal{E}$. The streams g_o and g_n are on the equal footing.

c) If X_t satisfies the (ACP)-property, and W_* is measurable and bounded, then $V_* = q^{-1}\mathcal{E}W_*$ is continuous (see Sato (1999), p.288-289). The case of unbounded functions satisfying conditions (2.6)-(2.7) can be reduced to the case of bounded functions, therefore V_* is continuous.

d) The statement of Lemma 4.2 and the remark above are valid under weaker regularity conditions on W: universal measurability suffices (for the definition, see Sato (1999), p.274). In the setting of Theorem 4.1, W_* turns out to be measurable (see Lemma 4.3).

Proof of Lemma 4.2. Notice that

$$V(g_o, g_n; x) = E^x\left[\int_0^{+\infty} e^{-qt} g_o(X_t)dt\right] + V(0, g_n - g_o; x) = q^{-1}\mathcal{E}g_o(x) + V(0, g; x),$$

where $g = g_n - g_o$, therefore the optimization of $V(g_o, g_n; x)$ is equivalent to the optimization of $V(0, g; x)$. Further, W satisfies (4.7)-(4.10) if and only if $\tilde{W} = W - g_o$ satisfies the same conditions with $g_o = 0$, $g_n = g$. On the strength of (4.7)-(4.8), $\tilde{W}$ is non-negative, a.e., and it is measurable, since W and g_0 are. Hence, equation (41.3) in Sato (1999) is applicable with $f = W$: for any stopping time τ,

$$\mathcal{E}\tilde{W}(x) = E^x\left[\int_0^{\tau} qe^{-qt}\tilde{W}(X_t)dt\right] + E^x\left[e^{-q\tau}\mathcal{E}\tilde{W}(X_\tau)\right]. \tag{4.11}$$

Using (4.7)-(4.8), and then (4.9)-(4.10), we derive from (4.11) the estimate

$$\mathcal{E}\tilde{W}(x) \geq E^x\left[e^{-q\tau}\mathcal{E}\tilde{W}(X_\tau)\right] \geq E^x\left[e^{-q\tau}\mathcal{E}g(X_\tau)\right].$$

Hence, $\mathcal{E}\tilde{W}(x) \geq qV(0,g;x)$. If we take $\tau = \tau_*$ and apply (4.7) and (4.9) to (4.11), we obtain

$$\mathcal{E}\tilde{W}(x) = E^x\left[e^{-q\tau_*}\mathcal{E}\tilde{W}(X_{\tau_*})\right] = E^x\left[e^{-q\tau_*}\mathcal{E}g(X_{\tau_*})\right] \leq qV(0,g;x).$$

Lemma has been proved.

Notice that equation (4.11), the key element of the proof, has a simple meaning: the EPV of a stream equals the EPV up to a stopping time τ plus the continuation value.

Now we can finish the proof of Theorem 4.1. As we have shown, it suffices to consider the case $g_o = 0$, $g_n = g$. First, we find W_* which generates the option value $V(0,g;x) = q^{-1}\mathcal{E}W_*(x)$. Applying $q - L$, we find $W_*(x) = (q-L)V(0,g;x)$. If $\tau_* = \tau^-_{h_*}$ is chosen as the stopping time, the option value is given by $qV(0,g;x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g(x)$ (see (3.1)), therefore, using $q^{-1}(q-L) = (\mathcal{E})^{-1} = (\mathcal{E}^+)^{-1}(\mathcal{E}^-)^{-1}$, we obtain $W_* = (\mathcal{E}^+)^{-1}\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g(x)$. Since $\mathbf{1}_{(-\infty,h_*]}$ is zero on $(h_*,+\infty)$, we can apply (2.20), and derive (4.7). For $x \leq h_*$,

$$\mathcal{E}W_*(x) = (\mathcal{E}^-\mathcal{E}^+g)(x) - \mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g(x) = \mathcal{E}g(x) - 0,$$

that is, (4.9) holds. For $x > h_*$, we notice that by the definition of h_*, $\mathcal{E}^+g$ is non-positive on $(h_*,+\infty)$, therefore $\mathcal{E}W_*(x) \geq (\mathcal{E}^-\mathcal{E}^+g)(x)$, and (4.10) holds. The last condition, (4.8), follows from equations (6.71)–(6.72) in Boyarchenko and Levendorskiĭ (2005), which state that W_* is non-increasing on $(-\infty,h_*)$, and $W_*(h_*-0) \geq 0$. Theorem 4.1 has been proved. □

For the future use, note the properties of W_* for general g_o and g_n:

Lemma 4.3. *Function $W_*(g_o,g_n;\cdot) = g_o + W_*(0,g;\cdot)$ is measurable, and if g_o is non-decreasing, then $W_*(g_o,g_n;\cdot)$ is non-decreasing as well.*

By symmetry, we obtain the following analogs of Theorem 4.1 and Lemma 4.3.

Theorem 4.4. *Suppose that functions g_o and g_n satisfy conditions (2.7), (2.6), and*

$$\text{function } g(x) := g_n(x) - g_o(x) \quad \text{is non} - \text{decreasing on } \mathbb{R}, \tag{4.12}$$

and

$$\lim_{x\to-\infty} g(x) < 0, \qquad \lim_{x\to+\infty} g(x) > 0. \tag{4.13}$$

Then: a) equation

$$\mathcal{E}^+g_o(h) = \mathcal{E}^+g_n(h) \tag{4.14}$$

has a unique solution, denote it h^*;

b) it is optimal to swap stream $g_o(X_t)$ for $g_n(X_t)$ when the stochastic factor X_t reaches h^* from below or crosses it;

c) the option value is given by

$$V(g_o,g_n;x) = q^{-1}\mathcal{E}^+\left\{\mathbf{1}_{[h_*,+\infty)}\mathcal{E}^-g_n + \mathbf{1}_{(-\infty,h_*)}\mathcal{E}^-g_o\right\}(x), \tag{4.15}$$

or equivalently,

$$V(g_o,g_n;x) = q^{-1}E[\mathbf{1}_{[h^*,+\infty)}(x+Y^+)g_n(x+Y^++Y^-)+\mathbf{1}_{(-\infty,h^*]}(x+Y^+)g_o(x+Y^++Y^-)]. \tag{4.16}$$

d) $V(g_o,g_n;\cdot)$ is generated by a measurable stream $W(g_o,g_n;\cdot)$: $V(g_o,g_n;\cdot) = q^{-1}\mathcal{E}W(g_o,g_n;\cdot)$.

e) If g_o is non-decreasing, then $W(g_o,g_n;\cdot)$ is non-decreasing as well.

Remark 4.3. Clearly, if $g(x) \ge 0$ for all x, then it is optimal to swap stream g_o for stream g_n the first moment one is allowed to.

4.2. **Embedded options: Russian dolls.** Consider a firm in a growing industry, which contemplates a multi-stage investment project. On each stage, an additional production facility can be added or a new technology adopted, etc. Assume that the number of stages is finite, say, N, the order of stages is fixed, and the investment is irreversible. After (investment) stage k but before stage $k+1$, the profit flow is $g_k(X_t)$, and the fixed investment cost on stage k is I_k. For $k=0$, set $g_0(X_t) \equiv 0$. The time τ^k of making investment on stage k is random; it is chosen by a firm to maximize the EPV of the project. Set $\tau^{N+1} = +\infty$, and denote by $V_k(X_t)$ the value of the firm for $t \in [\tau^k, \tau^{k+1}), k = 1, 2, \dots, N$. After stage N no further investment is expected, therefore $V_N(X_t) = q^{-1}\mathcal{E}g_N(X_t)$ is known. The firm needs to solve the following sequence of optimal stopping problems, for $k = N, N-1, \dots, 1$:

$(\mathbf{RD})^+$. *Find the optimal stopping time τ_k^* to exchange stream $g_{k-1}(X_t)$ for the instantaneous payoff $V_k(X_t) - I_k$.*

We will call this sequence of embedded options a *Russian doll*. After the completion of all N stages of investment, the firm can be associated with a Russian doll, containing a sequence of smaller dolls inside. We solve the problem of the expanding firm by backwards induction, that is by opening the Russian doll: in order to see the smaller doll (option), we must first remove (resolve) the current one. The Russian doll associated with the expansion of investment project will be called an *expanding Russian doll*.

On the other hand, a firm in a declining industry involved in a multistage contraction project can be viewed as a Russian doll stripped of larger dolls that contained the current one before the contraction had started. We call the multistage contraction option a *contracting Russian doll*. To obtain the solution in this situation, we divine in some way the exact characteristics of the smallest doll and then use this information to deduce the characteristics of the sequence of larger dolls (in other words, we assemble the Russian doll).

Assume that

(i) profit functions g_k are non-decreasing and satisfy (2.7) and (2.6);
(ii) for all $k = 1, 2, \dots, N$, the difference $g_k - g_{k-1}$ is non-decreasing, and for sufficiently large x, $g_k(x) - qI_k > g_{k-1}(x)$, which means that as the stochastic factor assumes larger values, the relative advantage of the next stage increases, and when X_t reaches a sufficiently high level, then it must be optimal to make step k investment.

Theorem 4.5. *Under conditions (i)–(ii), for $k = N, N-1, \dots, 1$, the following statements hold:*
a) functions $W_k^0 := (q-L)V_k$ and $W_k := W_k^0 - qI_k - g_{k-1}$ are defined on $\mathbb{R}$ (with a possible exception of one point), and are non-decreasing;
b) $\lim_{x\to+\infty} W_k(x) > 0$;
c) denote $h_k := \inf\{x \mid (\mathcal{E}^- W_k)(x) > 0\}$; then $\tau_k^* = \max\{\tau_{k-1}^*, \tau_{h_k}^+\}$ is the optimal time for stage k investment;
d) $V_{k-1} = q^{-1}\left(\mathcal{E}g_{k-1} + \mathcal{E}^+\mathbf{1}_{[h_k,+\infty)}\mathcal{E}^- W_k\right)$.

Note that c) means that it is optimal to make stage k investment when X_t reaches h_k from below or crosses it, the first time after stage $k-1$ investment; hence, it is possible that several investment stages will be simultaneous.

Proof. After stage $k-1$ is completed but stage k is not, the value of the firm, V_{k-1}, is the value of the option to swap stream g_{k-1} for stream $(q-L)(V_k - I_k)$. Conditions (i)–(ii) imply that $W_N = (q - L)(V_N - I_N) - g_{N-1} = g_N - qI_N - g_{N-1}$ is non-decreasing and positive in a neighborhood of $+\infty$. Thus, we obtain a) and b) for $k = N$. Now we prove that if a) and b) hold for $k = m$, $1 \le m \le N$, then c) and d) hold for the same k, and if c) and d) hold for $k = m$, $2 \le m \le N - 1$, then a) and b) hold for $k = m - 1$. Clearly, if W_m satisfies a) and b) then $\mathcal{E}^- W_m$ does, and $\mathcal{E}^- W_m$ can be locally constant in a neighborhood of $+\infty$ only. If $\lim_{x\to-\infty} \mathcal{E}^- W_m(x)$ is negative, then $h_m > -\infty$, and from Theorem 4.4, we conclude that c) and d) hold with the same $k = m$, and V_{m-1} is generated by a non-decreasing stream W^0_{m-1}: $V_{m-1} = q^{-1}\mathcal{E}W^0_{m-1}$. Function $W^0_{m-1} = (q - L)V_{m-1} = q^{-1}(q - L)(qV_{m-1})$ satisfies (4.7)-(4.8) with $g_o = g_{m-1}$, therefore $W^0_{m-1}(x) \ge g_{m-1}(x)$, and using (i)–(ii), we conclude that conditions a) and b) hold for $k = m - 1$. □

Now we consider a firm in a declining industry, which scraps its production facilities in a predetermined order; timing depends on a realization of uncertainty. Let C_k be the scrap value on stage k of disinvestment. Currently, the stream of profits is g_0; after stage k but prior to stage $k + 1$, it is g_k, and after the last stage, the firm disappears, and its stream of profits and value V_N is zero. The firm needs to solve the following sequence of optimal stopping problems, for $k = N, N - 1, \dots, 1$:

$(\mathbf{RD})^-$. *Find the optimal stopping time $\tau_{*,k}$ to exchange stream $g_{k-1}(X_t)$ for the instantaneous payoff $C_k - V_k(X_t)$.*

Assume that

(i) profit functions g_k are non-decreasing and satisfy (2.7) and (2.6);
(ii) for all $k = 1, 2$, the difference $g_k - g_{k-1}$ is non-increasing, and for sufficiently large negative x, $g_k(x) + qC_k > g_{k-1}(x)$, which means that as the stochastic factor tends to $-\infty$, the relative advantage of the next contraction increases, and when X_t reaches a sufficiently low level, then it must be optimal to make step k disinvestment.

The statement and proof of the following theorem are mirror reflections of Theorem 4.5 and its proof.

Theorem 4.6. *Under conditions (i)–(ii), for $k = N, N-1, \dots, 1$, the following statements hold:*

a) function $W^0_k := (q - L)V_k$ and $W_k := W^0_k + qC_k - g_{k-1}$ are defined on $\mathbb{R}$ (with a possible exception of one point), and are non-increasing;
b) $\lim_{x\to-\infty} W_k(x) > 0$;
c) denote $h_k := \sup\{x \mid (\mathcal{E}^+ W_k)(x) > 0\}$; then $\tau_{*,k} = \max\{\tau_{*,k-1}, \tau^-_{h_k}\}$;
d) $V_{k-1} = q^{-1}\left(\mathcal{E}g_{k-1} + \mathcal{E}^- \mathbf{1}_{(-\infty,h_k]}\mathcal{E}^+ W_k\right)$.

5. Capital expansion program

5.1. **Timing an investment of a marginal unit of capital.** Consider a monopoly whose production function depends only on capital: $Q = Q(K)$. (A generalization to the case of a production function with costlessly adjustable labor as in Abel and Eberly (1999) is straightforward but leads to more involved formulas below). For simplicity, assume that the inverse demand function is factorizable: $D_t = \bar{D}(Q_t)Z_t$, where Z_t is the exogenous demand shock. We assume that

(i) function $G(Q) := Q\bar{D}(Q)$ is differentiable, increasing, concave and satisfies the Inada conditions;
(ii) $Z_t = Z(X_t)$ is a non-decreasing function of a Lévy process X_t with the Lévy exponent Ψ;

(iii) function Z satisfies estimate

$$Z(x) \le c_1 e^{\gamma x}, \quad \forall\, x, \tag{5.1}$$

where $c_1 > 0$ and $\gamma \ge 0$ are independent of x, and the no-bubble condition holds:

$$q - \Psi(\gamma) > 0. \tag{5.2}$$

Remark 4.1. a) Under condition (i), when K_t units of capital is in place, the firm finds it optimal to produce the maximal amount $Q_t = Q(K_t)$, and therefore, the revenue flow is

$$R_t = Q(K_t)\bar{D}(Q(K_t))Z(X_t) = G(K_t)Z(X_t).$$

b) For a jump-diffusion with the Lévy exponent (2.2), (5.2) is equivalent to $\gamma < \beta_1^+$.

c) Conditions (5.1)-(5.2) guarantee that if the firm keeps the level of installed capital fixed: $K_t = K_0,\ \forall\, t$, then the EPV of the revenue flow is finite:

$$E\left[\int_0^{+\infty} e^{-qt} R(X_t)dt\right] \le \frac{c_1 G(K_0)}{q - \Psi(\gamma)} < \infty. \tag{5.3}$$

Should the firm decide to invest a unit of capital, it suffers the installation cost C; the investment is irreversible. The firm's objective is to choose the optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, X_t)\}_{t\ge 1}, K_0 = K, X_0 = x$, which maximizes the NPV of the firm:

$$V(K,x) = \sup_{\mathcal{K}} E^x\left[\int_0^{+\infty} e^{-qt}(Z(X_t)G(K_t) - qCK_t)dt\right]. \tag{5.4}$$

A similar situation was considered in Dixit and Pindyck (1996) and Abel and Eberly (1999) for the geometric Brownian motion model and extended by Boyarchenko (2004) for geometric Lévy processes. In these papers, $Z(X_t) = \exp X_t$, and therefore, condition (5.1) holds with $\gamma = 1$. As Dixit and Pindyck (1996) show, the value of the firm is infinite unless an additional restriction on the rate of growth of function $G(K)$ as $K \to +\infty$ is imposed, and this condition is too restrictive. We will show that if $Z(X_t)$ behaves as $\exp X_t$ up to a certain threshold but above the threshold the rate of growth of $Z(X_t)$ decreases then the restriction on the rate of growth of $G(K)$ can be relaxed. As a by-product, we will show that, as the optimal capital increases, the range within the monopoly price P_t fluctuates grows slower than in the standard geometric Lévy model. Moreover, we will demonstrate that this range may shrink as the demand shock reaches the intermediate region between the intervals of the fast exponential growth and of the slower growth. This means that the firm may find it optimal to simultaneously increase the capital stock and decrease the price of the output.

For the time being, to ensure that firm's value (5.4) were bounded, we impose a resource constraint: there exists $\bar{K} < \infty$ such that $K_t \le \bar{K},\ \forall t$. Later, we will show that if γ in (iii) is sufficiently small, then the resource constraint is redundant: the expected rate of growth of the optimal capital is not very large, and the value of the firm is finite even if the firm has unlimited access to capital. Notice that if the demand shock Z is bounded ($\gamma = 0$), then there exists $\bar{K}$ such that the firm would never want to choose $K_t > \bar{K}$.

It is well-known (see, for example, Dixit and Pindyck (1996)) that in order to determine the optimal capital expansion program, it is only necessary to decide when to invest at any given stock of capital. Equivalently, one needs to find the investment threshold $h(K)$, which is the boundary between two regions in the state variable space (K, x): the action and inaction ones. To derive the equation for the investment boundary, suppose first that every new investment can be made in chunks of capital, ΔK, only[2]. In this case, the firm has to suffer the cost $C\Delta K$, and the EPV of the profit gain due to this investment can be represented in the form of the EPV of the stream $g(X_t) = (G(K+\Delta K) - G(K))Z(X_t) - qC\Delta K$. From Subsection 3.4, we know that it is optimal to invest capital $C\Delta K$ the first time the price of the firm's output crosses the investment barrier $h(K;\Delta K)$ that satisfies (3.19). For g defined above, (3.19) can be written as

$$\mathcal{E}^- \left[(G(K+\Delta K) - G(K))\, Z(\cdot) - qC\Delta K \right](x) = 0,$$

or

$$(G(K+\Delta K) - G(K))\, \mathcal{E}^- Z(x) = qC\Delta K. \tag{5.5}$$

Dividing (5.5) by ΔK and passing to the limit as $\Delta K \to 0$, we obtain the following equation for the optimal investment threshold $h^* = h^*(K)$:

$$G'(K)(\mathcal{E}^- Z)(h^*) = qC, \tag{5.6}$$

or

$$G'(K) E\left[\int_0^{+\infty} e^{-qt} Z(h^* + \underline{X}_t) dt \big| X_0 = 0 \right] = C. \tag{5.7}$$

The last equation says that it is optimal to invest into a marginal unit of capital the first time the EPV of the marginal profit, calculated under the assumption that the underlying stochastic process $\{X_t\}$ is replaced by the infimum process $\{\underline{X}_t\}$, becomes non-negative[3].

Let $h = h(K;\Delta K)$ be a solution to (5.5). Then at the shock level x, the option value associated with the chunk of capital ΔK is

$$q^{-1}\mathcal{E}^+ \mathbf{1}_{[h,+\infty)} \left[(G(K+\Delta K) - G(K))\, (\mathcal{E}^- Z)(\cdot) - qC\Delta K \right](x).$$

As $\Delta K \to 0$, we have $h(K;\Delta K) \to h^*(K)$. Notice that capital accumulation extinguishes the option value of investment, this means that the option value is decreasing in K (for more detailed discussion, see Abel et al. (1996)). Therefore, dividing the above option value by ΔK and passing to the limit as $\Delta K \to 0$, we obtain the following formula for the marginal option value of capital:

$$V_K^{\mathrm{opt}}(K,x) = -q^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}(x) \left(G'(K)(\mathcal{E}^- Z)(x) - qC \right).$$

Substituting C from (5.6) into the above equation, we arrive at

$$qV_K^{\mathrm{opt}}(K,x) \quad = \quad -G'(K)\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}(x)[(\mathcal{E}^- Z)(x) - (\mathcal{E}^- Z)(h^*)].$$

Introduce the notation

$$w(x) = E\left[\int_0^{+\infty} qe^{-qt} \left(Z(x+\underline{X}_t) - Z(h^* + \underline{X}_t) \right) dt \big| \, X_0 = 0 \right].$$

Then

$$qV_K^{\mathrm{opt}}(K,x) = -G'(K)(\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} w)(x). \tag{5.8}$$

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[2]The authors are indebted for this simplifying trick to Mike Harrison; the initial proof (for geometric Lévy case) in Boyarchenko (2004) was more involved.

[3]For the rigorous justification of the limiting argument see Boyarchenko (2004).

Using independent random variables $Y^+ = \bar{X}_T$ and $Y^- = \underline{X}_T$ supported on the positive and negative half-axes, respectively (they were introduced in Subsection 2.2), we can write equations (5.6) and (5.8) in the form

$$G'(K)E[Z(h^* + Y^-)] = qC, \tag{5.9}$$

and

$$qV_K^{\mathrm{opt}}(K,x) = -G'(K)E[\mathbf{1}_{[h^*,+\infty)}(x+Y^+)(Z(x+Y^+ + Y^-) - Z(h^* + Y^-))]. \tag{5.10}$$

We have proved

Theorem 5.1. *Let conditions (i)–(iii) hold. Then the optimal capital expansion threshold $h^* = h^*(K)$ is a unique solution of any of equivalent equations (5.6), (5.7) and (5.9), and the marginal option value of capital is given by any of equivalent equations (5.8) and (5.10).*

Consider the case when X is a jump-diffusion process defined by (2.3). We use (2.16), and rewrite the equation for the threshold in the form

$$G'(K)\sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-)e^{-\beta_j^- y}Z(h^*+y)dy = qC; \tag{5.11}$$

the marginal option value of capital is

$$\begin{aligned} V_K^{\mathrm{opt}}(K,x) &= -G'(K)\sum_{j=1,2} a_j^+ \int_{h^*-x}^{+\infty} \beta_j^+ e^{-\beta_j^+ y} w(x+y)dy \\ &= -G'(K)\sum_{j=1,2} a_j^+ e^{\beta_j^+(x-h^*)} \int_0^{+\infty} \beta_j^+ e^{-\beta_j^+ y} w(h^*+y)dy, \end{aligned} \tag{5.12}$$

where

$$w(x) = q^{-1}\sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-)e^{-\beta_j^- y}\left(Z(x+y) - Z(h^*+y)\right)dy.$$

5.2. **Option value.** Integrating (5.12) w.r.t. K, we find the option value

$$V^{\mathrm{opt}}(K,x) = -\int_K^{\bar{K}} V_K^{\mathrm{opt}}(K',x)dK'. \tag{5.13}$$

If we want to remove the resource constraint $K \le \bar{K}$, we need to prove that the limit of the integral (5.13) exists as $\bar{K} \to +\infty$, and then the value of the firm is given by (5.13) with $\bar{K} = +\infty$. In the Appendix, we show that if (5.1) holds, then a sufficient condition for the convergence is

$$\int_1^{+\infty} G'(K')^{\beta_1^+/\gamma} dK' < +\infty. \tag{5.14}$$

In the geometric Lévy case, when $Z(X_t) = e^{\gamma X_t}$, this condition is necessary. In particular, if $G(K) = dK^\theta$ ($d > 0$, $\theta \in (0,1)$), then for the convergence of the integral in the case of the jump-diffusion process, we must have $\theta < 1 - \gamma/\beta_1^+$. In other words, θ must be sufficiently less than one, which means that the returns to capital must decrease sufficiently fast. As Dixit and Pindyck (1996) show in the geometric Brownian motion case, for typical parameters of a process, this condition requires for θ to be too small. If the jump component is not very strong, then β_1^+ is close to the one in the geometric Brownian motion case, and the same conclusion holds.

Now, suppose that up to a moderate level of demand, the demand shock is fitted well by a geometric jump-diffusion process with $\gamma = 1$, and $\theta \ge 1 - 1/\beta_1^+$. To ensure that the value of the firm be finite, we may assume that above a certain high level $\bar{Z}$ of the stochastic factor $Z(X_t)$, the rate of growth of $Z(X_t)$ slows down, and (5.1) holds with sufficiently small $\gamma > 0$ so that $\theta < 1 - \gamma/\beta_1^+$. Then the integral (5.14) converges, and the value of the firm is finite, even if the resource constraint is dropped. Finally, assume that Z is uniformly bounded from above: $Z(x) \le c_2$, which implies that the demand shocks are bounded. Then the LHS in (5.7) admits an upper bound via $G'(K)c_2q^{-1}$. Since G satisfies the Inada conditions, $G'(K) \to 0$ as $K \to +\infty$. Hence, for sufficiently large K, the LHS in (5.7) will be smaller than the RHS for any h^*, and it is not optimal to increase the capital stock above a certain level. The resource constraint becomes redundant.

5.3. **Examples.** Consider the Cobb-Douglas production function $Q_t = dK_t^\rho$, where $d, \rho > 0$, and the inverse demand function $P_t = Z_t Q_t^{-1/\epsilon}$, where $Z_t = Z(X_t)$ is the demand shock, and $\epsilon > 1$ is the elasticity of demand. Then $G(K) = d^{1-1/\epsilon}K^{\rho(1-1/\epsilon)}$, and the above results apply provided $\theta := \rho(1-1/\epsilon) \in (0,1)$, and Z satisfies condition (iii). We consider two families of functions Z; the process X_t is a jump-diffusion process with the Lévy exponent (2.2).

Example 5.1. First, consider the geometric Lévy case $Z(X_t) = e^{\gamma X_t}$, where $\gamma > 0$. Condition (5.2) is equivalent to $\gamma < \beta_1^+$. If there is no exogenous bound on the amount of capital available, then the value of the firm is finite iff $\theta = \rho(1-1/\epsilon) < 1 - \gamma/\beta_1^+$. This means that for a given $\gamma > 0$, either ρ or ϵ must be sufficiently small. However, if $\rho \le 1 - \gamma/\beta_1^+$, then the elasticity of demand may assume any value $\epsilon > 1$. The revenue flow is $R_t = (dK^\rho)^{1-1/\epsilon}e^{\gamma X_t}$, and equation (5.6) for the investment threshold becomes

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(\gamma)e^{\gamma h^*} = Cq. \tag{5.15}$$

The description of the optimal investment policy in terms of the demand shock is standard: when a point (X_t, K) remains to the left of the boundary (5.15) of the inaction region: $X_t \le h^*(K)$, the monopoly keeps the capital level $K_t = K$ fixed and increases or decreases the price of the output as the demand does; when the demand shock factor X_t crosses level $h^*(K)$, the firm increases the capital stock to the new level K' so that $X_t = h^*(K')$, and (X_t, K') is on the boundary of the inaction region. At this moment, the firm increases the price, decreases it, or keeps it fixed, if the production technology exhibits decreasing returns to scale, increasing returns to scale, or constant returns to scale, respectively[4]. Indeed, when the demand shock $Z(X_t)$ is at the investment threshold, the monopoly charges price

$$P^* = P^*(K, h^*(K)) = (dK^\rho)^{-1/\epsilon}Z(h^*(K)) = \frac{K^{1-\rho}}{d\rho(1-1/\epsilon)\kappa_q^-(\gamma)},$$

and the RHS increases in K if $\rho < 1$, decreases if $\rho > 1$, and remains constant if $\rho = 1$. The smaller the $\gamma > 0$, the larger is $\kappa_q^-(\gamma) = E[e^{\gamma Y^-}]$, and the lower is the output price at the moment of investment.

[4]Of course, we understand that the technology may exhibit increasing returns to scale only locally, for small levels of capital. We mention the price behavior for increasing returns to scale production function only because in numerical experiments we observe similar behavior for decreasing returns to scale technology and small rate of growth of the demand shock, when the demand is in a certain range.

Example 5.2. Consider the following demand shock. As $Z(X_t)$ remains below a certain critical value $\bar{Z}$, the dynamics of the stochastic factor is given by the geometric Lévy process:

$$Z(X_t) = \bar{Z}e^{X_t}, \quad X_t \le 0. \tag{5.16}$$

However, in the region above the critical level $\bar{Z}$, the rate of growth of $Z(X_t)$ slows down:

$$Z(X_t) = \bar{Z}[\gamma^{-1}(e^{\gamma X_t} - 1) + 1], \quad X_t > 0, \tag{5.17}$$

where $\gamma \in (0, 1)$. In the limit $\gamma \to 1$, we recover the standard geometric Lévy case; in the limit $\gamma \to 0$, the shock follows the geometric Lévy process below 0, and the Lévy process above 0.

Consider equation (5.7) for the investment threshold. Since function $Z = Z(x)$ is monotone, $(\mathcal{E}^- Z)(x)$ also is. Hence, (5.7) has a unique solution, $h^* = h^*(K)$. If $h^* \le 0$, then the LHS of (5.7) is independent of the values of $Z(x)$ for positive x, hence h^* is determined from the same equation as in the geometric Lévy case:

$$d^{1-1/\epsilon}\rho(1 - 1/\epsilon)K^{\rho(1-1/\epsilon)-1}E\left[\int_0^{+\infty} e^{-qt}\bar{Z}e^{h^*+\underline{X}_t}dt\right] = C, \tag{5.18}$$

which is

$$d^{1-1/\epsilon}\rho(1 - 1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(1)\bar{Z}e^{h^*} = qC. \tag{5.19}$$

From (5.19), it is evident that $h^* \le 0$ iff $d^{1-1/\epsilon}\rho(1 - 1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(1)\bar{Z} \ge qC$.

Let $d^{1-1/\epsilon}\rho(1 - 1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(1)\bar{Z} < qC$, then (5.19) has no non-positive solutions. Therefore, the investment threshold h^* is positive, and we have to use both (5.16) and (5.17). We calculate $(\mathcal{E}^- Z)(x)$ for $x > 0$:

$$\mathcal{E}^- Z(x) = \bar{Z}\left[\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma x} - \gamma^{-1}(1-\gamma) + \sum_{j=1,2} d_{\gamma,j}e^{\beta_j^- x}\right], \tag{5.20}$$

where $d_{\gamma,j}$ are positive constants (see the Appendix). The investment threshold is the solution to equation (5.6). Using (5.20), we write equation (5.6) in the form

$$d^{1-1/\epsilon}\rho(1 - 1/\epsilon)K^{\rho(1-1/\epsilon)-1}\bar{Z}\left[\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*} - \gamma^{-1}(1-\gamma) + \sum_{j=1,2} d_{\gamma,j}e^{\beta_j^- h^*}\right] = qC. \tag{5.21}$$

In the upper panel of Fig. 1, we plot the graph of $Z(x)$ for $\gamma = 0.999$ (which is close to the geometric Lévy case $\gamma = 1$), $\gamma = 0.6$ and $\gamma = 0.3$. In the middle panel, we plot the boundary of the inaction region in the (Z, K)-plane. Finally, in the lower panel, we plot the boundary of the inaction region in the (P, K)-plane. Here, as a natural technical device, we use the explicit parametrization of the curve (K, P^*) by h^*: $K = K(h^*)$ is found from (5.19) for $h^* \le 0$, and from (5.21) for $h^* > 0$, and after that we calculate $P^* = (dK^*)^{-1/\epsilon}Z(h^*) = (dK(h^*))^{-1/\epsilon}Z(h^*)$. We take $\rho = 0.9$ (decreasing returns to scale case). As $h^*(K) \le 0$ (which implies that K is below a certain level), the threshold is the same for all γ, and the boundary in the (P, K)-plane is upward sloping which means that each increase of capital stock is accompanied by an increase in the price of the output. For larger values of K, the boundary depends on γ, and it may be even locally downward sloping, which means that an *increase* in the capital stock may be accompanied by a *decrease* in the output price. The business returns to normality at sufficiently large levels of capital stock: once again, an increase in the capital stock is accompanied by an increase in the output price. To see this, we derive an approximate formula for the threshold

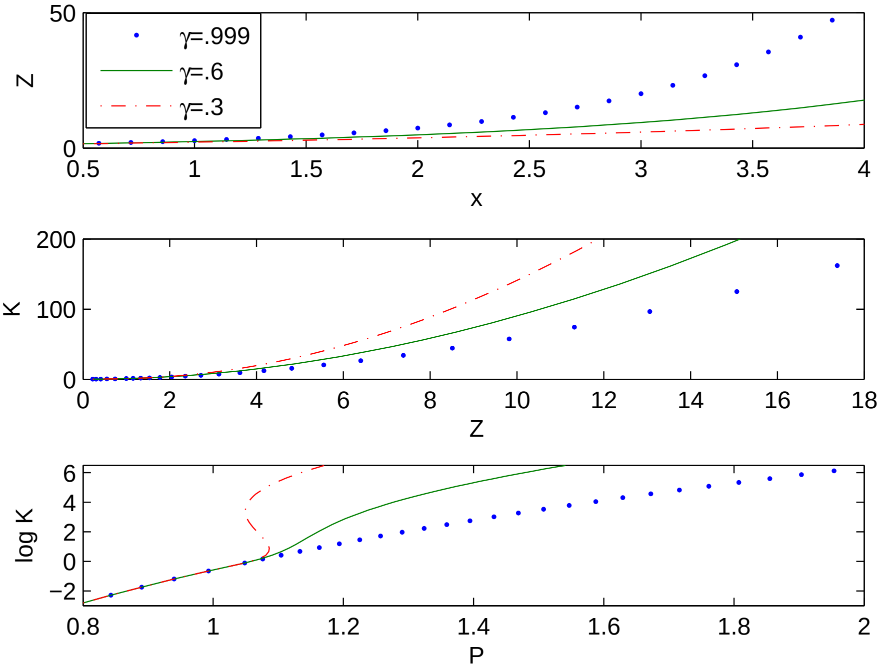

FIGURE 1. Upper panel: dependence of the demand shock $Z_t = Z(X_t)$ (Example 5.2) on the Lévy process. Middle panel: the boundary of the inaction region in (Z, K)-plane. Lower panel: the boundary of the inaction region in (P, K)-plane. Discount rate: $q = 0.08$. Marginal cost: $C = 2$. Elasticity of demand: $\epsilon = 2$. Parameters of the production function: $d = 1, \rho = 0.9$. Parameters of the Lévy process (diffusion with embedded downward jumps): $\sigma^2 = 0.2, b = -0.6, c^- = 0.10, \lambda^- = -2$.

in the region of large K. As $K \to \infty$, $\mathcal{E}^- Z(h^*) = qC/G'(K) \to \infty$, hence $e^{h^*(K)} \to \infty$, and $\mathcal{E}^- Z(h^*) \sim \bar{Z}\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*}$. Now we can write an approximate equation

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}q^{-1}\bar{Z}\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*} = C$$

instead of (5.6) and obtain

$$P^* = (dK^\rho)^{-1/\epsilon}Z(h^*) \sim (dK^\rho)^{-1/\epsilon}\frac{\bar{Z}}{\gamma}e^{\gamma h^*} \sim \frac{qC}{\kappa_q^-(\gamma)d\rho(1-1/\epsilon)}K^{1-\rho}.$$

The smaller the $\gamma > 0$, the larger is $\kappa_q^-(\gamma)$, and the lower is the output price at the moment of investment.

We see that the effect of the decrease of the monopoly price at the moment of investment is observed when the production function exhibits almost constant returns to scale ($\rho = 0.9$), and the demand shock grows slowly above a certain level ($\gamma = 0.3$). The same effect can be observed for smaller ρ but then the rate of growth of the demand shock must be smaller as well - see Fig.

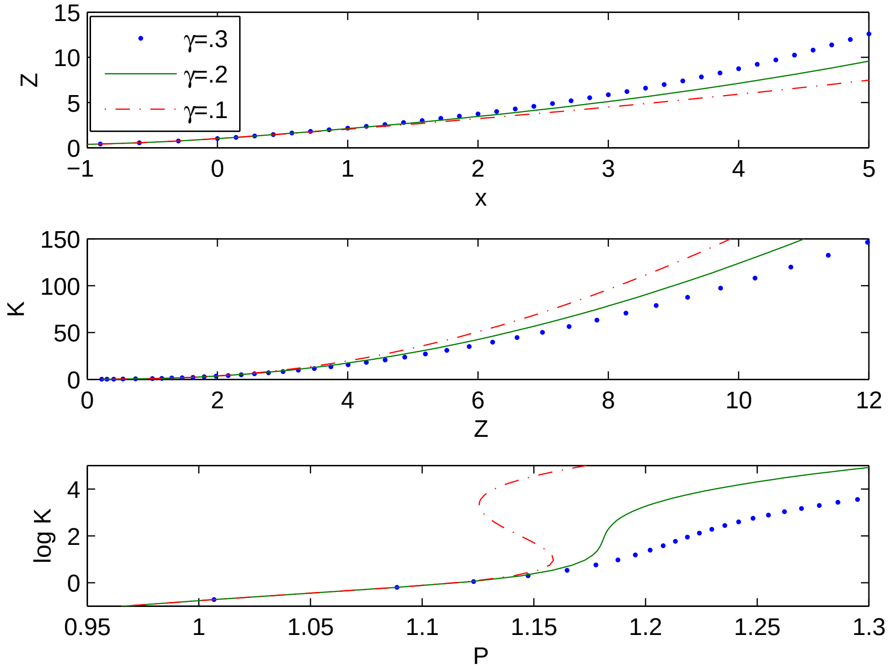

FIGURE 2. Upper panel: dependence of the demand shock $Z_t = Z(X_t)$ (Example 5.2) on the Lévy process. Middle panel: the boundary of the inaction region in (Z, K)-plane. Lower panel: the boundary of the inaction region in (P, K)-plane. Discount rate: $q = 0.08$. Marginal cost: $C = 2$. Elasticity of demand: $\epsilon = 2$. Parameters of the production function: $d = 1, \rho = 0.85$. Parameters of the Lévy process (diffusion with embedded downward jumps): $\sigma^2 = 0.2, b = -0.6, c^- = 0.10, \lambda^- = -2$.

2, where $\rho = 0.85$. The effect is not observed for $\gamma = 0.3$ anymore but it is observed for $\gamma = 0.1$.

6. New technology adoption

In this Section, we assume that the manager of a firm chooses not only the optimal capital stock, but also the optimal timing of an upgrade to the frontier technology. This model is more complicated than the ones of the previous Sections because it is driven by two factors: one characterizes the dynamics of the technology frontier, and the other incorporates all other shocks in the economy. Powerfully, the method of the paper preserves the tractability even in this two-factor model. Timing new technology adoption is one of the applications where it is essential to model a stochastic technology factor as a process with jumps, because the new technology is not introduced continuously. We believe that the most important component in the evolution of the technology frontier is a compound Poisson process with upward jumps, with possible inclusion of a small diffusion component. One may think about the diffusion component in the technological process as moderate innovations in technology, which may be caused by (or lead to) small fluctuations in non-technological uncertainty; in this case, the interaction between the technological factor and (small) innovations to non-technological factor is modelled as in the standard Gaussian model. However, major technological breakthroughs should be modelled as a jump process, and then it is natural to presume that if there is a correlation between technological and non-technological factors, it should be described by a bivariate jump process.

A natural assumption is that the capital adjustment when the same technology is in place is less costly than the adoption of the new technology; the extreme assumption is that the capital adjustment is costless.

6.1. **Model specification.** We follow fairly closely the setup of Abel and Eberly (2002). There are no costs of adjustment of the stock of capital, and the stock is chosen optimally, therefore we may concentrate solely on the timing of adoption of the frontier technology. Let A_t be the technology in place, and $\hat{A}_t$ be the frontier technology at date t. Suppose that the updating happens at stopping times $\tau_1 < \tau_2 < \cdots$, so that between the updates the level of technology remains constant: for $t \in [\tau_{i-1}, \tau)$, $A_t = A_{\tau_{i-1}}$. We take the inverse demand function $P_t = Z_t Q_t^{-1/\epsilon}$ as the primitive of the model, assume that the marginal cost of capital is constant (normalized to 1 for simplicity), and the production function is $Q_t = d_t K_t^\rho$, where $\rho > 0$, and d_t is the factor which is determined by the technology in place. Solving for the optimal level of capital between technology updates, we find $C_t = (\alpha\rho)^{-1} d_t^\alpha (\rho(\epsilon-1)/\epsilon Z_t)^\beta$, where $\alpha = (\epsilon-1)/(\epsilon-\rho(\epsilon-1))$ and $\beta = \epsilon/(\epsilon-\rho(1\epsilon-1))$ are positive constants. Hence, the firm's cash flow is $A_t S_t$, where $A_t = d_t^\alpha$ and $S_t = (\alpha\rho)^{-1}(\rho(\epsilon-1)/\epsilon Z_t)^\beta$.

Updating to the frontier technology is costly, and the cost of updating is proportional to the updated cash stream: $\theta A_{\tau_i} S_{\tau_i}$, $\theta \in (0,1)$. Let $V(A_{\tau_{i-1}}, \hat{A}_t, S_t)$ be the value of the firm net of the value of its capital stock for $t \in [\tau_{i-1}, \tau)$. Following Abel and Eberly (2002), we assume that the value admits a representation

$$V(A_{\tau_{i-1}}, \hat{A}_t, S_t) = A_{\tau_{i-1}} S_t V^1(\hat{A}_t / A_{\tau_{i-1}}), \tag{6.1}$$

and that updating occurs when the ratio $\hat{A}_t / A_{\tau_{i-1}}$ reaches a certain threshold, call it A^*.

In Abel and Eberly (2002), the technological factor $\hat{A}_t$ and non-technological factor S_t are modeled as geometric Brownian motions: $\hat{A}_t / A_{\tau_{i-1}} = e^{a_t}$, $S_t = e^{X_t}$, where (a_t, X_t) is a two-dimensional Gaussian process with the non-trivial correlation between components. We assume that $\hat{A}_t / A_{\tau_{i-1}} = e^{X_t^1}$, $S_t = e^{X_t^2}$, where $X_t = (X_t^1, X_t^2)$ is a two-dimensional Lévy process driven by compound Poisson processes and two independent standard Brownian motions W_t^1 and W_t^2. To be more specific, we model X as the solution to the stochastic differential equation

$$d\begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} + \sum_k \begin{bmatrix} 1 \\ \gamma_k \end{bmatrix} dJ_{c_k,\lambda_k;t}, \tag{6.2}$$

where $c_k > 0$, $\lambda_k > 0$, $\gamma_k \in \mathbb{R}$, and $J_{c,\lambda;t}$ denotes the compound Poisson process with the Lévy density $ce^{-\lambda x}\mathbf{1}_{(0,+\infty)}(x)$. We may identify $\sum_k J_{c_k,\lambda_k;t}$ as the jump component of the innovation process (creation of essentially new technologies), and then γ_k describe the impact of unexpected innovations on the dynamics of the non-technological factor. If $\gamma_k < 0$ (respectively, $\gamma_k > 0$), then a positive jump in the frontier technology is accompanied by a negative (respectively, positive) jump in the non-technological stochastic factor. The diffusion part of the process describes small fluctuations in the non-technological factor, and related fluctuations in minor technological improvements. If $\sigma_{12} = 0$, then σ_{21} describes the impact of the process of small technological innovations on small fluctuations in the non-technological uncertainty, and if $\sigma_{21} =$

0, then σ_{12} describes the impact of the latter on the former. The Lévy exponent of X_t, $\Psi(z) = \Psi(z_1, z_2)$, is defined by

$$E\left[e^{\langle z,\, X_t\rangle}\right] = E\left[e^{z_1 X_t^1 + z_2 X_t^2}\right] = e^{t\Psi(z)}.$$

For the process given by (6.2),

$$\Psi(z) = \frac{1}{2}||\Sigma' z||^2 + \langle b,\, z\rangle + \int_{\mathbb{R}^2\setminus\{0\}} \left(e^{\langle z,\, y\rangle} - 1\right) F(dy), \tag{6.3}$$

where $\Sigma = [\sigma_{j,k}]$; $b = (b_1, b_2)$ and $\Sigma\Sigma'$ are the drift and variance-covariance matrix of the Gaussian component of the process, and

$$F(dy) = \sum_k c_k \lambda_k e^{-\lambda_k y_1} \mathbf{1}_{[0,\,+\infty)}(y_1)\delta_0(y_2 - \gamma_k y_1) dy_1 \tag{6.4}$$

is the Lévy density. Here δ_0 is the one-dimensional Dirac delta-function.

W.l.o.g., set $\tau_{i-1} = 0$ and denote $\tau = \tau_i = \inf\{t > 0 |\, X_t^1 \geq h\}$, where $h = \log A^*$. Then the value of the firm satisfies

$$V(A_0, \hat{A}_t, S_t) = E_t\left[\int_t^\tau e^{-q(s-t)} A_0 S_s ds\right] + E_t\left[e^{-q(\tau-t)}\left(V(\hat{A}_\tau, \hat{A}_\tau, S_\tau) - \theta \hat{A}_\tau S_\tau\right)\right].$$

Substitute (6.1) into the last equation and divide it by $A_0 S_t$. Let

$$v(X_t) = V^1\left(e^{X_t^1}\right) = V^1(\hat{A}_t / A_0).$$

Notice that at the time of updating, $A_\tau = \hat{A}_\tau$, hence $V^1(\hat{A}_\tau / A_\tau) = V^1(1) = v(0)$. Now for $t \in [0, \tau)$, we have

$$v(X_t) = E_t\left[\int_t^\tau e^{-q(s-t) + X_s^2 - X_t^2} ds\right] + E_t\left[e^{-q(\tau-t) + X_\tau^2 - X_t^2} e^{X_\tau^1}(v(0) - \theta)\right]. \tag{6.5}$$

6.2. **One source of uncertainty.** First, we consider the case when only innovations to technology occur, i.e., the factor X_t^2 is constant. The underlying stochastic process is a one-dimensional Lévy process. Examining only technological innovations is not only instructive by itself, but as we will show it in the next Subsection, the general case reduces to this special case. Of course, the Lévy exponent of a one-dimensional process that appears after the reduction is made depends on the Lévy exponent of the initial two-dimensional process. In Subsection 6.3, we will discuss the impact of interaction between the two components of the process on the new technology adoption threshold.

Let h be the threshold for updating. The objective of the firm is to choose h so as to maximize the value

$$v(x; h) = E_t\left[\int_t^\tau e^{-q(s-t)} ds |\, X_t = x\right] + E_t\left[e^{-q(\tau-t)} e^{X_\tau}(v(0; h) - \theta) |\, X_t = x\right]. \tag{6.6}$$

To ensure that the value of the firm were finite, assume that X satisfies $r - \Psi(1) > 0$. In the Appendix, we show that it is possible to rewrite (6.6) in the form

$$v(x; h) = q^{-1} + \left(\mathcal{E}^+ \mathbf{1}_{[h,\,+\infty)}\left[\kappa_q^+(1)^{-1}(v(0; h) - \theta)e^{\cdot} - q^{-1}\right]\right)(x), \tag{6.7}$$

where $e^{\cdot}$ denotes the exponential function $x \mapsto e^x$. Introduce

$$v_{\text{opt}}(x; h) = v(x; h) - q^{-1}.$$

Recall that given the new technology is adopted at the threshold h, the value of the firm is

$$V(A_0, \hat{A}_t, S_t; h) = A_0 S_t v(X_t; h) = \frac{A_0 S_t}{q} + A_0 S_t v_{\rm opt}(X_t; h).$$

The first term, $A_0 S_t/q$, is the EPV of the stream of profits, which the firm will generate provided the current technology stays in place forever, and the second term is the option value of upgrading to the frontier technology. In order to find the option value, we rewrite (6.7) in terms of $v_{\rm opt}(x; h)$:

$$v_{\rm opt}(x; h) = \left(\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}\left[\kappa_q^+(1)^{-1}(v_{\rm opt}(0; h) + q^{-1} - \theta)e^{\cdot} - q^{-1}\right]\right)(x). \tag{6.8}$$

Suppose for a moment that we know the value $V_0 := v_{\rm opt}(0; h)$ at the moment of updating. Assuming that $V_0 + q^{-1} - \theta > 0$ (a sufficient condition is $q\theta < 1$, that is, the cost of updating is not too high), and arguing as in the proof of (3.19), we conclude that the optimal updating threshold h satisfies

$$\kappa_q^+(1)^{-1}(v_{\rm opt}(0; h) + q^{-1} - \theta)e^h - q^{-1} = 0. \tag{6.9}$$

Using (6.9), we can simplify (6.8) for $x < h$:

$$v_{\rm opt}(x; h) = e^{-h}\left(q^{-1}\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}\left(e^{\cdot} - e^h\right)\right)(x) = \left(q^{-1}\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}\left(e^{\cdot - h} - 1\right)\right)(x). \tag{6.10}$$

Equation (6.9) has two unknowns: h and $v_{\rm opt}(0; h)$, however we can add the second equation by letting $x = 0$ in (6.10):

$$v_{\rm opt}(0; h) = \left(q^{-1}\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}\left(e^{\cdot - h} - 1\right)\right)(0). \tag{6.11}$$

By substituting (6.11) into (6.9), and multiplying by $q\kappa_q^+(1)$, we obtain the equation for h:

$$e^h\left(\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}\left(e^{\cdot - h} - 1\right)\right)(0) + (1 - q\theta)e^h - \kappa_q^+(1) = 0. \tag{6.12}$$

We claim that if $q\theta < 1$, then this equation has a unique solution on $(0, +\infty)$. Indeed, as $h \to +\infty$, the LHS tends to $+\infty$, and at $h = 0$, the LHS is negative:

$$(\mathcal{E}^+(e^{\cdot} - 1))(0) + (1 - q\theta) - \kappa_q^+(1) = \kappa_q^+(1) - 1 + (1 - q\theta) - \kappa_q^+(1) = -q\theta < 0.$$

Hence, a solution exists, and to see that it is unique, it suffices to check that the LHS in (6.12) is convex. We will verify this, and obtain explicit formulas for h and $v_{\rm opt}(0; h)$ after we specify a process for the frontier technology.

Suppose that X is a diffusion process with exponentially distributed upward jumps. The Lévy density is

$$F(dy) = c\lambda e^{-\lambda y}\mathbf{1}_{(0,+\infty)}(y)dy, \tag{6.13}$$

where $c > 0$ and $\lambda > 1$ (the last inequality is necessary for the inequality $r - \Psi(1) > 0$ to hold, which ensures the finiteness of the value function). Then the Lévy exponent is $\Psi(z) = \sigma^2 z^2/2 + bz + cz/(\lambda - z)$, and the inequality $r - \Psi(z) > 0$ is satisfied provided $q > \sigma^2/2 + b + c/(\lambda - 1)$. The characteristic equation has three roots: $\beta^- < 0 < 1 < \beta_1^+ < \lambda < \beta_2^+$. The factor $\kappa_q^-(z)$ is defined by $\kappa_q^-(z) = \beta^-/(\beta^- - 1)$, and $\kappa_q^+(z)$ is given by (2.13) or (2.14). The value $v_{\rm opt}(x; h)$ satisfying (6.10) can be computed in exactly the same manner as the value $V^+(x)$ in (3.21):

$$v_{\rm opt}(x; h) = q^{-1}\sum_{j=1,2}\frac{a_j^+ e^{\beta_j^+(x-h)}}{\beta_j^+ - 1}, \text{ for } x < h,$$

and (6.12) assumes the form

$$\sum_{j=1,2} \frac{a_j^+ e^{(1-\beta_j^+)h}}{\beta_j^+ - 1} + (1-q\theta)e^h - \kappa_q^+(1) = 0. \tag{6.14}$$

Denote by $f(h)$ the LHS in (6.14). We have shown for the general case above that $f(h)$ changes sign on $(0, +\infty)$, and the root of (6.14) exists. To show the uniqueness of the root, we prove that f is convex:

$$f''(h) = \sum_{j=1,2} a_j^+(\beta_j^+ - 1)e^{(1-\beta_j^+)h} + (1-q\theta)e^h > 0.$$

6.3. **Two sources of uncertainty.** For simplicity, assume that there is only one term in the jump component. Set $c = c_k, \lambda = \lambda_k, \gamma = \gamma_k$, assume that $\gamma < \lambda - 1$, and denote by a_{jk} the entries of the variance-covariance matrix $\Sigma\Sigma'$. In the Appendix, we show that the new technology adoption threshold in the two-factor model (6.2) is the same as in the one-factor model with the characteristic exponent

$$\Psi^1(z_1) = \frac{a_{11}}{2} z_1^2 + b^1 z_1 + \frac{c^1 z_1}{\lambda^1 - z_1}, \tag{6.15}$$

where $b^1 = a_{12} + b_1$, $c^1 = c\lambda/(\lambda - \gamma)$, and $\lambda^1 = \lambda - \gamma$. To ensure that the value of the firm were finite, we need to impose two conditions ((A.9) and (A.10)), which in the case of one jump component assume the form

$$q^1 := q - \frac{a_{22}}{2} - b_2 - \frac{c\gamma}{\lambda - \gamma} > 0, \tag{6.16}$$

and

$$q - \frac{a_{11}}{2} - a_{12} - \frac{a_{22}}{2} - b_1 - b_2 - \frac{c(1+\gamma)}{\lambda - \gamma - 1} > 0. \tag{6.17}$$

Notice that both (6.16) and (6.17) imply that γ cannot be too close to λ, equivalently, if positive technological jumps are accompanied by vigorous positive jumps in the non-technological factor, then the value of the firm becomes infinite: the prospects are too good to be true. Probably, the advocates of the New Economy had in mind similar models for shocks in technology and non-technological uncertainty. We also need to require $1 - q^1\theta > 0$; if this condition is violated, then new technology adoption is never optimal.

If the Gaussian component in the dynamics of the technology frontier is non-trivial, then the characteristic equation has three roots $\beta^- < 0 < 1 < \beta_1^+ < \lambda < \beta_2^+$, and the equation for the technology adoption frontier is (cf. (6.14))

$$\sum_{j=1,2} \frac{a_j^+ e^{(1-\beta_j^+)h}}{\beta_j^+ - 1} + (1-q^1\theta)e^h - \kappa_{q^1}^+(1) = 0, \tag{6.18}$$

where a_j^+ and $\kappa_{q^1}^+(1)$ are defined by the same formulas as in Section 2 with q^1 in place of q. The existence and uniqueness of the solution h of (6.18) is proved in Subsection 6.2.

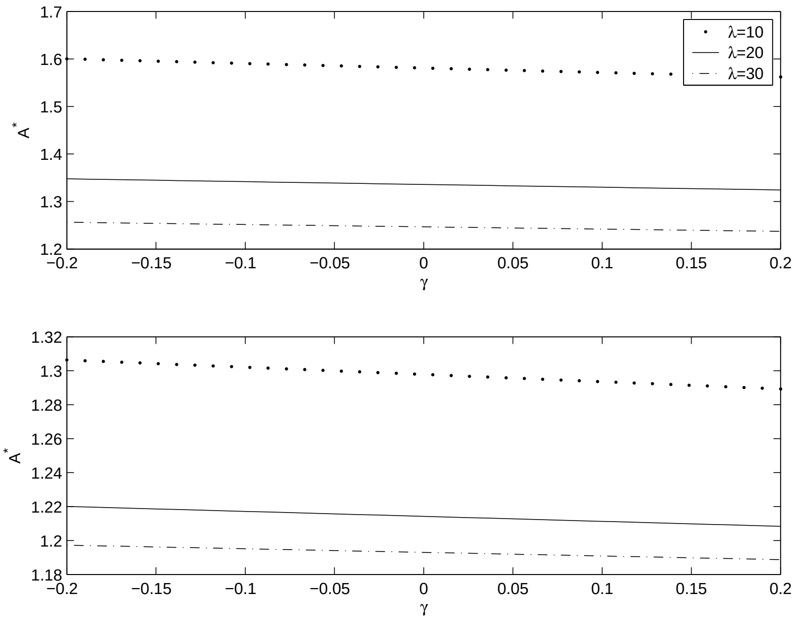

FIGURE 3. Adoption of new technology threshold A^* as a function of the correlation parameter γ, intensity of jumps c^+, and steepness parameter λ. The technology factor is compound Poisson. Parameters: $\theta = 5, q = 0.08, \sigma_{22}^2 = 0.10, b_1 = -0.01, b_2 = 0.00$. Upper panel: $c^+ = 0.25$; lower panel: $c^+ = 0.10$.

6.4. **Dependence of the new technology adoption threshold, A^*, on diffusion and jump uncertainty.** We start with the study of the dependence of A^* on the jump component when the technological process has no Gaussian component: $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. For the calculation of A^* in this case, see the Appendix. First, we fix the Gaussian component of the non-technological factor, σ_{22}, and change c, λ and γ (Fig. 3). Then we fix λ, and change c, σ_{22}, and γ (Fig. 4). The increase in c means that the total uncertainty of the technological factor increases, the increase in λ^{-1} means that the average jump size becomes larger, and the increase in σ_{22} means the increase in non-technological uncertainty. Finally, the increase in γ means that the correlation between the two factors goes up. In these figures, it is clearly seen (and the same effect is observed for other parameters' values) that the new technology adoption threshold is

(a) an increasing function of (c, λ^{-1}), that is, of the uncertainty in the technological factor, and average jump size;
(b) a decreasing function of σ_{22}, that is, of the uncertainty in the non-technological factor;
(c) a decreasing function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors.

Thus, the uncertainty in the technological factor and uncertainty in the non-technological one affect the threshold in opposite directions. The dependence on the technological uncertainty can be naturally explained in the framework of the record-setting news principles in Boyarchenko (2004) as follows. In a situation similar to the call option with an instantaneous (random) payoff, the record-setting good news principle applies, and the higher the uncertainty of good news, the higher is the threshold. Clearly, this is the situation with new technology adoption: once the new technology is in place, it remains fixed for a sizable time period. The feature (b)

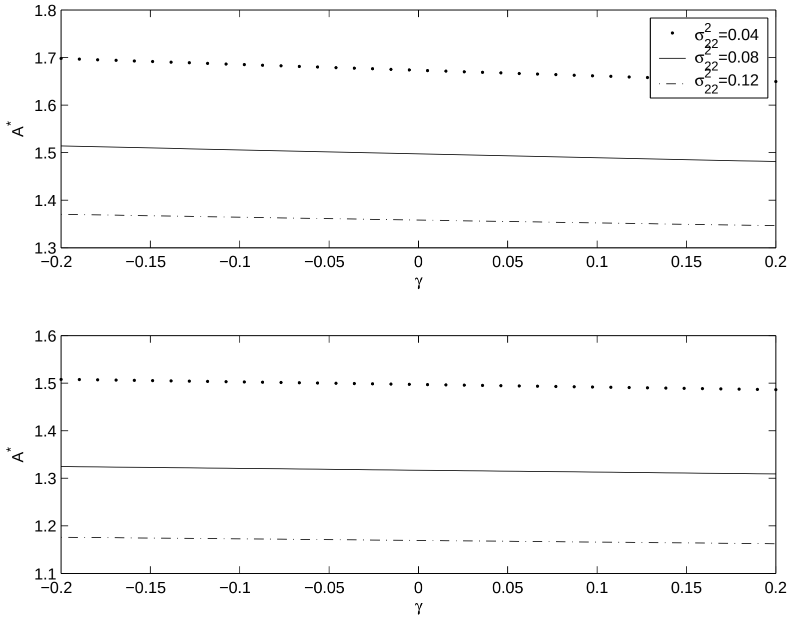

FIGURE 4. Adoption of new technology threshold A^* as a function of the correlation parameter γ, intensity of jumps c^+, and Gaussian uncertainty, σ_{22}. The technology factor is compound Poisson. Parameters: $\theta = 5, q = 0.08, \lambda = 15, b_1 = -0.01, b_2 = 0.00$. Upper panel: $c^+ = 0.25$; lower panel: $c^+ = 0.10$.

is not as transparent as (a). According to the record-setting news principles in Boyarchenko (2004), if the option gives the right to a *stream* of payoffs (a cash flow here), then the record-setting bad news principle applies, and the higher the uncertainty of bad news (the lower the trajectories of the infimum process), the higher is the threshold. It may seem that the increase in σ_{22} means the increase in the overall uncertainty in S_t, the non-technological factor, hence in the uncertainty of bad news, and so the threshold should increase. Notice, however, that the threshold is derived for the technological factor, but not for S_t, and the standard intuition may be non-applicable. If σ_{22} increases, then $b_2 + \sigma_{22}^2/2$, the rate of growth of S_t increases; therefore, the higher the expected rate of growth of S_t (hence, of the revenue), the sooner should the firm take the advantage of adoption of the frontier technology.

The reader may wonder if the difference between the ways the new technology factor and non-technological one influence the threshold is an artifact of the different ways these factors are modelled: pure jump process and diffusion process with embedded jumps, respectively. In Fig. 5, we demonstrate how the adoption threshold changes if we add the diffusion component to technological process so that the Gaussian uncertainty in the non-technological factor drives the Gaussian uncertainty in the technological factor (similar effects are observed when the latter driver the former). We also show the threshold when there is no Gaussian uncertainty in the technological factor. The conclusions (a)–(c) made above remain valid. The new technology adoption threshold is

(a) an increasing function of the uncertainty in the technological factor;
(b) a decreasing function of the uncertainty in the non-technological factor;

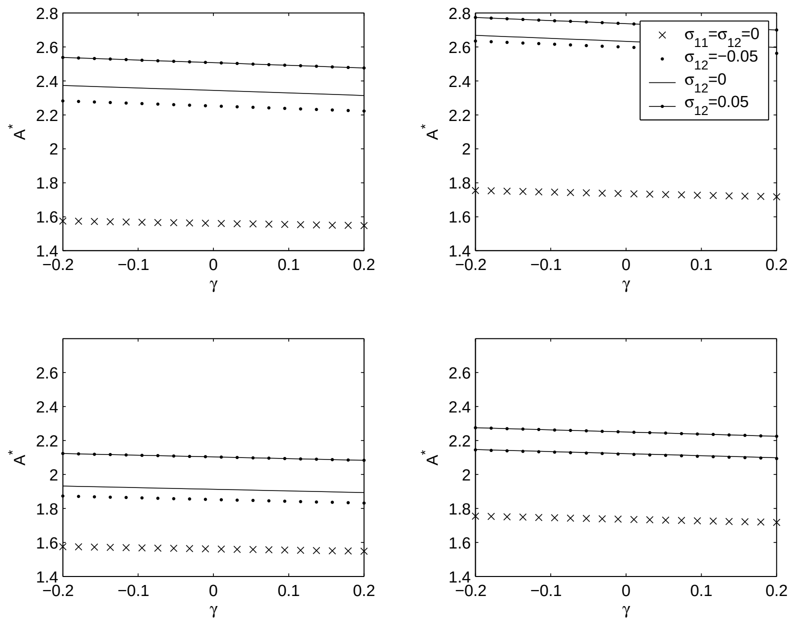

FIGURE 5. Adoption of new technology threshold A^* as a function of the correlation parameter γ, and the correlation coefficient σ_{12}; $\sigma_{21} = 0.0$ (demand/prices influence small improvements in the technology but not vice versa). Crosses: no Gaussian uncertainty in the technological factor: $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. Other parameters: $\theta = 5, q = 0.08, \lambda = 15, b_1 = -0.01, b_2 = 0.00$. Panels: (a) $c^+ = c^- = 0.2$; (b) $c^+ = 0.2, c^- = 0.1$; (c) $c^+ = 0.1, c^- = 0.2$; (d) $c^+ = c^- = 0.1$.

(c) a decreasing function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors;

(d) an increasing function of the covariance coefficients, σ_{12} and σ_{21}, between the Gaussian components in the technological and non-technological factors.

Notice the important difference between the impact of the "correlation" between the Gaussian and non-Gaussian sources of uncertainty on the threshold: A^* is a **decreasing** function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors, and an **increasing** function of the correlation coefficients σ_{12} and σ_{21} between the Gaussian components of technological and non-technological innovations. Hence, the interaction between Gaussian sources of uncertainty, and the one between non-Gaussian sources of uncertainty are not just qualitatively different: they are of **opposite signs**.

7. Conclusion

In the paper, we presented a general method for solving optimal stopping problems assuming that the underlying source of uncertainty can be represented as a function of the Brownian motion with embedded jumps. Our method uses the definition of the value of an option as the EPV of an instantaneous payoff or a stream of payoffs. If the payoff is instantaneous, we view it as the EPV of a stream of payoffs. Such a representation can be obtained in many situations. Of course, everyone knows how to calculate the EPV of a perpetual stream of payoffs that starts to accrue at a deterministic point in time. We show that the rational price of a payoff stream that starts to accrue at a random time (i.e., after the underlying stochastic variable X_t crosses a certain barrier) can also be obtained in terms of the EPV's of some payoff streams. In some cases, the EPV has to be calculated under the assumption that the underlying stochastic process is replaced by the infimum process $\underline{X}_t = \inf_{0\le s\le t} X_s$. In other instances, it becomes necessary to substitute the supremum process $\bar{X}_t = \sup_{0\le s\le t} X_s$ for the underlying process. Similar results hold for the value of a payoff stream that is lost at a random time. Clearly, one can price (real) perpetual American options using the EPV's of the payoff streams mentioned above or their combinations.

The form of the solution for the option value that we obtain suggests the following description of the optimal exercise strategy. If the payoff stream is a decreasing function of the underlying stochastic factor, then it is optimal to exercise the option the first time the EPV of the stream of payoffs calculated for the supremum process instead of the original stochastic process becomes non-negative. Similarly, if the payoff stream is an increasing function of the underlying stochastic factor, then it is optimal to exercise the option the first time the EPV of the stream of payoffs calculated for the infimum process instead of the original stochastic process becomes non-negative. This allows us to formulate a general optimal exercise rule: it is optimal to exercise the right for the stream of stochastic payoffs, g_t, the first time the EPV of the stream $\underline{g}_t = \inf_{0\le s\le t} g_s$ becomes non-negative. We call the last statement a *universal record-setting bad news principle*. This principle naturally generalizes and extends Bernanke's (1983) bad news principle and record-setting news principles spelled out in Boyarchenko (2004). The representation in the form of the EPV operators under supremum and infimum processes allows us to (relatively easily) prove not only theorems for basic types of investment/disinvestment problems, when the decision must be taken only once, but for certain sequences of embedded options (Russian dolls) of arbitrary length as well.

As additional applications of our methodology, we considered two models of monopolistic expansion. First, we calculated a capital expansion program for a monopoly which faces the demand uncertainty, and showed that an appropriate choice of the dependence of the inverse demand curve on the stochastic factor made it unnecessary to impose additional fairly stringent conditions on the production function, as in Dixit and Pindyck (1996). It is worth mentioning that the same choice leads to a lower investment threshold than in the geometric Brownian (more generally, Lévy) model.

We also solved a problem of new technology adoption, where the manager of a firm chooses not only the optimal capital stock, but also the optimal timing of an upgrade to the frontier technology. The model is driven by two factors: one characterizes the dynamics of the technology frontier, and the other incorporates all other shocks in the economy. Powerfully, the method of the paper preserves the tractability even in this two-factor model. We believe that it is natural to model the dynamics of the frontier technology as a process with upward jumps and not as a pure diffusion process. We analyze how the interaction between the two stochastic factors affects the process of new technology adoption, and show that the differences between the impact of the diffusion component and the impact of the jump component on the adoption threshold are not only quantitative but qualitative as well. This corroborates the conclusion made in Boyarchenko and Levendorskiĭ (1998) about a model with policy uncertainty and non-Gaussian uncertainty in prices: an interaction of two stochastic factors enhances the impact of jumps on the investment threshold.

Appendix A

Proof of (2.3) Computing the action of the infinitesimal generator (2.1) on e^{zx}, we obtain the exponent $\Psi(z)$ corresponding to the Lévy density (2.2):

$$\begin{aligned} Le^{zx} &= \left[\frac{\sigma^2}{2}z^2 + bz + c^+\lambda^+\int_0^{+\infty}\left(e^{(z-\lambda^+)y} - e^{-\lambda^+ y}\right)dy \right. \\ &\left. -c^-\lambda^-\int_{-\infty}^0\left(e^{(z-\lambda^-)y} - e^{-\lambda^- y}\right)dy\right]e^{zx} \\ &= \left(\frac{\sigma^2}{2}z^2 + bz + \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z}\right)e^{zx} = \Psi(z)e^{zx}. \end{aligned}$$

□

Formulas for $a_j^{\pm}$

$$a_1^+ = \frac{\beta_2^+}{\beta_2^+ - \beta_1^+}\cdot\frac{\lambda^+ - \beta_1^+}{\lambda^+}, \qquad a_2^+ = \frac{\beta_1^+}{\beta_1^+ - \beta_2^+}\cdot\frac{\lambda^+ - \beta_2^+}{\lambda^+}, \tag{A.1}$$

$$a_1^- = \frac{\beta_2^-}{\beta_2^- - \beta_1^-}\cdot\frac{\lambda^- - \beta_1^-}{\lambda^-}, \qquad a_2^- = \frac{\beta_1^-}{\beta_1^- - \beta_2^-}\cdot\frac{\lambda^- - \beta_2^-}{\lambda^-}. \tag{A.2}$$

If $g(x) = e^{zx}$, then

$$\int_0^{+\infty} e^{-\beta_j^+} e^{z(x+y)}dy = e^{zx}\int_0^{+\infty} e^{(z-\beta_j^+)y}dy = \frac{e^{zx}}{\beta_j^+ - z},$$

therefore

$$\mathcal{E}^+ e^{zx} = \kappa_q^+(z)e^{zx} = \sum_{j=1,2} a_j^+\frac{\beta_j^+}{\beta_j^+ - z}e^{zx}.$$

Thus, (2.15) is proved for exponential functions. By expanding an arbitrary (sufficiently regular) function g as a Fourier integral, we obtain (2.15). (2.16) is proved similarly.

Proof of (2.21)-(2.22) Set $c_\pm^1 = \frac{\lambda^\pm}{\beta_1^\pm\beta_2^\pm}$, $\quad c_\pm^0 = (\beta_1^\pm + \beta_2^\pm - \lambda^\pm)c_\pm^1$, $\quad b_\pm = 1 + c_\pm^0$. If $g(x) = e^{zx}$, then

$$\begin{aligned} c_+^1 g'(x) + c_+^0 g(x) + b_+\int_0^{+\infty}\lambda^+ e^{-\lambda^+ y}g(x+y)dy &= zc_+^1 e^{zx} + c_+^0 e^{zx} + b_+\frac{\lambda^+}{\lambda^+ - z}e^{zx} \\ &= \frac{\lambda^+}{\lambda^+ - z}\frac{\beta_1^+ - z}{\beta_1^+}\frac{\beta_2^+ - z}{\beta_2^+}e^{zx} = (\kappa_q^+(z))^{-1}e^{zx}. \end{aligned}$$

Thus, (2.21) is proved for exponential functions. By expanding an arbitrary (sufficiently regular) function g as a Fourier integral, we obtain (2.21). (2.22) is proved similarly.

Proof of (5.14). Let $Z(x)$ satisfy (5.1), then

$$w(x) \le c_1 E\left[\int_0^{+\infty} e^{-qt+\gamma(x+\underline{X}_t)}dt\right] \le c_1 q^{-1}\kappa_q^-(\gamma)e^{\gamma x} \le c_1 q^{-1}e^{\gamma x}.$$

Therefore

$$\int_0^{+\infty} e^{-\beta_j^+ y}w(h^* + y)dy \le c_1 q^{-1}e^{\gamma h^*}\int_0^{+\infty} e^{-\beta_j^+ y + \gamma y}dy = \frac{c_1 e^{\gamma h^*}}{q(\beta_j^+ - \gamma)},$$

and

$$V_K^{\mathrm{opt}}(K,x) \le \frac{c_1 G'(K) e^{\gamma h^*}}{q} \sum_{j=1,2} \frac{a_j^+ \beta_j^+}{\beta_j^+ - \gamma} e^{\beta_j^+(x-h^*)}.$$

Since $\gamma \in (0,1]$ and $1 < \beta_1^+ < \beta_2^+$, we obtain

$$V_K^{\mathrm{opt}}(K,x) \le D(x) G'(K) e^{(\gamma - \beta_1^+)h^*(K)}, \tag{A.3}$$

where $D(x)$ depends on $x \le h^*(K)$ but not on K. Next, we notice that if $W(X_t)$ is another demand shock such that $Z(x) \le W(x)$ for any x, then the corresponding thresholds are related as $h^*(K;Z) \ge h^*(K;W)$. This result follows immediately if one compares (5.11) for W

$$G'(K) \sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-) e^{-\beta_j^- y} W(h^* + y) dy = qC$$

with the one for Z:

$$G'(K) \sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-) e^{-\beta_j^- y} Z(h^* + y) dy \le qC.$$

For $W(x) = c_1 e^{\gamma x}$, we have from (5.11) $G'(K)\kappa_q^-(\gamma) e^{\gamma h^*(K,W)} = qC$, therefore the RHS in (A.3) admits a bound via $D_1(x) G'(K)^{\beta_1^+/\gamma}$, and we conclude that (5.14) is a sufficient condition for the convergence of the integral (5.13) with $\bar{K} = +\infty$. In the geometric Lévy case, we obtain that $V_K^{\mathrm{opt}}(K,x) = D_1(x) G'(K)^{\beta_1^+/\gamma}$, where $\gamma = 1$, therefore (5.14) is necessary as well. □

Proof of (5.20). W.l.o.g., $Z_c = 1$. We have

$$\begin{aligned}
\mathcal{E}^- Z(x) &= \sum_{j=1,2} a_j \int_{-\infty}^0 (-\beta_j^-) e^{-\beta_j^- y} Z(x+y) dy \\
&= \sum_{j=1,2} a_j e^{\beta_j^- x} \int_{-\infty}^x (-\beta_j^-) e^{-\beta_j^- y} Z(y) dy = \sum_{j=1,2} a_j e^{\beta_j^- x} f_j(x),
\end{aligned}$$

where

$$\begin{aligned}
f_j(x) &= \int_{-\infty}^0 (-\beta_j^-) e^{-\beta_j^- y} e^y dy + \int_0^x (-\beta_j^-) e^{-\beta_j^- y} (\gamma^{-1} e^{\gamma y} + (1-\gamma^{-1})) dy \\
&= \frac{-\beta_j^-}{1-\beta_j^-} + \frac{-\beta_j^-}{\gamma(\gamma - \beta_j^-)} (e^{(\gamma - \beta_j^-)x} - 1) - \frac{1-\gamma}{\gamma} (e^{-\beta_j^- x} - 1) \\
&= \frac{(-\beta_j^-)}{1-\beta_j^-} - \frac{(-\beta_j^-)}{\gamma(\gamma - \beta_j^-)} + \frac{1-\gamma}{\gamma} + \frac{(-\beta_j^-) e^{(\gamma - \beta_j^-)x}}{\gamma(\gamma - \beta_j^-)} - \frac{(1-\gamma) e^{(-\beta_j^-)x}}{\gamma(-\beta_j^-)}.
\end{aligned}$$

Using

$$\sum_{j=1,2} \frac{(-\beta_j^-) a_j}{\gamma(\gamma - \beta_j^-)} = \gamma^{-1} \kappa_q^-(\gamma), \quad \sum_{j=1,2} a_j = \kappa_q^-(0) = 1,$$

we obtain (5.20) with

$$d_{\gamma,j} = \frac{a_j(1-\gamma)}{(1-\beta_j^-)(\gamma - \beta_j^-)}.$$

Proof of (6.7) If $X_t = x$, then using (3.17),

$$\begin{aligned} E_t\left[\int_t^\tau e^{-q(s-t)}ds\right] &= E_t\left[\int_t^{+\infty} e^{-q(s-t)}ds\right] - E_t\left[\int_\tau^{+\infty} e^{-q(s-t)}ds\right] \\ &= q^{-1} - q^{-1}(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-1)(x) = q^{-1} - q^{-1}(\mathcal{E}^+\mathbf{1}_{[h,+\infty)})(x). \end{aligned}$$

Use the fundamental relationship between the infinitesimal generator and the EPV-operator (2.5) to write the payoff $e^{X_\tau}(v(0;h)-\theta)$ as the normalized EPV of a stream $g(x) = q^{-1}(q-L)e^x(v(0;h)-\theta)$, substitute $\mathcal{E}g(X_\tau)$ into (6.6), and apply (3.17) in order to write the second term in (6.6) as

$$\begin{aligned} \left(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-\right)g(x) &= \left(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-q^{-1}(q-L)(v(0;h)-\theta)e^{\cdot}\right)(x) \\ &= \left(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}(v(0;h)-\theta)e^{\cdot}\right)(x) \\ &= \left(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}(v(0;h)-\theta)\kappa_q^+(1)^{-1}e^{\cdot}\right)(x). \end{aligned}$$

(Here we used the Wiener-Hopf factorization formula (2.11) and (2.8).) Now it becomes possible to rewrite (6.6) in the form (6.7). □

Proof of (6.15). For the sake of brevity, assume that $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$; the proof in the general case is similar. For any $s \ge t$ and $z = (z_1, z_2)$,

$$\begin{aligned} E_t\left[e^{z_1X_s^1+z_2X_s^2}\right] &= e^{z_1X_t^1+z_2X_t^2+(s-t)\Psi(z_1,z_2)} \\ &= e^{z_2X_t^2+(s-t)\Psi(0,z_2)}e^{z_1X_t^1+(s-t)[\Psi(z_1,z_2)-\Psi(0,z_2)]}. \end{aligned} \tag{A.4}$$

It is easy to see that for a fixed z_2, $\Psi^1_{z_2}(z_1) = \Psi(z_1,z_2) - \Psi(0,z_2)$ is the Lévy exponent of a one-dimensional Lévy process. In particular, if $||\Sigma' z||^2 = \sigma_{22}^2 z_2^2$, then from (6.3), we obtain

$$\begin{aligned} \Psi(z) &= \frac{\sigma_{22}^2}{2}z_2^2 + b_1z_1 + b_2z_2 + \sum_k c_k\lambda_k\int_{y_1>0}\left(e^{z_1y_1+z_2y_2}-1\right)e^{-\lambda_ky_1}\delta_0(y_2-\gamma_ky_1)dy_1 \\ &= \frac{\sigma_{22}^2}{2}z_2^2 + b_1z_1 + b_2z_2 + \sum_k c_k\lambda_k\int_0^{+\infty}\left(e^{(z_1+z_2\gamma_k)y_1}-1\right)e^{-\lambda_ky_1}dy_1 \\ &= \frac{\sigma_{22}^2}{2}z_2^2 + b_2z_2 + \sum_k c_k\lambda_k\left[\frac{1}{\lambda_k-z_2\gamma_k-z_1} - \frac{1}{\lambda_k}\right]. \end{aligned}$$

Hence,

$$\Psi(z) = \frac{\sigma_{22}^2}{2}z_2^2 + b_1z_1 + b_2z_2 + \sum_k c_k\lambda_k\left[\frac{1}{\lambda_k-z_1-z_2\gamma_k} - \frac{1}{\lambda_k}\right], \tag{A.5}$$

and

$$\Psi^1_{z_2}(z_1) = b_1z_1 + \sum_k \frac{c_k\lambda_kz_1}{(\lambda_k-\gamma_kz_2)(\lambda_k-\gamma_kz_2-z_1)}, \tag{A.6}$$

which is the characteristic exponent of a pure jump process with exponentially distributed upward jumps, and the Lévy density which depends on z_2. Let $Q^1_{z_2}$ be the probability measure which corresponds to $\Psi^1_{z_2}$, that is,

$$E^{Q^1_{z_2}}\left[e^{z_1X_t^1}\right] = e^{t\Psi^1_{z_2}(z_1)}.$$

Then we can write (A.4) as

$$E_t\left[e^{z_1X^1_s+z_2X^2_s}\right]=e^{z_2X^2_t+(s-t)\Psi(0,z_2)}E_t^{Q^1_{z_2}}\left[e^{z_1X^1_s}\right].$$

Decomposing a sufficiently regular function $f(X^1_s)$ as a Fourier integral, we have

$$E_t\left[e^{z_2X^2_s}f(X^1_s)\right]=e^{z_2X^2_t+(s-t)\Psi(0,z_2)}E_t^{Q^1_{z_2}}\left[f(X^1_s)\right]. \tag{A.7}$$

Set $\Psi^1=\Psi^1_1, Q^1=Q^1_1$, and $q^1=q-\Psi(0,1)$, and apply (A.7) with $z_2=1$ to (6.5):

$$v(X^1_t)=E_t^{Q^1}\left[\int_t^\tau e^{-q^1(s-t)}ds\right]+E_t^{Q^1}\left[e^{-q^1(\tau-t)}e^{X^1_\tau}(v(0)-\theta)\right]. \tag{A.8}$$

If

$$q-\Psi(0,1)>0, \tag{A.9}$$

then (A.8) is of the same form as (6.6), which we have studied already. The condition for the value of the firm to be finite is

$$q^1-\Psi^1(1)=q^1-(\Psi(1,1)-\Psi(0,1))>0,$$

which is equivalent to

$$q-\Psi(1,1)>0. \tag{A.10}$$

Thus, we require both (A.9) and (A.10). If there is only one term in (6.4), then from (A.6), we derive (6.15). □

The case of a pure jump technological process

Assume first that there is no Gaussian component in the technological factor ($\sigma_{11}=\sigma_{12}=\sigma_{21}=0$), and $b^1=b_1<0$, that is, upward jumps in the frontier technology are followed by periods of decline in the effectiveness of innovations. Then the characteristic equation

$$q^1-\Psi^1(z_1)=0$$

has two roots $\beta^-<0<1<\beta^+$:

$$\beta^\pm=\frac{q^1+c^1+\lambda^1b_1\mp\sqrt{(q^1+c^1+\lambda^1b_1)^2-4q^1b_1\lambda^1}}{2b_1}, \tag{A.11}$$

(recall that we assume $b_1<0$), and $\kappa^+_{q^1}(z_1)=\beta^+(\lambda^1-z_1)/(\lambda^1(\beta^+-z_1))=\beta^+/\lambda^1+a^+\beta^+/(\beta^+-z_1)$, where $a^+=1-\beta^+/\lambda^1$. The equation for the technology adoption frontier is obtained in the same manner as (6.14), but there is no summation because there is a unique positive root:

$$\frac{a^+e^{(1-\beta^+)h}}{\beta^+-1}+(1-q^1\theta)e^h-\kappa^+_{q^1}(1)=0. \tag{A.12}$$

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