Heterogeneous Time Preferences and Interest Rates - The Preferred Habitat Theory Revisited

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Abstract

The influence of heterogeneous time preferences on the term structure is investigated. Motivated by the Preferred Habitat Theory of Modigliani and Sutch, a model for intertemporal preferences accounting for preferred habitats is proposed. In a heterogeneous world, preferred habitats can explain humps in the yield curve. Agents with a long habitat prefer long term bonds to shorter instruments as the Preferred Habitat Theory predicts.

Introduction

In their famous analysis of 'Operation Twist', Modigliani and Sutch (May 1966) develop what is since then called the 'Preferred Habitat Theory' of the term structure of interest rates. The main feature of said theory is that the investment horizon of investors should have a major influence on term premia of long term bonds. To give an example, if the majority of investors has a ten-year horizon, it seems plausible to assume higher prices and, hence, lower yields for zero-coupon bonds with ten years to maturity. Modigliani and Sutch use this argument in order to justify the introduction of long term rates into the econometric model employed to test the success of the economic policy 'Operation Twist'.

So far, there has not been the attempt to investigate systematically the influence of preferred habitats on interest rates in a general equilibrium model. The present paper aims at filling this gap. For this purpose, a continuous-time pure exchange economy with a financial market is studied where agents have different time preferences. A definition of the notion of 'Preferred Habitat' is given and a specific class

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of functions, the class of logistic densities, is proposed as a model for *Preferred Habitat Time Preferences*. The main goal of the present paper is to examine the influence of such time preferences on the term structure and to show that there is an influence on the shape of the yield curve, as the Preferred Habitat Theory predicts, if heterogeneous agents live in the economy.

The heterogeneity of agents just alluded to is an important aspect of the model. If homogeneity of agents is assumed as in most existing equilibrium models of the term structure (confer Cox, Ingersoll, and Ross (1985), Sun (1992), e.g.), agents do not trade and a preferred habitat effect does not exist. If, however, different types of agents are present in the economy, trade occurs and the form of the yield curve is altered. When the habitat of some agents is sharp enough, long-term interest rates around the habitat are lowered. In this sense, the Preferred Habitat Theory is able to explain humped yield curves.

In general, yields are composed of two summands: the first depends on the time preferences of agents, whereas the second is determined by the risk parameters of aggregate consumption and risk aversion of agents. I focus here on the first summand. It turns out that the larger is the share an agent plans to consume out of aggregate endowment at some time t, the higher is his contribution to the corresponding yield to maturity. Since the share is usually highest around a habitat of an agent, it follows that an agent sets the yields who correspond to his habitat. This interplay between different agents can lead to humped yield curves.

In a variation of the standard model, I study the portfolios chosen by agents forced to trade in bonds only. It is shown that agents with a long habitat tend to invest in long term bonds borrowing from impatient agents.

In their seminal paper, Cox, Ingersoll, and Ross (1981) perform a comparative statics analysis of equilibrium interest rates across different homogeneous economies and find no impact of time horizon (habitats) on interest rates. This is consistent with our finding that there is no influence of time preferences on the form of the yield curve in homogeneous economies and stresses again the importance of allowing for heterogeneity.

Among the few papers which explicitly model heterogeneity of agents without losing tractability are Constantinides and Duffie (1996) who allow for distinct endowment streams and Dumas (1989) as well as Wang (1996) who study economies populated by two classes of agents exhibiting distinct degrees of risk aversion. The paper of Wang (1996) is closest to ours in its setup since he also considers a pure exchange economy in continuous time. The way the growth of aggregate output is modeled is inspired by his approach.

The paper is organized as follows. The next section discusses the concept of a preferred habitat in an intertemporal utility framework. In Section 2, the general equilibrium model with a financial market is described. Section 3 derives the complete markets equilibrium. The term structure shape is analyzed in Section 4. In Section 5, a variation of the model is studied in which agents trade in long- and short-term bonds. The final section concludes.

1 Modelling Preferred Habitat Time Preferences

Think of a person who invests in the bond market with the specific goal of insuring herself against a decreasing standard of living upon retiring. Such an investor has a higher preference for consuming out of the portfolio when retiring than at the time the investment is made. She has a long habitat. The aim of the present section is to develop a type of intertemporal utility function which describes such people with a higher propensity to consume at a certain point in time H, their habitat, than at other times.

The typical intertemporal utility function over contingent consumption streams $(c_t(\omega))$ takes the form

$$U(c) = E\int_0^\infty e^{-\rho t} u(c_t)\, dt$$

for some *rate of time preference* $\rho > 0$[1]. This utility function is characterized by three properties: it satisfies the independence axiom, and it displays stationarity and impatience.

U satisfies the *axiom of cardinal independence* for states of nature as well as points in time because for any two processes c and d which coincide on a set $A \subset \Omega \times [0, \infty[$ one can arbitrarily alter the common value of c and d on A without changing the preference ordering: if, say, c is preferred to d, then for every c' $\tilde{c} = c' 1_A + c 1_{A^c}$ is preferred to $\tilde{d} = c' 1_A + d 1_{A^c}$,

$$U(\tilde{c}) > U(\tilde{d}) .$$

The independence axiom implies the existence of a time-additive expected utility representation for U:

$$U(c) = E\int_0^\infty u(t, c_t) dt ,$$

compare Debreu (1960). Since I will keep the independence axiom for states of nature (and, hence, the von Neumann-Morgenstern representation of the utility function), only the intertemporal aspect of the utility function U will be considered in the following.

For a deterministic consumption stream (c_t), the utility is

$$U(c) = \int_0^\infty e^{-\rho s} u(c_s)\, ds .$$

[1]For the heuristic discussion of this section, the horizon is set equal to ∞.

The preference ordering induced by U is *stationary* in the sense that the delayed utility U_t given by

$$U_t(c) = \int_t^\infty e^{-\rho s} u(c_s) ds$$

leads to the same ordering of (delayed) consumption streams:

$$U_t(c) = e^{-\rho t} U((c_{t+s})_{s \geq 0}) .$$

In particular, the rate of time preference ρ remains constant over time.

For $\rho > 0$ the agent is globally impatient. A positive rate of time preference implies that the long distant future does not matter much for current decisions. This is frequently used as a justification for the assumption of impatience.

My claim is that people with a (long) habitat do not have stationary time preferences nor are they globally impatient. If one has a habitat, for example, of ten years, then it is plausible to assume that one is patient up to said habitat and that one becomes very impatient at and after the habitat. As a consequence, one's rate of time preference changes over time. Thus, the preferences are not stationary. To model preferred habitat time preferences, the assumption of stationarity will be dropped. I will therefore allow for time-varying rates of time preference.

As far as impatience is concerned, it seems plausible to allow for periods of patience for agents with a long term habitat. Before their habitat, such agents will tend to delay consumption and will have a negative rate of time preference. I keep, however, the reasonable property that the very long distant future does not matter much for current decisions. I require the time preference density f to decrease exponentially in the long run: there is a $\bar{\rho} > 0$ with

$$f(t) \sim e^{-\bar{\rho} t} \qquad (t \to \infty) .$$

Strictly positive and smooth (C^2) functions f with finite integral,

$$\int_0^\infty f(t) dt < \infty ,$$

are henceforth called *time preference densities.* Without loss of generality, one may assume $\int f(t) dt = 1$.

Definition 1 *A time preference density f is said to display a preferred habitat in $h > 0$ iff*

- *f has a unique maximum in h;*
- *f decreases exponentially in the long run, that is, there is a* long-run rate of time preference $\bar{\rho} > 0$ *with*

$$\lim_{t \to \infty} f(t) e^{\bar{\rho} t} = 1 .$$

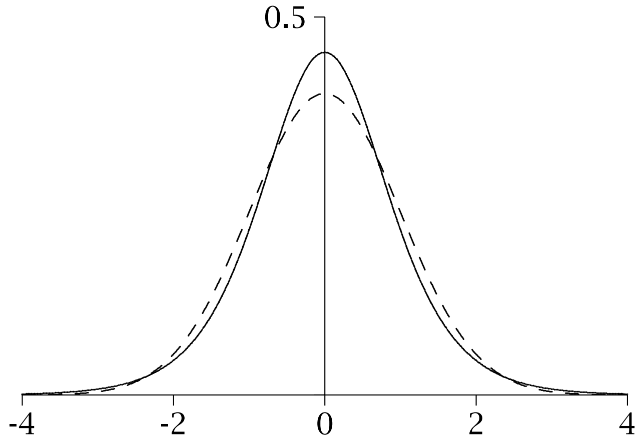

Figure 1: Density of the logistic distribution with mean 0 and variance 1. The dashed line represents the density of the standard normal distribution.

Note that the standard time preference density $\rho e^{-\rho t}$ is included in this definition. It displays a habitat at $h = 0$ and describes the short-run oriented individuals of the Hicksian world.

As a parametric class of time preference densities with a preferred habitat in h, I take the class of logistic densities,

$$f(t; h, \gamma) = \gamma \frac{\exp(-\gamma(t-h))}{(1+\exp(-\gamma(t-h)))^2},$$

see Figure 1. The corresponding distribution functions are

$$F(t; h, \gamma) = (1+\exp(-\gamma(t-h)))^{-1}.$$

h is the unique maximum, the habitat, and γ, the long-run rate of time preference, inversely relates to the variance of F, which is $\frac{\pi^2}{3\gamma^2}$. Note that the density is a bell-shaped curve as the normal density, but in contrast to the latter, it has fatter tails. The logistic densities will be the reference model for preferred habitat preferences and will be used to illustrate the results. I wish to stress that the structural results are independent of the specific form chosen for f.

Before concluding this discussion of time preferences, a final remark is in order. It is important not to confuse stationarity and time consistency. Individuals are said to act in a time inconsistent manner, if they wish to revise the plan chosen at time 0 at a later time $t > 0$[2]. Here, individuals' rates of time preference change over time. This change, though, is foreseen, and an adequate consumption plan is formulated in such a way as to avoid the necessity of a later revision. Their preferences are not stationary, but they act in a time consistent manner.

[2]Time preferences of the type "always discount the *next* period higher than the period following the next period" lead to time inconsistent behavior. Such time preferences were first studied by Strotz (1956).

2 The Model

The horizon of the economy is finite, $\bar{T} < \infty$. Uncertainty is modeled through a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ endowed with a one-dimensional Brownian motion W. The output K of the economy, or aggregate endowment, grows at rate X

$$\frac{dK_t}{K_t} = dX_t\,,$$

and the growth rate X is modeled as the solution of the stochastic differential equation

$$dX_t = (\phi_1 - \phi_2 X_t)dt + \xi dW_t\,,$$

with constant parameters $\phi_1 \in \mathbb{R}, \phi_2, \xi \in \mathbb{R}^+$. Two distinct cases are treated. For $\phi_2 = 0$, X is a Brownian motion with drift ϕ_1, and for $\phi_2 > 0$ X is a stationary Ornstein-Uhlenbeck process. It is useful to have the conditional distributions of the increments of X at hand. In the first case, $\phi_2 = 0$, these are

$$\mathcal{L}(X_{t+s} - X_t|\mathcal{F}_t) \quad = \quad N(\phi_1 s, \xi^2 s)\,. \tag{1}$$

For $\phi_2 > 0$, X is explicitly given by

$$X_{t+s} = e^{-\phi_2 s}\left(X_t + \int_t^{t+s} \phi_1 e^{\phi_2(u-t)}\,du + \int_t^{t+s} \xi e^{\phi_2(u-t)}\,dW_u\right)\,,$$

which leads to

$$\mathcal{L}(X_{t+s} - X_t|\mathcal{F}_t) \quad = \quad N\left(\left(\frac{\phi_1}{\phi_2} - X_t\right)\left(1 - e^{-\phi_2 s}\right), \frac{\xi^2\,(1 - e^{-2\phi_2 s})}{2\phi_2}\right)\,. \tag{2}$$

For shorter notation, the conditional mean of the increment is denoted by $\mu(s, X_t)$ and the conditional variance by $\sigma^2(s)$.

The asset market consists of a stock with price (S_t) paying the aggregate output K as a dividend and a market for borrowing and lending at a short rate (r_t). $\beta_t = \exp(\int_0^t r_u du)$ denotes the money market account. Stock price S and interest rate r are determined endogenously. Both processes S and r are assumed to be Itô-processes. This class is large enough to contain possible equilibrium prices.

The I agents have homogeneous expectations $P^i = P$ and constant relative risk aversion 1. Their endowment consists of s^i shares of the stock, $\sum s^i = 1$. Agents possibly differ in their time preferences f^i which are assumed to display a preferred habitat. Their utility function is thus

$$U^i(c) = E\int_0^{\bar{T}} f^i(t)\log(c_t)\,dt\,.$$

Without loss of generality, time preference densities are normalized to be probability densities. F^i denotes the associated distribution functions.

Taking stock price and interest rate as given, agents form a portfolio (θ^1, θ^2) in the stock and the money market account in order to finance their desired consumption stream c. The admissible portfolio/consumption policies are specified in the following

Definition 2 *A triple (θ^1, θ^2, c) of progressively measurable processes is called an portfolio/consumption policy with prices (S, β), if the following conditions hold true:*

$$E\int_0^{\bar{T}} c_t^2\, dt < \infty \quad (3)$$

$$\int_0^{\bar{T}} \left(\theta_u^1\right)^2 d[S]_u < \infty \qquad a.e. \quad (4)$$

$$\int_0^{\bar{T}} \theta_u^2 d\beta_u < \infty \qquad a.e. \quad (5)$$

The consumption/portfolio policy (θ^1, θ^2, c) is admissible for agent i with prices (S, β) if the value $V_t = \theta_t^1 S_t + \theta_t^2 \beta_t$ of the portfolio satisfies the budget and no ruin restrictions

$$V_0 = s^i S_0 \quad (6)$$

$$dV_t = \theta_t^1 (dS_t + K_t dt) + \theta_t^2 d\beta_t - c_t dt \qquad a.e. \quad (7)$$

$$V_t \geq 0 \qquad a.e. \quad (8)$$

The following concept of equilibrium is standard.

Definition 3 *A stock price S, an interest rate r and consumption portfolio policies $(\theta^{1i}, \theta^{2i}, c^i)_{i=1\ldots I}$ form an equilibrium if*

- *all markets clear,*

$$\sum \theta^{1i} = 1, \quad \sum \theta^{2i} = 0, \quad \sum c^i = K \quad (9)$$

- *for every agent i, c^i maximizes utility U^i over all admissible consumption/portfolio policies.*

3 Equilibrium

The Negishi method is used to derive the equilibrium. In a first step, efficient allocations are characterized by some vector $\lambda = (\lambda_1, \ldots, \lambda_I)$ of weights. This will also shed some light on the relevance of heterogeneous time preferences on consumption decisions. As a by-product, one obtains the utility of some representative agent in explicit form. By the Second Welfare Theorem, every efficient allocation can be supported as an Arrow-Debreu equilibrium where investors can trade arbitrary consumption streams at the initial date. The Arrow-Debreu consumption price is given by the marginal felicity of the corresponding representative agent. Using this marginal felicity as a state-price in the sense of Duffie (1992), one obtains candidates for equilibrium stock price and short rate. It remains to show that these candidates and the efficient allocation form a dynamic equilibrium (confer Duffie and Huang (1985)).

Definition 4 *For every vector of weights $\lambda \in \Delta = \{x \in \mathbb{R}^I_+ : \sum x_i = 1\}$ the utility of the representative agent corresponding to λ is*

$$U(c;\lambda) \quad = \quad \max_{\sum c^i = c} \sum \lambda_i U^i(c^i)\,. \tag{10}$$

As is well known, an allocation is Pareto efficient if and only if it solves the social welfare problem (10) for some λ. In our case, the problem can be solved explicitly.

Theorem 1 (Representative agent) *For every vector of weights $\lambda \in \Delta$, the representative agent's utility is given by*

$$U(c;\lambda) \quad = \quad E\int_0^{\bar{T}} f^\lambda(t)\log c_t\,dt + const.\,, \tag{11}$$

where the time preference density f^λ of the representative agent is

$$f^\lambda(t) \quad = \quad \sum_{j=1}^{I} \lambda^j f^j(t)\,.$$

The corresponding Pareto efficient allocation which solves the social welfare problem (10) is given by the sharing rules

$$x^i(c,f,\lambda)_t \quad = \quad \frac{\lambda^i f^i(t)}{\sum_j \lambda^j f^j(t)}\, c_t\,. \tag{12}$$

If aggregate consumption is efficiently allocated, agent i gets a share

$$\alpha^i_t := \frac{x^i(c,f,\lambda)_t}{c_t} = \frac{\lambda^i f^i(t)}{f^\lambda(t)}\,.$$

The share is the relative weight agent i places on the point in time t with respect to the "market's" weight $f^\lambda(t)$. This shows that at a habitat, an agent will in general consume a greater share of total endowment than at other times. However, the share increases only if the market's time preference density does not rise at the same time, that is, if there are other people who do not have a preferred habitat at this point.

Define the rate of time preference of the representative agent as

$$\rho(\lambda)_t = -\frac{\partial}{\partial t}\log f^\lambda(t) = -\frac{\frac{\partial}{\partial t} f^\lambda(t)}{f^\lambda(t)}\,.$$

In the same manner,

$$\rho_t^i = -\frac{\partial}{\partial t}\log f^i(t)$$

is the rate of time preference of agent i.

Lemma 1 *The rate of time preference of the representative agent is a convex combination of the agents' time preferences:*

$$\rho(\lambda)_t = \sum_i \alpha_t^i \rho_t^i\,,$$

where the weights α^i are the shares of consumption of the agents.

The natural candidate for an equilibrium consumption price and a state-price is the marginal felicity of some representative agent. As the following theorem shows, it is the vextor $\lambda^\star = s$ of initial shares owned by agents which characterizes the equilibrium.

Theorem 2 (Equilibrium) *An equilibrium is given by the stock price*

$$S_t = K_t\,\frac{1-F^s(t)}{f^s(t)}\,, \tag{13}$$

where F^s is the distribution function to f^s, interest rate

$$r_t = \rho(s)_t + \phi_1 - \phi_2 X_t - \xi^2\,, \tag{14}$$

consumption plans

$$c^i(s)_t = x^i(K,f,s)_t\,,$$

and portfolio strategies

$$\theta_t^{i1} = \frac{s^i(1-F^i(t))}{1-F^s(t)} \tag{15}$$

in the stock. There is no trading on the money market,

$$\theta_t^{i2} = 0\,.$$

The asset market given by the stock price and the money market account is complete.

3.1 Analysis of the short rate

The short rate r is the sum of the rate of time preference $\rho(s)$ and a component which does not depend on time preferences, but only on X and its risk parameters,

$$\phi_1 - \phi_2 X_t - \xi^2 .$$

The short rate is a linear function of the rate of growth X. Interestingly, a high current growth rate X_t has a negative impact on the short rate. This is due to the mean-reverting property of X. High values of X will force X back to its long-run level and the drift of X, $\phi_1 - \phi_2 X_t$ decreases.

Heterogeneity of time preferences has a deterministic influence on interest rates. The influence on the short rate is given by the rate of time preference ρ^s of the representative agent which is a weighted average of individuals' rates of time preference. If one denotes by

$$r_t^i = \rho_t^i + \phi_1 - \phi_2 X_t - \xi^2$$

the short rate which prevails in the homogeneous economy where only agent i lives, then it follows as an obvious corollary of Lemma 1

Corollary 1 (Conjecture of Dumas) *The short rate of the heterogeneous world is a convex combination of the short rates that prevail in the homogeneous worlds,*

$$r(s)_t = \sum_i \alpha_t^i r_t^i .$$

In particular,

$$\min_i r_t^i \leq r(s)_t \leq \max_i r_t^i .$$

The preceding corollary was conjectured in Dumas (1989) who studied the case of heterogeneous degrees of risk aversion among agents. In the case of heterogeneous time preferences, it turns out that the short rate of the heterogeneous world is in between the bounds given by the short rates of the homogeneous economies and the conjecture of Dumas holds true(which is not always the case, confer Wang (1996)).

Since the rate of output growth X is a Brownian motion for $\phi_2 = 0$ and an autoregressive Ornstein-Uhlenbeck process for $\phi_2 > 0$, so is the short rate if the rate of time preference $\rho(s)_t = \rho$ is constant. This type of model has been studied by Merton (1970) (for the Brownian case) and by Vasiček (1977) (for $\phi_2 > 0$). In general, $\rho(s)$ varies with time and the resulting short rate is as in the models proposed by Hull and White (1990).

3.2 Portfolios

The number of shares held in equilibrium is given by the fraction of the quantiles of the time preference distributions of agent i and the representative agent. The more weight an agent puts on the future, compared to the market's weight, the more she invests in the stock. Individuals with a long habitat will, therefore, initially defer consumption by investing in the stock (and re-investing the dividends obtained) in order to increase the number of shares of the stock in their portfolio. This strategy allows them to finance their higher consumption at their habitat. I repeat that this occurs only if there are other people who do the converse (have a short habitat). In a homogeneous economy, agents are forced to hold the stock at all times.

The fact that the portfolio strategies do not depend on the state of the world is due to the specific choice of logarithmic felicity functions[3].

4 Analysis of the Term Structure

Since the asset market is complete, zero-coupon bonds can be duplicated by trading in the risky asset and the money market account. It is well known, that the price of a zero-coupon bond with maturity T is given by the Euler formula

$$B_t^T = E\left[\frac{f^s(T)K_T^{-1}}{f^s(t)K_t^{-1}} \,\middle|\, \mathcal{F}_t \right].$$

Thus, the yield curve is determined. A calculation yields

Theorem 3 *Equilibrium bond prices are*

$$B_t^{s,t+\tau} = \frac{f^s(t+\tau)}{f^s(t)} \exp\left(-\mu(\tau, X_t) + \frac{\xi^2}{2}\tau + \frac{1}{2}\sigma^2(\tau) \right). \tag{16}$$

The yield curve is therefore given by

$$y_t^{s,t+\tau} = \frac{1}{\tau}\int_t^{t+\tau} \rho(s)_u \, du + \frac{\mu(\tau, X_t)}{\tau} - \frac{\xi^2}{2} - \frac{\sigma^2(\tau)}{2\tau}.$$

Like the short rate, the long-term yields consist of two summands - the first,

$$\frac{1}{\tau}\int_t^{t+\tau} \rho(s)_u \, du,$$

is due to the time preferences of the agents, whereas the second,

$$\frac{\mu(\tau, X_t)}{\tau} - \frac{\xi^2}{2} - \frac{\sigma^2(\tau)}{2\tau},$$

is caused by the dynamics of the growth rate X and risk aversion.

[3]For more general CRRA felicities, a risk coefficient must be included, see Riedel (1998).

4.1 Long-term Yields

Before I begin a more concrete study of the term structure, I give an interesting characterization of the long-rate[4] $y^s_\infty = \lim_{\tau\to\infty} y^{s,t+\tau}_t$.

Theorem 4 *Assume $\phi_2 > 0$. In homogeneous economies populated by agent i, the long yield is*

$$y^i_\infty = \bar{\rho}^i - \frac{\xi^2}{2}.$$

The long yield of the heterogeneous economies is determined by the lowest long yield which prevails in the homogeneous economies:

$$y^s_\infty = \min_i y^i_\infty = \min_i \bar{\rho}^i - \frac{\xi^2}{2}.$$

The long yield is therefore constant as it must be, if arbitrage is to be precluded (confer the important result in Dybvig, Ingersoll, and Ross (1996)). Again, one part is determined by time preferences and the other by the risk parameters of the model. The value $-\frac{\xi^2}{2}$ is the long-run value of the Vasiček-model, ignoring time preferences. The long-run behavior of rates of time preferences determines the second part of the long yield.

Note that the agent with the lowest long-run rate of time preference sets the long yield. In the present model, this is plausible since it is the agent with the highest long-run interest who has the highest impact on long-run interest rates. The phenomenon, however, is more general. For example, the same result holds in an economy where agents exhibit different degrees of risk aversion Wang (1996). Dybvig, Ingersoll, and Ross (1996) show that the long yield can never fall and is equal to the lowest value the long yield can possibly assume. Here, the lowest possible value for the long yield is determined by the lowest value the long yield can have in one of the homogeneous worlds populated by one type of agent only.

4.2 Logistic Time Preferences

Up to here, the results are general and do not depend on the specific choice of the time preference densities f^i. To illustrate the effect of preferred habitats, it is assumed from now on that there are two agents ($I = 2$) with logistic time preference densities. Agent 1 is impatient, that is his habitat is $h^1 = 0$ and agent 2 has a long habitat, $h^2 > 0$.

[4]By letting the horizon $\bar{T}$ of the economy tend to infinity, one obtains the yields $y^\tau_\cdot$ for all maturities and may therefore take the limit.

Let us first assume, that the rate of growth X is a Brownian motion, that is $\phi_2 = 0$. Then $\mu(\tau, x) = \phi_1 \tau$ and $\sigma^2(\tau) = \xi^2 \tau$. The yield curve is

$$y_t^{t+\tau} = \frac{1}{\tau} \int_t^{t+\tau} \rho(s)_u \, du + \phi_1 - \frac{\xi^2}{2} \, .$$

The part caused by the dynamics of X is constant and only the rates of time preference determine the shape of the yield curve. Interest rates are deterministic.

Let

$$f^i(t) = \frac{\exp(-\gamma^i(t - h^i))}{\left(1 + \exp(-\gamma^i(t - h^i))\right)^2} \qquad i = 1, 2$$

be the time preference densities of the agents and $F^i, i = 1, 2$ the corresponding distribution functions.

Lemma 2 *The rate of time preference of the logistic time preference densities is*

$$\rho^i(t) \;\; = \;\; \gamma^i(2F^i(t) - 1) \, . \tag{17}$$

In homogeneous economies, the short rate is $r_t^i = \rho^i(t) + const$. It is increasing towards the level $\gamma^i + const.$, where γ^i is the long-run rate of time preference of the time preference density as defined in Definition 1. As the short rate is rising, so is the yield curve, since it is the average over the future short rates. As long as only one type of agent is present, the yield curve is always increasing, regardless of the habitat of the agents.

The rate of time preference of an agent with a long habitat is the rate of an agent with a short habitat, shifted by the habitat h. Therefore, if the two long-run rates of time preference do not differ too much, the short rate in the homogeneous economy, where all agents display a short habitat, will be higher than the short rate in the homogeneous economy, where all agents have a long habitat. Hence, the shape of the yield curve does not change if one compares different homogeneous economies.

The shape of the yield curve changes only if different types of agents are present. By Lemma 1, in the heterogeneous economy, the rate of time preference $\rho(s)$ is a time-varying average of the individuals' rates ,

$$\rho(s)_t = \alpha_t \rho_t^1 + (1 - \alpha_t)\rho_t^2 \, ,$$

where $\alpha = \alpha^1$ is the share of consumption of agent 1 in equilibrium.

Two typical pictures of the mixed rate of time preference $\rho(s)$ are shown in Figures 2 and 3. Initially, the impatient agent 1 consumes a large share of the aggregate endowment, as 0 is his habitat. Hence, α is close to 1, and the mixed rate $\rho(s)$ is very close to the impatient agent's rate ρ^1. Later, the roles are reversed

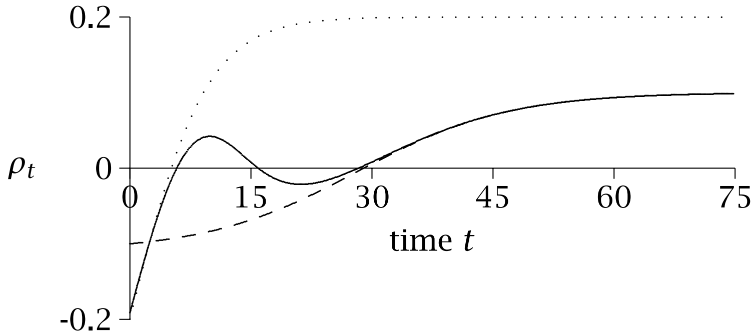

Figure 2: Rates of time preference. Agents have log-utility. Agent 1 is impatient, $h^1 = 0$. The habitat of agent 2 is $h^2 = 30$ and the long-run rates of time preference are $\gamma^1 = 0.2$ and $\gamma^2 = 0.1$. ρ^1 is represented by a dotted line and ρ^2 by a dashed line. The rate of time preference $\rho(\frac{1}{2})$ of the representative agent with $s_1 = \frac{1}{2}$ is represented as a solid line.

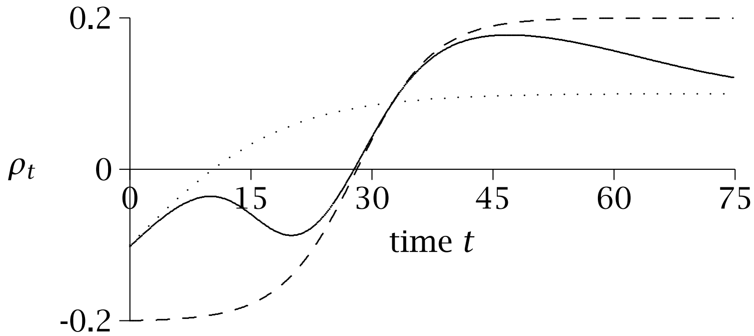

Figure 3: Rates of time preference. The parameter values are as in Figure 2, except for the long-run rates of time preferences, which are reversed: $\gamma^1 = 0.1$ and $\gamma^2 = 0.2$.

and agent 2 starts consuming a larger share of aggregate endowment. Around her habitat h_2, her share $1 - \alpha$ is close to 1, and $\rho(s)$ begins to approach rate ρ^2. If the long-run rates γ^i do not differ too much, this results in a local maximum for $\rho(s)$, since curve ρ^2 lies below curve ρ^1 .

In the long run, the behavior of $\rho(s)$ depends on the long-run rates of time preference γ^i. The limit value of the share

$$\alpha = \frac{s^1 f^1(t)}{s^1 f^1(t) + s^2 f^2(t)} = \left(1 + \left(\frac{s^2 f^2(t)}{s^1 f^1(t)}\right)\right)^{-1}$$

is determined by the limit of the likelihood quotient

$$\frac{f^2(t)}{f^1(t)} \sim \exp\left(-(\gamma^2 - \gamma^1)t\right) \quad (t \to \infty)\,.$$

Hence, α tends to 1 if $\gamma^1 < \gamma^2$ and to 0 if agent 2 is less impatient in the long run, $\gamma^2 < \gamma^1$. In the knife-edge case $\gamma^1 = \gamma^2$, both agents remain in the economy and

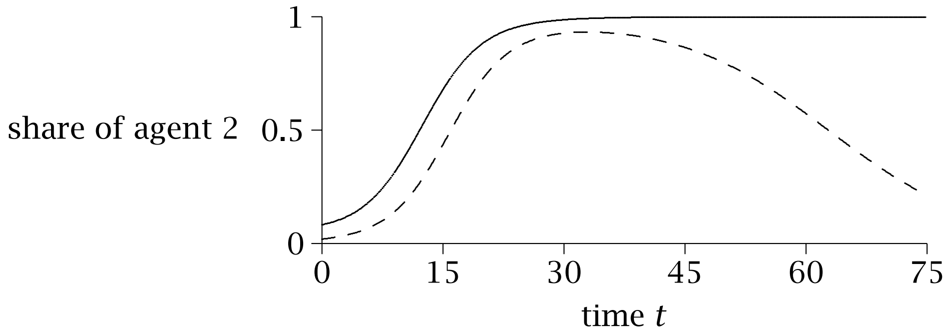

Figure 4: The consumption shares of the agent with the long habitat in the mixed economy. The habitat is in $h = 30$ and the long-run rates of time preference are $\gamma^1 = 0.2$ and $\gamma^2 = 0.1$ for the solid line and $\gamma^1 = 0.1$ and $\gamma^2 = 0.2$ for the dashed line.

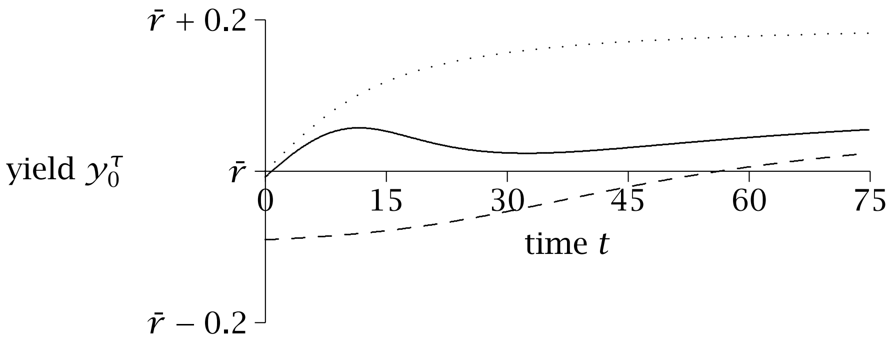

Figure 5: The yield curves. The parameter values are as in Figure 2. In addition, $\phi_2 = 0$ is assumed. The yields fluctuate around the level $\bar{r} = \phi_1 - \xi^2$. The yield curves of the homogeneous economies, y^0 and y^1 are represented by dotted and dashed lines, respectively.

$\alpha_t \to \left(1 + \frac{s^2}{s^1}\right)^{-1}$. Typical curves of the share $1 - \alpha$ of agent 2 are given in Figure 4. The share starts at a relatively low level and increases up to the habitat. If agent 2 is less impatient, she also dominates in the long run. Her share continues to increase towards 1. Otherwise, her share decreases again to the long-run level of zero.

This has the following impact on the mixed rate of time preference: Near 0 the mixed rate $\rho(s)$ follows the rate ρ^1 because agent 1 dominates, in the sense that she consumes a large part of the aggregate endowment. Later, around the habitat, agent 2 takes over. Therefore, the rate of time preference $\rho(s)$ decreases to the lower rate ρ^1, attains a minimum, and then follows the curve ρ^1. If agent 2, with the long habitat, is less impatient in the long run, the mixed rate $\rho(s)$ stays close to ρ^2 after the habitat. This case is depicted in Figure 2. If $\gamma^1 < \gamma^2$, $\rho(s)$ again approaches ρ^1, see Figure 3.

The resulting yield curves, which are the average, up to a constant, of the rates of time preference, are shown in Figures 5 and 6. The averaged curve is naturally smoother than the original one. One sees a certain overshooting of the yields prior

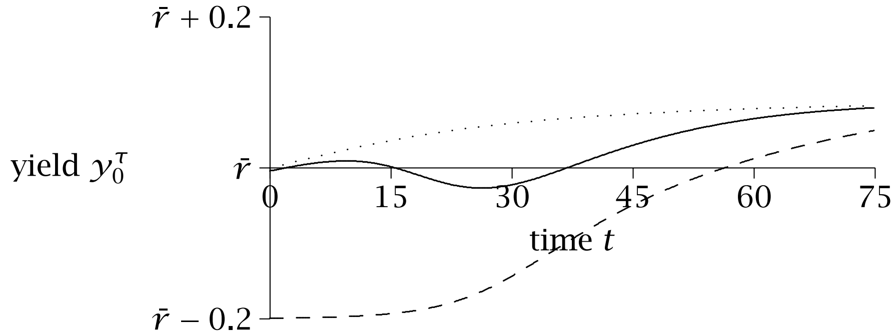

Figure 6: The yield curves. The parameter values are as in Figure 3. In addition, $\phi_2 = 0$ is assumed. The yields fluctuate around the level $\bar{r} = \phi_1 - \xi^2$. The yield curves of the homogeneous economies, y^0 and y^1 are represented by dotted and dashed lines, respectively.

to the habitat and lower yields around the habitat - a hump appears in the yield curve. Therefore, in the heterogeneous economy, where people actually trade in equilibrium, preferred habitats for consumption create humps in the yield curve.

In order to isolate the effects of preferred habitats on the yield curve, the case of deterministic interest rates ($\phi_2 = 0$) has been analyzed. In general, interest rates are stochastic. As already noted, apart the effect of time preferences, a Vasiček-type model is the outcome. It is well known that three types of yield curves, increasing, decreasing and single-humped, are possible in this case. The presence of heterogeneous preferred habitats causes an additive shift of the 'Vasiček'-curve and can lead to additional humps.

5 The Demand for Long-Term Bonds

Preferred habitats produce humps in the yield curve when there are different types of agents present in the economy. The resulting curves seem to indicate that there is a high demand for long-term bonds whose maturity correspond to a specific habitat. One may argue, though, that agents do not trade in bonds, since the market spanned by the stock and the money market account is complete. The question is whether they would buy long-term bonds if needed. For this reason, a variation of the model is studied, in which the risky stock is replaced with a long-term bond. Agents use the bond market to finance their (excess) demand, instead of the stock market, as in the preceding sections.

The variation of the model goes as follows. There is no longer trade in the stock. Instead, agents consider their endowment $e_t^i = s^i K_t$ as an exogenously given income stream. To finance their excess demand $\epsilon_t^i = c^i(s)_t - e_t^i$, they use the bond market. Trade occurs in the money market account β and in the long term bond $B^{\bar{T}}$, whose maturity is the horizon $\bar{T}$ of the economy. To ensure completeness of this financial market, assume $\phi_2 > 0$, which yields stochastic interest rates.

The next theorem is devoted to the dynamics of the long-term bond $B^{\bar{T}} = B^{s,\bar{T}}$.

Theorem 5 *Bond price dynamics are*

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \left(\xi\sigma_t^{\bar{T}} + r_t\right) dt + \sigma_t^{\bar{T}} dW_t \tag{18}$$

where the volatility is

$$\sigma_t^{\bar{T}} = \xi\left(1 - e^{-\phi_2(\bar{T}-t)}\right) .$$

Since the volatility of the long term bond $B^{\bar{T}}$ is strictly positive, the financial market is complete. Agents can therefore finance their equilibrium excess demand ϵ_t^i by trading in bonds. The corresponding portfolio strategy can be stated in closed form.

Theorem 6 *To finance their excess demand ϵ^i, the agents hold*

$$\theta_t^{1i} = \frac{s^i(F^s(t) - F^i(t))}{B_t^{\bar{T}} f^s(t) K_t^{-1}\left(1 - e^{-\phi_2(\bar{T}-t)}\right)} \tag{19}$$

shares of the long-term bond and

$$\theta_t^{2i} = -\frac{s^i(F^s(t) - F^i(t))e^{-\phi_2(\bar{T}-t)}}{\beta_t f^s(t) K_t^{-1}\left(1 - e^{-\phi_2(\bar{T}-t)}\right)} \tag{20}$$

shares of the money market account.

In particular, agent i invests in the long-term bond if and only if she places more weight on the future than the market does:

$$1 - F^i(t) > 1 - F^s(t) .$$

If people use the bond market to finance their demand, long-term bonds are indeed bought by those who have a long habitat. Again, it is worth to be pointed out that the behavior of the agents depends on the preferences of the other agents or the ”market”- you lend long and borrow short if you are more interested in distant payments than the average agent.

6 Concluding Remarks

This article studies a continuous-time pure exchange economy populated by agents with different time preferences. It is shown that preferred habitats influence the behavior of agents and the shape of the term structure when different types of agents are present in the economy. Agents whose time horizon is longer than the average time horizon of the market participants invest in long-term bonds whereas their impatient counterparts do the converse and prefer short-term instruments to long ones. If there is a sharp habitat of a significantly large class of investors, humps appear in the yield curve.

Our results can be seen as supporting the traditional Preferred Habitat Theory formulated by (Modigliani and Sutch May 1966). Their intuition that time preferences should play a major role in term structure theory is confirmed in our model.

Appendix: Proofs

Proof of Theorem 1 : Since the expected utility functionals are time-additive, one can maximize pointwise for every t and ω:

$$\max_{\sum c_t^i = c_t} \sum \lambda^i f^i(t) \log(c_t^i) \,.$$

The solution is

$$x^i(c, f, \lambda)_t \quad = \quad \frac{\lambda^i f^i(t)}{\sum_j \lambda^j f^j(t)}\, c_t \,. \tag{21}$$

It follows that

$$U(c; \lambda) = E \int_0^{\bar{T}} f^\lambda(t) \log c_t \, dt + \int_0^{\bar{T}} \sum \lambda^i \log \left(\frac{\lambda^i f^i(t)}{f^\lambda(t)} \right) dt \,.$$

Since the second summand is independent of the consumption stream c, it plays no role in utility maximization. □

Proof of Lemma 1 : By direct calculation,

$$\begin{aligned} \rho(\lambda)_t \quad &= \quad -\frac{\frac{\partial}{\partial t} f^\lambda(t)}{f^\lambda(t)} \\ &= \quad -\sum \frac{\lambda^i f^i(t)}{f^\lambda(t)} \frac{\frac{\partial}{\partial t} f^i(t)}{f^i(t)} = \sum \alpha_t^i \rho_t^i \,. \end{aligned}$$

□

Proof of Theorem 2 : The consumption market clears since (c^i) is an efficient allocation. The clearing of the financial market is clear from the definitions of θ^i.

It remains to show that the strategy θ^i finances the consumption plan c^i when the stock price S is given by (13). First, c^i is clearly square-integrable, since so is aggregate endowment c, and (3) holds true. Since θ^{i1} is continuous, (4) is satisfied. Set $V^i_t = \theta^{i1}_t S_t$. Then $V^i_0 = s^i S_0$ and the initial budget constraint (6) is satisfied. Moreover, $V^i_t \geq 0$, since S and θ^{i1} are nonnegative, and (8) is also satisfied. For the intertemporal budget constraint (7), note first that from (13)

$$\begin{aligned} dS_t &= \frac{1-F^s(t)}{f^s(t)} dK_t - \frac{(f^s(t))^2 + (1-F^s(t))\frac{\partial}{\partial t} f^s(t)}{(f^s(t))^2} K_t dt \\ &= \frac{1-F^s(t)}{f^s(t)} K_t \left(\mu_t dt + \xi dW_t\right) - K_t dt - \rho(s)_t \frac{1-F^s(t)}{f^s(t)} K_t dt \\ &= \mu_t S_t dt + \xi S_t dW_t - K_t dt + \rho(s)_t S_t dt \,, \end{aligned}$$

hence

$$dS_t + K_t dt = \left(\mu_t + \rho(s)_t\right) S_t dt + \xi S_t dW_t \,. \tag{22}$$

On the other hand, the dynamics of V^i are

$$\begin{aligned} dV^i_t &= \frac{s^i\left(1-F^i(t)\right)}{f^s(t)} dK_t - \frac{s^i f^i(t) f^s(t) + s^i\left(1-F^i(t)\right)\frac{\partial}{\partial t} f^s(t)}{\left(f^s(t)\right)^2} K_t dt \\ &= \frac{s^i\left(1-F^i(t)\right)}{f^s(t)} \left(\mu_t K_t dt + \xi K_t dW_t\right) - \alpha^i_t K_t dt \\ &\quad + \frac{s^i\left(1-F^i(t)\right)}{f^s(t)} \rho(s)_t K_t dt \\ &= \theta^{i1}_t \frac{1-F^s(t)}{f^s(t)} \left((\mu_t + \rho(s)_t) S_t dt + \xi S_t dW_t\right) - c^i_t dt \,, \end{aligned}$$

which by (22) is equal to

$$= \theta^{i1}_t \frac{1-F^s(t)}{f^s(t)} \left(dS_t + K_t dt\right) - c^i_t dt \,.$$

Hence, θ^i finances c^i.

The asset market is complete because S has nondegenerate volatility $\xi > 0$. □

Remark 1 *The above proof of the equilibrium relations does not show how the equilibrium prices and strategies are to be found. In general, this is done by using the Negishi method. First, one computes an Arrow-Debreu equilibrium. Here , it is given by the efficient allocation $x(c; s)$ and the consumption price $\psi(s)_t = f^s(t) c_t^{-1}$, which is the marginal felicity of the associated representative agent. Using $\psi(s)$*

as a state-price in the sense of Duffie (1992), one obtains the candidate stock price via $S_t = \psi(s)_t^{-1} E\left[\int_t^{\bar{T}} K_u \psi_u du \middle| \mathcal{F}_t \right]$ *and the candidate for the short rate via* $r_t = -E\left[\frac{d\psi(s)_t}{\psi(s)_t} \middle| \mathcal{F}_t \right]$.

Proof of Theorem 3 : By the usual Euler formula,

$$\begin{aligned} B_t^{s,T} &= E\left[\frac{f^s(T) K_T^{-1}}{f^s(t) K_t^{-1}} \middle| \mathcal{F}_t \right] \\ &= \frac{f^s(T)}{f^s(t)} \exp(\frac{\xi^2}{2}(T-t)) E\left[\exp(-(X_T - X_t)) \middle| \mathcal{F}_t \right]. \end{aligned}$$

One therefore has to compute the conditional Laplace transform of a normal random variable, which yields

$$B_t^{s,T} = \frac{f^s(T)}{f^s(t)} \exp\left(\frac{\xi^2}{2}(T-t) - \mu(T-t, X_t) + \frac{1}{2}\sigma^2(T-t) \right)$$

□

Proof of Theorem 4 : Since $\mu(\tau, x)$ and $\sigma^2(\tau)$ are bounded, one has

$$\frac{\mu(\tau, X_t)}{\tau} - \frac{\xi^2}{2} - \frac{\sigma^2(\tau)}{2\tau} \to -\frac{\xi^2}{2}$$

as τ tends to infinity. $\rho(i)_t$ is a continuous function which converges to $\bar{\rho}^i$, thus

$$\frac{1}{\tau} \int_t^{t+\tau} \rho(i)_u \, du \to \bar{\rho}^i .$$

Because of $f^j(t) \sim e^{-\bar{\rho}^j t}$, the consumption share of agent i satisfies

$$\alpha_t^i \sim \frac{s^i e^{-\bar{\rho}^i t}}{\sum s^j e^{-\bar{\rho}^j t}} = s^i \left(\sum s^j e^{(\bar{\rho}^i - \bar{\rho}^j)t} \right)^{-1} .$$

If $\bar{\rho}^i > \min_j \bar{\rho}^j$, then $e^{(\bar{\rho}^i - \bar{\rho}^j)t} \to \infty$ for some j, hence $\alpha_t^i \to 0$. Agents whose long run rate of time preference $\bar{\rho}^i$ is not minimal do not contribute to the long run value of $\rho(s)$. Hence, $\rho(s)_t \to \min_j \bar{\rho}^j$. This in turn implies

$$\frac{1}{\tau} \int_t^{t+\tau} \rho(s)_u \, du \to \min_j \bar{\rho}^j$$

and $y_\infty^s = \min_j y_\infty^j$ follows. □

Proof of Lemma 2 : F^i solves the differential equation

$$f^i(t) = \gamma^i F^i(t)(1 - F^i(t)) . \tag{23}$$

The logarithmic derivative is therefore

$$\begin{aligned}
\frac{\partial}{\partial t}\log f^i(t) &= \frac{\partial}{\partial t}\left(\log \gamma^i + \log F^i(t) + \log(1-F^i(t))\right)\\
&= \frac{f^i(t)}{F^i(t)} - \frac{f^i(t)}{1-F^i(t)}
\end{aligned}$$

and by applying (23), one obtains

$$= \gamma^i(1-F^i(t)) - \gamma^i F^i(t) = \gamma^i(1-2F^i(t)).$$

□

Proof of Theorem 5 : By (16), the bond price is a function of the growth rate X_t and time t:

$$B_t^{\bar{T}} = \exp\left(-\int_t^{\bar{T}} \rho_u du - \mu(\bar{T}-t, X_t) + \frac{\xi^2}{2}(\bar{T}-t) + \frac{\sigma^2(\bar{T}-t)}{2}\right),$$

with $\mu(\bar{T}-t,x) = \left(\frac{\phi_1}{\phi_2} - x\right)\left(1-e^{-\phi_2(\bar{T}-t)}\right)$ and $\sigma^2(\bar{T}-t) = \frac{\xi^2\left(1-e^{-\phi_2(\bar{T}-t)}\right)}{2\phi_2}$. The partial derivatives are

$$\begin{aligned}
\frac{\partial}{\partial t}B_t^{\bar{T}} &= B_t^{\bar{T}}\left(\rho_t + (\phi_1 - \phi_2 X_t)\, e^{-\phi_2(\bar{T}-t)} - \frac{\xi^2}{2} - \frac{\xi^2}{2}e^{-2\phi_2(\bar{T}-t)}\right)\\
\frac{\partial}{\partial x}B_t^{\bar{T}} &= B_t^{\bar{T}}\left(1-e^{-\phi_2(\bar{T}-t)}\right)\\
\frac{\partial^2}{\partial x^2}B_t^{\bar{T}} &= B_t^{\bar{T}}\left(1-e^{-\phi_2(\bar{T}-t)}\right)^2.
\end{aligned}$$

Itôs formula yields

$$\begin{aligned}
\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} &= \left(\rho_t + (\phi_1 - \phi_2 X_t)\, e^{-\phi_2(\bar{T}-t)} - \frac{\xi^2}{2} - \frac{\xi^2}{2}e^{-2\phi_2(\bar{T}-t)}\right)dt\\
&\quad + \left(1-e^{-\phi_2(\bar{T}-t)}\right)dX_t + \frac{1}{2}\left(1-e^{-\phi_2(\bar{T}-t)}\right)^2 d[X]_t\\
&= \left(\rho_t + (\phi_1 - \phi_2 X_t)\, e^{-\phi_2(\bar{T}-t)} - \frac{\xi^2}{2} - \frac{\xi^2}{2}e^{-2\phi_2(\bar{T}-t)}\right)dt\\
&\quad + \left(1-e^{-\phi_2(\bar{T}-t)}\right)(\phi_1 - \phi_2 X_t)dt + \xi\left(1-e^{-\phi_2(\bar{T}-t)}\right)dW_t\\
&\quad + \frac{\xi^2}{2}\left(1-e^{-\phi_2(\bar{T}-t)}\right)^2 dt\\
&= \left(\rho_t + \phi_1 - \phi_2 X_t - \frac{\xi^2}{2} - \frac{\xi^2}{2}e^{-2\phi_2(\bar{T}-t)} + \frac{\xi^2}{2}\left(1-e^{-\phi_2(\bar{T}-t)}\right)^2\right)dt\\
&\quad + \xi\left(1-e^{-\phi_2(\bar{T}-t)}\right)dW_t\\
&= \left(\rho_t + \phi_1 - \phi_2 X_t - \xi^2 e^{-\phi_2(\bar{T}-t)}\right)dt + \xi\left(1-e^{-\phi_2(\bar{T}-t)}\right)dW_t\\
&= (r_t + \xi\sigma_t^{\bar{T}})dt + \sigma_t^{\bar{T}}dW_t,
\end{aligned}$$

with $\sigma_t^{\bar{T}} = \xi\left(1 - e^{-\phi_2(\bar{T}-t)}\right)$. □

Proof of Theorem 6 : Let $V_t^i = \theta_t^{1i} B_t^{\bar{T}} + \theta_t^{2i}\beta_t$ be the value of the portfolio formed by $(\theta^{1i}, \theta^{2i})$. One has $V_0^i = s^i$, hence the initial budget constraint is satisfied. In order to check that the strategy $(\theta^{1i}, \theta^{2i})$ finances the excess demand ϵ^i, one has to show that the intertemporal budget constraint

$$dV_t^i = \theta_t^{1i} dB_t^{\bar{T}} + \theta_t^{2i} d\beta_t - \epsilon_t^i dt$$

holds true. Now,

$$V_t^i = \frac{s^i\left(F^s(t) - F^i(t)\right)}{f^s(t)} K_t\,,$$

which yields

$$\begin{aligned} dV_t^i &= \frac{s^i\left(F^s(t) - F^i(t)\right)}{f^s(t)} dK_t + K_t d\frac{s^i\left(F^s(t) - F^i(t)\right)}{f^s(t)} \\ &= V_t^i dX_t + \frac{s^i(f^s(t) - f^i(t))f^s(t) - s^i(F^s(t) - F^i(t))\frac{\partial}{\partial t}f^s(t)}{\left(f^s(t)\right)^2} K_t dt \\ &= V_t^i\left(\phi_1 - \phi_2 X_t\right)dt + V_t^i \xi dW_t + (s^i K_t - \alpha_t^i K_t)dt + \rho_t^s V_t^i dt \\ &= V_t^i \xi dW_t + V_t^i\left(\rho_t^s + \phi_1 - \phi_2 X_t\right)dt - \epsilon_t^i dt\,. \end{aligned} \qquad (24)$$

Note that

$$\theta_t^{1i} = \frac{V_t^i}{B_t^{\bar{T}}\left(1 - e^{-\phi^2(\bar{T}-t)}\right)}$$

and

$$\theta_t^{2i} = \frac{-V_t^i e^{-\phi^2(\bar{T}-t)}}{B_t^{\bar{T}}\left(1 - e^{-\phi^2(\bar{T}-t)}\right)}\,.$$

Therefore,

$$\begin{aligned} \theta_t^{1i} dB_t^{\bar{T}} + \theta_t^{2i} d\beta_t - \epsilon_t^i dt &= V_t^i \xi dW_t + V_t^i \frac{\xi\sigma_t^{\bar{T}} + r_t}{1 - e^{-\phi^2(\bar{T}-t)}} dt + \frac{V_t^i e^{-\phi^2(\bar{T}-t)}}{1 - e^{-\phi^2(\bar{T}-t)}} r_t dt - \epsilon_t^i dt \\ &= V_t^i \xi dW_t + V_t^i\left(\xi^2 + r_t\right)dt - \epsilon_t^i dt\,. \end{aligned} \qquad (25)$$

Because of $\xi^2 + r_t = \rho_t^s + \phi_1 - \phi_2 X_t$, (24) and (25) are equal and the proof is done. □

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