Practical guide to real options in discrete time II

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Abstract. This paper is an extended version of the paper "Practical Guide to Real Options in Discrete Time" (http://econwpa.wustl.edu:80/eps/fin/papers/0405/0405016.pdf), where a general, computationally simple approach to real options in discrete time was suggested. We explicitly formulate conditions of the general theorems for basic types of real options, and explain our method in detail for the case of transition density given by exponential functions on each half-axis. To demonstrate that the discrete time approach can be more analytically tractable than the continuous time one, we consider timing of investment with lags, and a model of gradual capital expansion. We obtain simple formulas for the expected values of capital stock in every time period; in continuous time models, a much more sophisticated technique is needed.

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1 Introduction

The goal of the paper is to provide a general framework for pricing of real options in discrete time; since option pricing cannot be separated from the problem of the optimal exercise time of an option, we solve for the optimal timing as well. Valuation of real options and optimal exercise strategies explicitly derived in the paper are relevant to many practical situations where individuals make at least partially irreversible decisions. Simple real options cover such situations as timing an investment of a fixed size, scrapping of a production unit, capital expansion program, etc. Embedded real options are relevant in such cases as new technology adoption, timing an investment (partially) financed by debt with an embedded option to default in the future, human capital acquisition, and many others (see, for example, [17], papers in the volume [11], and the bibliography therein). Usually, continuous time models are used. We believe that the discrete time approach has certain advantages over the continuous time approach to modeling of real options.

First of all, discrete time is more natural for economics, and it allows one to obtain analytical results in some situations, where continuous time models are either not applicable, or do not lead to simple analytical results. In Section 3, the discrete time approach allows us to incorporate in a more tractable way such important aspects of economic reality as "time-to-build", and demonstrate analytically that some general claims made in [4], in the framework of the Gaussian model, are not quite correct. In Section 4, in a model of gradual capital expansion, we obtain simple formulas for the expected values of capital stock in every time period; in continuous time models, a much more sophisticated technique is needed (for comparison, see a Gaussian model in [3] and a non-Gaussian model in [7]). There are also natural questions, such as the rates of job creation and job destruction (see, e.g., [15] and [13]), which cannot be addressed in the Brownian motion setting under the standard assumption that labor is instantly adjustable, because the rates will be infinite. In the discrete time model, one can obtain finite closed form solutions. Second, for many important variables, the data are available only quarterly or even annually. If new pieces of information arrive quarterly, it does not make much sense to assume that investment decisions can be taken any moment; discrete time models with time periods of one day, week or month seem to be more reasonable. Some options – for example, renewing a labor contract or awarding a grant to a research project – can be exercised only at certain dates (typically, once a year), and therefore, a discrete time model is appropriate as well. From the point of view of econometrics, when only a list of observations at equally spaced moments in time is available, it is easier to fit a transition density to the data in a discrete time model than in a continuous time one. Finally, the most challenging task in the case of real options (and perpetual American options) is to determine the optimal exercise strategy. The proof of optimality of the exercise price is simpler in discrete time than in continuous time, and it does not require sophisticated results from the theory of stochastic processes in continuous time (for comparison, see Boyarchenko and Levendorskiĭ (2004)).

In fact, the appearance of continuous time models in economics was mainly due to the success of these models in finance, and their tractability. Notice, however, that even in finance, the success of simple models is gone. Relatively simple models use the (geometric) Brownian motion as the underlying stochastic process (see, for example, [19]). However, these models proved to be rather inaccurate, and have been amended in many ways, none of which being as tractable as the Brownian motion model. In particular, nowadays processes with jumps are widely used (see, for instance, [12] and [18]). The normality of commodity price processes is rejected by the data as well (see, for example, [16] or [23]). Assume, for the moment, that we are willing to sacrifice the accuracy of a model for its simplicity, and decide to use the Brownian motion model. The apparent advantage is a well-known manageable scheme: with the help of Ito's lemma write down a second order differential equation for the value of an option, employ economic arguments to add appropriate boundary conditions, such as value matching and smooth pasting, and, using the general solution to the differential equation, reformulate the problem as a system of algebraic equations. In elementary situations, a closed form solution can be derived, and a simple exercise strategy results; in other situations, numerical procedures are available.

Unfortunately, this scheme is uncomplicated only when one considers really simple options. If a part of the option value comes from instantaneous payoffs due at certain future dates, a closed form solution to the optimal exercise problem is no longer available. If we consider an option whose value comes from different streams in different regions of the state space, the resulting system of algebraic equations may involve too many unknowns, and it becomes messy indeed. If we consider embedded options, then there is no general result about the optimal exercise rule, apart from the heuristic smooth pasting condition, and it is not even clear whether the formal solution satisfying the smooth pasting condition exists. In addition, if one incorporates jumps, the intuitive justification for the smooth pasting condition in [17] is lost, and there is no reason to believe that this principle always holds[2]. In fact, it may fail as it was shown in [8] and [9].

To summarize, for a broad range of applications, continuous time real options models lose much of their tractability. This paper provides a tractable alternative approach for pricing of real options assuming that an option is the right to abandon or acquire a stream of stochastic payoffs $g(X_t)$, and the underlying stochastic process X_t is a random walk (under a risk neutral measure chosen by the market) on the real line. We solve models with simple real options in three steps:

(1) assume that the optimal exercise price is known and write down the Bellman equation for the value of an option (say, option to invest or option to default etc.);

(2) solve the Bellman equation explicitly;

(3) find the optimal exercise boundary using the explicit form of the value

[2] We are grateful to Avinash Dixit for pointing out to us that this principle may not hold in a discrete time model.

function in terms of the expected present value operators (EPV-operators).

The first step of the above algorithm is straightforward. The second step is well known in analysis as the Wiener-Hopf factorization method. In this paper, we explain the method in a simple setup when the transition density is modeled as exponential distributions on the negative and positive half-axis. The form of the value function allows one to guess the optimal exercise boundary quite easily, and if one accepts the hypothesis that the optimal exercise rule can be described in terms of a certain threshold, then the proof of optimality requires no knowledge of stochastic analysis and optimal stopping theory. The proof of optimality in the class of all stopping times is more involved but the representation of the solution in the form of the EPV-operators allows us to give a short proof (see Appendix A). A sufficient (but by no means necessary) condition for our methodology to work is that the payoff stream is a monotone function of the underlying stochastic factor. If the payoff is instantaneous, we represent its value as the EPV of a stream of stochastic payoffs, and use the general scheme.

Note that the standard approach in probability literature uses the direct link of the Winer-Hopf factorization to the first passage problem for a Lévy process or random walk, and it can be directly applied for standard payoffs of the put and call options in the geometric Lévy model or Lévy model (see, e.g., [1] and the bibliography therein); each type of payoffs should be treated individually, the proofs are long and technically involved, and it is difficult to adjust the standard approach to payoffs of a general form.

The results of the paper are valid for general random walks but it may become difficult to apply the Wiener-Hopf factorization formula. However, one of the advantages of the discrete time setting is that the transition density of a random walk can be approximated by exponential polynomials on each half-axis with desired accuracy and simplicity, although there is certainly a trade-off between the two. The family of transition densities given by exponential polynomials is fairly flexible, and such densities can account for fat tails and skewness observed in empirical distributions of commodity prices. At the same time, modeling with these distributions is tractable. The computations reduce to finding roots of a polynomial and straightforward algebraic manipulations. To find the optimal exercise boundary, one only needs to find a unique zero of a monotone function.

In the case of embedded options (i.e., options on options), the conventional approach requires one to solve a system of non-linear equations. A solution to a non-linear system is not unique. Therefore there is no guarantee that a numerical procedure gives the correct solution. As opposed to the standard approach, we use the argument similar to the backward induction in discrete time models with finite time horizon. We solve for the value and optimal exercise time (if necessary) of the most distant option in a sequence of embedded options, then we move to the second to last option, etc. For each option in the sequence, we just repeat the three steps described earlier.

The rest of the paper is organized as follows. In Section 2 we show how to derive the EPV of a payoff stream in the following model situations: (i) a perpetual stream, (ii) a stream that is lost at a random time, and (iii) a stream that starts to accrue at a random time. For the last two cases we also determine the optimal time to abandon or to acquire the stream. Finally, we calculate the expected waiting time for the option to be exercised. In the next two sections, we demonstrate the advantages of our approach in two situations. In Section 3, we consider a model of irreversible investment with construction lags. We study how the investment threshold changes when it takes time to build a project, and how it depends on uncertainty. In Section 4, a model of gradual capital expansion is studied, and in appendices, we provide the proof of optimality in the class of all stopping times, and derive analytical formulas for a more general class of random walks than in the main body of the text.

2 Model situations

2.1 Perpetual stream of payoffs

We assume that the underlying stochastic process X_t is a random walk (under a risk-neutral measure chosen by the market) on the real line, that is, $X_t = X_0 + Y_1 + Y_2 + \cdots + Y_t$, where $Y_1, Y_2, \ldots$ are independently identically distributed random variables on a probability space Ω, and X_0 is independent of $Y_1, Y_2, \ldots$. This process specification implies that the dates when observations and/or decisions to exercise options can be made are equally spaced, and time periods are normalized to one. Let the random walk X have the transition density, p (the method of the paper can be applied to random walks on lattices as well). Let g be the payoff function, and $q \in (0,1)$ be the discount factor per period. The transition operator, T, is defined by

$$(Tg)(x) = E^x[g(X_1)] \equiv E[g(X_1)|\, X_0 = x].$$

Given p, one calculates the EPV of a stochastic payoff tomorrow:

$$E^x[qg(X_1)] = q(Tg)(x) = q\int_{-\infty}^{+\infty} p(y)g(x+y)dy.$$

To compute the EPV of a stochastic payoff t periods from now, we use the Markov property of a random walk: $E^x[q^t g(X_t)] = q^t(T^t g)(x)$. Let

$$M(z) = E\left[e^{zY_1}\right] = \int_{-\infty}^{+\infty} e^{zy}p(y)dy$$

be the moment generating function of Y_1. If $g(x) = e^{zx}$, then the transition operator acts on $g(x) = e^{zx}$ as follows $(Tg)(x) = M(z)e^{zx}$. By the law of iterated expectations, we have

$$E^x[q^t g(X_t)] = (qM(z))^t\, e^{zx}.$$

The next step is to calculate the normalized EPV of a perpetual stream of payoffs $g(X_t)$, which we denote $\mathcal{E}g(x)$:

$$(\mathcal{E}g)(x) = (1-q)E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = (1-q)\sum_{t=0}^{\infty} q^t(T^t g)(x).$$

The normalization is convenient because

$$(\mathcal{E}\mathbf{1})(x) = (1-q)E^x\left[\sum_{t=0}^{\infty} q^t \mathbf{1}(X_t)\right] = (1-q)\sum_{t=0}^{\infty} q^t = 1.$$

In order to find $u(x) = (\mathcal{E}g)(x)$, we write the Bellman equation

$$u(x) = (1-q)g(x) + qE^x[u(X_1)],\ \forall\ x$$

in the form

$$(1-q)^{-1}(I - qT)u(x) = g(x),\ \forall\ x. \tag{2.1}$$

One can view (2.1) as an infinite system of algebraic linear equations with an infinite matrix (operator)

$$A = (1-q)^{-1}(I - qT).$$

To solve the linear system, one needs to invert the matrix (operator) A; the inverse A^{-1} will be denoted $\mathcal{E}$. The norm of the operator qT (in $L^\infty(\mathbf{R})$) is $q < 1$, hence the operator A is invertible in $L^\infty(\mathbf{R})$, and we have

$$u(x) = (\mathcal{E}g)(x) = (A^{-1}g)(x), \tag{2.2}$$

for a bounded measurable g. The same formula can be used with some unbounded g. For instance, if $g(x) = e^{zx}$ and $1 - qM(z) > 0$, then

$$u(x) = (\mathcal{E}g)(x) = (1-q)\sum_{t=0}^{\infty}(qM(z))^t e^{zx} = \frac{(1-q)e^{zx}}{1-qM(z)}, \tag{2.3}$$

and

$$(Ag)(x) = a(z)g(x), \tag{2.4}$$

where

$$a(z) = (1-q)^{-1}(1 - qM(z)). \tag{2.5}$$

Sufficient conditions for $(\mathcal{E}g)(x)$ and $Ag(x)$ to be finite are: g is measurable, and

$$|g(x)| \leq C\exp(\sigma^+ x), \quad x \geq 0, \tag{2.6}$$

$$|g(x)| \leq C\exp(\sigma^- x), \quad x \leq 0, \tag{2.7}$$

where constant C is independent of x, and $\sigma^\pm$ satisfy

$$1 - qM(\sigma) > 0, \quad \forall\ \sigma \in [\sigma^-, \sigma^+]. \tag{2.8}$$

These conditions are necessary if g is monotone on each half-axis.

Below, we introduce a factorization of the operator A, which will allow us to solve the Bellman equation in entry and exit problems. We explain the construction in a simple case when the transition density is of the form

$$p(y) = c^+\lambda^+ e^{-\lambda^+ y}\mathbf{1}_{(0,+\infty)}(y) + c^-(-\lambda^-)e^{-\lambda^- y}\mathbf{1}_{(-\infty,0]}(y), \tag{2.9}$$

where $c^+, c^- > 0$, and $\lambda^- < 0 < \lambda^+$. Here parameter c^+ is the probability of an upward jump per time period. If an upward jump has happened, then the probability of a jump from the current state x into an interval $[x+y,\, x+y+dy]$ is $\lambda^+ e^{-\lambda^+ y}dy$. Parameters c^- and λ^- admit a similar interpretation. If we want to have a continuous p, we must require that $c^+\lambda^+ + c^-\lambda^- = 0$, and then the normalization requirement $M(0) = 1$ leads to $c^+ = \lambda^-/(\lambda^- - \lambda^+)$, and $c^- = \lambda^+/(\lambda^+ - \lambda^-)$. We have constructed a two-parameter family of probability densities. The moment generating function is

$$M(z) = \frac{c^+\lambda^+}{\lambda^+ - z} + \frac{c^-\lambda^-}{\lambda^- - z} = \frac{-\lambda^-\lambda^+}{\lambda^+ - \lambda^-}\left[\frac{1}{\lambda^+ - z} - \frac{1}{\lambda^- - z}\right],$$

hence

$$M(z) = \frac{\lambda^-\lambda^+}{(\lambda^+ - z)(\lambda^- - z)}. \tag{2.10}$$

It is easy to see that $a(z)$ defined by (2.5) has zeros $\beta^+ = \beta^+(q) \in (0, \lambda^+)$ and $\beta^- = \beta^-(q) \in (\lambda^-, 0)$, which are the roots of the quadratic equation

$$z^2 - (\lambda^+ + \lambda^-)z + (1-q)\lambda^+\lambda^- = 0. \tag{2.11}$$

We find

$$\beta^\pm(q) = 0.5 \cdot \left(\lambda^+ + \lambda^- \pm \sqrt{(\lambda^+ + \lambda^-)^2 - 4(1-q)\lambda^-\lambda^+}\right). \tag{2.12}$$

Factorize $a(z)$ as

$$a(z) = (1-q)^{-1}\frac{(\beta^+ - z)(\beta^- - z)}{(\lambda^+ - z)(\lambda^- - z)}.$$

Since

$$a(0) = (1-q)^{-1}\frac{\beta^+\beta^-}{\lambda^+\lambda^-} = 1,$$

we may write

$$a(z) = a^+(z)a^-(z),$$

where

$$a^+(z) = \frac{\lambda^+(\beta^+ - z)}{\beta^+(\lambda^+ - z)}, \quad a^-(z) = \frac{\lambda^-(\beta^- - z)}{\beta^-(\lambda^- - z)}.$$

Introduce

$$\kappa_q^+(z) = a^+(z)^{-1} = \frac{\beta^+(\lambda^+ - z)}{\lambda^+(\beta^+ - z)} \tag{2.13}$$

and

$$\kappa_q^-(z) = a^-(z)^{-1} = \frac{\beta^-(\lambda^- - z)}{\lambda^-(\beta^- - z)}, \tag{2.14}$$

and notice that

$$\frac{1-q}{1-qM(z)} = a(z)^{-1} = a^+(z)^{-1}a^-(z)^{-1} = \kappa_q^+(z)\kappa_q^-(z). \tag{2.15}$$

Next, define operators $A^{\pm}$ and $\mathcal{E}^{\pm}$ as follows. For $g(x) = e^{zx}$,

$$A^{\pm}g(x) = a^{\pm}(z)g(x), \quad \mathcal{E}^{\pm}g(x) = a^{\pm}(z)^{-1}g(x). \tag{2.16}$$

Evidently,

$$a^{+}(z) = \frac{\lambda^{+}}{\beta^{+}} + \frac{\beta^{+} - \lambda^{+}}{\beta^{+}} \cdot \frac{\lambda^{+}}{\lambda^{+} - z},$$

and

$$\int_{0}^{+\infty} \lambda^{+} e^{-\lambda^{+} y} e^{z(x+y)} dy = \frac{\lambda^{+}}{\lambda^{+} - z} e^{zx}.$$

Hence, A^{+} acts on exponential g as follows

$$(A^{+}g)(x) = \frac{\lambda^{+}}{\beta^{+}} g(x) + \frac{\beta^{+} - \lambda^{+}}{\beta^{+}} \int_{0}^{+\infty} \lambda^{+} e^{-\lambda^{+} y} g(x+y) dy. \tag{2.17}$$

Similarly,

$$(A^{-}g)(x) = \frac{\lambda^{-}}{\beta^{-}} g(x) + \frac{\beta^{-} - \lambda^{-}}{\beta^{-}} \int_{-\infty}^{0} (-\lambda^{-}) e^{-\lambda^{-} y} g(x+y) dy, \tag{2.18}$$

$$(\mathcal{E}^{+}g)(x) = \frac{\beta^{+}}{\lambda^{+}} g(x) + \frac{\lambda^{+} - \beta^{+}}{\lambda^{+}} \int_{0}^{+\infty} \beta^{+} e^{-\beta^{+} y} g(x+y) dy, \tag{2.19}$$

$$(\mathcal{E}^{-}g)(x) = \frac{\beta^{-}}{\lambda^{-}} g(x) + \frac{\lambda^{-} - \beta^{-}}{\lambda^{-}} \int_{-\infty}^{0} (-\beta^{-}) e^{-\beta^{-} y} g(x+y) dy. \tag{2.20}$$

Note that we can use (2.17)–(2.20) to define $A^{\pm}g(x)$ and $\mathcal{E}^{\pm}g(x)$ for any measurable g that satisfies (2.6)–(2.7). It is obvious that $\mathcal{E}^{+} = (A^{+})^{-1}$, and $\mathcal{E}^{-} = (A^{-})^{-1}$ and vice versa, and (2.15) can be written as

$$\mathcal{E} = A^{-1} = \mathcal{E}^{+}\mathcal{E}^{-} = \mathcal{E}^{-}\mathcal{E}^{+}. \tag{2.21}$$

Using (2.21), we can find the solution to the Bellman equation (2.3) in two steps: first, calculate $g_1(x) = (\mathcal{E}^{+}g)(x)$, and then

$$u(x) = (\mathcal{E}^{-}g_1)(x) = \mathcal{E}^{-}\mathcal{E}^{+}g(x). \tag{2.22}$$

Alternatively, we may calculate $g_2(x) = (\mathcal{E}^{-}g)(x)$ first, and then

$$u(x) = (\mathcal{E}^{+}g_2)(x) = \mathcal{E}^{+}\mathcal{E}^{-}g(x). \tag{2.23}$$

We conclude this subsection by the list of important properties of operators $\mathcal{E}^{\pm}$ and $A^{\pm}$. The first statement is immediate from (2.17)–(2.20).

Proposition 2.1 *(a) If $g(x) = 0 \ \forall\ x \geq h$, then $\forall\ x \geq h$*

$$(\mathcal{E}^{+}g)(x) = 0, \quad (A^{+}g)(x) = 0. \tag{2.24}$$

(b) If $g(x) = 0 \ \forall\ x \leq h$, then $\forall\ x \leq h$

$$(\mathcal{E}^{-}g)(x) = 0, \quad (A^{-}g)(x) = 0. \tag{2.25}$$

Denote by Y^+ and Y^- random variables on $\mathbf{R}_+$ and $\mathbf{R}_-$, respectively, with the distributions

$$F^+(dy) = \frac{\beta^+}{\lambda^+}\delta(dy) + \frac{\lambda^+ - \beta^+}{\lambda^+}\beta^+ e^{-\beta^+ y}\mathbf{1}_{(0,+\infty)}dy,$$

and

$$F^-(dy) = \frac{\beta^-}{\lambda^-}\delta(dy) + \frac{\lambda^- - \beta^-}{\lambda^-}(-\beta^-) e^{-\beta^- y}\mathbf{1}_{(-\infty,0)}dy,$$

where $\delta(dy)$ is a unit point mass at zero. $F^\pm(dy)$ are probability distributions since $\beta^+/\lambda^+ > 0$, $(\lambda^+ - \beta^+)/\lambda^+ > 0$, $\beta^+/\lambda^+ + (\lambda^+ - \beta^+)/\lambda^+ = 1$, $\beta^-/\lambda^- > 0$, $(\lambda^- - \beta^-)/\lambda^- > 0$, and $\beta^-/\lambda^- + (\lambda^- - \beta^-)/\lambda^- = 1$.

Using (2.17)–(2.20) once again, we obtain

Proposition 2.2 *The operators $\mathcal{E}^\pm$ admit the representations*

$$(\mathcal{E}^+ g)(x) = E[g(x + Y^+)], \tag{2.26}$$

$$(\mathcal{E}^- g)(x) = E[g(x + Y^-)]. \tag{2.27}$$

Corollary 2.3 *a) If $g(x) \ge 0\ \forall\, x$, then $(\mathcal{E}^+ g)(x) \ge 0$, $\forall\, x$. If, in addition, there exists x_0 such that $g(x) > 0\ \forall\, x > x_0$, then $(\mathcal{E}^+ g)(x) > 0\ \forall\, x$.*

b) If $g(x) \ge 0\ \forall\, x$, then $(\mathcal{E}^- g)(x) \ge 0$, $\forall\, x$. If, in addition, there exists x_0 such that $g(x) > 0\ \forall\, x < x_0$, then $(\mathcal{E}^- g)(x) > 0\ \forall\, x$.

c) If g is monotone, then $\mathcal{E}^+ g$ and $\mathcal{E}^- g$ are also monotone.

2.2 Payoff stream that is lost when a threshold is crossed from above

Assume that the payoff stream $g(X_t)$ is a continuous non-decreasing function of X_t, the typical example being a firm facing demand uncertainty and a constant variable cost. Let G be the rate of output, and C the variable cost. For high levels of the log-price of the firm's output, X_t, the profit flow $g(X_t) = Ge^{X_t} - C$ is positive, and for low levels, it is negative. Should the (log) price fall sufficiently low, to a certain level h, it may become optimal to cease production. Fix h, a candidate for the exit threshold (the optimal choice of h will be analyzed in the next subsection), and denote by $V(x; h)$ the value of the firm with this choice of the exit threshold. In the region $x > h$, the value of the firm, $V(x; h)$, obeys the Bellman equation

$$V(x; h) = g(x) + qE^x[V(X_1; h)].$$

After the exit, the value is zero:

$$V(x; h) = 0, \quad x \le h. \tag{2.28}$$

It is convenient to solve the problem for the normalized value function $\mathcal{V}(x; h) = (1 - q)V(x; h)$:

$$(1-q)^{-1}(I - qT)\mathcal{V}(x; h) = g(x), \ x > h, \tag{2.29}$$

$$\mathcal{V}(x; h) = 0, \quad x \le h. \tag{2.30}$$

The Bellman equation (2.29) is similar to the Bellman equation (2.1) for the values of the firm which exists forever, but unlike the latter, the former holds for $x > h$ only.

For a set U, denote by $\mathbf{1}_U$ the indicator function of U and the multiplication operator by the same function. The next theorem, which explains the essence of the Wiener-Hopf factorization method, states that $\mathcal{V}$ can be calculated by a formula which is similar to (2.22); the new element is the multiplication-by-$\mathbf{1}_{(h,+\infty)}$-operator, which must be inserted between the factors $\mathcal{E}^-$ and $\mathcal{E}^+$.

Theorem 2.4 *Assume that conditions (2.6)–(2.8) hold. Then a solution to the problem (2.29)–(2.30) in the class of measurable functions satisfying (2.6)–(2.7) exists. It is unique and given by*

$$\mathcal{V}(x;h) = (\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g)(x). \tag{2.31}$$

Remark 2.1. a) From the technical point of view, the calculation of the solution to the problem (2.29)–(2.30) is no more difficult than the calculation of the value of the firm which never stops producing:

(1) calculate $g_1 = \mathcal{E}^+g$:

$$g_1(x) = \frac{\beta^+}{\lambda^+}g(x) + \frac{\lambda^+ - \beta^+}{\lambda^+}\int_0^{+\infty}\beta^+e^{-\beta^+y}g(x+y)dy;$$

(2) set $g_2(x) = g_1(x)$ for $x > h$, and $g_2(x) = 0$ for $x \le h$;

(3) calculate $\mathcal{V} = \mathcal{E}^-g_2$:

$$\mathcal{V}(x) = \frac{\beta^-}{\lambda^-}g_2(x) + \frac{\lambda^- - \beta^-}{\lambda^-}\int_{-\infty}^0(-\beta^-)e^{-\beta^-y}g_2(x+y)dy.$$

Notice that now we may not inverse the order of application of $\mathcal{E}^+$ and $\mathcal{E}^-$; the inverse order appears when we solve the problem for a stream which is abandoned as X_t reaches a certain threshold h from below; and then we use the indicator function $\mathbf{1}_{(-\infty,h)}$ instead of $\mathbf{1}_{(h,+\infty)}$.

b) The representation of the normalized EPV in the form (2.31) is convenient for the choice of the optimal exercise boundary. Using the (independent) random variables Y^+ and Y^-, we can write (2.31) in another form

$$\mathcal{V}(x;h) = E\left[\mathbf{1}_{(h,+\infty)}(x+Y^-)g(x+Y^-+Y^+)\right]. \tag{2.32}$$

Proof of Theorem 2.4. Here we will use properties stated in Proposition 2.1. Rewrite equation (2.29) as

$$(A\mathcal{V})(x;h) = g(x) + g^-(x),\ \forall\, x,$$

where $g^- \in L^\infty(\mathbf{R})$ vanishes for $x > h$. Equivalently,

$$(A^+A^-\mathcal{V})(x;h) = g(x) + g^-(x), \quad \forall\ x. \tag{2.33}$$

Multiply (2.33) by the inverse $\mathcal{E}^+$ to A^+:

$$(A^-\mathcal{V})(x;h) = (\mathcal{E}^+g)(x) + (\mathcal{E}^+g^-)(x), \quad \forall\ x. \tag{2.34}$$

Since $g^-(x) = 0$ for $x > h$, we have $\mathcal{E}^+ g^-(x) = 0$, $x > h$, on the strength of (2.24). Therefore

$$(A^- \mathcal{V})(x; h) = (\mathcal{E}^+ g)(x), \quad \forall\, x > h. \tag{2.35}$$

Multiply (2.35) by $\mathbf{1}_{(h,+\infty)}$. Since $\mathcal{V}(x; h) = 0,\ x \le h$, we have $(A^- \mathcal{V})(x; h) = 0,\ x \le h$ (see (2.25)). Thus, the LHS in (2.35) does not change, and (2.35) becomes

$$(A^- \mathcal{V})(x; h) = \mathbf{1}_{(h,+\infty)}(x) \mathcal{E}^+ g(x),\ \forall\, x. \tag{2.36}$$

Now it remains to apply the inverse $\mathcal{E}^- = (A^-)^{-1}$ to obtain (2.31). Note that (2.28) holds in view of (2.25). Theorem 2.4 has been proved.

Now we are in a position to determine the optimal exit threshold, h_*. Assume that

$$g(+\infty) > 0 \quad \text{and} \quad g(-\infty) < 0 \tag{2.37}$$

(one limit or both may be infinite; in the example of a firm under the demand uncertainty, only $g(+\infty)$ is infinite). Set $w = \mathcal{E}^+ g$. From (2.31), we have

$$\mathcal{V}(x; h) = E[(\mathbf{1}_{(h,+\infty)} w)(x + Y^-)], \tag{2.38}$$

where Y^- is the random variable on $\mathbf{R}_-$ defined after Proposition 2.1. Clearly, the larger the value of the product $\mathbf{1}_{(h,+\infty)} w$, the larger is the value $\mathcal{V}(x; h)$. Hence, the optimal choice of h should replace all negative values of w by zero, and leave positive ones as they are. By assumption, g is continuous and non-decreasing, therefore, by Corollary 2.3, w is also continuous and non-decreasing. Properties (2.37) ensure that w changes sign. Moreover, w can be locally constant only if $g(x)$ is constant above a certain level x_0, but even then w is strictly increasing on $(-\infty, x_0)$. Hence there exists a unique zero, h_*, of function w, and $w(x) < 0,\ \forall\, x \le h_*,\ w(x) > 0,\ \forall\, x > h_*$. We have proved

Theorem 2.5 *Let the payoff stream $g(X_t)$ be an increasing function of X_t, that satisfies (2.6)–(2.8). Then the optimal exit threshold is a unique solution to*

$$w(h_*) = (\mathcal{E}^+ g)(h_*) = E[g(h_* + Y^+)] = 0, \tag{2.39}$$

where Y^+ is the random variable on $\mathbf{R}_+$ defined after Proposition 2.1, and the normalized rational value of the stream is given by (2.31) with $h = h_*$.

We would like to emphasize that here uncertainty is modeled as a random walk, nevertheless equation (2.39) for the optimal exercise boundary is of almost the same form as in many fields of economics, where uncertainty is modeled as random draws from a stationary probability distribution. Also we stress that contrary to the Brownian motion model, the exercise boundary in the present model is not necessarily the price at which the option is exercised. It is optimal to exercise the option the first time τ such that $X_\tau \le h_*$.

2.3 Payoff stream that is lost when a threshold is crossed from below

Consider the case of a decreasing g; an example is the profit flow of a firm with uncertainty on supply side. The price of the firm's output, P, is constant and the variable cost C follows a geometric random walk. The instantaneous profit $g(X_t) = PG - e^{X_t}$ is a decreasing function of X_t, and it is positive for low levels of X_t and negative for high levels of X_t. It may be optimal to exit should the cost level become too high. Assuming that the exit threshold is given, writing down the Bellman equation, and repeating all the steps in Subsection 2.2 with $(-\infty, h)$, $\mathcal{E}^-$ and $\mathcal{E}^+$ in place of $(h, +\infty)$, $\mathcal{E}^+$ and $\mathcal{E}^-$, respectively, we obtain

Theorem 2.6 *Let the payoff stream $g(X_t)$ be a decreasing function of X_t, that satisfies (2.6)–(2.8). Then the optimal exit threshold is a solution to*

$$(\mathcal{E}^- g)(h^*) = E[g(h^* + Y^-)] = 0, \tag{2.40}$$

and the normalized rational value of the stream is

$$\mathcal{V}(x, h^*) = (\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)} \mathcal{E}^- g)(x). \tag{2.41}$$

2.4 Alternative interpretation of the optimal exit rule

In Appendix C, we prove that the operators $\mathcal{E}^\pm$ admit another interpretation as normalized EPV-operators under supremum and infimum processes $\bar{X}_t = \max_{0 \le s \le t} X_s$ and $\underline{X}_t = \min_{0 \le s \le t} X_s$, respectively:

$$\mathcal{E}^+ g(x) = (1-q) E \left[\sum_{t \ge 0} q^t g(\bar{X}_t) \mid X_0 = x \right], \tag{2.42}$$

$$\mathcal{E}^- g(x) = (1-q) E \left[\sum_{t \ge 0} q^t g(\underline{X}_t) \mid X_0 = x \right]. \tag{2.43}$$

Now, if the payoff stream is a non-decreasing function of the underlying stochastic factor, then the optimal exit rule can be formulated as follows: starting from the current value $X_0 = x$, consider all sample paths of the process X_t, and along each sample path, disregard all *temporary drops* of X_t. Then calculate the EPV of the stream, and if it is non-positive, give up the right for the stream. It looks as if a firm's manager contemplating an exit is too optimistic. However, the same manager becomes too pessimistic when making investment decisions. See Subsection 2.5, where the investment rule is expressed in terms the infimum process for the profit flow.

If the payoff stream is a non-increasing function of the underlying factor, then the exit decision is stated in terms of the infimum process for the underlying process X_t. However, in terms of the payoff stream $g_t = g(X_t)$, the exit rule is stated in terms of the supremum process, as in the case of a non-decreasing g.

2.5 Payoff stream that starts to accrue at random time

Suppose that we want to price the stream of payoffs that starts to accrue after the underlying stochastic factor, X_t, crosses a certain barrier, h. First, we let the payoff stream g to be an increasing function of X_t, a typical example being investment into a firm specific technology under demand uncertainty. Let h be the investment threshold, and τ_h the random time when X_t reaches h or crosses it from below the first time. The value of the option to invest is

$$v(x) = E^x \left[\sum_{t=\tau_h}^{\infty} q^t g(X_t) \right].$$

We can represent $v(x)$ in the form

$$v(x) = E^x \left[\sum_{t=0}^{\infty} q^t g(X_t) \right] + E^x \left[\sum_{t=0}^{\tau_h - 1} q^t g^1(X_t) \right], \tag{2.44}$$

where $g^1(X_t) = -g(X_t)$ is a decreasing function of X_t. The first term in (2.44) is independent of h, and the second term is the value of the stream that is abandoned at random time τ_h. Therefore the optimal exercise rule for this option is: invest when the current level x of X satisfies $(\mathcal{E}^- g^1)(x) \le 0$, equivalently $(\mathcal{E}^- g)(x) \ge 0$. Therefore the optimal investment threshold h^* is the solution to the equation $\mathcal{E}^- g(h^*) = 0$. Multiplying (2.44) by $(1-q)$, and using (2.41) with g^1 instead of g, and (2.21), we obtain

$$(1-q)v(x) = \mathcal{E}g(x) + (\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)} \mathcal{E}^- g^1)(x) = (\mathcal{E}^+\mathcal{E}^- - \mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)} \mathcal{E}^-) g(x).$$

Finally, since $\mathbf{1} - \mathbf{1}_{(-\infty,h^*)} = \mathbf{1}_{([h^*,+\infty)}$, we derive

$$\mathcal{V}(x; h^*) := (1-q)v(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} \mathcal{E}^- g(x). \tag{2.45}$$

Theorem 2.7 *Let the payoff stream be an increasing measurable function satisfying (2.6)–(2.8). Then it is optimal to exercise the right for the stream when* $(\mathcal{E}^- g)(X_t)$ *becomes non-negative, and the normalized option value is given by (2.45), where* h^* *is the solution to the equation* $\mathcal{E}^- g(h^*) = 0$.

Now, suppose that the payoff stream is a decreasing function of the underlying stochastic factor, a typical example being investment under supply uncertainty. Straightforward modification of the reasoning above gives

Theorem 2.8 *Let the payoff stream be a decreasing function satisfying (2.6)–(2.8). Then it is optimal to exercise the right for the stream when* $(\mathcal{E}^+ g)(X_t)$ *becomes non-negative, and the normalized option value is*

$$\mathcal{V}(x, h_*) = \mathcal{E}^- \mathbf{1}_{(-\infty,h_*]} \mathcal{E}^+ g(x), \tag{2.46}$$

where h_* *is the solution to the equation* $(\mathcal{E}^+ g)(h_*) = 0$.

2.6 General random walks

For a general random walk, the operators $\mathcal{E}^{\pm}$ are *defined* by (2.42)–(2.43), in terms of the supremum and infimum processes, and they can be applied to a measurable stream g that satisfies (2.6)– (2.8). Propositions 2.1, 2.2 and Corollary 2.3, hence, the main theorems above are valid. The proof of Theorem 2.4 in the general case contains additional subtle points (see [10] and [9]) but the proof of Theorem 2.5 can be repeated word by word. Explicit formulas for the action of $\mathcal{E}^{\pm}g$, when the probability densities of positive and negative jumps given by exponential polynomials, are given in Appendix B.

2.7 Expected waiting time

Consider an option to acquire a stream of payoffs $g(X_t)$, where g is an increasing function. Assume that the current value $X_0 = x < h^*$, and consider the waiting time R_x till the option will be exercised. This is the random variable defined by $R_x = \min\{t > 0 \mid X_t \geq h^*\}$. The expected waiting time can be calculated as follows:

$$\begin{aligned} E[R_x] &= E^x\left[\sum_{t=0}^{\infty} \mathbf{1}_{(-\infty,h^*)}(\bar{X}_t)\right] = \lim_{q\to 1-0} E^x\left[\sum_{t=0}^{\infty} q^t \mathbf{1}_{(-\infty,h^*)}(\bar{X}_t)\right] \\ &= \lim_{q\to 1-0} \frac{1}{1-q}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}(x) = \lim_{q\to 1-0} \frac{1}{1-q}\int_0^{h^*-x} p_q^+(y)dy, \end{aligned}$$

where p_q^+ is the probability density of $Y^+ = Y^+(q)$. Notice that here one must consider the process under the historical measure and not a risk-neutral one, and q above is just an auxiliary parameter needed for computational purposes. If the transition density is given by exponential polynomials on each of half-axis, then the limit can be easily calculated. In particular, if the transition density is given by (2.19), we obtain for $x < h^*$:

$$\frac{1}{1-q}\int_0^{h^*-x} p_q^+(y)dy = \frac{(1 - e^{-\beta^+(q)(h^*-x)}) + e^{-\beta^+(q)(h^*-x)}\beta^+(q)/\lambda^+}{1-q},$$

where $\beta^+ = \beta^+(q)$ is given by (2.12). Both terms in the numerator are positive, therefore, if $\beta^+(q)/(1-q)$ is unbounded as $q \to 1$, the limit is clearly infinite, and hence, the expected waiting time is infinite. From (2.12), we find that $E[R_x] < +\infty$ iff $\lambda^- + \lambda^+ < 0$. If $\lambda^- + \lambda^+ < 0$, we obtain

$$\beta^+(q) = \frac{\lambda^+\lambda^-}{\lambda^+ + \lambda^-}(1-q) + O((1-q)^2),$$

and therefore

$$E[R_x] = \frac{1}{m}(h^* - x + 1/\lambda^+), \tag{2.47}$$

where $m = 1/\lambda^+ + 1/\lambda^- = E[X_1 - X_0]$. Condition $\lambda^- + \lambda^+ < 0$ admits a clear interpretation: the expected waiting time is finite iff the drift of the underlying factor, m, is positive, and if it is positive, then (2.47) says that the expected waiting time is inversely proportional to the drift. It is also proportional to the distance to the barrier *plus a constant term* $1/(m\lambda^+) = 1/(1-c^-/c^+) > 0$, which increases with the frequency of downward jumps.

3 Investment lags

Typically, models of irreversible investment assume that a project is brought on line immediately after the decision to invest is made. In fact, in many instances investments take time, which is referred to as "time-to-build", "construction lag", and "gestation period". In [4], it is shown *numerically* that conventional results on the effect of price uncertainty on investment are weakened or reversed if there are lags in investment. That model is set in continuous time and the underlying stochastic factor follows a geometric Brownian motion. We are going to demonstrate similar effects *analytically* in discrete time, and correct certain general claims made in [4].

Let the project completion take n periods after the decision to invest has been made. When the project is completed, the firm will produce 1 unit of output every period and sell the output at the spot price $P = e^x$. The marginal cost of production, w is constant. The fixed cost of production, I, has to be paid in equal installments during the construction period. The present value of the deterministic stream of payoffs I/n that accumulates for n periods is $(1-q)^{-1}(1-q^n)I/n$. Clearly, such a value is generated by a perpetual stream $(1-q^n)I/n$. The future value (at date $t = n$) of this stream is $q^{-n}(1-q^n)I/n$. If the investment is made at the spot price $P = e^x$, then the expected firm's profit at date $t = n$ will be

$$E^x[e^{X_n}] - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n} = M(1)^n e^x - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n}.$$

Discounting n periods back, we may write the payoff flow as

$$g(x) = q^n \left(M(1)^n e^x - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n} \right).$$

The investment threshold, h^*, is defined by (2.40):

$$(\mathcal{E}^- g)(h^*) = q^n \left(M(1)^n \kappa_q^-(1) e^{h^*} - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n} \right) = 0,$$

equivalently,

$$\kappa_q^-(1) e^{h^*} = M(1)^{-n}(w + I(q^{-n}-1)/n). \tag{3.1}$$

It is natural to assume that $M(1) > 1$, which means that the expected revenue increases with time. Thus, the first factor on the RHS decreases with the investment lag. Since $(q^{-n}-1)/n = \sum_{j\ge1}(-\log q)^j n^{j-1}/j!$, the second factor on the RHS increases with n, and so one may expect that the overall effect of the investment lag is ambiguous. Since $qM(1) < 1$ (otherwise, the value of the project is infinite), we conclude that in the region of very large investment lags, the investment threshold increases with n, and the intuition is clear: part of the investment cost is suffered in the first period, and although it is the n-th part of the total cost, this part will outweigh the potential benefits which will be exponentially discounted over a long time interval. For moderate investment lags, the situation is more interesting. Assume that the time period in the model is not very large (say, a day, week or month). Then the discount factor per period, q, is close to one. Assume further that n is not very large so that the product $-n \log q$ is small (less than 1/3, say; for reasonable values of the discount factor, this means that the lag is 3-4 years or smaller). Then the product $n \log M(1)$ is also small because $qM(1) < 1$, and we can use the Taylor formula and derive an approximation to the RHS in (3.1) of the form

$$w + I(-\log q) + n[-(w + I(-\log q)) \log M(1) + (\log q)^2 I/2].$$

We see that if the investment lag is moderate, then the investment threshold is an increasing or decreasing function in n depending on the sign of the difference $(\log q)^2 I/2 - (w+I(-\log q)) \log M(1)$. For instance, if the prospects are not very good: $\log M(1)$ is much smaller than $-\log q$, and the fixed cost I is relatively large with respect to the variable cost w, then the investment threshold may increase when the lag increases. However, if the prospects are bright: $\log M(1) > -\log q/2$, then the investment threshold decreases for all w and I. We conclude that depending on the characteristics of the project, "time-to-build" increases or decreases the investment threshold.

The effect of uncertainty (measured by the variance of X) can be described in a simpler fashion. For any length of the construction period, there exists a critical value of the variance of the underlying stochastic factor such that for all the variances below the critical value, the investment threshold increases if the uncertainty measured by the variance increases. For all the variances higher than the critical value, the investment threshold decreases in uncertainty, so that it may even drop below the certainty investment barrier. Thus, a general claim [4], p.617: "Unless abandonment is possible, an increase in uncertainty always delays investments" is not quite correct.

Let the transition density for X be given by (2.9). For simplicity assume that $\lambda^+ = -\lambda^-$ (positive and negative jumps, on average, have the same size), and set $l = (\lambda^+)^{-1} = -(\lambda^-)^{-1}$. (This is the average size of jumps. Since $\lambda^+ > 1$, we have $l \in (0,1)$.) We have $E[Y_1] = (\lambda^+)^{-1} + (\lambda^-)^{-1} = 0$, and $\mathrm{var}(X_1) = (\lambda^+)^{-2} + (\lambda^-)^{-2} = 2l^2$. Clearly, the bigger the size of an average jump, the larger is the variance. In other words, uncertainty increases if jumps become bigger on average. We rewrite (3.1) as

$$M(1)^n \kappa_q^-(1) e^{h^*} = w + (q^{-n} - 1)I/n. \tag{3.2}$$

In (3.2), the RHS is independent from l. We only need to study the product $M(1)^n \kappa_q^-(1)$ as a function of l. From (2.10),

$$M(1) = \frac{\lambda^+ \lambda^-}{(\lambda^+ - 1)(\lambda^- - 1)} = \frac{1}{1 - l^2}.$$

Next, using (2.12), we find $\beta^- = -\sqrt{1-q}/l$, so that from (2.14)

$$\kappa_q^-(1) = \frac{(\lambda^- - 1)\beta^-}{\lambda^-(\beta^- - 1)} = \frac{1+l}{1 - 1/\beta^-} = \frac{1+l}{1 + l/\sqrt{1-q}}.$$

Straightforward calculations show that $M(1)^n \kappa_q^-(1)$ is decreasing in l (hence h^* is increasing in l) on the interval where

$$\frac{2n}{\sqrt{1-q}} l^2 + (2n + \frac{1}{\sqrt{1-q}} - 1)l - 1/\sqrt{1-q} + 1 < 0. \tag{3.3}$$

Given the "construction lag", n, (3.3) specifies the interval for the level of uncertainty, where the conventional intuition concerning the behavior of the investment threshold applies. When the critical level $l_{\rm up}$ (the positive root of the quadratic polynomial on the LHS) is crossed, the investment threshold starts to decrease with uncertainty. We must observe the condition $\beta^+ = \sqrt{1-q}/l > 1$, therefore we need $l_{\rm up} < \sqrt{1-q}$. It can be shown that this condition is satisfied if $4n(-\log q)$ is of order 1. Since $n(-\log q) = Tr$, where r is the discount rate in the corresponding continuous time model, and T is the investment lag in years, the effect of the decrease of the investment threshold with the further increase of uncertainty can be observed for investment lags of several years.

Since $M(1) \to +\infty$ as $l \to 1$, we see that for any given n, there exists another critical value $l_c > l_{\rm up}$ such that for all $l \in (l_c, 1)$, $(M(1))^n \kappa_q^-(1) > 1$, i.e., the investment threshold drops below the investment threshold in the case of no uncertainty ($l = 0$). This possibility can realize only when lags are large and/or future is discounted heavily.

4 Incremental capital expansion and expected capital stock

Consider an operating firm whose production function depends only on capital: $G(K) = dK^\theta$, where $d > 0$ and $\theta \in (0,1)$. A similar situation was considered in [2] for a two-period model of partially reversible investment, in [17] for the geometric Brownian motion model, and in [6] for Lévy processes. At each time period t, the firm receives $e^{X_t}G(K_t)$ from the sales of its product, and, should it decide to increase the capital stock, suffers the installation cost $C \cdot (K_{t+1} - K_t)$. The firm's objective is to chose the optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, X_t)\}_{t\geq 1}, K_0 = K, X_0 = x$, which maximizes the NPV of the firm:

$$V(K,x) = \sup_{\mathcal{K}} E^x \left[\sum_{t\geq 0} q^t (e^{X_t} G(K_t) - C(K_{t+1} - K_t)) \right]. \tag{4.1}$$

Here we treat the current log price x and capital stock K as state variables, and $\mathcal{K}$ as a sequence of control variables. Due to irreversibility of investment, $K_{t+1} \geq K_t$, $\forall t$. To ensure that firm's value (4.1) is bounded, we assume that $qM(1) < 1$, and $\beta^+ > 1/(1-\theta)$, where β^+ was defined by (2.12).

Formally, the manager has to choose both the timing and the size of the capital expansion. However, it is well-known (see, for example, [17]) that for each level of the capital stock, it is only necessary to decide when to invest. The manager's problem is equivalent to finding the boundary (the investment threshold), $h(K;C)$, between two regions in the state variable space (K,x): inaction and action ones. For all pairs (K,x) belonging to the inaction region, it is optimal to keep the capital stock unchanged. In the action region, investment becomes optimal. To derive the equation for the investment boundary, suppose first that every new investment can be made in chunks of capital, ΔK, only. In this case, the firm has to suffer the cost $C\Delta K$, and the EPV of the revenue gain due to the investment of a chunk of capital can be represented in the form of the EPV of the stream $g(X_t) = qM(1)(G(K+\Delta K) - G(K))e^{X_t} - (1-q)C\Delta K$. Thus, the multi-shot investment problem reduces to the one-shot problem studied above[3]. On the strength of (2.40), the optimal exercise boundary is determined from the equation $(\mathcal{E}^- g)(x) = 0$, which can be written as

$$qM(1)(G(K+\Delta K) - G(K))\kappa_q^-(1)e^x = (1-q)C\Delta K. \tag{4.2}$$

Dividing by ΔK in (4.2) and passing to the limit, we obtain the equation for the optimal threshold, $h^* = h^*(K)$:

$$qM(1)\kappa_q^-(1)G'(K)e^{h^*} = C(1-q),$$

which for the given form of production function reduces to

$$qM(1)\kappa_q^-(1)\theta d K^{\theta-1}e^{h^*} = C(1-q). \tag{4.3}$$

Set

$$B = \frac{qM(1)\kappa_q^-(1)\theta d}{1-q},$$

then the optimal exercise price is

$$e^{h^*} = e^{h^*(K)} = \frac{CK^{1-\theta}}{B}. \tag{4.4}$$

The rigorous justification of this limiting argument can be made exactly as in the continuous time model in [6]. Let $h = h(K;\Delta)$ be the solution to (4.2). Then, on the strength of (2.45), the option value associated with the increase of the capital by ΔK, at the price level $e^x < e^{h^*}$, is

$$(1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(x)\left(qM(1)(G(K+\Delta K) - G(K))\kappa_q^-(1)e^x - (1-q)C\Delta K\right).$$

As $\Delta K \to 0$, we have $h = h(K;\Delta) \to h^*(K)$; therefore, dividing by ΔK and passing to the limit, we obtain the formula for the derivative of the option value of future investment opportunities w.r.t. K:

$$\begin{aligned} V_K^{\text{opt}}(K,x) &= (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(x)\left(qM(1)G'(K)\kappa_q^-(1)e^x - (1-q)C\right) \\ &= \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(x)\left(BK^{\theta-1}e^x - C\right). \end{aligned} \tag{4.5}$$

[3]The authors are indebted for this simplifying trick to Mike Harrison; our initial proof was more involved.

Equations (4.4) and (4.5) imply together that

$$V_K^{\mathrm{opt}}(K,x) = Ce^{-h^*}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(x)\left(e^x - e^{h^*}\right). \tag{4.6}$$

Let $u(x) = \mathbf{1}_{[h^*,+\infty)}(x)\left(e^x - e^{h^*}\right)$. Then for $x < h^*$,

$$\begin{aligned}
(\mathcal{E}^+u)(x) &= \frac{\beta^+(\lambda^+ - \beta^+)}{\lambda^+}\int_0^{+\infty} e^{-\beta^+ y}\mathbf{1}_{[h^*,+\infty)}(x+y)\left(e^{x+y} - e^{h^*}\right)dy \\
&= \frac{\beta^+(\lambda^+ - \beta^+)}{\lambda^+}\left[e^x\int_{h^*-x}^{+\infty} e^{(1-\beta^+)y}dy - e^{h^*}\int_{h^*-x}^{+\infty} e^{-\beta^+ y}dy\right] \\
&= \frac{(\lambda^+ - \beta^+)}{\lambda^+}\cdot\frac{e^{\beta^+(x-h^*)+h^*}}{\beta^+ - 1}.
\end{aligned}$$

Now we may substitute $(\mathcal{E}^+u)(x)$ into (4.6) and using (4.4) derive

$$V_K^{\mathrm{opt}}(K,x) = \frac{(\lambda^+ - \beta^+)C}{\lambda^+(\beta^+ - 1)}e^{\beta^+(x-h^*)} = \frac{(\lambda^+ - \beta^+)C}{\lambda^+(\beta^+ - 1)}\left(\frac{B}{C}\right)^{\beta^+}K^{\beta^+(\theta-1)}e^{\beta^+ x}.$$

Integrating w.r.t. K, we derive the option value of investment opportunities:

$$\begin{aligned}
V^{\mathrm{opt}}(K,x) &= \frac{(\lambda^+ - \beta^+)C}{\lambda^+(\beta^+ - 1)}\left(\frac{B}{C}\right)^{\beta^+} e^{\beta^+ x}\int_K^{+\infty} k^{\beta^+(\theta-1)}dk \\
&= \frac{(\lambda^+ - \beta^+)CK^{1-\beta^+(1-\theta)}}{\lambda^+(\beta^+ - 1)(\beta^+(1-\theta) - 1)}\left(\frac{B}{C}\right)^{\beta^+} e^{\beta^+ x}
\end{aligned} \tag{4.7}$$

Given the spot price $P = e^x$, the value of the firm with the capital stock K is the EPV of the stream of revenues, calculated under the assumption that the capital stock remains constant in the future, plus the option value of investment opportunities:

$$V(K,x) = \frac{dK^\theta e^x}{1 - qM(1)} + V^{\mathrm{opt}}(K,x). \tag{4.8}$$

Our next goal is to determine the optimal amount of investment and the dynamics of the capital stock. As it was stressed in [20], the benchmark models of uncertainty introduced in [17] do not suggest specific predictions about the level of investment. Since the investment rule itself is not observable, one has to use the data on investment and capital stock to evaluate investment models. In [3] the behavior of the capital stock of a new born firm in a Gaussian model is examined. Paper [7] does the same for the case when the uncertainty is modeled as a Lévy process. In both cases, fairly sophisticated mathematical techniques are used. Below, we obtain the recurrent formulas for the expected value of capital at any time period in the future by using the elementary calculus.

Direct calculations show at the moment of entry, the firm will install the stock of capital given by:

$$BK^{\theta-1}e^x = C, \tag{4.9}$$

and the firm's value is a function of the spot price only:

$$V(x) = C\delta \left(\frac{B}{C}\right)^{\frac{1}{1-\theta}} e^{\frac{x}{1-\theta}}, \tag{4.10}$$

where

$$\delta = \frac{\kappa_q^+(1)}{qM(1)\theta} + \frac{\lambda^+ - \beta^+}{\lambda^+(\beta^+ - 1)(\beta^+(1-\theta) - 1)}.$$

Let I be the fixed cost of entry, which can be viewed as a deterministic stream of expenditures $(1-q)I$ to which the investor commits at the time of entry. By (2.40), it is optimal to enter when $(\mathcal{E}^-(V(\cdot) - (1-q)I))(x) = 0$, equivalently

$$C\delta \left(\frac{B}{C}\right)^{\frac{1}{1-\theta}} \kappa_q^-(1/(1-\theta)) e^{\frac{x}{1-\theta}} = (1-q)I,$$

therefore the price that triggers new entry is

$$e^{h_e} = \left[\frac{(1-q)I}{C\delta\kappa_q^-(1/(1-\theta))}\right]^{1-\theta} \frac{C}{B}. \tag{4.11}$$

Denote the moment of entry $t = 0$. Since investment is irreversible, the capital stock cannot be decreased, and it is increased when (and only when) the supremum process $\bar{X}_t$ jumps. Therefore, after the entry, the capital stock dynamics is given by

$$K_t = \left(\frac{B}{C}\right)^{\frac{1}{1-\theta}} e^{\frac{\bar{X}_t}{1-\theta}} = K_0 e^{\frac{\bar{X}_t - X_0}{1-\theta}}.$$

The expected capital stock at time $t > 0$ is

$$E[K_t] = K_0 E\left[e^{\frac{\bar{X}_t - X_0}{1-\theta}}\right] = K_0 E\left[e^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right].$$

Using equations (2.16) and (2.42), we may write

$$\sum_{t=0}^{\infty} q^t E\left[e^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right] = \frac{\kappa_q^+(1/(1-\theta))}{1-q}. \tag{4.12}$$

Equation (4.12) tells us that in order to find the expected stock of capital at any time t, one needs to know the coefficients c_t of the Taylor series of the function $\kappa_q^+(1/(1-\theta))/(1-q)$: if $\kappa_q^+(1/(1-\theta))/(1-q) = 1 + \sum_{t=1}^{\infty} c_t q^t$, then

$$E[K_t] = K_0 E\left[e^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right] = c_t K_0.$$

To find c_t, recall that

$$\kappa_q^+\left(\frac{1}{1-\theta}\right) = \frac{\left(\lambda^+ - \frac{1}{1-\theta}\right)\beta^+(q)}{\lambda^+\left(\beta^+ - \frac{1}{1-\theta}\right)} = \frac{\lambda^+(1-\theta) - 1}{(1-\theta)\lambda^+}\left[1 + \frac{1}{\beta^+(q)(1-\theta) - 1}\right], \tag{4.13}$$

where $\beta^+(q)$ is given by (2.12). We can write

$$\beta^+(q) = \frac{\lambda^+ + \lambda^-}{2} + \frac{\lambda^+ - \lambda^-}{2}\left(1 + \frac{4\lambda^+\lambda^-}{(\lambda^+ - \lambda^-)^2} q\right)^{1/2}. \qquad (4.14)$$

Set $A = 2\lambda^+\lambda^-/(\lambda^+ - \lambda^-)^2$, then the square root on the RHS can be written as

$$(1 + 2Aq)^{1/2} = 1 + \sum_{t=1}^{\infty} a_t q^t, \qquad (4.15)$$

where $a_1 = 1$, and for $t > 1$, $a_t = A^t(-1)^{t-1}(2t-3)(2t-5)\cdots 3 \cdot 1/t!$. Using (4.14) and (4.15), we derive $\beta^+(q) = \lambda^+ + 0.5(\lambda^+ - \lambda^-)\sum_{t=1}^{\infty} a_t q^t$, and

$$\beta^+(1-\theta) - 1 = \left(\lambda^+(1-\theta) - 1\right)\left(1 + \gamma \sum_{t=1}^{\infty} a_t q^t\right), \qquad (4.16)$$

where $\gamma = (\lambda^+ - \lambda^-)(1-\theta)/\left(\lambda^+(1-\theta) - 1\right)/2$. Next, we define $b_1, b_2, \ldots$, by

$$\left(1 + \gamma \sum_{t=1}^{\infty} a_t q^t\right)^{-1} = 1 + \sum_{t=1}^{\infty} b_t q^t. \qquad (4.17)$$

Straightforward computations show that b_t can be calculated recurrently

$$b_t = -\gamma \sum_{k=1}^{t} a_k b_{t-k}, \qquad (4.18)$$

where $b_0 \equiv 1$. Substituting (4.17) into (4.16), and (4.16) into (4.13), we obtain the Taylor series for $\kappa_q^+(1/(1-\theta))$:

$$\kappa_q^+\left(1/(1-\theta)\right) = 1 + (\lambda^+(1-\theta))^{-1} \sum_{t=1}^{\infty} b_t q^t.$$

Finally, we write

$$\frac{\kappa_q^+\left(1/(1-\theta)\right)}{1-q} = \kappa_q^+\left(1/(1-\theta)\right)\left(1 + \sum_{t=1}^{\infty} q^t\right) = 1 + \sum_{t=1}^{\infty} c_t q^t,$$

where

$$c_t = 1 + \frac{1}{\lambda^+(1-\theta)} \sum_{n=1}^{t} b_n = E\left[e^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right] \qquad (4.19)$$

are the coefficients in the formula for the expected value of the capital at time t.

A Optimality in the class of all stopping times

We consider the option to acquire a stream g, which is an increasing function of the stochastic factor. Denote by G the normalized EPV of the stream g. Then $\mathcal{V}(x, h^*)$ is the value of the perpetual option with the instantaneous payoff G. According to Lemma on p.1364 in [14], to prove the optimality of $\mathcal{V}(x, h^*)$ given by (2.45), it suffices to check the following two conditions:

$$\mathcal{V}(x, h^*) \geq \max\{G(x), 0\}, \quad \forall\, x \tag{A.1}$$

(the option value is non-negative, and not less than the payoff), and

$$(1-q)^{-1}(I - qT)\mathcal{V}(x, h^*) \geq 0, \quad \forall\, x, \tag{A.2}$$

that is, the discounted value tomorrow is not higher than the value today. By our choice of h^*, $w(x) = \mathcal{E}^- g(x) \geq 0$ for all $x \geq h^*$, therefore $\mathcal{V}(x, h^*) = \mathcal{E}^+ \mathbf{1}_{[h^*, +\infty)} w(x) \geq 0$ for all x. Further,

$$\mathcal{V}(x, h^*) = \mathcal{E}^+\mathcal{E}^- g(x) - \mathcal{E}^+ \mathbf{1}_{(-\infty, h^*)} w(x) = \mathcal{E} g(x) - v(h^*; x) = G(x) - v(h^*; x),$$

where $v(h^*; x) = \mathcal{E}^+ \mathbf{1}_{(-\infty, h^*)} w(x)$. Due to the choice of h^*, we have $w(x) < 0$ $\forall\, x < h^*$, so that $v(h^*; x) \leq 0$ for all x, and hence $\mathcal{V}(h^*; x) \geq G(x)$ for all x. We conclude that (A.1) holds. Under a very weak regularity assumption on p, it is proved in [10] that for any h,

$$\mathcal{V}(x, h) = qT\mathcal{V}(x, h), \qquad x < h. \tag{A.3}$$

Using more elaborate arguments as in [9] and [8] in a continuous time model, one can prove that (A.3) holds if the probability density p exists. Thus, (A.2) holds on $(-\infty, h^*)$, and it remains to verify (A.2) for $x \geq h^*$. Introduce $W(x) = (1-q)^{-1}(I - qT)\mathcal{V}(h^*; x)$. Using equalities $\mathcal{E} = (1-q)(I - qT)^{-1}$ and $\mathcal{E} = \mathcal{E}^+\mathcal{E}^-$, we obtain

$$W(x) = (\mathcal{E}^+\mathcal{E}^-)^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*, +\infty)} \mathcal{E}^- g(x) = (\mathcal{E}^-)^{-1} \mathbf{1}_{[h^*, +\infty)} \mathcal{E}^- g(x), \tag{A.4}$$

and also

$$\begin{aligned} W(x) \quad &= \quad (1-q)^{-1}(I - qT)\mathcal{E}^+\mathcal{E}^- g(x) - (1-q)^{-1}(I - qT)\mathcal{E}^+ \mathbf{1}_{(-\infty, h^*])} \mathcal{E}^- g(x) \\ &= \quad g(x) + (1-q)^{-1}(-I + qT)\mathcal{E}^+ \mathbf{1}_{(-\infty, h^*)} \mathcal{E}^- g(x). \end{aligned}$$

But if a function u vanishes outside $(-\infty, h)$, then $\mathcal{E}^+ u(x) = 0, x \geq h$, as well. Therefore, for $x \geq h^*$,

$$W(x) = g(x) + q(1-q)^{-1} T\mathcal{E}^+ \mathbf{1}_{(-\infty, h^*)} \mathcal{E}^- g(x). \tag{A.5}$$

By our standing assumption, g is non-decreasing, hence from (A.5), $W(x)$ is non-decreasing on $[h^*, +\infty)$. We apply $\mathcal{E}^-$ to (A.4) and obtain

$$\mathbf{1}_{[h^*, +\infty)} \mathcal{E}^- g(x) = \mathcal{E}^- W(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t W(x + \underline{X}_t)\right]. \tag{A.6}$$

Suppose that $W(h^*) < 0$. Then there exists $h_1 > h^*$ such that $W(x) \le 0$ for all $x \in (h^*, h_1)$. It follows that for the same x, the RHS in (A.6) is non-positive. But for these x, the LHS is positive by the very definition of h^*. Hence, our assumption $W(h^*) < 0$ is false, and since W is non-decreasing on $[h^*, +\infty)$, the condition (A.2) follows, and the proof of optimality is finished.

B Transition densities given by exponential polynomials

B.1 The case of three exponentials

In this Subsection, we demonstrate how to obtain the transition density of a desired shape. The density (2.9) has a kink (and maximum) at the origin. If we want to have a smooth p (and allow for the maximum to be not at the origin), we need to use more than two exponential functions. Suppose that we want to model a density which has the maximum on the positive half-axis. Then we use one exponential on the negative half-axis, and two on the positive one:

$$(c_1^+\lambda_1^+ e^{-\lambda_1^+ x} - c_2^+\lambda_2^+ e^{-\lambda_2^+ x})\mathbf{1}_{(0,+\infty)}(x) - c^-\lambda^- e^{-\lambda^- x}\mathbf{1}_{(-\infty,0]}(x), \tag{B.1}$$

where c_1^+, c_2^+ and c^- are positive, and $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$. Later in this Subsection, we show that for any choice of $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$, equation (B.1) with c^-, c_1^+, c_2^+ given by simple formulas (B.6) defines a probability density, which has the maximum on the positive half-axis. See Figure 1 for an example. Similarly, one can construct a 3-parameter family of probability densities which have the maximum on the negative half-axis. Should one wish to have a smooth probability density which has the maximum at the origin, one must use two exponential functions on each half-line or exponential polynomials of the form $(ax+b)e^{\gamma x}$.

The moment generating function of the probability density (B.1) is

$$M(z) = \frac{c_1^+\lambda_1^+}{\lambda_1^+ - z} - \frac{c_2^+\lambda_2^+}{\lambda_2^+ - z} + \frac{c^-\lambda^-}{\lambda^- - z}.$$

At the end of this subsection we will show that the *fundamental rational function* $1 - qM(z) = 0$ has 3 real roots, call them β^-, β_1^+ and β_2^+. We have

$$\lambda^- < \beta^- < 0 < \beta_1^+ < \lambda_1^+ < \lambda_2^+ < \beta_2^+. \tag{B.2}$$

Clearly, $(1-q)/(1-qM(z))$ can be represented in the form

$$\frac{1-q}{1-qM(z)} = \frac{a_1^+}{\beta_1^+ - z} + \frac{a_2^+}{\beta_2^+ - z} + \frac{a^-}{\beta^- - z}, \tag{B.3}$$

where $a_{+,j} = (1-q)/(qM'(\beta_j^+))$, $j = 1,2$, $a^- = (1-q)/(qM'(\beta^-))$. Hence,

$$(\mathcal{E}g)(x) = \sum_{j=1,2} a_j^+ \int_0^\infty e^{-\beta_j^+ y} g(x+y)dy - a^- \int_{-\infty}^0 e^{-\beta^- y} g(x+y)dy. \tag{B.4}$$

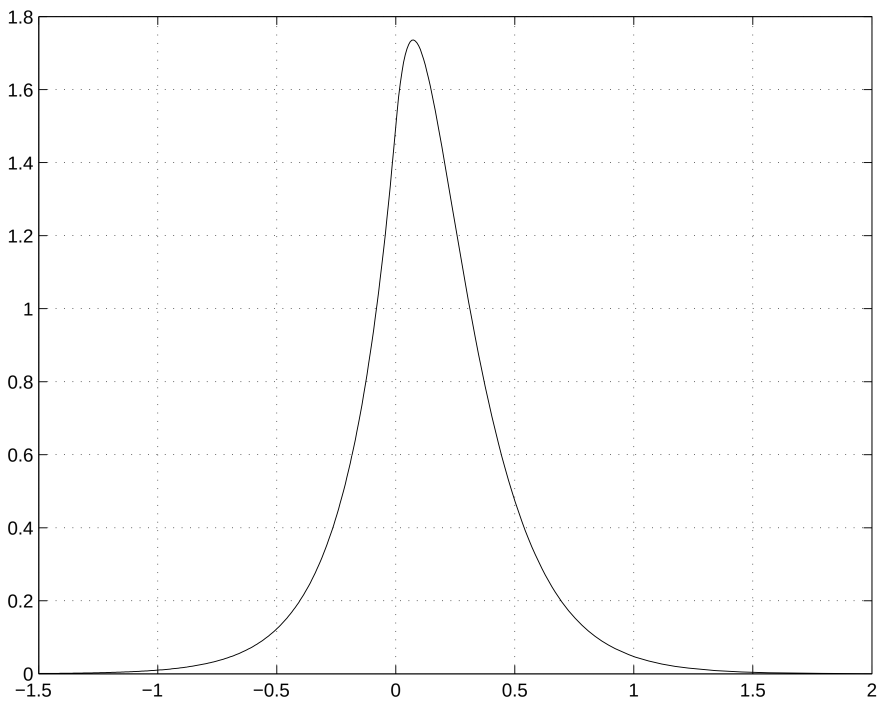

Figure 1:

Similarly, we can consider a probability density given by linear combinations of two or more exponents on each of the half-axis. If we use two exponentials for each, we have two roots $\beta_j^\pm$, $j = 1, 2$, of the characteristic equation $1 - qM(z) = 0$ on each half-axis, and (B.2)–(B.4) change in the straightforward manner. One can also use more than two exponentials on each axis, and obtain more elaborate probability densities.

Now we show that any choice $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$ defines a probability density. Three conditions: $\int_{-\infty}^{+\infty} p(x)dx = 1$, p is continuous at 0, and p is smooth at 0, give a linear system of three equations

$$\begin{aligned} c_1^+ - c_2^+ + c^- &= 1, \\ c_1^+\lambda_1^+ - c_2^+\lambda_2^+ + c^-\lambda^- &= 0, \\ c_1^+(\lambda_1^+)^2 - c_2^+(\lambda_2^+)^2 + c^-(\lambda^-)^2 &= 0. \end{aligned} \tag{B.5}$$

Using Cramer's rule, it is easy to find a unique solution (c_1^+, c_2^+, c^-) to (B.5):

$$c^- = \frac{\lambda_1^+\lambda_2^+}{(\lambda_1^+ - \lambda^-)(\lambda_2^+ - \lambda^-)}, \quad c_j^+ = \frac{-\lambda^-\lambda_k}{(\lambda_2^+ - \lambda_1^+)(\lambda_j - \lambda^-)}, \tag{B.6}$$

where $j \neq k \in \{1, 2\}$. It is easily seen that c_1^+, c_2^+ and c^- are positive, and that p is positive as well.

The roots of $1 - qM(z)$ are found as follows. Clearly, $1 - qM(z)$ has 3 roots at most. As $z \to \lambda^- + 0$, $1 - qM(z) \to -\infty$, and the same holds as $z \to \lambda_1^+ - 0$, and as $z \to \lambda_2^+ + 0$. Under condition $q \in (0, 1)$, $1 - qM(0) = 1 - q > 0$, and $1 - qM(+\infty) = 1 > 0$. Hence, on each of the intervals $(\lambda^-, 0)$ $(0, \lambda_1^+)$, and $(\lambda_2^+, +\infty)$, $1 - qM(z)$ changes sign. Therefore, on each of these three intervals, there is exactly one root, which we have called β^-, β_1^+ and β_2^+, respectively.

B.2 General scheme for the computation of $\mathcal{E}^+$ and $\mathcal{E}^-$

Step 1. Calculate the moment-generating function $M(z)$, and consider the rational function $1 - qM(z)$. Find the roots of the denominator, $\lambda_j^\pm$, and the numerator, $\beta_j^\pm$, with their multiplicities (sign "+" for the roots on the positive axis, sign "-" for the ones on the negative axis).

Step 2. Define

$$\kappa_q^+(z) = \prod_j \frac{\lambda_j^+ - z}{\lambda_j^+} \prod_k \frac{\beta_k^+}{\beta_k^+ - z}, \tag{B.7}$$

$$\kappa_q^-(z) = \prod_j \frac{\lambda_j^- - z}{\lambda_j^-} \prod_k \frac{\beta_k^-}{\beta_k^- - z}. \tag{B.8}$$

Step 3. If all the roots $\beta_k^\pm$ are simple, we represent $\kappa_q^+(z)$ and $\kappa_q^-(z)$ in the form

$$\kappa_q^+(z) = \kappa_q^+(\infty) + \sum_k \frac{a_k^+}{\beta_k^+ - z}, \quad \kappa_q^-(z) = \kappa_q^-(\infty) - \sum_k \frac{a_k^-}{\beta_k^- - z}, \tag{B.9}$$

where

$$a_k^+ = \prod_j \frac{\lambda_j^+ - \beta_k^+}{\lambda_j^+} \beta_k^+ \prod_{l \neq k} \frac{\beta_l^+}{\beta_l^+ - \beta_k^+}, \quad a_k^- = -\prod_j \frac{\lambda_j^- - \beta_k^-}{\lambda_j^-} \beta_k^- \prod_{l \neq k} \frac{\beta_l^-}{\beta_l^- - \beta_k^-}.$$

The case of multiple roots can be treated similarly.
Step 4. For a continuous g satisfying (2.6), we can calculate

$$(\mathcal{E}^+ g)(x) = \kappa_q^+(\infty) g(x) + \sum_k a_k^+ \int_0^{+\infty} e^{-\beta_k^+ y} g(x+y) dy, \tag{B.10}$$

and for a continuous g satisfying (2.7), we can find

$$(\mathcal{E}^- g)(x) = \kappa_q^-(\infty) g(x) + \sum_k a_k^- \int_{-\infty}^0 e^{-\beta_k^- y} g(x+y) dy. \tag{B.11}$$

Under condition (2.8), all the roots $\beta_k^\pm$ are outside $[\sigma^-, \sigma^+]$, therefore (2.6) and (2.7) ensure the convergence of the integrals in (B.10) and (B.11).

C Proof of (2.42)–(2.43)

It suffices to consider g of the form $g(x) = e^{zx}$. Define $\tilde{\kappa}_q^\pm(z)$ by

$$\tilde{\kappa}_q^+(z) = (1-q) E\left[\sum_{t \geq 0} q^t e^{z\bar{X}_t} \mid X_0 = 0\right], \quad \tilde{\kappa}_q^-(z) = (1-q) E\left[\sum_{t \geq 0} q^t e^{z\underline{X}_t} \mid X_0 = 0\right].$$

We need to show that $\kappa_q^+(z) = \tilde{\kappa}_q^+(z)$ and $\kappa_q^-(z) = \tilde{\kappa}_q^-(z)$. We know that (2.15) holds, and the Wiener-Hopf factorization formula [22] states that

$$(1-q)/(1-qM(z)) = \tilde{\kappa}_q^+(z) \tilde{\kappa}_q^-(z). \tag{C.1}$$

Comparing (2.15) and (C.1), we find $\tilde{\kappa}_q^+(z)/\kappa_q^+(z) = \kappa_q^-(z)/\tilde{\kappa}_q^-(z)$. The LHS is analytic in the right complex half-plane Re$z > 0$, and the RHS - in the left complex half-plane. The LHS and RHS equal 1 at 0, and continuous up to the boundary Re$z = 0$. One of the basic facts of the complex analysis (Morera's theorem) implies that both sides equal 1 in the right and left complex half-planes, respectively.

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