Princeton University

# CONSISTENCY PROBLEMS FOR JUMP-DIFFUSION MODELS

LI CHEN ERHAN BAYRAKTAR H. VINCENT POOR

Abstract. In this paper we examine a consistency problem for a multi-factor jump diffusion model. First we bridge a gap between a jump-diffusion model and a generalized Heath-Jarrow-Morton (HJM) model, and bring a multi-factor jump-diffusion model into the HJM framework. By applying the drift condition for a generalized arbitrage-free HJM model, we derive the general consistency condition for a jump-diffusion model. Then we consider the case that the forward rate function has a separable structure, and obtain a specific version of the general consistency condition. In particular, we provide the necessary and sufficient condition for a jump-diffusion model to be affine, which generalizes the result in [10]. Finally we discuss the Nelson-Siegel type of forward curve structure, and give the necessary and sufficient condition for the consistency of this class of models in the jump-diffusion case.

## 1. Introduction of the arbitrage-free Condition for Generalized HJM Models

The purpose of this paper is to study the consistency problems for multi-factor jump-diffusion term structure models of interest rates. Previous works ([4], [12], [13], [14]) focus on the diffusion models without considering jumps. Because the jump-diffusion models usually provide a better characterization of the randomness in the financial market than diffusion models (see [1], [19]), there is an upsurge in modeling the dynamics of interest rates with jumps (e.g. [3], [11], [16], [20]). Therefore it is necessary to clarify the consistency conditions for a jump-diffusion model.

Consider a Heath-Jarrow-Morton model (HJM, [15]) in the presence of a marked point process. The dynamics of the forward curve can be given by

$$dr(t,T) = \alpha(t,T)dt + \sigma(t,T)dB_t + \int_J \rho(t,T,y)\mu(dt,dy), \tag{1.1}$$

where $B$ is a Brownian motion and $\mu(dt,dy)$ is a random measure on $\mathbb{R}_+ \times J$ with the compensator $\nu(t,dy)dt$. With the Musiela's parameterization ([17]), (1.1) can be rewritten as

$$dr_t(\tau) = \left(\frac{\partial}{\partial \tau} r_t(\tau) + \alpha(t,t+\tau)\right)dt + \sigma(t,t+\tau)dB_t + \int_J \rho(t,t+\tau,y)\mu(dt,dy),$$

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This version April 17, 2003 (first draft)
Contacts: {lichen, ebayrakt, poor}@princeton.edu.
We are indebted to Erhan Cinlar and Damir Filipović for the helpful discussion.

where $\tau = T - t$ denotes the time to maturity. Therefore the price of a zero-coupon bond can be given by

$$P(t,\tau) = e^{-\int_0^\tau r_t(u)du}. \tag{1.2}$$

A measure $\mathbb{Q}$ is said to be a local martingale measure if the discounted bond price

$$D(t,\tau) = \frac{P(t,\tau)}{e^{-\int_0^t r_s(0)ds}}$$

is a $\mathbb{Q}$-local martingale, for each $\tau \in \mathbb{R}_+$. It is well known that the existence of an equivalent local martingale measure implies the absence of arbitrage (see e.g. [8]).

Under regularity conditions, Björk et.al. [5] gives the following lemma for the arbitrage-free condition of the generalized HJM model defined by (1.1).

**Lemma 1.1.** *An equivalent local martingale measure exists, if and only if and the forward rate dynamics under this measure specified by (1.1) satisfies the following relation for* $\forall\ 0 \le t < T$.

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \int_J \rho(t,T,y)e^{-\int_t^T \rho(t,u,y)du}\nu(t,dy). \tag{1.3}$$

*Proof.* See [5], Theorem 3.13 and Proposition 3.14.

Lemma 1.1 gives the drift condition for a generalized HJM model, which generalizes the traditional arbitrage-free condition for diffusion HJM models. This result provides us a way to derive the consistency conditions for a multi-factor model with jumps.

The remainder of the paper is organized as follows. In Section 2, we bring a jump-diffusion model into the generalized HJM framework, and derive the consistency condition for the coefficient functions of the model. In Section 3, we discuss a class of separable term structure models. In particular, the affine term structure is investigated and the sufficient and necessary conditions for a jump-diffusion model to be affine are derived. A typical non-separable term structure model, namely Nelson-Siegal term structure is examined in Section 4. Brief concluding remarks are made in Section 5.

### 1.1. Basic Notation.

First we introduce the notation that will be frequently used in the paper as shown in Table 1.

## 2. The Consistency Conditions for Multi-factor Jump-Diffusion Models

Consider a multi-factor term structure model $\mathcal{M}$ with forward rates of the form:

$$r_t(\tau) = G(\tau, X_t),$$

and the state process $X_t$ follows a general Itô's process with state space $(D,\mathcal{D})$ under a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ satisfying the usual conditions such that

$$dX_t = b(X_t)dt + c(X_t)dW_t + \int_E N(dt,dz)k(X_{t^-},z), \quad X_0 = x, \tag{2.1}$$

Table 1. Summary of Notation

| Notation | Implications |
|---|---|
| $X$ | A time-homogeneous Itô's process with jumps |
| $(D, \mathcal{D})$ | The state space $D := \mathbb{R}^n$ and its Borel $\sigma$-algebra $\mathcal{D} := \mathcal{B}(D)$ |
| $C(D)$ | The Banach space of continuous functions on $D$ |
| $bD$ | The Banach space of bounded Borel-measurable functions on $D$ |
| $C_b(D)$ | The Banach space consisting of all bounded continuous functions on $D$ |
| $C^k(D)$ | The space of $k$-times differentiable functions $f$ on the interior of $D$ such that all partial derivatives of $f$ up to order $k$ are continuous |
| $\mathbb{R}_+$, $(\mathbb{R}_{++})$ | The set of positive (strictly positive) real numbers |
| $\mathcal{G}$ | The infinitesimal generator of $X$ |
| $Leb$ | The Lebesgue measure on $\mathbb{R}_+$ |
| $\langle \cdot, \cdot \rangle$ | The inner product in the vector space $\mathbb{R}^n$ |
| $\nabla f$ | The gradient of the function $f$ on $D$ |

where $G : \mathbb{R}_+ \times D \mapsto \mathbb{R}$ is a deterministic function, $W_t$ is an $n$-dimensional standard $\mathbb{P}_x$-Brownian motion and $N(\cdot, \cdot)$ is a Poisson random measure independent of $W$ with mean measure as $Leb \times \psi$ on $\mathbb{R}_+ \times E$. Here it is assumed that $(\mathcal{F}_t)$ is generated by $W$ and $N$.

**Definition 2.1.** *A multi-factor jump-diffusion model $\mathcal{M}$ is said be consistent if the induced dynamics of the forward rates $G$ satisfies the relation (1.3).*

**Assumption 2.1.** *For the sake of simplicity for analysis, it is assumed that*

- *The function $G \in C^{1,2}(\mathbb{R}_+ \times D)$;*
- *The functions $b : D \mapsto \mathbb{R}^n$ and $c : D \mapsto \mathbb{R}^{n \times n}$ are deterministic and continuous, and $k : D \times E \mapsto \mathbb{R}^n$ is deterministic and bounded continuous;*
- *$\psi$ is a finite measure on $E$ such that $\psi(E) < \infty$;*
- *(2.1) has a unique strong solution $X_t(x)$ in $D$.*

By Itô's formula, the dynamics of the forward rates $G(\cdot, X_t)$ can be derived as follows.

$$
\begin{aligned}
(2.2) \qquad G(\cdot, X_t) \quad &= \quad G(\cdot, x_0) + \sum_{i=1}^{n} \int_0^t \partial_{x_i} G(\cdot, X_{s^-}) b_i(X_{s^-}) ds \\
&+ \quad \sum_{i=1}^{n} \int_0^t \partial_{x_i} G(\cdot, X_{s^-}) \sum_{j=1}^{n} c_{i,j}(X_{s^-}) dW_t^j \\
&+ \quad \sum_{i,j=1}^{n} \int_0^t a_{i,j}(X_{s^-}) \partial_{x_i} \partial_{x_j} G(\cdot, X_{s^-}) ds \\
&+ \quad \int_0^t \int_E N(ds, dz) [G(\cdot, X_{s^-} + k(X_{s^-}, z)) - G(\cdot, X_{s^-})],
\end{aligned}
$$

where $a(X_t) = \frac{1}{2} c(X_t) c(X_t)^T$ is a semi-positive definite matrix, which denotes the diffusion.

**Lemma 2.1.** *Under Assumption 2.1, the infinitesimal generator $\mathcal{G}$ of $X$ has the following generic form:*

$$\begin{aligned}(2.3)\quad \mathcal{G}f(x) &= \sum_{i,j=1}^{n} a_{i,j}(x)\partial_{x_i}\partial_{x_j} f(x) + \sum_{i=1}^{n} b_i(x)\partial_{x_i} f(x) \\ &\quad + \int_D [f(x+\xi) - f(x)]L(x, d\xi), \quad \forall\ x \in D,\ f \in C_b^2(D).\end{aligned}$$

*where $L(\cdot,\cdot)$ is a Markov kernel on $(D, \mathcal{D})$ which is defined by*

$$L(x, B) = \psi\{z \in E : k(x, z) \in B\}, \quad \forall\ B \in \mathcal{D}. \tag{2.4}$$

*Furthermore, if $L(x, D) > 0$, for each $x \in D$, then $L(\cdot,\cdot)$ can be rewritten as*

$$L(x, d\xi) = \lambda(x)Q(x, d\xi), \tag{2.5}$$

*where $\lambda(\cdot)$ represents the jump intensity of the process $X$ and $Q(\cdot,\cdot)$ represents the probability kernel of the jump magnitude.*

*Proof.* By Itô's formula, it is easy to deduce that for $\forall\ x \in D,\ f \in C_b^2(D)$

$$\begin{aligned}(2.6)\quad \mathbb{E}_x f(X_t) &= f(x) + \sum_{i=1}^{n} \int_0^t \mathbb{E}_x \left[\partial_{x_i} f(X_{s^-}) b_i(X_{s^-}) ds\right] \\ &+ \sum_{i,j=1}^{n} \int_0^t \mathbb{E}_x \left[a_{i,j}(X_{s^-})\partial_{x_i}\partial_{x_j} f(X_{s^-}) ds\right] \\ &+ \mathbb{E}_x \left[\int_0^t \int_E N(ds, dz)(f(X_{s^-} + k(X_{s^-}, z)) - f(X_{s^-}))\right].\end{aligned}$$

By the definition of the Poisson random measure $N$, the last term on the right hand side (RHS) of (2.6) can be written as

$$\mathbb{E}_x \left[\int_0^t \int_E ds\psi(dz)(f(X_{s^-} + k(X_{s^-}, z)) - f(X_{s^-}))\right] = \int_0^t ds \int_D (f(x+\xi) - f(x))L(x, d\xi),$$

where $L(\cdot,\cdot)$ is defined in (2.4). Therefore we can rewrite (2.6) as

$$\begin{aligned}(2.7)\quad P_t f(x) &= f(x) + \sum_{i=1}^{n} \int_0^t \mathbb{E}_x \left[\partial_{x_i} f(X_{s^-}) b_i(X_{s^-}) ds\right] \\ &+ \sum_{i,j=1}^{n} \int_0^t \mathbb{E}_x \left[a_{i,j}(X_{s^-})\partial_{x_i}\partial_{x_j} f(X_{s^-}) ds\right] \\ &+ \int_0^t ds \mathbb{E}_x \left[\int_D (f(X_{s^-} + \xi) - f(X_{s^-}))L(X_{s^-}, d\xi)\right],\end{aligned}$$

where $(P_t)$ denotes the transition semigroup of the Markov process $X$. By (2.7) and the bounded convergence theorem, we derive that

$$\begin{aligned}\tilde{\mathcal{G}}f(x) &= \lim_{t\downarrow 0}\frac{P_tf(x)-f(x)}{t}\\ &= \sum_{i,j=1}^{n} a_{i,j}(x)\partial_{x_i}\partial_{x_j}f(x)+\sum_{i=1}^{n} b_i(x)\partial_{x_i}f(x)\\ &\quad +\int_D [f(x+\xi)-f(x)]L(x,d\xi),\quad \forall\, x\in D,\ f\in C_b^2(D),\end{aligned}$$

where $\tilde{\mathcal{G}}$ is called the weak generator of $X$. Since $X$ is a Feller process, by Lemma 31.7 in [21], we have $\mathcal{G}=\tilde{\mathcal{G}}$, which proves the first argument of Lemma 2.1. If $L(x,D)>0$, for each $x\in D$, since $\psi(E)<\infty$, then $0<L(x,D)<\infty$. Therefore by simply defining $\lambda(x)=L(x,D)$ and $Q(x,d\xi)=\frac{L(x,d\xi)}{\lambda(x)}$, we complete the proof.

**Remark 2.1.** *Because $X$ is a Feller process, the process*

$$f(X_t)-f(x)-\int_0^t \mathcal{G}f(X_{s^-})ds$$

*is a $\mathbb{P}_x$ martingale for each $f\in C_b^2(D)$. In particular, if $\mathcal{G}f=0$, then $f(X_t)$ is a martingale.*

**Lemma 2.2.** *If we define $N'(\cdot,\cdot)$ as a random measure on $(\mathbb{R}_+,D)$ given by*

$$N'([0,t],B,\omega)=\int_0^t\int_E N(ds,dz,\omega)1_{\{z:\ k(X_{s^-}(\omega),z)\in B\}},\quad \forall\, B\in\mathcal{D}, \tag{2.8}$$

*then $N'(dt,d\xi)$ has a compensator $dtL(X_{t^-},d\xi)$.*

*Proof.* For each $B\in\mathcal{D}$ and $s,t\in\mathbb{R}_+$, we have

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{s+t}(N'(du,B)-duL(X_u,B))|\mathcal{F}_s\right] &= \int_0^s (N'(du,B)-duL(X_{u^-},B))\\ &\quad +\mathbb{E}_{X_s}\left[\int_s^{s+t}(\int_E N(du,dz)1_{\{z:\ k(X_{u^-},z)\in B\}}\right.\\ &\quad\quad -duL(X_{u^-},B))]\\ &= \int_0^s (N'(du,B)-duL(X_{u^-},B)).\end{aligned}$$

The second derivation comes from the definition of the filtration , the independence condition for Poisson random measure $N$ and (2.4). By monotone class theorem, we can deduce that for each $f\in\mathcal{D}_+$,

$$M_t^f=\int_0^t\int_D [N'(du,d\xi)f(\xi)-duL(X_{s-},d\xi)f(\xi)].$$

is a martingale, and since $L(X_{t^-},\cdot)$ is predictable, therefore we finish our proof.

Now since we derive the dynamics of the induced forward rates, by fitting (2.2) into Lemma 1.1, we can derive the following theorem of the consistency condition for a jump diffusion model.

**Theorem 2.1.** *Under Assumption 2.1, a multi-factor jump-diffusion model $\mathcal{M}$ is consistent, if and only if for each $(\tau, x) \in \mathbb{R}_+ \times D$, given the forward rates curve $G(\tau, X_t)$, the coefficients $a(x)$, $b(x)$ and $L(x, \cdot)$ satisfy the following constraint.*

$$
\begin{aligned}
-\partial_\tau G(\tau, x) &+ \sum_{i=1}^n \partial_{x_i} G(\tau, x) b_i(x) + \sum_{i,j=1}^n a_{i,j}(x) \partial_{x_i} \partial_{x_j} G(\tau, x) \\
&= 2 \sum_{i,j=1}^n a_{i,j}(x) \partial_{x_i} G(\tau, x) \int_0^\tau \partial_{x_j} G(u, x) du \\
&\quad - \int_D \delta_0(x, \tau, \xi) L(x, d\xi),
\end{aligned} \tag{2.9}
$$

*where* $\delta_0(x, \tau, \xi) = [G(\tau, x + \xi) - G(\tau, x)] e^{-\int_0^\tau (G(u, x+\xi) - G(u, x)) du}$.

*Proof.* Since $r_t(\tau) = G(\tau, X_t)$, by the Musiela's parameterization and Lemma 1.1, it follows from (2.2) that

$$
\begin{aligned}
\sum_{i=1}^n \int_0^t \partial_{x_i} G(\tau, X_{s^-}) b_i(X_{s^-}) ds &+ \sum_{i,j=1}^n \int_0^t a_{i,j}(X_{s^-}) \partial_{x_i} \partial_{x_j} G(\tau, X_{s^-}) ds \\
&= 2 \sum_{i,j=1}^n \int_0^t a_{i,j}(X_{s^-}) \partial_{x_i} G(\tau, X_{s^-}) \int_0^\tau \partial_{x_j} G(u, X_{s^-}) du ds \\
&\quad - \int_0^t ds \int_D \delta_0(X_{s^-}, \tau, \xi) L(X_{s^-}, d\xi) + \int_0^t \partial_\tau G(\tau, X_s) ds.
\end{aligned} \tag{2.10}
$$

Under Assumption 2.1, we notice the there exist at most finite jumps for the process $X$ during the time 0 to $t$, for each $t > 0$, therefore without loss of generality, we don't distinguish between $X_{s^-}$ and $X_s$ in (2.10), and replace both by $x$, therefore we can obtain (2.9) by differentiating on both sides of (2.10) with respect to $t$.

**Assumption 2.2.** *Now it is further assumed that the jump intensity $\lambda(\cdot)$ is a continuous function on $D$, and the jump kernel $Q(x, \cdot)$ defined by (2.5) is independent of $x$, which means that*

$$
L(x, d\xi) = \lambda(x) Q(d\xi). \tag{2.11}
$$

**Remark 2.2.** *The models with the jump measure defined by (2.11) include two specific classes: pure diffusion models ($\lambda(\cdot) = 0$), and the models driven by Lévy processes or more precisely, compound Poisson processes (the intensity $\lambda$ is a constant).*

Now we can derive the following characterization theorem for a multi-factor jump-diffusion model $\mathcal{M}$.

**Theorem 2.2.** *Under Assumptions 2.1 and 2.2, the jump measure $Q(\cdot)$ can be freely chosen only subject to the regularity condition:*

$$
\int_D \delta_0(x, \tau, \xi) Q(d\xi) < \infty, \quad \forall\ (\tau, x) \in \mathbb{R}_+ \times D. \tag{2.12}
$$

*If the forward rate curve $G(\cdot, x)$ satisfies the condition that the functions $\partial_{x_i} G(\cdot, x)$, $\partial_{x_i} \partial_{x_j} G(\cdot, x)$ and $\int_D \delta_0(x, \cdot, \xi) Q(d\xi)$ are linearly independent for all $1 \leq i, j \leq n$*

*and for all $x$ in some dense set $D_0 \subset D$, then the drift $a(\cdot)$, diffusion $b(\cdot)$ and jump intensity $\lambda(\cdot)$ of the state process $X$ are uniquely determined by $G$.*

*Proof.* Set $M = (n+1) + (n+1)n/2$ and choose a sequence $0 \leq \tau_1 < \tau_2 <, ..., < \tau_M$, such that by the linear independence condition, we know the $M \times M$ matrix with $i$th row constructed by

$$\left(\partial_{x_1} G(\tau_i, x), ..., \partial_{x_1} G(\tau_i, x), \partial_{x_1}\partial_{x_1} G(\tau_i, x), ..., \partial_{x_n}\partial_{x_n} G(\tau_i, x), \int_D \delta_0(x, \tau_i, \xi) Q(d\xi)\right),$$

for each $i = 1, 2, ..., n$, is invertible. Therefore $a(x)$, $b(x)$ and $\lambda(x)$ are uniquely determined by the arbitrage-free condition (2.9), for each $x \in D_0$. Because of the continuity of $a$, $b$ and $\lambda$, the extensions to the state space $D$ are unique. This completes the proof of Theorem 2.2.

Now by applying Theorem 2.2, we can discuss several specific cases. For simplicity, throughout the following sections, it is assumed that the space $E$ is $\mathbb{R}_+$, and Assumptions 2.1 and 2.2 are satisfied.

## 3. Separable Term Structure Models

In this section, we consider the forward rate curve $G(\tau, x)$ has a separable structure[1]

$$G(\tau, x) = \sum_{k=1}^{m} h_k(\tau)\phi_k(x). \tag{3.1}$$

where the function $h_k : \mathbb{R}_+ \mapsto \mathbb{R}$ is a deterministic function for each $k = 1, ..., m$. Therefore according to Theorem 2.1, we have the following consistency conditions.

**Proposition 3.1.** *A separable term structure model (3.1) is consistent, if and only if the following equation holds.*

$$\begin{aligned} \sum_{k=1}^{m}(h_k(\tau) - h_k(0))\phi_k(x) \quad = \quad & \sum_{i=1}^{n} \Gamma_i(\tau, x) b_i(x) \\ & + \sum_{i,j=1} a_{i,j}(x)(\Lambda_{i,j}(\tau, x) - \Gamma_i(\tau, x)\Gamma_j(\tau, x)) \\ & + \lambda(x)\Psi(H(\tau), x), \quad \forall\ (\tau, x) \in \mathbb{R}_+ \times D, \end{aligned} \tag{3.2}$$

[1]This class of models has been investigated by Filipović [14] in the diffusion case.

*where for* $\forall\ 1 \le i,j \le n$ *and* $v \in \mathbb{R}^m$,

$$\begin{aligned}
\Gamma_i(\tau,x) &:= \sum_{k=1}^{m} H_k(\tau)\frac{\partial \phi_k(x)}{\partial x_i},\\
\Lambda_{i,j}(\tau,x) &:= \sum_{k=1}^{m} H_k(\tau)\frac{\partial^2 \phi_k(x)}{\partial x_i \partial x_j},\\
\Psi(v,x) &:= \int_D \left(1 - e^{\sum_{k=1}^m v_k(\phi_k(x+\xi)-\phi_k(x))}\right) Q(d\xi)\\
\text{with} \quad & H_i(\tau) := \int_0^\tau h_i(u)du,\\
& H(\tau) := (H_1(\tau), H_2(\tau), ..., H_m(\tau))^T,\\
& \phi(x) := (\phi_1(x), \phi_2(x), ..., \phi_m(x))^T.
\end{aligned}$$

*Moreover, if we assume the functions*

$$\Psi(H(\cdot),x), \quad \Gamma_i(\cdot,x), \quad \Lambda_{i,j}(\cdot,x) - \Gamma_i(\cdot,x)\Gamma_j(\cdot,x), \quad \forall\ 1 \le i,j \le n$$

*are linearly independent for all* $x \in D_0$*, then* $a$*,* $b$*,* $\lambda$ *are uniquely determined by* $h$*,* $\phi$ *and the measure* $Q$*.*

*Proof.* All the results can be derived from Theorem 2.1 and 2.2 directly.

Now on setting

$$\begin{aligned}
B_k(x) &:= \sum_{i=1}^{n} b_i(x)\frac{\partial \phi_k(x)}{\partial x_i} + \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 \phi_k(x)}{\partial x_i \partial x_j},\\
A_{k,l}(x) = A_{l,k}(x) &:= \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial \psi_k(x)}{\partial x_i}\frac{\partial \psi_l(x)}{\partial x_j}, \quad \forall 1 \le k,l \le m,
\end{aligned}$$

it follows from (3.2) that

$$\begin{aligned}
\sum_{k=1}^{m}(h_k(\tau)-h_0(0))\phi_k(x) &= \sum_{k=1}^{m} H_k(\tau)B_k(x) - \sum_{k,l=1}^{m} H_k(\tau)H_l(\tau)A_{k,l}(x)\\
&\quad +\lambda(x)\Psi(H(\tau),x), \quad \forall\,(\tau,x) \in \mathbb{R}_+ \times D.
\end{aligned} \tag{3.3}$$

Therefore once we know $(a(\cdot), b(\cdot), \lambda(\cdot), Q(\cdot), (h_i(0)_{0\le i\le n})$ and $\phi(\cdot)$, we can derive the regularity conditions for $H(\cdot)$. This is clarified by the following proposition.

**Proposition 3.2.** *Suppose that the functions* $\phi_1, ..., \phi_m$ *are linearly independent. Then the coefficient functions* $H_1, ..., H_m$ *solve a system of ODEs*

$$\frac{dH_k(\tau)}{d\tau} = R_k(H(\tau)), \quad 1 \le k \le m, \tag{3.4}$$

*where* $R_k$ *has the form*

$$R_k(v) = \theta_k + \langle \beta_k, v\rangle - \langle \alpha_k v, v\rangle + \gamma_k(v) \quad v \in \mathbb{R}^m \tag{3.5}$$

*with* $\theta_k = h_k(0)$*.*

*Proof.* Choose $m$ mutually distinct points $(x^l)_{1\le l\le m}$ in $D$ such that the $m \times m$ matrix $\left(\phi_k(x^l)\right)$ is invertible. Then by multiplying the inverse matrix on both sides of (3.3) and setting $\theta_k = h_k(0)$, we can obtain (3.5). It is easy to see $R_i(\cdot)$ has the form of (3.9) by appropriately setting $\alpha_k$, $\beta_k$ and $\gamma_k$.

**Remark 3.1.** *Notice that giving $(a(\cdot), b(\cdot), \lambda(\cdot), Q(\cdot), (h_i(0))_{0\le i\le n})$ is equivalent to giving a multi-factor short rate model, therefore Proposition 3.2 provides a way to solve the forward rate structure by applying the consistent requirements. Moreover it implies a necessary condition for a model to be consistent: the existence of the solution of the ODE system (3.4).*

3.1. **Affine Term Structure Models.** Now we will take a look at the simplest class of models, namely the affine term structure models, where the forward rate curve $G$ is defined as

$$G(\tau, x) = h_0(\tau) + \sum_{i=1}^{n} h_i(\tau)x_i, \quad \forall\ (\tau, x) \in \mathbb{R}_+ \times D, \tag{3.6}$$

Therefore if we set

$$\phi_0(0) = 1, \quad \phi_i(x) = x_i, \quad 1 \le i \le n,$$

according to Theorem 3.1, we can derive the following consistent condition for affine term structure models:

$$\begin{aligned} h_0(\tau) - h_0(0) + \sum_{i=1}^{n} x_i(h_i(\tau) - h_i(0)) \quad &= \quad \sum_{i=1}^{n} H_i(\tau)b_i(x) - \sum_{i,j=1}^{n} a_{i,j}(x)H_i(\tau)H_j(\tau) \\ &\quad +\lambda(x)(1 - \Psi(H(\tau)), \end{aligned} \tag{3.7}$$

where $H(\cdot) = (H_1(\cdot), ..., H_n(\cdot))^T$ and $\Psi(v) = \int_D e^{-\langle v,\xi\rangle} Q(d\xi)$, which is the Laplace transform of the probability measure $Q$.

According to Proposition 3.1, we have the following results.

**Proposition 3.3.** *If the functions $a(x)$, $b(x)$, $\lambda(x)$ are affine, and $Q(\cdot)$ satisfies (2.12), then the term structure of forward rates $G(\cdot, x)$ is affine. On the other hand, if $G(\tau, x)$ is an affine function with respect to $x$ as defined by (3.6), and*

$$H_1, ..., H_n, H_1H_1,\ H_1H_2, ..., H_nH_n,\ 1 - \Psi(H)$$

*are linearly independent functions, then the functions $a(\cdot)$, $b(\cdot)$ and $\lambda(\cdot)$ are affine.*

*Proof.* The first part is well established by many literatures (e.g. [11]). Basically one can show that given a jump-diffusion model with the drift, diffusion and jump intensity being affine functions, the price of a zero-coupon bond price has an exponential affine form and the coefficient functions solve a series of generalized Riccati equations, and thus the term structure is affine. The second part can be deduced by the solution property of the linear equation (3.7). Since the left hand side of (3.7) is affine and the coefficient matrix is invertible and independent of $x$, therefore the solution $a(x)$, $b(x)$ and $\lambda(x)$ must be affine functions of $x$. This completes the proof.

Proposition 3.3 provides a necessary and sufficient condition for a jump-diffusion model to be affine, which generalizes the result proposed by Duffie and Kan in [10] for diffusion models.

Now it is assumed that $a(\cdot)$, $b(\cdot)$, $\lambda(\cdot)$ and $Q(\cdot)$ are given as affine functions, and $(h_i(0))_{0\le i\le n}$ are known. The following corollary can be directly derived from Proposition 3.2.

**Corollary 3.1.** *Under the consistency condition (3.7), if $a(\cdot)$, $b(\cdot)$, $\lambda(\cdot)$ and $Q(\cdot)$ satisfy Assumption 2.1 and 2.2, the coefficient functions $(H_i(\cdot))_{0\le i\le n}$ can be determined from a system of ODEs shown as follows. For $\forall\ k = 0, 1, ...n$,*

$$\frac{dH_k(\tau)}{d\tau} = R_k(H(\tau)), \tag{3.8}$$

*where $R_k$ has the form*

$$R_k(v) = \theta_k + \langle \beta_k, v\rangle + \langle \alpha_k v, v\rangle + \gamma_k \int_D \left(1 - e^{-\sum_{j=1}^n v_j \xi_j}\right) Q(d\xi), \tag{3.9}$$

*where $\theta_k = h_k(0)$.*

**Remark 3.2.** *The above system of ODEs (3.8) and (3.9) is called generalized Riccati equations (GREs). The existence and uniqueness of GREs have been studied in* [9].

Generally speaking[2], the only consistent polynomial term structure models[3] are the affine term structure models in the jump-diffusion case. Actually under some regularity conditions, Filipović [14] has demonstrated that affine and quadratic Gaussian models are the only two possible consistent models that can produce separable polynomial term structure in the diffusion case. Chen and Poor [7] show that in order to retain the quadratic term structure, the state process $X_t$ can only follow a so-called quadratic Gaussian process that does't allow jumps. Therefore, it implies that importing jumps into the underlying state process of a term structure model may yield a better fit to real market curves, whereas the model can not yield an analytically tractable forward curve structure. Due to this factor, several alternative approaches have been adopted by researchers. One way is to mix affine jump diffusion models with quadratic Gaussian models originally proposed by Piazzesi [20], in which the jumps are linked to the announcement of target interest rates by the Federal Reserve. Another approach is to apply a special Lévy process to drive the dynamics of the state variables (see [2]). Then pricing bonds and other derivatives can be achieved by approximating (see [6], [16]).

## 4. The Nelson-Siegel Curves

In this section, we discuss a typical non-separable term structure model, namely the Nelson-Siegel curve family (see [18]). This curve family has been studied in [12], and it turns out there does not exist a non-trivial consistent diffusion model with the Nelson-Siegel forward curve. This is not an inspiring result in view of the widespread applications of the Nelson-Siegel family in financial industry.

[2]one can always find some pathological examples to produce polynomial term structure models, e.g. see [14].

[3]This means that $\phi_k(x)$ defined in (3.1) is a polynomial function of $x$, for each $1 \le k \le m$.

The Nelson-Siegel forward curves can be given by the form:

$$G(\tau,x) = x_1 + (x_2 + x_3\tau)e^{-x_4\tau}, \quad (x_1,x_2,x_3)\in\mathbb{R}^3, \quad x_4>0. \tag{4.1}$$

Let us redefine $D := \mathbb{R}^3 \times \mathbb{R}_{++}$ in this section. By (4.1), it is straightforward to deduce that

$$\partial_\tau G(\tau,x) = (x_3 - x_4x_2 - x_3x_4\tau)e^{-x_4\tau}, \tag{4.2}$$

$$\nabla_x G(\tau,x) = (1, \quad e^{-x_4\tau}, \quad \tau e^{-x_4\tau}, \quad -\tau(x_2+x_3\tau)e^{-x_4\tau})^T, \tag{4.3}$$

$$\nabla_x(\partial_{x_4} G(\tau,x)) = e^{-x_4\tau}(0, \quad -x_4, \quad -x_4\tau, \quad \tau^2(x_2+x_3\tau))^T, \tag{4.4}$$

$$\text{and} \quad \frac{\partial^2}{\partial x_i \partial x_j} G(\tau,x) = 0, \quad 1\le i,j\le 3. \tag{4.5}$$

By applying (4.2)-(4.5) into (2.9), we can derive the consistency condition. Notice if we move the terms $2\sum_{i,j=1}^n a_{i,j}(x)\partial_{x_i}G(\tau,x)\int_0^\tau \partial_{x_j}G(u,x)du$ to the LHS of (2.9), for the Nelson-Siegel curve, (2.9) can be generally written as

$$q_0(\tau,x) + q_1(\tau,x)e^{-x_4\tau} + q_2(\tau,x)e^{-2x_4\tau} = \lambda(x)\int_D \delta_0(x,\tau,\xi)Q(d\xi), \tag{4.6}$$

where $q_0(\tau,\cdot)$, $q_1(\tau,\cdot)$ and $q_2(\tau,\cdot)$ are polynomial functions of $\tau$. Since the consistency condition requires (4.6) to be true for all $(\tau,x)\in\mathbb{R}_+\times D$, therefore we have the following proposition.

**Proposition 4.1.** *Under some regularity conditions, a non-trivial jump-diffusion model with Nelson-Siegel forward curve is consistent if and only if the expectation of $\delta_0$ under the measure $Q$ has the form:*

$$\delta(\tau,x) = (p_0(\tau,x) + p_1(\tau,x)e^{-x_4\tau} + p_2(\tau,x)e^{-2x_4\tau}), \tag{4.7}$$

*where $p_0(\tau,\cdot)$, $p_1(\tau,\cdot)$ and $p_2(\tau,\cdot)$ are polynomial functions of $\tau$ with the degree $d_0\le 1$, $d_1\le 3$ and $d_2\le 4$, respectively and*

$$\begin{aligned}\delta(\tau,x) &= \lambda(x)\mathbb{E}^Q[\delta_0(x,\tau,\xi)] \\ &= \lambda(x)\int_D \delta_0(x,\tau,\xi)Q(d\xi).\end{aligned}$$

*Proof.* By the consistent equation (4.6), first we need to prove that if $\lambda(x)=0$, there does not exist a non-trivial consistent model. Since $\lambda(x)=0$ implies that

$$q_0(\tau,x) = q_1(\tau,x) = q_2(\tau,x) = 0, \quad \forall(\tau,x)\in\mathbb{R}_+\times D. \tag{4.8}$$

Therefore by a careful calculation, one can show that (4.8) implies the diffusion $a(x)$ is zero. A complete proof can be found in [12]. Therefore there does not exist a non-trivial model such that $\lambda(x)=0$, which completes the proof of the necessary condition. Now we assume that (4.7) is true, and assuming that

$$p_0(\tau,x) = p_0^0(x) + p_1^0(x)\tau \tag{4.9}$$

$$p_1(\tau,x) = \sum_{i=0}^3 p_i^1\tau^i \tag{4.10}$$

$$p_2(\tau,x) = \sum_{i=0}^3 p_i^2\tau^i. \tag{4.11}$$

Then by letting $a_{12}(\cdot) = a_{13}(\cdot) = a_{23}(\cdot) = 0$. Then the other parameters $a_{i,j}(\cdot)$ in $a$ and $b$ are uniquely determined by $p_0(\tau, x)$, $p_1(\tau, x)$ and $p_2(\tau, x)$. To be more precise, the further calculation will give the following results.

- $p_4^2(\cdot)$ uniquely determines $a_{4,4}(\cdot)$;
- $p_3^2(\cdot)$ uniquely determines $a_{3,4}(\cdot)$;
- $p_2^2(\cdot)$ and $p_1^2(\cdot)$ uniquely determine $a_{2,4}(\cdot)$ and $a_{3,3}(\cdot)$;
- $p_0^2(\cdot)$ uniquely determines $a_{2,2}(\cdot)$;
- $p_3^1(\cdot)$ uniquely determines $a_{1,4}(\cdot)$;
- $p_2^1(\cdot)$ uniquely determines $b_4(\cdot)$;
- $p_1^1(\cdot)$ uniquely determines $b_3(\cdot)$;
- $p_0^1(\cdot)$ uniquely determines $b_2(\cdot)$;
- $p_1^0(\cdot)$ uniquely determines $a_{1,1}(\cdot)$;
- $p_0^0(\cdot)$ uniquely determines $b_1(\cdot)$.

The requirement of $a$ to be a semi-positive definite matrix imposes the regularity condition on the coefficient functions of polynomials $p_0(\tau, x)$,$p_1(\tau, x)$ and $p_2(\tau, x)$. This completes the proof.

## 5. Conclusion

Motivated by the discussion of consistency problems for diffusion models, this article investigated this issue in the jump-diffusion case. Different from the diffusion case, here we have four elements to consider: the drift, diffusion, jump intensity and jump size measure. This difference seems to give us more freedom for making a model to be consistent. We have shown that the jump size measure $Q$ can be chosen freely, and once given the jump size measure, under the regularity condition, the drift, diffusion and jump intensity are uniquely determined by the consistent requirement.

For separable term structure models, in addition to the consistency condition given by Proposition 3.1, we also derive a necessary condition by the existence of the solution for the ODEs defined in (3.4). This indicates that once given the short rate model and the functions $\phi(\cdot)$, you can solve the term structure of the forward rates by these ODEs. Therefore the price of a zero-coupon bond can be derived by (1.2).

It has been demonstrated that there does not exist a non-trivial diffusion model with the Nelson-Siegel-type forward curve. However, because of the freedom of choosing the jump size measure, an appropriate choice of $Q$ will possibly produce the structure of $\delta(\tau, x)$ as defined in (4.7) and (4.9)-(4.11). This possibility perhaps can be interpreted by the inherent limitation of the Nelson-Siegel family. Because the structure of a Nelson-Siegel curve is too simple to capture a daily forward curve, there exists a large discontinuity in the estimated time series of the state processe $X = (X_1, X_2, X_3, X_4)$. This implies that the dynamics of $X$ comprises jumps, which can not be captured by diffusion models. Therefore, heuristically speaking, this is why that the diffusion models can not yield Nelson-Siegel curves, whereas jump-diffusion models can.

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