# Merging discrete measurements

Antonio Quesada[†]

Departament d'Economia, Universitat Rovira i Virgili, Avinguda de la Universitat 1, 43204 Reus, Spain

2nd April 2005

121.3

**Abstract**
A merging function synthesizes a vector of numbers (representing measurements, scores or quantitative opinions) into a single number (representing a consensus or collective measurement, score or quantitative opinion). Assuming that all the involved numbers are drawn from a discrete set, it is shown that projection functions are the only merging functions satisfying three properties satisfied by the arithmetic mean (defined for real numbers). Another projection result is obtained under alternative assumptions when merging functions are assumed to transform matrices of numbers from a discrete set to a vector of numbers from the discrete set.

*JEL Classification*: D71, C80

*Keywords*: Aggregation of measurements; merging functions; interdependent measurements; social choice.

[†] E-mail address: aqa@urv.net. Financial support from the Spanish *Ministerio de Educación y Ciencia* under research project SEJ2004-07477 is gratefully acknowledged.

## 1. Introduction

A general formal problem considered in many disciplines consists of associating a unique number with an ordered set of numbers, so that the number integrates, summarizes, synthesizes, represents, aggregates or merges the numbers from the ordered set. The procedures accomplishing this task are called “merging functions” in Aczél and Roberts (1989).

The archetypal merging functions are probably those generating some form of mean value: the arithmetic mean, the weighted arithmetic mean, the geometric mean, the weighted geometric mean… Conditions under which merging functions of this sort arise can be found in Aczél (1966, pp. 234-240), Aczél and Saaty (1983) and Aczél and Roberts (1989, pp. 236-241).

There are nonetheless cases in which it is not possible to resort to mean values. This may occur when the merging function has to select its output from a discrete set, because, for instance, the numbers to be aggregated are measured in terms of an indivisible unit (Euro Cents or persons) or are used as a device to name objects or options (in which case merging function are procedures to select the objects or options).

This paper is motivated by the problem of aggregating measurements, scores, data or quantitative opinions when the inputs and output of merging functions belong to a discrete set. The two main results (Propositions 2.4 and 3.6) make evident the difficulties of obtaining a reasonable merging procedure that satisfies certain plausible requirements: in both cases, under the interpretation that individuals provide the data to be merged, the only admissible type of merging procedure consists of selecting one of the individuals and let the aggregation be determined by that individual. In other words, the merging functions must be projection functions. In the social choice literature, this kind of result is interpreted as a dictatorial result, since the collective measurement, score or opinion always coincides with the measurement, score or opinion of a given individual.

The results are obtained in two different but related settings. In the first one (Section 2), merging functions as defined above are considered and it is shown that only projection functions satisfy three apparently plausible conditions. First, agreement: merging the same score must yield that score as output. Second, monotonicity: if no score is lowered then the score generated by the merging function does not decrease. And third, a property of independence: if, starting from certain scores, a certain change in the scores does not alter the summarizing score then, no matter from which scores one starts, the effect of those changes on the summarizing score is null.

The second setting (Section 3) deals with the case in which the $n$ individuals do not report a single measurement, score or quantitative opinion but a vector of such values, each value being a measurement, score or quantitative opinion referred to a different object from a given set of $m$ objects. If the merging procedures in Section 2 deal with the transformation of $1 \times n$ vectors into numbers, the merging procedures in Section 3 deal simultaneously with $m$ such transformations, so that they map an $m \times n$ vector into an $m \times 1$ vector (the entries in the output vector representing the collective score attributed to the $m$ objects). It is shown that only projection functions satisfy the following three conditions.

First, an allocation condition according to which all the individual score vectors as well as the collective score vector must distribute a total score $c$ among all the objects. This condition makes sense, for example, when scores are proportions or when the underlying problem is just one of allocating a certain amount among the objects (which can then be viewed as tasks). Second, a decentralization condition establishing that the aggregation takes place object by object: to compute the score of an object the only information that is relevant is given by the scores individuals ascribe to the object. And third, a bound condition stating that the collective score of an object cannot be greater than the maximum score that some individual assigns to the object.

On the one hand, Proposition 2.4 appears to alert us to the danger of presuming the existence of satisfactory aggregation procedures without properly specifying the domain over which the aggregation takes place. In Economics, for instance, variables (like production or prices) typically range over a continuum, though the conceptually correct choice seems to be that they should range over a discrete set. In this respect, Proposition 2.4 may suggest that aggregation problems in that context could create more inconveniences than expected. On the other hand, the message of Proposition 3.6 perhaps lies in stressing the difficulties of trying to reduce one type of aggregation problems to simpler aggregation problems when some connection exists between the parts of the main aggregation problem. This reading connects Proposition 3.6 to one of the reference results in social choice theory, Arrow’s (1963, p. 97) theorem.

## 2. Merging measurements of a single object

Let $N = \{1, \dots , n\}$ be a finite subset of the set of natural numbers, with $n \geq 2$, and $D$ a subset of the set $Z$ of integers having at least three members and being closed in the following sense: for all $x \in Z$, $y \in Z\backslash\{x\}$ and $z \in Z\backslash\{x, y\}$, if $x \in D$, $y \in D\backslash\{x\}$ and $x < z < y$ then $z \in D$. A merging function is a mapping $f : D^n \to D$.

The general interpretation is: (i) that there is some underlying object having a measurable property; (ii) that $D$ defines the set of possible values of the measurements; (iii) that $n$ measures $(\xi_1, \dots , \xi_n) \in D^n$ have been obtained; and (iv) that $f$ represents a procedure to obtain a representative measure $f(\xi_1, \dots , \xi_n)$. In more specific contexts, members of $N$ may designate individuals, members of $D$ quantitative opinions about some issue (for instance, utility values associated with some collective decision) and $f$ yield a "social" opinion or a "consensus" value.

For $x \in D$ and $y \in D\backslash\{x\}$, the expression "$x$ covers $y$" means "$y < x$ and there is no integer $z$ such that $y < z < x$". For $i \in N$ and $\xi = (\xi_1, \dots , \xi_n) \in D^n$, $\xi_i$ designates the $i$th component of $\xi$. For $i \in N$, $x \in D$, $y \in D\backslash\{x\}$ and $\xi \in D^n$: (i) $(x^i, y^{-i})$ denotes the member $\xi$ of $D^n$ such that $\xi_i = x$ and, for all $j \in N\backslash\{i\}$, $\xi_j = y$; and (ii) $(x^i, \xi_{-i})$ denotes the member $\zeta$ of $D^n$ such that $\zeta_i = x$ and, for all $j \in N\backslash\{i\}$, $\zeta_j = \xi_j$. The aim of this section is to characterize the set of merging functions satisfying the three conditions stated next.

A1. For all $x \in D$, $f(x, \dots , x) = x$.

A1 is the agreement assumption in Aczél and Roberts (1989, p. 218). The interpretation according to which merging functions are rules to integrate measurements into one representative measurement makes A1 a plausible requirement: if all the measurements are equal to a certain value $x$, which is the best summarizing measurement but $x$ itself?

A2. For all $\alpha, \beta, \gamma \in D^n$, if $f(\alpha) = f(\beta)$ and, for all $i \in N$, $\gamma_i \geq \alpha_i$ then $f(\gamma) \geq f(\beta)$.

A2 is a monotonicity property: if $f$ yields the same value for two measurement profiles then, every change that does not decrease any value in one of the profiles, cannot induce $f$ to yield a value smaller than the value that corresponds to the other profile. This establishes a requirement of non-negative response: increasing the values to be merged cannot reduce the resulting merged value.

A3. For all $\alpha, \beta, \gamma, \delta \in D^n$, if $f(\alpha) = f(\beta)$ and, for all $i \in N$, $\gamma_i - \delta_i = \alpha_i - \beta_i$ then $f(\gamma) = f(\delta)$.

A3 can be viewed as a condition trying to reduce the complexity of the merging function. The motivation is as follows. Suppose that $f$ yields the same value under two different measurement profiles $\alpha$ and $\beta$. This fact can be interpreted in the sense that $f$ neutralizes (or considers irrelevant) the changes in passing from $\alpha$ to $\beta$. By A3, if this happened once then it happens always: by applying the same absolute changes to any $\gamma$ in order to obtain a new profile $\delta$, the effects on the resulting merged value are innocuous, so that $f(\delta) = f(\gamma)$.

A3 bears some resemblance to the independence type conditions in social choice theory, such as the condition of independence of irrelevant alternatives in Arrow's (1963, p. 97) theorem. According to this sort of assumptions, the aggregation of several items is carried out by combining the aggregation of parts of those items. Independence then refers to the fact that the aggregation of one of the parts does not depend on the aggregation of the other parts. In this sense, A3 makes the effect on $f$ of a change in the measurements independent of the initial measurements: if given a measurement profile $\alpha$, $f(\alpha)$ itself is the result of increasing $x$ units one of the measurements while decreasing $y$ units another measurement, with the rest of measurements held constant, then the same changes leave the value of $f$ unaltered no matter the initial profile $\alpha$ considered.

A3 is arguably not an uncontroversial requirement. It is nonetheless worth noticing that the arithmetic mean satisfies A3, as well as A1 and A2. In this respect, the main result in this section provides an answer to the question of what type of merging functions is consistent with A1, A2 and A3 on discrete and closed domains.

Lemma 2.1 states that the value of a merging function satisfying A1 and A2 at profile $\alpha$ is bounded above by the maximum of the values in $\alpha$ and bounded below by the minimum of these values.

**Lemma 2.1.** If $f : D^n \to D$ satisfies A1 and A2 then, for all $\alpha \in D^n$, $min\{\alpha_1, \dots, \alpha_n\} \leq f(\alpha) \leq max\{\alpha_1, \dots, \alpha_n\}$.

*Proof*. Let $\alpha \in D^n$, $x := max\{\alpha_1, \dots, \alpha_n\}$ and $y := min\{\alpha_1, \dots, \alpha_n\}$. If $f(\alpha) = z > x$ then, by A2, $f(x, \dots, x) \geq z$, whereas, by A1, $f(x, \dots, x) = x$: contradiction. If $f(\alpha) < y$ then, by A1, $f(y, \dots, y) = y$ and, given this, by A2, $f(\alpha) \geq y$: contradiction.■

Lemma 2.2 shows that, for merging functions satisfying A1, A2 and A3, there is a situation in which $f$ disregards all but one of the measurements. In particular, by interpreting $N$ as a set of individuals, Lemma 2.2 holds that $f$ attributes some individual $i$ the power to impose some assessment $x$ when the rest of individuals declare as assessment the smallest value larger than $x$.

**Lemma 2.2.** If $f : D^n \to D$ satisfies A1, A2 and A3 then there are $i \in N$, $x \in D$ and $y \in D\backslash\{x\}$ such that $y$ covers $x$ and $f(x^i, y^{-i}) = x$.

*Proof*. Suppose not: for all $i \in N$, $x \in D$ and $y \in D\backslash\{x\}$, if $y$ covers $x$ then $f(x^i, y^{-i}) \neq x$. By Lemma 2.1,

$$\text{for all } i \in N, x \in D \text{ and } y \in D\backslash\{x\}, \text{ if } y \text{ covers } x \text{ then } f(x^i, y^{-i}) = y. \tag{1}$$

Choose $i \in N$, $x \in D$ and $y \in D\backslash\{x\}$ such that $y$ covers $x$. By (1), $f(x^i, y^{-i}) = y$. By A1, $f(x^i, y^{-i}) = f(y^i, y^{-i})$. Consequently, by A3,

$$\begin{gathered}\text{for all } k \in N, \alpha \in D^n \text{ and } \beta \in D^n, \text{ if } \beta_k \text{ covers } \alpha_k \\ \text{and, for all } j \in N\backslash\{k\}, \beta_j = \alpha_j \text{ then } f(\alpha) = f(\beta).\end{gathered} \tag{2}$$

Taking $f(x^i, y^{-i}) = f(y^i, y^{-i})$ as the base of an induction argument, choose $G \subset N$ with $i \in G$ and assume that $f(x^G, y^{-G}) = f(y^i, y^{-i})$. Let $j \in N\backslash G$. By (2), $f(x^{G\cup\{j\}}, y^{-(G\cup\{j\})}) = f(y^G, y^{-G})$. Since, by A1, $f(y^G, y^{-G}) = y$ it follows that, for all non-empty $G \subset N$ and $j \in N\backslash G$, $f(x^G, y^{-G}) = y$ implies $f(x^{G\cup\{j\}}, y^{-(G\cup\{j\})}) = y$. Accordingly, by starting with $f(x^i, y^{-i}) = y$ and successively adding members of $N$ to $\{i\}$, there is some $k \in N$ such that $f(x^{-k}, y^k) = y$, which contradicts (1).■

Lemma 2.3 establishes that, for merging functions satisfying A1, A2 and A3, the power attributed to the individual in Lemma 2.2 in the specific case there defined cannot be shared with other individuals.

**Lemma 2.3.** Let $f : D^n \to D$ satisfy A1, A2 and A3. If there are $i \in N$, $x \in D$ and $y \in D\backslash\{x\}$ such that $y$ covers $x$ and $f(x^i, y^{-i}) = x$ then, for all $j \in N\backslash\{i\}$, $f(x^j, y^{-j}) = y$.

*Proof*. Suppose there are $i \in N$, $j \in N\backslash\{i\}$, $x \in D$ and $y \in D\backslash\{x\}$ such that $y$ covers $x$ and $f(x^i, y^{-i}) = x = f(x^j, y^{-j})$. Case 1: some $z \in D$ covers $y$. By A1, $f(x^j, y^{-j}) = x$ yields $f(x^j, y^{-j}) =$

$f(x^j, x^{-j})$. By A3, for all $\alpha \in D^n$ and $\beta \in D^n$, if $\beta_j = \alpha_j$ and, for all $k \in N\backslash\{j\}$, $\beta_k$ covers $\alpha_k$ then $f(\alpha) = f(\beta)$. Given this, it follows from $f(x^i, y^{-i}) = x = f(x^i, x^{-i})$ that $f(y^{\{i,j\}}, z^{-\{i,j\}}) = f(x^i, x^{-i})$, where $z$ covers $y$. By A1, $f(y^{\{i,j\}}, z^{-\{i,j\}}) = x < min\{y, z\}$, contradicting Lemma 2.1. Case 2: no $z \in D$ covers $y$. Since $f(x^i, y^{-i}) = x = f(x^j, y^{-j})$, by A1 and A3, $f(v^i, x^{-i}) = v = f(v^j, x^{-j})$, where $x$ covers $v$. Given that $y$ covers $x$, case 1 applies, so that $f(v^i, x^{-i}) = v$ implies $f(v^j, x^{-j}) \neq v$. Hence, by A3, $f(x^i, y^{-i}) = x$ implies $f(x^j, y^{-j}) \neq x$: contradiction.■

**Proposition 2.4.** Merging function $f : D^n \to D$ satisfies A1, A2 and A3 if, and only if, there exists $i \in N$ such that, for all $\xi \in D^n$, $f(\xi) = \xi_i$.

*Proof*. "⇒" By Lemmas 2.2 and 2.3, there is $i \in N$ such that, for all $j \in N\backslash\{i\}$, if $y$ covers $x$ then $f(x^j, y^{-j}) = y$. Therefore, by A3,

$$\text{for all } j \in N\backslash\{i\},\ \xi \in D^n \text{ and } z \in D, \text{ if } \xi_j \text{ covers } z \text{ then } f(z^j, \xi_{-j}) = f(\xi). \tag{3}$$

To show that, for all $\alpha \in D^n$, $f(\alpha) = \alpha_i$ let $x = \alpha_i$. Define $J = \{j \in N\backslash\{i\}: \alpha_j > x\}$ and $K = \{k \in N\backslash\{i\}: \alpha_k < x\}$. Choose first any $j \in J$ and let $(x_1, \dots, x_r)$ be the sequence of members of $D$ such that $x_1 = x$, $x_r = \alpha_j$ and $x_t$ is covered by $x_{t+1}$, for $t \in \{1, \dots, r-1\}$. By A1, $f(x, \dots, x) = x$ and, by successive application of (3), $x = f((x_1)^j, x^{-j}) = f((x_2)^j, x^{-j}) = \dots = f((x_r)^j, x^{-j})$. Applying the same reasoning to the rest of members of $J$, it follows that $f(\beta) = x$, where, for all $j \in J$, $\beta_j = \alpha_j$ and, for all $j \in N\backslash J$, $\beta_j = x$. Proceeding in a similar vein with the members of $K$, choose $k \in K$ and let $(y_1, \dots, y_s)$ be the sequence of members of $D$ such that $y_1 = x$, $y_s = \alpha_k$ and $y_t$ covers $y_{t+1}$, for $t \in \{1, \dots, s-1\}$. Given $f(\beta) = x$, by successive application of (3), $x = f((y_1)^k, \beta_{-k}) = f((y_2)^k, \beta_{-k}) = \dots = f((y_s)^k, \beta_{-k})$. Applying the same reasoning to the rest of members of $K$, it follows that $f(\alpha) = x$. "⇐" If, for some $i \in N$ and all $\xi \in D^n$, $f(\xi) = \xi_i$ then it is easy to verify that $f$ satisfies A1, A2 and A3.■

By Proposition 2.4, the only merging functions satisfying A1, A2 and A3 are projection functions. In the social choice literature, such functions are called "dictatorial", as one of the individuals completely determines the outcome of the merging procedure. In terms of measurements, imposing A1, A2 and A3 on a merging function $f$ amounts to simplifying extraordinarily the way $f$ operates: the synthesizing value is always one given entry of the measurement profile.

**Remark 2.5.** No condition in the set {A1, A2, A3} is redundant in Proposition 2.4. First, the non-dictatorial $f : D^n \to D$ such that, for some $x \in D$ and all $\alpha \in D^n$, $f(\alpha) = x$ satisfies

A2 and A3 but not A1. Second, with $D = \{1, 2, 3\}$, the non-dictatorial $f : D \times D \to D$ such that $f(1, 1) = f(2, 3) = f(3, 2) = 1$, $f(2, 2) = f(1, 3) = f(3, 1) = 2$ and $f(3, 3) = f(1, 2) = f(2, 1) = 3$ satisfies A1 and A3 but not A2. And third, the non-dictatorial $f : D^n \to D$ such that, for all $\alpha \in D^n$, $f(\alpha) = max\{\alpha_1, \dots , \alpha_n\}$ satisfies A1 and A2 but not A3.

**Remark 2.6.** As the proofs of Lemmas 2.3 and 2.4 make evident, Proposition 2.4 holds if A3 is replaced by the weaker condition A3′.

A3′. If $f(\alpha) = f(\beta)$ and $N = \{i \in N: \alpha_i = \beta_i\} \cup \{i \in N: \alpha_i \text{ covers } \beta_i\}$ then $f(\delta) = f(\gamma)$ provided $\{i \in N: \delta_i = \gamma_i\} = \{i \in N: \alpha_i = \beta_i\}$ and $\{i \in N: \delta_i \text{ covers } \gamma_i\} = \{i \in N: \alpha_i \text{ covers } \beta_i\}$.

**Remark 2.7.** Proposition 2.4 holds if A3 is replaced by A3″, since A3″ implies A3′.

A3″. If $f(\alpha) = f(\beta)$ and, for all $i \in N$, $\delta_i - \gamma_i \leq \alpha_i - \beta_i$ then $f(\delta) \leq f(\gamma)$.

## 3. Merging interdependent measurements of several objects

In some cases, the aggregate measurement of the property of some object may depend on the measurements of other objects. As an illustration, consider the situation in which $n$ individuals report utility values associated with a set of $m$ public projects and a procedure $F$ must determine the collective value of each project on the basis of the reported utility values. In this case, the input of $F$ consists of an $m \times n$ matrix of entries $\xi^i_j$ representing the utility individual $i$ ascribes to project $j$, whereas the output is a vector whose $m$ entries represent the collective value attributed to each project. For this problem, the procedure could just decentralize the aggregation by resorting to $m$ merging functions $f_1, \dots , f_m$, so that $f_j$ takes row $j$ of the matrix and yields the $j$th entry of the vector of collective values.

In could nonetheless be that the $m$ merging functions are subject to some form of interdependence. For example, suppose that, instead of declaring utility values, individuals reveal how they prefer a given amount $c$ of money to be allocated among the $m$ projects. Thus, the columns of the matrix as well as the output vector must all add up to $c$. In consequence, the merging functions $f_j$ cannot be freely chosen. In general, a situation like this would arise when measurements are expressed as proportions and the aggregate values must also be proportions. The aim of this section is to determine the extent to which such problems can be handled by several merging functions acting simultaneously.

Let $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$ be finite subsets of the set of natural number, with $n \geq 2$ and $m \geq 3$. For some natural number $c$, define $D = \{0, 1, \dots, c\}$. Members of $D^{m\times n}$ can be viewed as matrices, so that, for $k \in M$ and $i \in N$, $\xi_k$ is the $k$th row, $\xi^i$ is the $i$th column and $\xi^i_k$ is the value simultaneously in the $k$th row and the $i$th column. With $E = \{\xi \in D^{m\times n}$: for all $i \in N$, $\xi^i_1 + \dots + \xi^i_m = c\}$, a merging* function is a mapping $F : E \rightarrow D^m$ satisfying B0. For $k \in M$, merging* function $F$ induces $m$ functions $F_k : E \rightarrow D$ such that, for all $\xi \in E$, $F_k(\xi)$ is the $k$th value in $m$-tuple $F(\xi)$.

B0. For all $\xi \in E$, the sum of the values in $F(\xi)$ is $c$; that is, $F_1(\xi) + \dots + F_m(\xi) = c$.

The interpretation is as follows: (i) members of $M$ designate the objects assigned score; (ii) $n$ is the number of scores associated with each object (each member of $N$ could be viewed as an individual reporting the scores he associates with each object); (iii) $D$ collects the possible scores, which range from 0 to a maximum score of $c$; (iv) each multidimensional score $\xi^i$ is such that the sum of the scores of all the $m$ objects must add up to $c$; and (v) $f$ represents a procedure to obtain a representative score $F_k(\xi)$ for each object $k \in M$.

To clarify the notation, let $c = 100$, $m = 4$ and $n = 3$. Let $\xi$ be the member of $E$ in which the first score profile is (50, 5, 20, 25), so 50 is the first score of the first object, 5 the first score of the second object, and so on. That is, $\xi^1 = (50, 5, 20, 25)$ and this constitutes the first column of the matrix $\xi$. If $\xi$ is such that $\xi^2 = (30, 15, 20, 35)$ and $\xi^3 = (0, 100, 0, 0)$ then $\xi_1 = (50, 30, 0)$ is the first row of the matrix $\xi$ and determines the set of scores that the first object receives. Similarly, $\xi_2 = (5, 15, 100)$, $\xi_3 = (20, 20, 0)$ and $\xi_4 = (25, 35, 0)$ define the scores that receive, respectively, objects 2, 3 and 4. Finally, that $\xi^3_2 = 100$ expresses the fact that object 2 receives score 100 in the third scoring profile. This section is concerned with the following question: what merging* functions $F$ are consistent, in the sense of B1, with the use of merging functions?

B1. For every $k \in M$ there is a merging function $f_k : D^n \rightarrow D$ such that $F_k(\xi) = f_k(\xi_k)$.

By B1, merging* function $F$ defines a separable procedure by means of which, for each object $k \in M$, a merging function $f_k$ determines the score of that object taking into account only the scores of that object. To a certain extent, B1 expresses the aim of reducing merging* functions to merging functions.

B2. For all $k \in M$ and $\xi \in E$, $F_k(\xi) \leq max\{\xi^1_k, \dots, \xi^n_k\}$.

B2 can be interpreted as a sort of "capacity constraint": the summarizing score of an object cannot be larger than the maximum of the scores the object receives. The main result in this section asserts that merging* functions satisfying B1 and B2 must be projection functions and, accordingly, the summarizing profile of scores always coincides with a fixed column of scores. If members of $N$ represent individuals and, hence, $\xi^i$ defines how individual $i$ ascribes scores to the different objects, the main result states that the summarizing profile of scores always coincides with the score profile of a given individual.

By B1, a merging* function $F$ is decomposed into $m$ merging functions $f_k$. Lemma 3.1 next asserts that, if B2 is assumed in addition to B1, the merging functions satisfy the agreement property A1 from Section 2.

**Lemma 3.1.** If merging* function $F : E \to D$ satisfies B1 and B2 then, for all $k \in M$ and $x \in D$, $f_k(x, \dots , x) = x$.

*Proof*. Choose $k \in M$, $q \in M\backslash\{k\}$ and $x \in D$. Consider the member $\xi$ of $E$ such that $\xi_k = (x, \dots , x)$, $\xi_q = (c - x, \dots , c - x)$ and, for all $p \in M\backslash\{k, q\}$, $\xi_p = (0, \dots , 0)$. By B2, $F_q(\xi) \leq c - x$ and, for all $p \in M\backslash\{k, q\}$, $F_p(\xi) = 0$. Therefore, since $F_1(\xi) + \dots + F_m(\xi) = c$, $F_k(\xi) \geq x$. But, by B2, $F_k(\xi) \leq x$ and, consequently, $F_k(\xi) = x$. Thus, by B1, $f_k(\xi_k) = x$.■

For $i \in N$, $x \in D$, $y \in D\backslash\{x\}$ and $G \subseteq N$: (i) as in Section 2, $(x^i, y^{-i})$ abbreviates the $n$-tuple $(z_1, \dots , z_n)$ such that, for all $j \in N$, $z_j = x$ if $j = i$ and $z_j = y$ if $j \neq i$; and $(x^G, y^{-G})$ abbreviates the $n$-tuple $(z_1, \dots , z_n)$ such that, for all $j \in N$, $z_j = x$ if $j \in G$ and $z_j = y$ if $j \in N\backslash G$.

Lemma 3.2 is a result similar to Lemma 2.2: interpreting $N$ as a set of individuals, if merging* function $F$ satisfies B1 and B2 then there are some object $k \in M$ and individual $i \in N$ such that, when all but $i$ associate the lowest score 0 to the object and $i$ associates the second lowest score 1, the collective score coincides with $i$'s score.

**Lemma 3.2.** If merging* function $F : E \to D$ satisfies B1 and B2 then there are $i \in N$ and $k \in M$ such that $f_k(1^i, 0^{-i}) = 1$.

*Proof*. Suppose not: for all $i \in N$ and $k \in M$, $f_k(1^i, 0^{-i}) \neq 1$. By B2,

$$\text{for all } i \in N \text{ and } k \in M,\ f_k(1^i, 0^{-i}) = 0. \tag{4}$$

Choose $i \in N$. Let $\xi \in E$ be such that $\xi_1 = \xi_2 = (1^i, 0^{-i})$, $\xi_3 = ((c-2)^i, c^{-i})$ and, for all $p \in M\backslash\{1, 2, 3\}$, $\xi_p = (0, \ldots, 0)$. Given B1, by B2, $f_p(\xi_p) = 0$ for all $p \in M\backslash\{1, 2, 3\}$. By (4), $f_1(\xi_1) = f_2(\xi_2) = 0$. Consequently, $f_3(\xi_3) = c$. Let $\zeta \in E$ differ from $\xi$ only in that $\zeta_1 = (0, \ldots 0)$ and $\zeta_2 = (2^i, 0^{-i})$. As $f_3(\xi_3) = c$, it follows from B1 that $f_3(\zeta_3) = c$. By B0, $f_2(\zeta_2) = 0$. Thus, given (4) and B2, $f_2(x^i, 0^{-i}) = 0$ for all $x \in \{0, 1, 2\}$. Now, the aim is to show that,

$$\text{for all } x \in D,\ f_2(x^i, 0^{-i}) = 0. \tag{5}$$

To that end, choose $x \in D\backslash\{0, 1, 2\}$ and, arguing inductively, suppose that, for all $y \in \{0, 1, \ldots, x-1\}$, $f_2(y^i, 0^{-i}) = 0$. It has to be shown that $f_2(x^i, 0^{-i}) = 0$. With $\varphi \in E$ differing from $\xi$ only in that $\varphi_2 = ((x-1)^i, 0^{-i})$ and $\varphi_3 = ((c-x)^i, c^{-i})$, it follows from B2 that, for all $p \in M\backslash\{1, 2, 3\}$, $f_p(\varphi_p) = 0$. By (4), $f_1(\varphi_1) = 0$. By the induction hypothesis, $f_2(\varphi_2) = 0$. As a result, by B0, $f_3(\varphi_3) = c$. Let $\eta \in E$ differ from $\varphi$ only in that $\eta_1 = (0, \ldots 0)$ and $\eta_2 = (x^i, 0^{-i})$. Since $f_3(\varphi_3) = c$, by B1, $f_3(\eta_3) = c$. By B0, $f_2(\eta_2) = 0$ and (5) is proved. Taking $f_2(c^i, 0^{-i}) = 0$ as the base of another induction argument, choose $G \subset N$ containing $i$ and assume that $f_2(c^G, 0^{-G}) = 0$. It will be shown that, for all $j \in N\backslash G$, $f_2(c^{G\cup\{j\}}, 0^{-(G\cup\{j\})}) = 0$. To this end, let $j \in N\backslash G$ and $\mu \in E$ differ from $\xi$ only in that $\mu_1 = (c^G, 0^{-G})$, $\mu_2 = (0^{G\cup\{j\}}, c^{-(G\cup\{j\})})$ and $\mu_3 = (c^j, 0^{-j})$. By B2, for all $p \in M\backslash\{1, 2, 3\}$, $f_p(\mu_p) = 0$. By the induction hypothesis, $f_1(\mu_1) = 0$. By (5) and B1, $f_3(\mu_3) = 0$. By B0, $f_2(\mu_2) = c$, which is what has to be proved. In view of this, there must be $k \in N\backslash\{i\}$ such that $f_2(c^k, 0^{-k}) = c$, contradicting (5).■

Lemma 3.3 extends the result in Lemma 3.2 to any score that $i$ could report.

**Lemma 3.3.** If merging* function $F : E \to D$ satisfies B1 and B2, and there are $i \in N$, $x \in D$ and $k \in M$ such that $f_k(x^i, 0^{-i}) = x$ then, for all $k \in M$, $f_k(x^i, 0^{-i}) = x$.

*Proof*. Assume $f_k(x^i, 0^{-i}) = x$. Let $q \in M\backslash\{k\}$. To prove that $f_q(x^i, 0^{-i}) = x$, choose $p \in M\backslash\{k, q\}$ and let $\xi \in E$ be such that $\xi_k = (x^i, 0^{-i})$, $\xi_p = ((c-x)^i, c^{-i})$ and, for all $s \in M\backslash\{k, p\}$, $\xi_s = (0, \ldots, 0)$. By B2, $f_s(\xi_s) = 0$, for all $s \in M\backslash\{k, p\}$. Since $f_k(x^i, 0^{-i}) = x$, $f_k(\xi_k) = x$ and, by B0, $f_p(\xi_p) = c - x$. Given this and $\zeta \in E$ that differs from $\xi$ only in that $\zeta_k = (0^i, 0^{-i})$ and $\zeta_q = (x^i, 0^{-i})$, it follows that $f_p(\zeta_p) = c - x$ and, for all $s \in M\backslash\{p, q\}$, $f_s(\zeta_s) = 0$. Thus, $f_q(x^i, 0^{-i}) = x$.■

Lemma 3.4 extends the result in Lemma 3.3 to all the objects.

**Lemma 3.4.** If merging* function $F : E \to D$ satisfies B1 and B2 then

$$\text{there is } i \in N \text{ such that, for all } k \in M \text{ and } x \in D,\ f_k(x^i, 0^{-i}) = x. \tag{6}$$

*Proof*. By B2, for all $i \in N$ and $k \in M$, $f_k(0^i, 0^{-i}) = 0$. By Lemma 3.2, there are $i \in N$ and $k \in M$ such that $f_k(1^i, 0^{-i}) = 1$. Choose $x \in D\backslash\{0, 1\}$ and, arguing inductively, suppose that, for all $y \in \{0, 1, \dots , x - 1\}$, $f_k(y^i, 0^{-i}) = y$. To prove that $f_k(x^i, 0^{-i}) = x$, choose $q \in M\backslash\{k\}$ and $p \in M\backslash\{k, q\}$. Let $\xi \in E$ satisfy: for all $s \in M\backslash\{k, p, q\}$, $\xi_s = (0, \dots , 0)$; $\xi_p = (1^i, 0^{-i})$; $\xi_k = ((x - 1)^i, 0^{-i})$; and $\xi_q = ((c - x)^i, c^{-i})$. By B2, $f_s(\xi_s) = 0$, for all $s \in M\backslash\{k, p, q\}$. By the induction hypothesis, $f_k(\xi_k) = x - 1$. By the induction hypothesis and Lemma 3.3, $f_p(\xi_p) = 1$. Thus, by B0, $f_q(\xi_q) = c - x$. Given this and $\zeta \in E$ that differs from $\xi$ only in that $\zeta_k = (x^i, 0^{-i})$ and $\zeta_p = (0^i, 0^{-i})$, it follows that $f_q(\zeta_q) = c - x$ and, for all $s \in M\backslash\{q, k\}$, $f_s(\zeta_s) = 0$. By B0, $f_k(x^i, 0^{-i}) = x$. Consequently, there are $i \in N$ and $k \in M$ such that, for all $x \in D$, $f_k(x^i, 0^{-i}) = x$. This and Lemma 3.3 imply (6).■

By Lemma 3.5, individual $i$ in (6) can also impose the null score: for each object, whenever $i$ ascribes score 0 to that object, the collective score is also 0.

**Lemma 3.5.** If merging* function $F : E \to D$ satisfies B1, B2 and (6) then, for all $\xi \in E$ and $k \in M$, $\xi^i_k = 0$ implies $f_k(\xi_k) = 0$.

*Proof*. Let $i$ be the member of $N$ from (6). Suppose $\xi \in E$ and $k \in M$ are such that $\xi^i_k = 0$. With $q \in M\backslash\{k\}$ and $p \in M\backslash\{k, q\}$, consider the $\zeta \in E$ satisfying: for all $s \in M\backslash\{k, p, q\}$, $\zeta_s = (0, \dots , 0)$; $\zeta_q = (c^i, 0^{-i})$; $\zeta^i_p = \zeta^i_k = 0$; and, for all $j \in N\backslash\{i\}$, $\zeta^j_p = c - \xi^j_k$ and $\zeta^j_k = \xi^j_k$. By (6), $f_q(\zeta_q) = c$. By B0, this implies $f_k(\zeta_k) = 0$ and this, by B1, $f_k(\xi_k) = 0$.■

**Proposition 3.6.** If merging* function $F : E \to D$ satisfies B1 and B2 then $F$ is dictatorial in the sense that there exists $i \in N$ such that, for all $\xi \in E$, $F(\xi) = \xi^i$.

*Proof*. By Lemma 3.4, (6) holds. Let $\xi \in E$ and $k \in M$. It must be shown that $f_k(\xi_k) = \xi^i_k$. Choose $q \in M\backslash\{k\}$, $p \in M\backslash\{k, q\}$ and $\zeta \in E$ such that: $\zeta_k = \xi_k$; for all $s \in M\backslash\{k, q, p\}$, $\zeta_s = (0^i, 0^{-i})$; $\zeta_q = ((c - \xi^i_k)^i, 0^{-i})$; $\zeta^i_p = 0$; and, for all $j \in N\backslash\{i\}$, $\zeta^j_p = c - \xi^j_k$. By B2, $f_s(\zeta_s) = 0$, for all $s \in M\backslash\{k, q, p\}$. By (6), $f_q(\xi_q) = c - \xi^i_k$. By Lemma 3.5, $f_p(\xi_p) = 0$. Therefore, by B0, $f_k(\xi_k) = \xi^i_k$.■

Proposition 3.6 is a result analogous to Proposition 2.4 in that the merging* function is a projection function. In the present case, the vector of collective scores always coincides with one and the same vector of individual scores. In this respect, if members of $N$ are regarded as experts providing their opinion as to how resources must be allocated or efforts distributed, B1 and B2 amount to disregarding all but one of the experts.

**Remark 3.7.** No condition in the set {B0, B1, B2} is redundant in Proposition 3.6. First, the non-dictatorial $F : E \rightarrow D$ such that, for all $k \in M$ and $\xi \in E$, $F_k(\xi) = max\{\xi^1{}_k, \dots , \xi^n{}_k\}$ satisfies B1 and B2 but not B0. Second, the non-dictatorial $F : E \rightarrow D$ such that, for some $i \in N$, some $j \in N\backslash\{i\}$, some $\zeta \in E$ with $\zeta^i \neq \zeta^j$, and all $\xi \in E\backslash\{\zeta\}$, $F(\xi) = \xi^i$ and $F(\zeta) = \zeta^j$ satisfies B0 and B2 but not B1. And third, when $c/m \in D$, the non-dictatorial $F : E \rightarrow D$ such that, for all $\xi \in E$ and $k \in M$, $F_k(\xi) = c/m$ satisfies B0 and B1 but not B2.

## References


Aczél, J. (1966). *Lectures on functional equations and their applications*. New York: Academic Press.

Aczél, J., & Roberts, F. S. (1989). On the possible merging functions. *Mathematical Social Sciences*, 17, 205–243.

Aczél, J., & Saaty, T. L. (1983). Procedures for synthesizing ratio judgements. *Journal of Mathematical Psychology*, 27, 93–102.

Arrow, K. (1963). *Social choice and individual values*, 2nd edition. New York: Wiley.