

Options Pricing with Arithmetic Brownian Motion and its Implication for Risk-Neutral Valuation

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Abstract: Risk-neutral valuation is used widely in derivatives pricing. It is shown in this paper, however, that the naïve approach of simply setting the growth rate of the underlying security to risk-free interest rate, which happens to work for a geometric Brownian motion (GBM) process, fails to work when the underlying price follows the arithmetic Brownian motion (ABM). Therefore, the formal approach using a martingale measure should be used instead when the underlying process is not a GBM.

Risk-neutral is an important concept in modern derivatives pricing and is well-known to students of finance. According to Hull (2003), the insight of risk-neutral valuation comes as a by-product from the celebrated Black-Scholes partial differential equation (PDE), in which the growth rate of the underlying security, the only variable in the Black-Scholes framework that is related to investors' risk preferences, of a derivative does not appear at all. As a result, derivatives can be priced in any world, including of course the risk-neutral world, where all securities are expected to grow by the risk-free interest rate.

The following three-step procedure of risk-neutral valuation is outlined by Hull (2003). First, set the expected return of the underlying to risk-free rate; second, compute the expected payoff of the derivative at maturity; third, discount the result of Step Two by the risk-free rate.

Risk-neutral valuation is a simple yet powerful tool. It is one of the two major methods developed to price derivatives since the breakthrough of Black and Scholes, which as the other major method tries to solve a Black-Scholes like PDE (Neftci, 2000). "But we should not overstretch the economic analogy – within our one-factor market all tradables are instantaneously perfectly correlated," Baxter and Rennie (1996) cautioned. In this article, we show that the simple three-step procedure outlined above by Hull may lead to incorrect results for a stochastic process as simple as the arithmetic Brownian motion (ABM).

Risk-Neutral Valuation: The Informal Approach

According to the procedure mentioned in the introduction, one can price a derivative by replacing the growth rate of the underlying (or the drift term) in the price model of

the underlying by the risk-free interest rate and take the expectation of the discounted payoff of the derivative. Assume that the stock price, as an underlying, follows the arithmetic Brownian motion (ABM):

$$dS_t = \sigma dW_t + \mu dt ,$$

where σ and μ are constant volatility and growth rate (or drift), respectively, of the underlying stock. Further assume a cash bond model,

$$dB_t = B_t r dt ,$$

where r is the (constant) risk-free interest rate.

In a typical risk-neutral valuation, μ is set to r in equation :

$$dS_t = \sigma dW_t^* + r dt$$

and we obtain by integration:

$$S_T = S_t + r(T - t) + \sigma(W_T^* - W_t^*).$$

Note that $(W_T^* - W_t^*)$ is distributed as conditional normal with a zero mean and a variance of $T - t$, or $N(0, T - t)$. The price of a European call option is therefore:

$$\begin{aligned} c_t &= e^{-r(T-t)} E_Q[\max(S_T - K, 0) | S_t] \\ &= e^{-r(T-t)} \sigma \sqrt{T-t} [dN(d) + n(d)] \end{aligned}$$

Where Q indicates the risk-neutral measure, K is the strike of the call option,

$$d = \frac{S_t + r(T-t) - K}{\sigma \sqrt{T-t}},$$

$N(d)$ as usual is the cumulative distribution function (cdf) of

the standard normal, and $n(d)$ is the probability density function (pdf) of the standard normal.

If the risk-free rate is zero, we arrive at the same result as was discussed extensively by Crack (2004a).

Risk-Neutral Valuation: The Formal Approach

The formal approach of risk-neutral valuation also involves three steps. First, transform the discounted stock process into a martingale under a new measure Q using Girsanov theorem; second, represent the underlying stock SDE by this new Q -Brownian motion; third, price any derivative by taking the expectation of the discounted payoff with respect to this Q -martingale measure (Baxter and Rennie, 1996).

Using the same models represented by equations (1) and (2), we form the discounted stock process $Z_t = B_t^{-1}S_t$, the SDE of which is then:

$$dZ_t = d(B_t^{-1}S_t) = B_t^{-1}[\sigma dW_t + (\mu - rS_t)dt].$$

Let $\sigma dW_t' = \sigma dW_t + (\mu - rS_t)dt$, then Z_t is a martingale under the new W_t' measure (Girsanov theorem). Put W_t' into equation (1), we obtain the risk-neutral SDE:

$$dS_t = \sigma dW_t' + rS_t dt.$$

Note that the drift parameter is rS_t , rather than simply r as in equation (1). The stock price at option's maturity is obtained by solving equation (1),

$$S_T = e^{r(T-t)} [S_t + \sigma e^{rt} \int_t^T e^{-rs} dW_s'].$$

Note that the Ito integral $\int_t^T e^{-rs} dW_s'$ is another Brownian motion with a mean of zero and a variance of Σ^2 , where $\Sigma^2 = \frac{e^{-2rt} - e^{-2rT}}{2r}$ and is computed using the Ito

isometry (Øksendal, 1995). Clearly the ABM model for the underlying security is more complicated mathematically than the geometric Brownian motion (GBM), as in the Black-Scholes framework, in pricing derivatives. Finally again:

$$\begin{aligned} c_t &= e^{-r(T-t)} E_Q[\max(S_T - K, 0) | S_t] \\ &= e^{rt} \sigma \Sigma [dN(d) + n(d)] \end{aligned}$$

where $d = \frac{S_t - Ke^{-r(T-t)}}{\sigma \Sigma e^{rt}}.$

Black-Scholes Like PDE

The call price in (3) is apparently not the same as that in (1). Therefore the informal and formal approaches of risk-neutral valuation do lead to different results when the underlying price process follows an ABM. The question we have to address now is: which price is correct?

To settle this problem, we have to resort to the other major method for pricing derivatives, namely the Black-Scholes PDE approach, because in principle, the risk-neutral and PDE methods should yield the same result (Neftci, 2000). Formally, we follow Black-Scholes by deriving a PDE and then solving the PDE to obtain the price for a call option.

Given the stock model in equation (1), a Black-Scholes like PDE can be obtained

by forming a suitable portfolio of the derivative and the underlying, applying Ito's lemma, and eliminating the random Brownian movements:

$$\frac{\partial c}{\partial t} + rS \frac{\partial c}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial S^2} = rc.$$

Note that first, compared with the Black-Scholes equation, the only difference is the missing of the S^2 factor in front of the second-order partial derivative; second, the growth rate of the underlying stock drops out of the PDE, just as in the original Black-Scholes equation. Therefore, risk-neutral valuation still applies in the case of ABM.

It is difficult to solve equation (9) to obtain a fair price for a European call option. Fortunately, we do not have to do this here, however. Remember that we already have two formulas for the call price. We can simply plug them into equation (9) to see which or any is a solution of it. A correct derivative price should solve equation (9), just as in the case of Black-Scholes.¹

It turns out that, after some length algebraic manipulations that are omitted, we show that price (10) satisfies the PDE (9), while (11) does not.² Therefore, (11) from the informal approach is not a correct price. This result is reassuring and expected, since intuitively, the formal approach should provide us with the correct answer. Note that when the risk-free rate is set to zero, the prices (10) and (11) do become the same, which is expected since in this case the SDEs (1) and (2) are the same to begin with.

Conclusion

Risk-neutral valuation is very useful in derivatives pricing and is shown to work no matter the underlying security follows a GBM, an ABM, or (probably) any other processes. It is found out, however, that it does not yield a correct price for derivatives by simply setting the growth rate of the underlying to the risk-free rate when the price model of the underlying is an ABM. The simple procedure of setting the underlying's expected return to risk-free rate for GBM could well be coincident and not universal.³

We conclude that the procedure of risk-neutral valuation as outlined by Hull (2003), in which the growth rate in the underlying SDE is simply replaced by the risk-free interest rate, only works for price models with GBM. For ABM or other processes, the martingale risk-neutral procedure, as was done in the formal approach

¹ As was pointed out by Dr. Timothy F. Crack (private communications), it is necessary, but not sufficient, for the solution (8) to satisfy the PDE. This solution is correct, however, since Crack derived the same result by solving the PDE directly with the boundary conditions (Crack, 2004b).

² Putting (5) into the left-hand side of equation (9), one obtains instead:

$$\frac{\partial c}{\partial t} + rS \frac{\partial c}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial S^2} = rc + rN(d)(S-1)e^{-r(T-t)}$$

³ Applying the formal approach to GBM, we derive the following risk-neutral SDE:

$$dS_t = \sigma S_t dW_t' + rS_t dt,$$

which happens to be the same as the real world SDE, if r is replaced simply by the growth rate.

of this article, should be utilized.

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