Monopoly Market with Externality: an Analysis with Statistical Physics and Agent Based Computational Economics

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Abstract

In this paper, we explore the effects of localised externalities introduced through interaction structures upon the properties of the simplest market model: the discrete choice model with a single homogeneous product and a single seller (the monopoly case).

The resulting market is viewed as a complex interactive system with a communication network. Our main goal is to understand how generic properties of complex adaptive systems can enlighten our understanding of the market mechanisms when individual decisions are inter-related. To do so we make use of an ACE (Agent based Computational Economics) approach, and we discuss analogies between simulated market mechanisms and classical collective phenomena studied in Statistical Physics.

More precisely, we consider discrete choice models where the agents are subject to local positive externality. We compare two extreme special cases, the McFadden (McF) and the Thurstone (TP) models. In the McF model the individuals' willingness to pay are heterogeneous, but remain fixed. In the TP model, all the agents have the same homogeneous part of willingness to pay plus an additive random (logistic) idiosyncratic characteristic. We show that these models are formally equivalent to models studied in the Physics literature, the McF case corresponding to a 'Random Field Ising model' (RFIM) at zero temperature, and the TP case to an Ising model at finite temperature in a uniform (non random) external field. From the physicist's point of view, the McF and the TP models are thus quite different: they belong to the classes of, respectively, 'quenched' and 'annealed' disorder, which are known to lead to very different aggregate behaviour. This paper explores some consequences for market behaviour.

Considering the optimisation of profit by the monopolist, we exhibit a new 'first order phase transition': if the social influence is strong enough, there is a regime where, if the mean willingness to pay increases, or if the production costs decreases, the optimal solution for the monopolist jumps from a solution with a high price and a small number of buyers, to a solution with a low price and a large number of buyers.

1 Introduction

Following Kirman [10, 11], we view the market as a complex interactive system with a communication network. We consider a market with discrete choices [1], and explore

the effects of localised externalities (social influence) on its properties. We focus on the simplest case: a single homogeneous product and a single seller (monopoly). On the demand side, customers are assumed to be myopic and non strategic. The only cognitive agent in the process is the monopolist, who determines the price in order to optimise his profit. We obtain interesting and complex phenomena, that arise due to the interactions between heterogeneous agents.

Agents are assumed to have idiosyncratic willingness-to-pay (IWP) which are described by means of random variables. We consider two different models: on the one hand, the IWP are randomly chosen and remain fixed, on the other hand, the IWP present independent temporal fluctuations around a fixed (homogeneous) value. The former case is known by physicists as a model with *quenched disorder*, whereas the latter corresponds to an *annealed disorder*. In both cases we assume that the heterogeneous preferences of the agents are drawn from a same (logistic) distribution. The equilibrium states of the two models generally differ, except in the special case of homogeneous interactions with complete connectivity. In this special situation, which corresponds to the *mean-field approximation* in physics, the expected aggregate steady-state is the same in both models.

In the following, we first present the demand side, and then consider the optimisation problem left to the monopolist, who is assumed to know the demand model and the distribution of the IWP over the population, but cannot observe the individual (private) values.

2 Simple models of discrete choice with social influence

We consider a set Ω_N of N agents with a classical linear IWP function [17]. Each agent $i \in \Omega_N$ either buys ($\omega_i = 1$) or does not buy ($\omega_i = 0$) one unit of the single given good of the market. A rational agent chooses ω_i in order to maximise his *surplus function* V_i:

$$\max_{\omega_i \in \{0,1\}} V_i = \max_{\omega_i \in \{0,1\}} \omega_i (H_i + \sum_{k \in \vartheta_i} J_{ik} \omega_k - P), \tag{1}$$

where P is the price of one unit and H_i represents the idiosyncratic preference component. Some other agents k, within a subset $\vartheta_i \subset \Omega_N$, such that $k \in \vartheta_i$, hereafter called neighbours of i, influence agent i's preferences through their own choices ω_k. This social influence is represented here by a weighted sum of these choices. Let us denote J_{ik} the corresponding weight *i.e.* the marginal social influence on agent i, of the decision of agent $k \in \vartheta_i$. When this social influence is assumed to be positive ($J_{ik} > 0$), it is possible, following Durlauf [4], to identify this external effect as a *strategic complementarity* in agents' choices [3].

For simplicity we consider here only the case of *homogeneous* influences, that is, identical positive weights $J_{ik} = J_\vartheta$ and identical neighbourhood structures ϑ of size n, for all the agents. That is,

$$J_{ik} = J_\vartheta \equiv J/n > 0 \quad \forall i \in \Omega_N, \ k \in \vartheta_i \tag{2}$$

2.1 Psychological *versus* economic points of view

Depending on the nature of the idiosyncratic term H_i, the discrete choice model (1) may represent two quite different situations. Following the typology proposed by Anderson *et al.* [1], we distinguish a "psychological" and an "economic" approach to individual choices. Within the psychological perspective (Thurstone [23]), the utility has a *stochastic* aspect because "there are some qualitative fluctuations from one occasion to the next... for a given stimulus" (this point of view will be referred to hereafter as the *TP-model*). On the contrary, for McFadden [12] each agent has a willingness to pay that is *invariable* in time, but may differ from one agent to the other. In a "risky" situation the seller cannot observe each specific IWP, but knows its statistical distribution over the population (we call this perspective the *McF-model*). Accordingly, the TP and the McF perspectives differ in the nature of the individual willingness to pay.

In the TP model, the idiosyncratic preference has two sub-components: a constant deterministic term H (the same for all the agents), and a time- and agent-dependent additive term $\epsilon_i(t)$ ($H_i = H + \epsilon_i$). The $\epsilon_i(t)$ are i.i.d. random variables of zero mean; in the simulations they are refreshed at each time step (asynchronous updating). Agent i decides *to buy* according to the conditional probability

$$P(\omega_i = 1 | z_i(P, H)) = \mathcal{P}(\epsilon_i > z_i(P, H)) = 1 - F(z_i(P, H)), \tag{3}$$

with

$$z_i(P, H) = P - H - J_\vartheta \sum_{k \in \vartheta_i} \omega_k, \tag{4}$$

where $F(z_i) = \mathcal{P}(\epsilon_i \leq z_i)$ is the cumulative distribution of the random variables ϵ_i. In the standard TP model, the agents make repeated choices, and the time varying components $\epsilon_i(t)$ are drawn at each time t from a logistic distribution with zero mean, and variance $\sigma^2 = \pi^2/(3\beta^2)$:

$$F(z) = \frac{1}{1 + \exp(-\beta\, z)}. \tag{5}$$

In the McF model, the private idiosyncratic terms H_i are randomly distributed over the agents, but remain fixed during the period under consideration. There are no temporal variations: the ϵ_i are strictly zero. In analogy with the TP model, it is useful to introduce the following notation: $H_i = H + \theta_i$, and to assume that the θ_i are logistically distributed with zero mean and variance $\sigma^2 = \pi^2/(3\beta^2)$ over the population. This assumption implies:

$$\lim_{N \to \infty} \frac{1}{N} \sum_i \theta_i = 0 \ \text{ and } \ \lim_{N \to \infty} \frac{1}{N} \sum_i H_i = H. \tag{6}$$

Thus, the correspondence between models is better the larger the number of agents, and is strict only in the limit of an infinite population. In practice, the simulations presented in the following sections show that the theoretical predictions are already verified for population sizes of the order of some hundreds.

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For a given distribution of choices in the neighbourhood ϑ_i, and for a given price, the customer's behaviour is deterministic. An agent buys if:

$$\theta_i > P - H - J_\vartheta \sum_{k \in \vartheta_i} \omega_k. \tag{7}$$

2.2 'Annealed' *versus* 'quenched' disorder

Since we have assumed isotropic (hence symmetric) interactions, there is a strong relation between these models and Ising type models in Statistical Mechanics, which is made explicit if we change the variables $\omega_i \in \{0, 1\}$ into *spin* variables $s_i \in \{\pm 1\}$ through $\omega_i = \frac{1+s_i}{2}$. All the expressions in the present paper can be put in terms of either s_i or ω_i using this transformation. In the following we keep the encoding $\omega_i \in \{0, 1\}$.

In physics, the TP model corresponds to a case of *annealed disorder*. Having a time varying random idosyncratic component is equivalent to introducing stochastic dynamics for the Ising spins. In the particular case where $F(z)$ in (3) is the logistic distribution, we obtain an Ising model in a uniform (non random) external field $H - P$, at temperature $T = 1/\beta$. The McF model has fixed heterogeneity; it is analogous to a *Random Field Ising Model* (RFIM) at zero temperature, that is, with deterministic dynamics. The RFIM belongs to the class of *quenched disorder* models: the values H_i are equivalent to random time-independent local fields. The assumption of *strategic complementarity* in economics corresponds to having ferromagnetic couplings in physics (that is, the interaction J between Ising spins is positive). In this case, the spins s_i (and consequently the choices ω_i) all tend to take the same value. This "agreement" among agents may be broken by the influence of the heterogeneous external fields H_i. Due to the random distribution of H_i over the network of agents, the resulting organisation may be complex. Thus, from the physicist's point of view, the TP and the McF models are quite different: uniform field and finite temperature in the former, random field and zero temperature in the latter. The properties of disordered systems have been and still are the subject of numerous studies in statistical physics. They show that annealed and quenched disorder can lead to very different behaviours.

The TP model is well understood. In the case where the agents are situated on the vertices of a 2-dimensional square lattice, and have four neighbours each, there is an exact analysis of the model for $P = P_n$ (the unbiased case, (33), see below) due to Onsager [14]. Even if an analytical solution of the optimization problem (1) for an arbitrary neighbourhood does not exist, the *mean field* analysis gives approximate results that become exact in the limiting situation where every agent is a neighbour of every other agent (*i.e.* all the agents are interconnected through weights (2)). On the contrary, the properties of the McF model are not yet fully understood. A number of important results have been published in the physics literature since the first studies of the RFIM by Aharony and Galam [6, 7] (see also [5], [21]). Several variants of the RFIM have already been used in the context of socio-economic modelling [8, 15, 2, 24].

2.3 Static *versus* dynamic points of view

Hereafter, we restrict our investigation to the McF model in the "global" externality case, considering *homogeneous interactions* and *full connectivity*, which is equivalent to the *mean field* model in physics. Within this general framework, we are interested in two different perspectives. First we consider a static point of view computing the set of possible economic equilibria, solving the equality between demand and supply. This will allow us to analyse in section 4 the optimal strategy of the monopolist, as a function of the model parameters.

Next (section 5) we consider the market's dynamics assuming myopic and non strategic agents: based on the observation of the behaviour of the other agents at time $t-1$, each agent decides at time t to buy or not to buy. We show that, in general, the market converges towards the static equilibria of the preceding section, except for a precise range of the parameter values where interesting static as well as dynamic features are observed.

These two kinds of analysis correspond in Physics to the study of the thermal equilibrium properties within the *statistical ensemble* framework on the one hand, and the out of equilibrium dynamics (which, in most cases, approaches the static equilibrium through a relaxation process) on the other hand, respectively.

3 Aggregate demand

As discussed in the preceding section, we consider the full connectivity case in the limit of a very large number of agents. *The penetration rate η defined as the fraction of customers that choose to buy at a given price*, $\eta \equiv lim_{N\to\infty} \sum_{k=1}^{N} \omega_k/N$, can be approximated by the social influence term of the agents' surplus function (eq. (1)): $\eta \approx \sum_{k\in\vartheta} \omega_k/(N-1)$. In the large N limit, equation (7) may thus be replaced by

$$\theta_i > P - H - J\,\eta. \tag{8}$$

For the following discussion it is convenient to identify the *marginal customer*, indifferent between buying and not buying. Let $H_m = H + \theta_m$ be his IWP. This *marginal customer* has zero surplus ($V_m = 0$), that is:

$$\theta_m = P - H - J\,\eta \equiv z. \tag{9}$$

Thus, an agent i buys if $\theta_i > \theta_m$, and does not buy otherwise. Equations (8) and (9) allow us to obtain η as a fixed point:

$$\eta = 1 - F(z) \tag{10}$$

where $z = P - H - J\,\eta$ depends on P, H, and η. Note that this (macroscopic) equation is formally equivalent to the (microscopic) individual expectation that $\omega_i = 1$ in the TP case (3). Using the logistic distribution for θ_i, we have:

$$\eta = \frac{1}{1 + \exp{(+\beta\,z)}} \tag{11}$$

Equation (10) allows us to define η as an implicit function of the price through

$$\Phi(\eta, P) \equiv \eta(P) + F(P - H - J \, \eta(P)) - 1 = 0. \tag{12}$$

The shape of this (implicit) demand curve can be evaluated using the implicit derivative theorem:

$$\frac{d\eta(P)}{dP} = \frac{-\partial\Phi/\partial P}{\partial\Phi/\partial\eta} = \frac{-f(z)}{1 - J \, f(z)} \tag{13}$$

where $f(z) = dF(z)/dz$ is the probability density.

Since for a given P, equation (11) defines the penetration rate η as a fixed-point, inversion of this equation gives an *inverse demand function*:

$$P^d(\eta) = H + J \, \eta + \frac{1}{\beta} \ln \frac{1 - \eta}{\eta} \tag{14}$$

At given values of β, J and H, for most values of P, (11) has a unique solution $\eta(P)$.

However for $\beta J > 4$, there is a range of prices

$$P_1(\beta J, \beta H) < P < P_2(\beta J, \beta H) \tag{15}$$

such that, for any P in this interval, (11) has two stable solutions and an unstable one. The limiting values P_1 and P_2 are the particular price values obtained from the condition that eq. (11) has one degenerate solution:

$$\eta = 1 - F(z), \text{ and } \frac{d(1 - F(z))}{d\eta} = 1.$$

This gives $\beta J \eta(1 - \eta) = 1$ (together with eq. (11)). This equation has two solutions, $\eta_2 \le 1/2 \le \eta_1$,

$$\eta_i = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{\beta J}}\right] \; ; \; i \in \{1, 2\} \tag{16}$$

Note that η_i depends only on βJ. Then, the limiting prices P_i are equal to the inverse demands associated with these values η_i, that is, from (14):

$$P_i = H + J\eta_i + \frac{1}{\beta} \ln[\frac{1 - \eta_i}{\eta_i}] \; ; \; i \in \{1, 2\}. \tag{17}$$

Note that these limiting prices are not necessarily positive.

It is interesting to note that the set of equilibria is the same as what would be obtained if agents had rational expectations about the choices of the others: if every agent had knowledge of the distribution of the H_i, he could compute the equilibrium state compatible with the maximisation of his own surplus, taking into account that every agent does the same, and take his decision (to buy/not to buy) accordingly. For $\beta J < 4$ every agent could thus anticipate the value of η to be realized at the price P, and make his choice according to (8). For $\beta J > 4$, however, if the price is set within the interval $[P_1, P_2]$, the agents are unable to anticipate which equilibrium will be realized, even though the one with the largest value of η should be preferred by every one (it is the Pareto dominant equilibrium).

4   Supply side

On the supply side, we consider a monopolist facing heterogeneous customers in a risky situation where the monopolist has perfect knowledge of the functional form of the agents' surplus functions and the maximisation behaviour (1). He also knows the statistical (logistic) distribution of the idiosyncratic part of the reservation prices (H_i). In the special case of "global" externality, where the interactions are the same for all customers, as in equation (2), as just seen, the TP model and the McF one have the same equilibrium states. Indeed, the monopolist cannot observe any *individual* reservation price. He observes only the result of the individual choices (to buy or not to buy). As a result, the conditional probability for an agent taken at random by the monopolist to be a customer is formally equivalent in both cases. Accordingly, hereafter we limit ourselves to the case where the demand follows the McF model, in the limit of full connectivity. The social influence on each individual decision is then close to $J\,\eta$, and the fraction of customers η is observed by the monopolist. That is, for a given price, the expected number of buyers is given by equation (10).

4.1   Profit maximisation

Let C be the monopolist cost for each unit sold, so that

$$p \equiv P - C \tag{18}$$

is his profit *per unit*. Since $P - H = (P - C) - (H - C)$, defining

$$h \equiv H - C, \tag{19}$$

we can rewrite z in (10) as:

$$z = p \; - \; h \; - \; J\,\eta. \tag{20}$$

Hereafter we write all the equations in terms of p and h.

Since each customer buys a single unit of the good, the monopolist's total expected profit is $p\,N\,\eta$. Thus, in this mean field case, the monopolist's profit is proportional to the total number of customers. He is left with the following maximisation problem:

$$p_M = \arg\max_p \Pi(p), \tag{21}$$

where $N\,\Pi(p)$ is the expected profit, with:

$$\Pi(p) \equiv p\,\eta(p), \tag{22}$$

and $\eta(p)$ is the solution to the implicit equation (10). If there is no discontinuity in the demand curve $\eta(p)$ (hence for $\beta J \leq 4$), p_M satisfies $d\Pi(p)/dp = 0$, which gives $d\eta/dp = -\eta/p$ at $p = p_M$. Using the implicit derivative (13), we obtain at $p = p_M$:

$$\frac{f(z)}{1 - Jf(z)} = \frac{\eta}{p}, \tag{23}$$

where z, defined in (20), has to be taken at $p = p_M$.

Because the monopolist observes the demand level η, we can use equation (10) to replace $1 - F(z)$ by η. After some manipulations, equation (23) gives an inverse supply function $p^s(\eta)$:

$$p^s(\eta) = \frac{1}{\beta(1-\eta)} - J\eta \tag{24}$$

We obtain p_M and η_M as the intersection between supply (24) and demand (14):

$$p_M = p^s(\eta_M) = p^d(\eta_M), \tag{25}$$
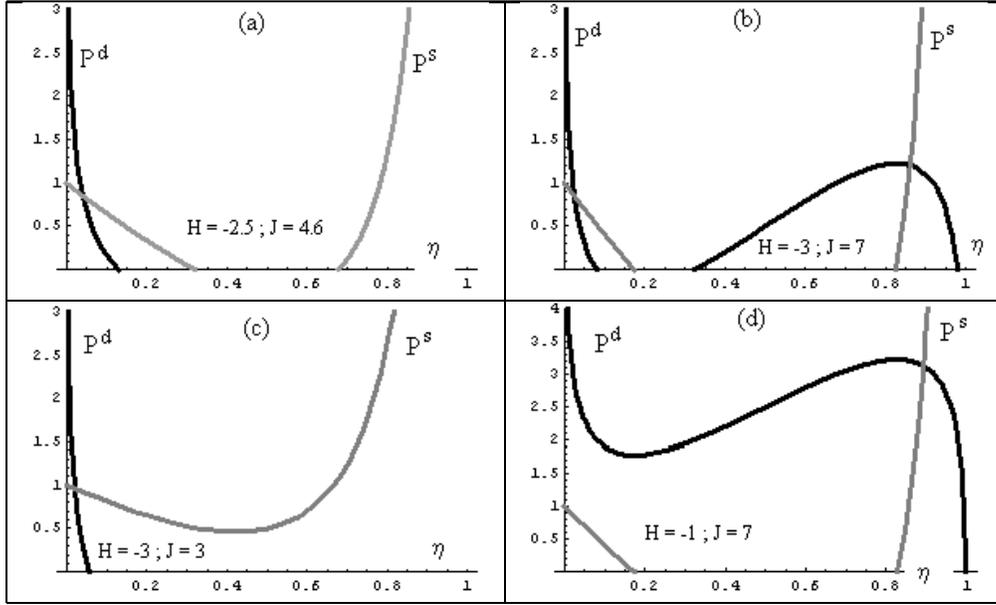
where $p^d \equiv P^d - C$.

Figure 1:   Inverse supply and demand curves $P^s(\eta)$ and $P^d(\eta)$, for different values of H and J ($\beta = 1$), corresponding to the points labelled (a) to (d) in the phase diagram (figure 3), all in the parameter region where the average willingness to pay is negative (which means weak consumer interest for this commodity). The social effects and the inhomogeneity allow us to find equilibrium prices (the intersection between the demand (black) and the supply (grey) curves).
(a) corresponds to the *coexistence region* between two local market equilibria in (figure 3), but one of them, corresponding to a negative price solution is not relevant (not shown). (b) corresponds to the *coexistence region*; in this case, the high η market equilibrium is the optimal one. (c) lies in the region with only one market equilibrium, with few buyers (small η). (d) corresponds to a large social effect, the single market equilibrium has large η and a high price.

The (possibly local) maxima of the profit are the solutions of (25) for which

$$\frac{d^2\Pi}{dp^2} < 0. \tag{26}$$

After some manipulations, one gets the expression for the second derivative of the profit:

$$\frac{d^2\Pi}{dp^2} = -2\frac{\eta}{p}\left[1 + \frac{2\eta - 1}{2\beta p(1-\eta)^2}\right] \tag{27}$$

It is clear from this expression that the solutions with $\eta > 1/2$ are local maxima. For $\eta < 1/2$, condition (26) reads

$$\frac{1 - 2\eta}{2\beta p(1 - \eta)^2} < 1. \tag{28}$$

Making use of the above equations, this can also be rewritten as

$$2\beta J\eta(1 - \eta)^2 < 1. \tag{29}$$

For $\beta J > 4$, the monopolist has to find $p = p_M$ which realises the programme :

$$p_M : \max\{\Pi_-(p_-^M), \Pi_+(p_+^M)\} \tag{30}$$

$$p_+^M = arg \max_p \Pi_+(p) \equiv p \, \eta_+(p), \tag{31}$$

$$p_-^M = arg \max_p \Pi_-(p) \equiv p \, \eta_-(p) \tag{32}$$

where the subscripts $+$ and $-$ refers to the solutions of (11) with a fraction of buyers being more, respectively less, than $1/2$.

To illustrate the behaviour of these equations, we represent several examples of inverse supply and demand curves in figure 1. These curves correspond to different market configurations, obtained for different distributions of the IWP, for the value $\beta = 1$ of the logistic parameter. In the absence of externality ($J = 0$, dashed lines) the case $h = 2$ corresponds to a strong positive average of the population's IWP. The population is neutral for $h = 0$, and $h = -1$ means that, on average, the agents are not willing to buy. In all three cases, the supply curve shrinks for increasing values of the externality weight J. When the penetration rate is low, the monopolist must lower the price to attract new customers: the second term in the inverse supply function (24) dominates over the first one, and the supply curve bottoms-out. Conversely, when the penetration rate is high, the positive effects of the externality are dominant and the supply curve grows faster, more than proportionally to the price decrease. In the same figures we represented the inverse demand and supply curves for the threshold value $\beta J = 4$ (in the figure, $\beta = 1$, $J = 4$), beyond which the demand curve has a minimum at: $\eta = 0.5$. For larger values of the external effect, the supply curve is discontinuous, with a decreasing component at low penetration rates and an increasing component for large penetration rates. For $h = -1.9$ (strong aversion against the product), equation (11) has three fixed points, but only two are stable equilibria of the consumers' demand.

4.2   Phase transition in the monopolist's strategy

In this section we analyse and discuss the solution of the optimal supply-demand static equilibria, that is, the solutions of equations (25) and (29). As might be expected, the result for the product βp_M depends only on the two parameters βh and βJ. That is, the variance of the idiosyncratic part of the reservation prices fixes the scale of the important parameters, and in particular that of the optimal price.

Let us first discuss the case where $h > 0$. It is straightforward to check that in this case there is a single solution η_M. It is interesting to compare the value of p_M
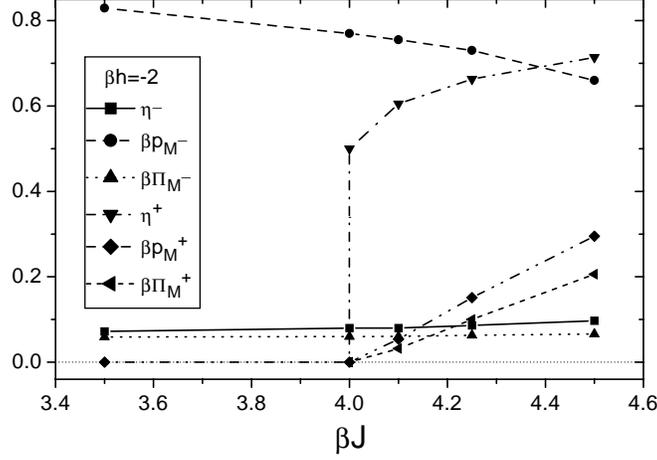
Figure 2: Fraction of buyers η, optimal price βp_M and monopolist profit Π_M, as a function of the social influence, for $\beta h = -2$. The superscripts $-$ and $+$ refer to the two solutions of equations (25) that are relative maxima.
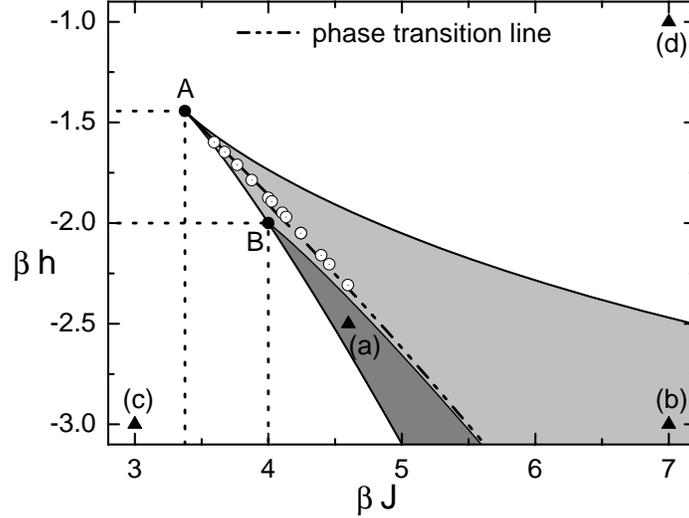

Figure 3: Phase diagram in the plane $\{\beta J, \beta h\}$: the grey region represents the domain in the parameter space where coexist two maxima of the monopolist's profit, a global one (the optimal solution) and a local one. Inside this domain, as βJ and/or βh increase, there is a (first order) transition where the monopolist's optimum jumps from a high price, low penetration rate solution $\eta = \eta_-$ to one with low price, large $\eta = \eta_+$. The circles on the transition line have been obtained numerically, the smooth curves are obtained analytically (see the Appendix and [13] for details). The points (a) to (d) correspond to the inverse supply and demand curves represented in figure 1. In the white region, for $\beta J < 27/8$, the fraction of buyers, η, increases continuously from 0 to 1 as βh increases from $-\infty$ to $+\infty$ (c-d). At the singular point A, ($\beta J = 27/8$, $\beta h = -3/4 - \log(2)$), $\eta_+ = \eta_- = 1/3$. At point B ($\beta J = 4$, $\beta h = -2$), the local maximum with η large appears with a null profit and $\eta_+ = 1/2$. In the dark-grey region below B, this local maximum exists with a negative profit, being thus non viable for the monopolist (a).

with the value p_n corresponding to the neutral situation on the demand side. The latter corresponds to the *unbiased* situation where the willingness to pay is neutral on average: there are as many agents likely to buy as not to buy ($\eta = 1/2$). Since the expected willingness to pay of any agent i is $h + \theta_i + J/2 - p$, its average over the set of agents is $h + J/2 - p$. Thus, the neutral state is obtained for

$$p_n = h + J/2. \tag{33}$$

To compare p_M with p_n, it is convenient to rewrite equation (14) as

$$\beta(p^d - p_n) = \beta J(\eta - 1/2) + \ln[(1 - \eta)/\eta]. \tag{34}$$

This equation gives $p^d = p_n$ for $\eta = 0.5$, as it should. For this value of η, equation (24) gives $p^s = p_n$ only if $\beta(h + J) = 2$: for these values of J and h, the monopolist maximises his profit when the buyers represent half of the population. When $\beta(h + J)$ increases above 2 (decreases below 2), the monopolist's optimal price decreases (increases) and the corresponding fraction of buyers increases (decreases).

Finally, if there are no social effects ($J = 0$) the monopolist optimal price is a solution of the implicit equation:

$$p_M = \frac{1}{\beta \, F(p_M - h)} = \frac{1 + \exp\left(-\beta(p_M - h)\right)}{\beta}. \tag{35}$$

The value of βp_M lies between 1 and $1 + \exp(\beta h)$. Increasing β lowers the optimal price: since the variance of the distribution of willingness to pay gets smaller, the only way to keep a sufficient number of buyers is to lower the prices.

Consider now the case with $h < 0$, that is, on average the population is not willing to buy. Due to the randomness of the individual's reservation price, $H_i = H + \theta_i$, the surplus may be positive but only for a small fraction of the population. Thus, we would expect that the monopolist will maximise his profit by adjusting the price to the preferences of this minority. However, this intuitive conclusion is not supported by the solution to equations (25) when the social influence represented by J is strong enough. The optimal monopolist's strategy shifts abruptly from a regime of high price and a small fraction of buyers to a regime of low price with a large fraction of buyers as βJ increases. Such a discontinuity can actually be expected for $\beta J > 4$, that is when the demand itself has a discontinuity. But, quite interestingly, the transition is also found in the range $27/8 < \beta J < 4$, that is, in a domain of the parameter space $(\beta J, \beta h)$ where the demand $\eta(p)$ is a smooth function of the price.

Such a transition is analogous to what is called a first order phase transition in physics [22]: the fraction of buyers jumps at a critical value of the control parameter $\beta J_c(\beta h)$ from a low to a high value. Before the transition, above a value $\beta J_-(\beta h) < \beta J_c(\beta h)$ equations (25) already present several solutions. Two of them are local maxima of the monopolist's profit function, and one corresponds to a local minimum. The global maximum is the solution corresponding to a high price with few buyers for $\beta J < \beta J_c$, and that of low price with many buyers for $\beta J > \beta J_c$. Figure 2 presents these results for the particular value $\beta h = -2$, for which it can be shown analytically that $\beta J_- = 4$, and $\beta J_c \approx 4.17$ (determined numerically).

The detailed discussion of the full phase diagram in the plane $\{\beta J, \beta h\}$, shown on Figure 3, is presented elsewhere [13].

5   Dynamic features

In this class of models, the individual threshold of adoption implicitly embodies the number of agents each individual considers sufficient to modify his behaviour, as underlined in the field of social science [20, 9]. We briefly discuss here some dynamic aspects, considering a market dynamics with myopic customers: an agent takes its decision at time t based on the observation of the behaviour of other agents at time $t - 1$. The adoption by a single agent in the population (a "direct adopter") may then lead to a significant change in the whole population through a chain reaction of "indirect adopters" [17].

Within the McF model, the dynamics for the fraction of adopters in the large N limit is then given by

$$\eta(t) = 1 - F(P - H - J\,\eta(t - 1)) \tag{36}$$

and $\eta(t)$ converges to a solution of the fixed point equation (10). As we have seen, given J, H and P, two stable and one unstable fixed points appear in (10) for values of β large enough (small σ). The stable solutions correspond to two possible levels of η at a given price (Figure 4a). Varying the price smoothly, a transition may be observed between these phases. The jump in the number of buyers occurs at different price values according to whether the price increases or decreases, leading to *hysteresis loops.* In some cases, the number of customers evolves through a series of clustered flips (between $\omega_i = 1$ and $\omega_i = 0$), called avalanches. For small values of β (large σ), there is a single fixed point for all values of P, and no hysteresis at all ([17, 18]).

a - Hysteresis in the trade-off between prices
and customers: upstream (black)
and downstream (grey) trajectories

b - Chronology and sizes of induced effects
in an avalanche at the phase
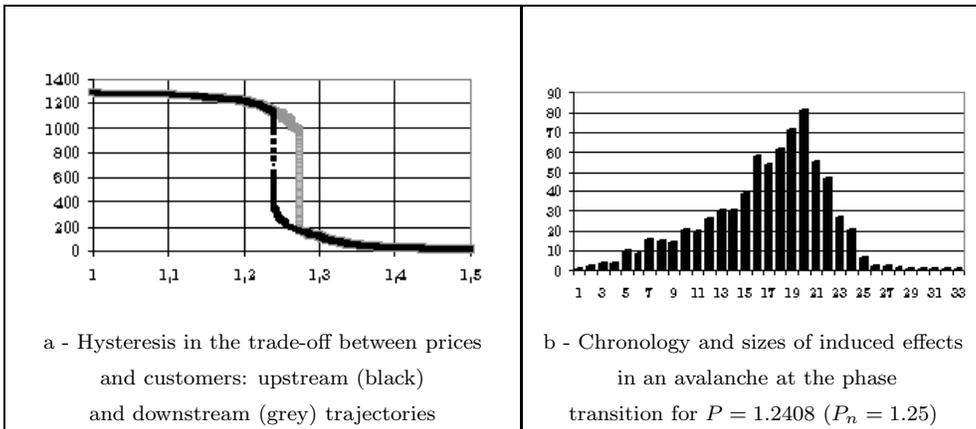transition for $P = 1.2408$ ($P_n = 1.25$)

Figure 4: Discontinuous phase transition (full connectivity, synchronous activation regime; source: Phan *et al.* [17]; parameters: $N = 1296$, $H = 1$, $J = 0.5$, $\beta = 10$).

The curves in Figure 4a, represent the number of customers as a function of the price, obtained through a simulation of the whole demand system. The black (grey) curve corresponds to the "upstream" (downstream) trajectory, when prices decrease (increase) in steps of 10^{-4}, within the interval $[0.9, 1.6]$. We observe a *hysteresis* phenomenon with discontinuous transitions around the theoretical point of symmetry, $P_n = 1.25$. Typically, along the downstream trajectory (with increasing prices,

grey curve) the externality effect induces a strong resistance of the demand system against a decrease in the number of customers. In both cases, large avalanches occur at the so-called "first order phase transition". Figure 4b represents the sizes of these dramatic induced effects as a function of time (see [17, 18] for more details).

6 Conclusion

In this paper, we have compared two extreme special cases of discrete choice models, the McFadden (McF) and the Thurstone (TP) models, in which the individuals bear a local positive social influence on their willingness to pay, and have random heterogeneous idiosyncratic preferences. In the McF model the latter remain fixed, and give rise to a complex market organisation. For physicists, this model with fixed heterogeneity belongs to the class of 'quenched' disorder models; the McF model is equivalent to a 'Random Field Ising Model' (RFIM). In the TP model, all the agents share a homogeneous part of willingness to pay, but have an additive, time varying, random (logistic) idiosyncratic characteristic. In physics, this problem corresponds to a case of 'annealed' disorder. In the TP model, the random idiosyncratic component is equivalent to having a stochastic dynamics, because each agent decides to buy according to the logit choice function, making this model formally equivalent to an Ising model at temperature $T \neq 0$ in a uniform (non random) external field. From the physicist's point of view, the McF and TP models are quite different: random field and zero temperature in the McF case, uniform field and non zero temperature in the TP case. An important result in statistical physics is that quenched and annealed disorders can lead to very different behaviours. In this paper we have discussed some consequences on the market's behaviour.

Considering that the monopolist optimises his own profit, we have exhibited a new 'first order phase transition': when the social influence is strong enough, there is a regime where, upon increasing the mean willingness to pay, or decreasing the production costs, the optimal monopolist's solution jumps from one with a high price and a small number of buyers, to one with a low price and a large number of buyers.

We have only considered fully connected systems: the theoretical analysis of systems with finite connectivity is more involved, and requires numerical simulations. The simplest configuration is one where each customer has only two neighbours, one on each side. The corresponding network, which has the topology of a ring, has been analysed numerically by Phan *et al.* [17]) who show that the optimal monopolist's price increases both with the degree of the connectivity graph and the range of the interactions (in particular, in the case of small worlds). Buyers' clusters of different sizes may form, so that it is no longer possible to describe the externality with a single parameter, like in the mean field case. Further studies in computational economics are required in order to explore such situations.

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Appendix: Phase Diagram

In this Appendix we detail the derivation of the phase diagram in the plane (βJ, βh), presented in figure 3. The phase diagram shows the domain in the parameter space where coexist two maxima of the monopolist's profit, one global maximum (the optimal solution) and one local maximum. Inside this domain there is a (first order) transition line where, as βJ and/or βh increases, the optimal solution jumps from a solution '-' with a low value $\eta = \eta_-$ to a solution '+' with a large value $\eta = \eta_+$, η being the fraction of buyers. In figure 3, the circles 'o' are points on the transition line obtained numerically, all the other curves being obtained analytically as explained below.

In the following we measure J and h in units of $1/\beta$, which is equivalent to say that without loss of generality we set $\beta = 1$.

To explain the phase diagram in more detail, we write the equations as follows. It is convenient to parametrise every quantity/curve as functions of η. First the (*per unit*) profit p is given by

$$p = \frac{1}{1 - \eta} - J\eta \tag{37}$$

(hence the profit $\Pi = p(\eta)\eta$), and η is fixed point of the equation

$$\eta = G(h, J, \eta) \tag{38}$$

with

$$G(h, J, \eta) = \frac{1}{1 + \exp(-h - 2J\eta + \frac{1}{1-\eta})} \tag{39}$$

We will also make use of an alternative form of (38), (39), that is

$$h = -2J\eta + \frac{1}{1 - \eta} + \log(\frac{\eta}{1 - \eta}) \tag{40}$$

One can also show that the condition for having a maximum, (27), is equivalent to

$$\frac{d\Pi}{d\eta} > 0. \tag{41}$$

As we will see, there are two singular points of interest:

A: $J = 27/8$, $h = -3/4 - \log(2)$;
B: $J = 4$, $h = -2$.

Let us describe the phase diagram considering that, at fixed J, one increases h starting from some low (strongly negative) value.

If $J < 27/8$, the optimal solution changes continuously, the fraction of buyers increasing with no discontinuity from a low to a high value as h increases. More generally, outside the domain delimited by the lower and upper curves in figure 3 there is a unique solution of the optimisation of the profit.

For $J > 27/8$, as h increases one will first hit the lower curve on the phase diagram, $h = h_-(J)$. On this line, a local maximum of the profit appears, corresponding to a value $\eta = \eta_+ > 1/3$. As shown in figure 5a, the curve $y = G(h, J, \eta)$ intersects $y = \eta$ at some small value $\eta = \eta_-$ and is tangent to it at $\eta = \eta_+$. For $h_-(J) < h < h_+(J)$ $y = G(h, J, \eta)$ has three intersects with the diagonal $y = \eta$, $h = h_+(J)$ being the upper curve on the phase diagram. The stability analysis shows that the two extreme intersects correspond to maxima of the profit, giving the solutions $\eta = \eta_-$ and $\eta = \eta_+$. On the upper curve $h = h_+(J)$, it is the solution with a small value of η which disappears, with $y = G(h, J, \eta)$ becoming tangent to $y = \eta$ for $\eta = \eta_-$, see figures 5c1 and 5c2. These lower and upper curves are obtained by writing that the second derivative of the profit with respect to p is zero, giving

$$2J\eta(1 - \eta)^2 = 1 \tag{42}$$

Together with (40) this gives the curves parametrized by η,

$$
\begin{aligned}
J &= \frac{1}{2\eta(1 - \eta)^2} \\
h &= -\frac{1}{(1 - \eta)^2} + \frac{1}{1 - \eta} + \log\left(\frac{\eta}{1 - \eta}\right)
\end{aligned}
\tag{43}
$$

the lower curve $h_-(J)$ corresponding to the branch $\eta = \eta_+ \in [1/3,\ 1]$, and the upper curve $h_+(J)$ corresponding to the branch $\eta = \eta_- \in [0,\ 1/3]$.

The two curves merge at the singular point A, at which $\eta_+ = \eta_- = 1/3$, $J = 27/8, h = -3/4 - \log(2)$. Expanding the above equations (43) near $\eta = 1/3$ shows that the two curves are cotangent at A, with a slope $-2/3$. This common tangent is thus also tangent to the transition line at A. A straight segment of slope $-2/3$ starting from A is plotted on the phase diagram, figure 3, and one can see that this is a very good approximation of the transition line for $J < 4$.

On the lower curve, for $J > 4$, the local maximum with $\eta = \eta_+$ appears with a negative profit (zero profit at point B where $\eta_+ = 1/2$). The profit becomes positive on the curve starting at point B, on which the profit is zero with $\eta_+ > 1/2$. This curve is obtained by writing $p = 0$, $\eta_+ > 1/2$, that is,

$$\eta_+ = \frac{1}{2}\left[1 + \sqrt{1 - \frac{4}{J}}\right] \tag{44}$$

and h is obtained as a function of J, replacing η in (40) by the above expression (44). In this domain of negative profit for the local maximum, the distance to the transition line (at a given value of J) is equal to the amount by which the production cost per unit of good must be lowered in order to make the solution viable.

In the domain $J > 4$, the transition line, computed numerically, appears to be just above this null profit line. This suggests making an expansion with p small for the solution η_+, and for η small for the solution η_-. The transition is obtained when $\Pi_+ = \Pi_-$, as explained below, and this gives the full curve of figure 3 which is a very good approximation of the transition line for large values of J (or small values of h, typically $h < -4.5$).

Let us first consider the neighbourhood the point B at which $p_+ = 0$, $\eta_+ = 1/2$, and the second derivative of the profit is zero for this '+' local solution. Expanding near $J = 4$, h just above $h_-(4) = -2$ (p small), one gets the behaviour of the '+' solution:

$$
\begin{aligned}
\epsilon &\equiv h + 2, \ \ 0 < \epsilon << 1 \\
\eta_+ &= \frac{1}{2}(1 + \sqrt{\frac{\epsilon}{2}}) \\
p_+ &= \epsilon + o(\epsilon^{3/2}) \\
\Pi_+ &= \frac{\epsilon}{2} + o(\epsilon^{3/2})
\end{aligned}
\tag{45}
$$

The singular, square-root, behaviour of η is specific to point B. For any $J > 4$, just above the null curve η_+ increases linearly with $\epsilon \equiv h - h_-(J)$, but with a similar behaviour as for $J = 4$ for the price and the profit, at lowest order in ϵ: denoting by $\eta_+^0(J)$ the value $\eta_+(h_-(J))$,

$$
\begin{aligned}
0 &< \epsilon = h - h_-(J) << 1 \\
\eta_+ &= \eta_+^0(J) + \epsilon \frac{\eta_+^0(1 - \eta_+^0)^2}{1 - 2J\eta_+^0(1 - \eta_+^0)^2} \\
p_+ &= \epsilon, \\
\Pi_+ &= \eta_+^0(J) \, \epsilon.
\end{aligned}
\tag{46}
$$

One can see from the expression of η_+ in (46) how the singularity at point B appears: the coefficient of ϵ diverges when condition (42) is fulfilled, that is when the solution is marginally stable, which is the case at B.

Similarly one can get the behaviour of the '+' solution near point B at $h = -2$, increasing J from $J = 4$, as shown in figure 2

$$
\begin{aligned}
\eta_+ &= \frac{1}{2}(1 + \sqrt{\frac{J - 4}{2}}) \\
p_+ &= \frac{1}{2}(J - 4) + \frac{\sqrt{2}}{12}(J - 4)^{3/2} \\
\Pi_+ &= \frac{1}{4}(J - 4) + \frac{1}{3\sqrt{2}}(J - 4)^{3/2}
\end{aligned}
\tag{47}
$$

Coming back to the behaviour at a given value of J, one can get an approximation of the '-' solution. The fixed point equation for η for given values of J and h, is

$$
\eta = H(\eta) \equiv 1/\left[1 + \exp\left(1 - h - 2J\eta + \frac{\eta}{1 - \eta}\right)\right]
\tag{48}
$$

The '-' solution corresponding to a small value of η can be found by iterating $\eta(k + 1) = H(\eta(k))$ starting with $\eta(0) = 0$, and $\eta(k)$ is an increasing sequence of approximations of η_-. The lowest non trivial order is then given by

$$
\eta_-^0 = H(0) = 1/\left[1 + \exp\left(1 - h\right)\right]
\tag{49}
$$

which is indeed small for h strongly negative. At the next order

$$\eta^1_- = H(\eta^0_-) \tag{50}$$

Taking η^0_- as the small parameter, the expansion of η^1_- gives

$$\eta^1_- = \eta^0_-(1 + (2J - 1)\eta^0_-) \tag{51}$$

and this gives the corresponding approximations for the price and the profit,

$$\begin{aligned} p^1_- &= 1 - (J - 1)\eta^0_- \\ \Pi^1_- &= \eta^0_- + J(\eta^0_-)^2. \end{aligned} \tag{52}$$

It is clear from the above equation that the dependency on J is weak since η^0_- is small, in agreement with the exact behaviour computed numerically, shown from figure 2.

Now we consider the neighbourhood of $(J, h_-(J))$, that is $h = h_-(J) + \epsilon$. Denoting by $\eta_0(J)$ the value of η^0_- at $h = h_-(J)$, $\eta^0_-(h) = \eta_0(J) + \epsilon\eta_0(J)(1 - \eta_0(J))$. Replacing the profit for the '-' solution, and writing that at the transition the two solutions '+' and '-' give the same profit, one gets the following approximation for the value $\epsilon_c(J)$ of ϵ at the transition (hence the value of h at the transition, $h_c(J) = h_-(J) + \epsilon_c(J)$):

$$\epsilon_c(J) = \frac{\eta_0(J)}{\eta_+(J)} \tag{53}$$

where $\eta_+(J)$ is given by equation (46). It is this curve $h_c(J) = h_-(J) + \epsilon_c(J)$ which is plotted in figure 3 for $J > 4$.

Figure 5: Functions $y = G(h, J, \eta), y = \eta$, price $p(\eta)$ and profit $\Pi(\eta)$ $(+-)$.
The intersects of $y = G(h, J, \eta)$ with $y = \eta$ give the extrema of the profit; the (possibly local) maxima are those for which $d\Pi/d\eta > 0$. Shown here are marginal cases where for one solution $d\Pi/d\eta = 0$, that is $y = G(h, J, \eta)$ is tangent to $y = \eta$ (points on the lower or upper curves of the phase diagram, figure 3).

(a) $J = 27/8 \ h = -3/4 - \log(2)$

(b) $J = 4 \ h = -2$

(c1) Point on the lower curve, $\eta < 1/2$

(c2) Point on the lower curve, $\eta > 1/2$

(d1) Point on the upper curve, $J < 4$

(d2) Point on the upper curve, $J > 4$