Non Cooperatives Stackelberg Networks[1]

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Abstract

Noncooperative network-formation games in oligopolies analyze optimal connection structures that emerge when linking represent the appropriation of cost-reducing one-way externalities. These models reflect situations where one firm access to another firm's (public or private) information and this last cannot refuse it. What would happen if decisions are sequential? A model of exogenous Stackelberg leadership is developed and first-mover advantages are observed and commented.

Keywords: non cooperative games, network formation strategies, Stackelberg equilibrium.

JEL Classification System: C70, D43, L13

1. Introduction

Networks of collaboration in oligopolies have been empirically found in several industries. Firms make alliances for cost-reducing technologies development. Goyal and Joshi (2003) study models of emergence of networks of collaboration among firms that compete in quantities (Cournot) and prices (Bertrand). They model linking benefits as bidirectional externality that helps both agreement signers to reduce production costs. This is the usual modeling option for mutual consent contracts where technical information and collaboration are shared. However, the monodirectional (one-way) externality case can also be modeled. Billand and Bravard (2004) use Goyal and Joshi's basic structure but allow only for one-way externality flow. This way of modeling externality flow is meaningful for firms that access another firms' cost-reducing public or private information without reciprocity. Optimal topologies found are, for the lowest-cost investment infrastructure, complete (Cournot) and star (Bertrand) networks and for the highest-cost investment infrastructure the empty network. Cournot market also allows for intermediate topologies (see Table 1 for comparing these two papers' findings).

[1] This work is a recapitulation of Chapter 4 of my PhD thesis dissertation (Larrosa, 2005). I would like to acknowledge previous comments and corrections by Fernando Tohmé, Leandro Arozamena, Diego Caramuta and Federico Weinschelbaum and from participants in Segundo Congreso Nacional de Estudiantes de Posgrado and Seminario Interno de Posgrado at UNS. All errors are mine.

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Our paper adds the influence of sequential decision in connections' structure formation. Suppose a market with n firms. Suppose that, exogenously given[3], a firm can move first by establishing links with other firms in the market and the $n-1$ follower firms observe this and then choose their own connection structure. Later on they will compete in quantities or prices. This is the game setting, initially formulated by Stackelberg (1934), that we are going to solve and analysis.

Table 1. Optima topologies

Authors	Type of Externality Flow	Type of Market Competition	Optimum Topology
Goyal & Joshi (2003)	Bidirectional	Quantity Competition (Cournot market)	• Complete network.
		Price Competition (Bertrand market)	• Empty network.
Billand & Bravard (2004)	Monodirectional	Quantity Competition (Cournot market)	• For highest investment cost: Empty network • For intermediate investment cost: Firms make connections but neither complete nor empty networks are observed. • For lowest investment cost: Complete network
		Price Competition (Bertrand market)	• For highest investment cost: Empty network • For lowest investment cost: Centered sponsored star network

The main results could be summarized as follows. In Stackelberg single-leader-rest-followers quantity competition equilibrium, leader firm obtains as least as much benefits as any follower. In Stackelberg single-leader-rest-followers price competition equilibrium, leader firm is the only who obtains benefits. More specifically, depending on the value fixed investment cost optimum topology varies. If the cost is low enough, optimum topology is the complete network (where leader firm connects to every follower firm and each follower firm connect to each other and to the leader firm) if there's quantity competition, and the leader-firm-sponsored star network when there's price competition. If the cost is sufficiently high, for both types of competition, optimum topology is the empty network. For intermediate costs, quantity

[3] Some academic literature criticizes the exogeneity of leader firm's selection process, on a priori all-equal firms. Some works model role selection in a previous stage of the game where firms could choose whether they are going to move first or later. Endogenous models of Stackelberg competition are Amir and Grilo (1999) and van Damme and Hurkens (1999).

competition allows for optimum architectures that are neither the empty nor the complete but there are connections among firms.

This paper is organized as follows. Section 2 describes the model and definitions. Section 3 presents the results in the quantity competition market and price competition market. Section 4 ends the paper with the conclusions.

2.1 Framework and model

We follow Billand and Bravard (2004) modeling. Networks represent in this framework the externality benefit of information accessing (technical, technological, legal, marketing, management practices and the like). This information allows for firms who initiates and maintains the link to reduce production costs by adopting more efficient management practices. When externality flow is asymmetric (technically, by using directed graphs), firms who form links access to linked firm information and link formation cannot be refused. This modeling approach comprises situations like:

1. Access to firms' public information by
    a. Surfing competitors' web sites for acknowledging their products, pricing policies, market prestige and the like for benchmarking.
    b. Analyzing balance sheets of stock exchange's firms for determining their economic and financial performance,
    c. Consulting the Patent Register for competitors' new products,
    d. Reading business magazines that report competitors' best practices.
2. Access to firms' private information by
    a. Accessing through illegal ways other firms' information, like industrial espionage, among others.

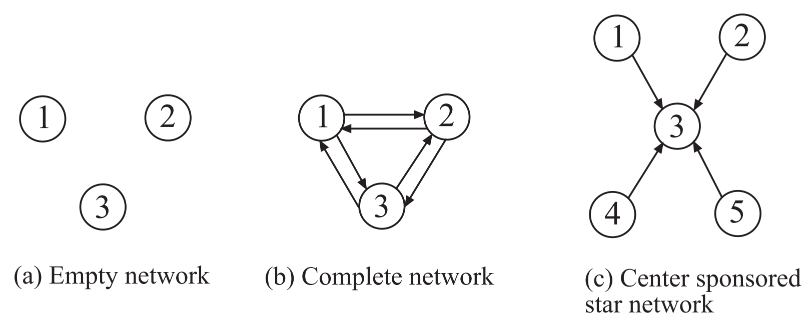
Next we define concepts to be applied later on. Let $N=\{1,\ldots,i,j,\ldots,n\}$ with $n\geq 3$, be the a set of firms. For any $i,j\in N$, $g_{i,j}=1$ means that a firm *i* has formed a direct link with firm *j*, $g_{i,j}=0$ in any other case. From here, we denote $g_i=\left(g_{i,1},\ldots,g_{i-1,0},g_{i+1,1},\ldots,g_{i,j},\ldots,g_{i,n}\right)$ to firm *i*'s link vector.

A network $g=\left\{\left(g_{i,j}\right)_{i\in N,j\in N}\right\}$ is a formal description of the directed links that exists among firms. Let *G* be the set of directed networks without loops (we suppose that a firm cannot form a link with itself).

We suppose that a link $g_{i,j}=1$ allows firm *i* to access to *j*'s information but not viceversa. We focus on one way resources flow. Let $N_i(g)=\{j|g_{i,j}=1\}$ be the set of firms *j* such that *i* obtains externalities from *j*. Let $n_i(g)$ be the cardinal of $N_i(g)$. We frequently refer to all other firms distinct from *i* as *i*'s opponents and will be noted as $-i$. We note $n_{-i}(g)=\sum_{i\neq j} n_j(g)$ as the number of links in the network *g* excluding those links generated by firm *i*. $n_{-i}(g)$ can be interpreted as the number of externalities that benefits to all other firms except firm *i*.

We define main network topologies that will be used extensively thru our work. A network *g* is *complete* if for every pair of firms *i* and *j*, there exists a link from *i* to *j*. Complete network is denoted as g^c. A network *g* is a *center sponsored star* if and only if there is a firm *i* such that *i* has formed one link with every firm *j*, and every $j \neq i$ has formed no link at all (Figure 1). Center sponsored star network is denoted as g^s. A network *g* is *empty* if there is no firm that has formed any link. This network is denoted as g^e.

Figure 1. Network architectures



2.2 Linking and cost reduction

We suppose that establishing a link requires a fixed investment cost given by $\delta > 0$. We suppose that firms are initially symmetric with a nonnegative fixed cost γ_0 and identical cost functions[4]. We consider that establishing a link is a way of cost reduction. More specifically, we suppose that firm's marginal and average variable cost of a generic firm $i \in N$ has the same functional form:

$$c_i(n_i(\mathrm{g}))=\gamma_0-\gamma n_i(\mathrm{g}), \quad (1)$$

[4] γ_0 can be thought as the production cost under isolation.

where $\gamma_0, \gamma \in R_+^*$ such that $\gamma_0 > \gamma(n-1)$. A network g induces an average variable cost vector given by the following function: $c(n_i(\mathrm{g})) = \{c_1(n_i(\mathrm{g})), c_2(n_i(\mathrm{g})), \ldots, c_i(n_i(\mathrm{g})), \ldots, c_n(n_i(\mathrm{g}))\}$.

2.3 Equilibrium networks

A network $\mathrm{g} \in G$ is said to be an equilibrium if, leaving constant the set of link formed by another firms, any firm that has a connection to any other firm in $\mathrm{g} \in G$ has an incentive of keeping that link. Moreover, any firm that is not connected to another firm in $\mathrm{g} \in G$ has no incentive to form a link with this other firm. Let g' be a network where *i* is the only firm that has the same links in g.

We define $\Pi_i(n_i(\mathrm{g}), n_{-i}(\mathrm{g}))$ as net benefits of firms $i \in N$. A network *g* is an equilibrium network, if for all *i*, we have that:

$$\Pi_i(n_i(\mathrm{g}), n_{-i}(\mathrm{g})) \geq \Pi_i(n_i(\mathrm{g}'), n_{-i}(\mathrm{g}')),\ \forall \mathrm{g}' \in G \qquad (2)$$

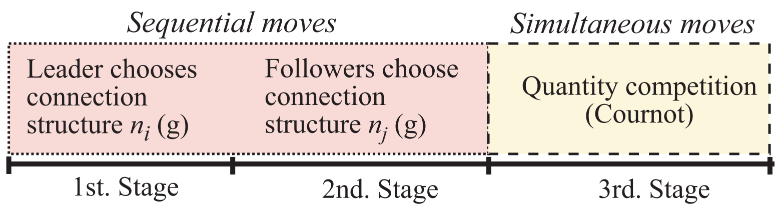
Next we begin with the description of the competition structure.

3. Stackelberg networks

Quantity competition *á la* Stackelberg is represented by a three stages game. In the first stage, only leader firm moves by choosing who to connect with; in the second stage, follower firms observe leader firm choice and make their own connection decision. Finally, in the third stage, firms simultaneously compete in quantities or prices. As leader firm moves first, it will envisage follower firms' behavior and maximize benefits accordingly by using backward-induction. That will be the solving method for the games to come.

Then Stackelberg oligopoly is represented as a three stages game: First, leader firm choose the number of firms to link with; second, followers firm observe leader firm connections and establish their own linking decisions, and; third, there's quantity competition. Game timing is a sequential connections game following by simultaneous (Cournot) quantity game at the end (Figure 2).

Figure 2. Game Setting



Single leader rest followers is a market where two types of firms exist: the leader firm i with $i \neq j$ that for convenience we noted $i \in [1]$ and there exists $(n-1)$ firms labeled with the j subindex such that $j \neq i$ and $j \in [2,\ldots,n]$ that we will call the set of followers firms. Two kind of market competition will be modeled: quantity and price competition. Next section begins with quantity competition definitions and modeling.

3.1 Stackelberg quantity competition

Let q_i be the quantity produced by firm i and p the market price. We model a market with homogeneous products and quantity competition. We suppose a linear inverse demand function:

$$p = \begin{cases} \alpha - \sum_{i \in N} q_i & \text{if } \sum_{i \in N} q_i < \alpha \\ 0 & \text{if } \sum_{i \in N} q_i \geq \alpha \end{cases}, \text{ with } \alpha > 0 \tag{3}$$

We suppose that the nonnegative production condition (NPC) $(\alpha - \gamma_0) > \gamma(n-1)$ verifies. Given this schema we postulate the following Lemma:

Lemma 1. *Given any network* $\mathrm{g} \in G$ *and suppose it is verified (1), (3) and the NPC. Suppose a single leader rest followers market with quantity competition. Then, Stackelberg leader firm's reaction function* q_i^* *is* $q_i^*\left(n_i(\mathrm{g}), n_j(\mathrm{g})\right) = \dfrac{(\alpha - \gamma_0) + \gamma n n_i(\mathrm{g}) - (n-1)\gamma n_j(\mathrm{g})}{n+1}$. *If all the above verifies, Stackelberg follower firm's reaction function* q_j^* *is* $q_j^*\left(n_i(\mathrm{g}), n_j(\mathrm{g})\right) = \dfrac{(\alpha - \gamma_0) + 2\gamma n_j(\mathrm{g}) - \gamma n_i(\mathrm{g})}{n+1}$.

***Sketch of the Proof*:** Using (1) and (3) and solving by backward induction the Stackelberg game gives leader firm's reaction function $q_i^*(\cdot)$ and follower firm's reaction function $q_j^*(\cdot)$.

Equilibrium benefits are defined by:

$$\Pi_i\left(n_i(\mathrm{g}), n_j(\mathrm{g})\right) = \left(q_i^*\left(n_i(\mathrm{g}), n_j(\mathrm{g})\right)\right)^2 - \delta n_i(\mathrm{g}) \qquad (4)$$

Following this we postulate:

> **Proposition 1**. *Suppose that NPC verifies and there is quantity competition among firms in a single leader rest followers Stackelberg market. Suppose that it is verified (1) and (3). Then, in an equilibrium network* **g*** *for the* ***leader firm*** *i,* $n_i\left(g^*\right) \in \{0, n-1\}$. *More precisely:*
>
> 1. *if* $\delta < n\gamma \frac{2(\alpha-\gamma_0)+\gamma\left(n\gamma\left(2\left(1-n^3\right)+7\left(n^2-1\right)\right)\right)}{(n+1)^2}$, *then the complete network* $\mathbf{g^c}$ *is the only equilibrium network;*
> 2. *if* $\delta > n\gamma \frac{2(\alpha-\gamma_0)+(n-2)(n-1)\gamma}{(n+1)^2}$, *then the empty network* $\mathbf{g^e}$ *is the only equilibrium network;*
> 3. *if* $\delta \in \left(n\gamma \frac{2(\alpha-\gamma_0)+n\gamma\left(2\gamma\left(1-n^3\right)+7n\gamma(n-1)\right)}{(n+1)^2}, n\gamma \frac{2(\alpha-\gamma_0)+(n-2)(n-1)\gamma}{(n+1)^2}\right)$, *then in a equilibrium network* $\mathbf{g}^*$, *leader firm i connects in such a way that* $n_i\left(\mathrm{g}^*\right) \in \{2, n-2\}$.

Proof: See Appendix I.

For the follower firm case, which works with the analogous benefit function $\Pi_j\left(n_i(\mathrm{g}), n_j(\mathrm{g})\right) = \left(q_j^*\left(n_i(\mathrm{g}), n_j(\mathrm{g})\right)\right)^2 - \delta n_j(\mathrm{g})$, it is established the following proposition:

> **Proposition 2**. *Suppose that NPC verifies and there is quantity competition among firms in a single leader rest followers Stackelberg market. Suppose also that (1) and (3) verify. Then, in an equilibrium network* ***g**** *for all* ***follower firm*** *j,* $n_j\left(g^*\right) \in \{0, n-1\}$. *More precisely:*
>
> 1. *if* $\delta < 4\gamma \frac{(\alpha-\gamma_0)}{(n+1)^2}$, *then the complete network* $\mathbf{g^c}$ *is the only equilibrium network;*
> 2. *if* $\delta > 4\gamma \frac{(\alpha-\gamma_0)+\gamma(n-1)}{(n+1)^2}$, *then the empty network* $\mathbf{g^e}$ *is the only equilibrium*

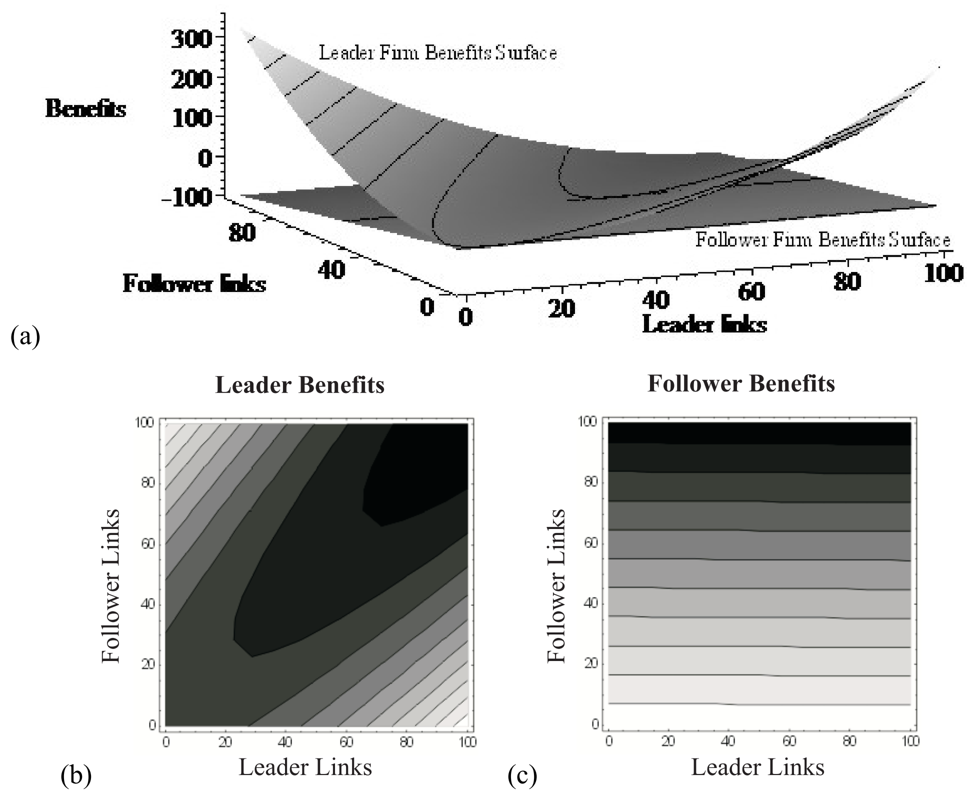
network;

3. *if* $\delta \in \left(4\gamma \frac{(\alpha-\gamma_0)}{(n+1)^2}, 4\gamma \frac{(\alpha-\gamma_0)+\gamma(n-1)}{(n+1)^2}\right)$, *then in an equilibrium network* $\mathbf{g}^*$ *there are* x *firms* j, $x \in \{2,\ldots,n-2\}$, *such that* $n_j\left(\mathrm{g}^*\right) = n-1$ *and* $n - x$ *firms* j *such that* $n_j\left(\mathrm{g}^*\right) = 0$.

Sketch of the Proof: It is analogous to Proposition 1 (Appendix I) but with the follower firm's reaction function.

Example 1. We reproduce Example 1 from Billard and Bravard (2004: 598) so we define α = 200, γ_0 = 50, γ = 0.2, n = 100 and δ = 1. Figure 3.a represents benefit surface for a leader and follower firm given this initial values and increasing link quantities for both, from 0 to 100. It is shown the exponential benefits for leader firm if she does not coordinate with follower firm on initiated links quantities choice.

Figure 3. Benefit surfaces and contour plots (*n* =100)



Leader firm obtains, at least, the same benefits that any follower. If both agree in the number of links to be established, both will obtain the same amount of benefits. Disagreement in the numbers of established links reports higher benefits to leader firm.

However, an interesting aspect arises. We can present bi-dimensional information of Figure 3.a by plotting the functional form contours given the Example 1's values. This is done in Figures 3.b and 3.c where in Figure 3.a contour plots are presented. Axis represents number of formed links by leader (horizontal) and follower (vertical) firms and lighter gray scale indicates higher benefits surface and darker gray scale represent lower benefits surface. Model supposes that connection infrastructure investment cost is exogenous. So, as leader firm moves first she will choose the number of connections that would maximizes her benefits. Given Example 1's data this is observed in $n_i(\mathrm{g}) = 99$ (Figure 3.b). In the second stage of the game, follower firms will observe this choice and will choose their optimal connections' strategy. This is observed in Figure 3.c in the lighter color contour plot, that is $n_j(\mathrm{g}) = 0$. This way, these connections' strategies sustain a leader firm sponsored star network.

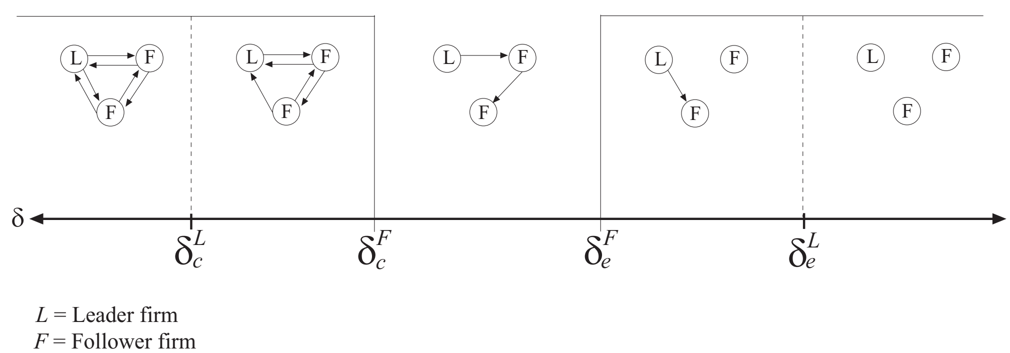
Another interesting analysis appears when we sort threshold values of connections infrastructure investment costs that determines optimum connections' infrastructure for both players. So, let $\delta_c^L = n\gamma \frac{2(\alpha-\gamma_0)+\gamma\left(n\gamma\left(2\left(1-n^3\right)+7\left(n^2-1\right)\right)\right)}{(n+1)^2}$ be the value of δ to which leader firm (*L*) decides for embarking in complete network g^c connections' strategy and let $\delta_e^{\mathrm{L}} = n\gamma \frac{2(\alpha-\gamma_0)+(n-2)(n-1)\gamma}{(n+1)^2}$ be the threshold value that determines if leader firm decides to adopt the empty network g^e connections' policy. Let also $\delta_c^F = 4\gamma \frac{(\alpha-\gamma_0)}{(n+1)^2}$ and $\delta_e^F = 4\gamma \frac{(\alpha-\gamma_0)+\gamma(n-1)}{(n+1)^2}$ be the δ threshold values analogous for the follower firm (*F*). Next we postulate:

> **Corollary 1**. *Given the above definitions, δ threshold values for a quantity competition Stackelberg market are sorted, form highest to lowest, in the following way:* $\delta_e^L > \delta_e^F > \delta_c^F > \delta_c^L$.

Proof: Trivial. □

The interval of sorted threshold values δ let us observe that leader firm values are significantly separated between each others. The same range of values for the follower firm strategy choice is narrower. Figure 4 represents a network with one leader and two follower firms over the range of δ threshold values. From left to right we depart from the lowest δ value that coincides as best response for leader and follower firm for full connectivity among them, which means that $\mathbf{g}^c$ (complete network) becomes optimal. However, just as the threshold value overcomes δ_c^L, leader firm has no incentive to play $\mathbf{g}^c$ and stops connecting to all follower firms. Though, on the interval $\delta_c^L < \delta < \delta_c^F$, follower firms still play complete network as best response. Just on the interval $\delta_c^F < \delta < \delta_e^F$ neither follower nor leader firms have incentives to play complete connections so connections exist but are neither complete nor empty. Once $\delta > \delta_e^F$, follower firms find optimum to keep themselves disconnected while leader firm still find optimum neither full nor empty connections. Finally, when $\delta > \delta_e^L$ g^e topology is the best decision for both.

Figure 4. Threshold values and optimal topologies for type of firm



It can be noted that leader firm requires for a much larger investment cost for choosing isolation as optimal response. Given that, it could be more likely that a leader firm play any form of connections' strategy compared to any follower firm. Under this conjecture, leader firm sponsored star networks would emerge as observed optimal response.

3.1.2 Comparing results with Cournot outcomes

Comparing optimal quantities produced under Cournot or Stackelberg competition usually arises the microeconomic interrogative: how each model's optimal quantities compare?

Given that both models used the same cost and demand functions, comparison should be direct. As shown in specific comparative literature (Dastidar 2004: 559, for a focused description and Vives, 1999 for a more general treatment) among classical oligopoly models we should verify that Cournot's optimal quantity should be an intermediate value between optimal leader firm quantity and optimal follower firm quantity in the Stackelberg case.

We define $q^C = \frac{(\alpha-\gamma_0)+n\gamma n_i(g)-\gamma\sum_{j\neq i} n_j(g)}{n+1}$ as firm's reaction function in Cournot market (see Billand & Bravard, 2004), $q_L^S = \frac{(\alpha-\gamma_0)+\gamma n n_i(g)-(n-1)\gamma n_j(g)}{n+1}$ as leader firm's reaction function in Stackelberg competition and $q_F^S = \frac{(\alpha-\gamma_0)+2\gamma n_j(g)-\gamma n_i(g)}{n+1}$ as the analogous follower firm's reaction function. Effectively we obtain that $q^C = q_L^S$, while $q_F^S = q_L^S$ if and only if $n_i(g) = n_j(g)$ given $q_L^S - q_S^S = \gamma(n_i(g) - n_j(g))$. Under equilibrium, well identified optimal topologies are the complete and empty network, where in both $n_i(g) = n_j(g)$ verifies, leader firm and follower firms coordinate in the number of links established. As suggested before, in intermediate network configurations, leader firm will obtain higher benefits that under coordination. So if the number of links is uncoordinated it's better for the leader firm. Under any other case, leader firm earns as much as any follower firm.

It is interesting to remember that information is a public good in this setting, so if leader firm connects first to any number of firms that do not prevent follower firms to connect using the same or any other strategy.

Next, we develop the price competition variant under the Stackelberg setting.

3.2 Stackelberg price competition

New definitions are required. Let $D(p) = \alpha - p$ be the market demand function. For price competition case we define demand faced by firms as:

$$d_i(p_i)\begin{cases} D(p_i) & \text{if } p_i < p_j, \forall j \\ D(p_i)/k & \text{if } p_i = p_j, \forall j, \text{ with equality for } k \text{ firms} \\ 0 & \text{if } p_i > p_j, \quad \text{for some } j \neq i \end{cases} \tag{5}$$

Total net benefits for firm i is given by:

$$\Pi_i(n_i(g), n_{-i}(g)) = d_i(p_i)(p_i - c_i(n_i(g))) - \delta n_i(g) \tag{6}$$

Game setting remains similar in the first two sequential decisions stages but now in the third competition is in prices (it analogous with Figure 3 representation but in the third stage there is price competition). What is the optimum price and what topology sustains it?

Definitely, leader firm would expects follower firms play Bertrand price on the second stage, so leader firm's manager should anticipate this movement and play accordingly. On the third stage, firms compete in prices.

We suppose that demand faced by firm *i* if it fixes price p_i is given by (5). Firm *i*'s total net benefit flows are determined by (6). Network equilibrium under price competition in Stackelberg market is given by the following lemma:

> ***Lemma 2.*** *Suppose a single leader rest followers Stackelberg market. Suppose that (1), (5), (6) and the NPC are verified. In a price competition market, in equilibrium there only one firm who establish connections to all the other firms and that unique firm is the leader firm.*

Proof: See Appendix II. In words, in the first stage leader firm sets a price based on other firms' connections strategy profile. As cost function is decreasing on number of links, prices quickly tend to fixed cost. Then, leader firm anticipates follower firms will play Bertrand equilibrium price so she fixes the minimum feasible amount and follower firms will be out of the connection market in the second stage.

Once established that by moving first leader firm obtains an advantage, only rests to determine optimum market topologies. For that to be accomplished, we postulate the following proposition:

> **Proposition 3.** *Suppose that equations (1), (5), (6) and the NCP are verified. Suppose that is a price competition Stackelberg market. Then the only firm who establish links is the leader firm and for that firm it is verified that:*
>
> 1. *if* $\delta > \gamma(\alpha - \gamma_0)$, *the empty network,* g^e, *is the only equilibrium network;*
> 2. *if* $\delta < \gamma(\alpha - \gamma_0)$, *the leader-firm sponsored star network,* g^s, *is the only equilibrium network.*

Proof: See Appendix II.

Paradoxically there are few examples of Stackelberg price competition models in the economic literature. A good exception is Dastidar (2004) who finds out that in duopoly sequential price competition leader firm gets a higher market share at a lower price and follower firm gets a smaller market share but a higher price. In equilibrium, both earn equal profits. Neither leader nor follower firm get advantage under this setting. In our case, on the contrary, leader firm gets all. As reflected in Goyal and Joshi (2003), in price competition markets competition is so intense that connection's probability among firms becomes smaller.

We end this paper with the conclusions.

4. Conclusions

To move faster towards getting competitors information could be translated in higher benefits. That would be the main finding of this paper. This is another example of first mover advantage (Gal-Or, 1985). In this case, firms look for allocate resources in economic intelligence investment. Moving first gives to early movers a benefit that could be understood as the benefits of spying the competence. This is translated in copying competitors' best practices that, at the same time, it is transformed in the adoption of lower cost production techniques.

It is interesting to note that in our model late movers (followers) are not restricted by early mover choices but by the exogenous given cost in connections infrastructure. Connecting firms (spying on them) behaves as a public good. If the leader firm connects to any number of followers this will not constraint future follower firms choices of connection. This is something that deserves a better modeling as future research path.

Finally, leader firm has a wider range of threshold values for adopting optimal topologies. This is another advantage that entails greater versatility for leader firm connections' structure choice.

Other paths of future research comprehends: (i) endogeneizing connections structure's investment cost for dealing with the possibility that firms could modify production structure for adapting themselves in the connection market competition; (ii) endogeneizing the process of selecting leader firm role assignment as suggested by footnote 3's quotations.

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Appendix I. Quantity competition

Proof of Proposition 3: Demonstration is in three parts. First we show that in an equilibrium network g^*, for leader firm *i*, we have that $n_i\left(g^*\right)\in\{0,n-1\}$.

(1) Let **g*** be an equilibrium network where at least leader firm *i* is such that $n_i\left(g^*\right)\in\{2,n-2\}$. Let suppose that $n_i\left(g^*\right)=k$. We show that if *i* has no incentive to sever a link, then it has an interest in forming a link (and inversely). We know that in an equilibrium network a firm never has an incentive for severing a link. So we have that: $\Pi_i\left(k,n_j\left(g^*\right)\right)-\Pi_i\left(k-1,n_j\left(g^*\right)\right)>0$, that is $n\gamma\left(\frac{2(\alpha-\gamma_0)+(2k-1)n\gamma-2(n-1)\gamma n_j(g)}{(n+1)^2}\right)-\delta>0$, and then $\delta<n\gamma\frac{2(\alpha-\gamma_0)+(2k-1)n\gamma-2(n-1)\gamma n_j(g)}{(n+1)^2}=A$. In the same manner, in an equilibrium network a firm never has incentives to form new links. That is to say that, $\Pi_i\left(k,n_j\left(g^*\right)\right)-\Pi_i\left(k+1,n_j\left(g^*\right)\right)>0$, that is $\frac{-n\gamma\left(2\alpha+(1+2k)n\gamma\right)+2n\gamma\left((n-1)\gamma n_j(g)+\gamma_0\right)}{(n+1)^2}+\delta>0$ and then $\delta<n\gamma\frac{2(\alpha-\gamma_0)+n\gamma(2k+1)-2(n-1)\gamma n_j(g)}{(n+1)^2}=B$. This way, we must have that $A-B>0$. Which never verifies given $A-B=-\frac{2(\gamma n)^2}{(n+1)^2}<0$. Then, if a leader firm *i* has formed *k* links such that $k\in\{2,\ldots,n\}$ then is never in equilibrium. In equilibrium, a leader firm forms none or *n* – 1 links with its followers.

(2) This is a two parts demonstration. In the first part we are going to see that a firm once has formed a complete network has no incentives in sever links. In 2.a we show that if $\delta < n\gamma \frac{2(\alpha-\gamma_0)+(n-2)(n-1)\gamma}{(n+1)^2}$ then $\mathbf{g}^c$ is an equilibrium network and in 2.b we show that there's no other network that could be an equilibrium network.

a) A leader firm has no incentives in sever link under complete network $\mathbf{g}^c$ configuration. Then we have to probe that: $\Pi_i\left(n-1,(n-1)^2\right)-\Pi_i\left(n-1-k,(n-1)^2\right)>0$. In fact we arrive to $\frac{-n\gamma\left(-2\alpha+2\gamma_0+\gamma\left(n\left(k+2(n-2)^2\right)-2\right)\right)}{(n+1)^2}-\delta>0$ which verifes that $\delta < n\gamma\frac{2(\alpha-\gamma_0)+n\gamma\left(2\gamma\left(1-n^3\right)+8n\gamma(n-1)-kn\gamma\right)}{(n+1)^2}$. If this inequality verifies for $k=n-1$ then it verifies for all *k*. So we have that $\delta < n\gamma\frac{2(\alpha-\gamma_0)+\gamma\left(n\gamma\left(2\left(1-n^3\right)+7\left(n^2-1\right)\right)\right)}{(n+1)^2}$. This result will be necessary next.

b) We show now that an equilibrium network $\mathbf{g} \neq \mathbf{g}^c$ is not an equilibrium network. In (1) we proved that in equilibrium a leader firm form link with *all* or *none* of the follower firms. For confirming this outcome, we are going to prove if there's is a chance that a leader firm could establish connections with every firm less one or may be with a cluster of firms and this would be an equilibrium outcome. We establish that a contradiction by supposing that there exists an equilibrium network $\mathbf{g}^*$ such that the leader firm establish no contacts. As it is an equilibrium network it should be check that: $\Pi_i\left(0,n_j\left(\mathrm{g}^*\right)\right)-\Pi_i\left(k,n_j\left(\mathrm{g}^*\right)\right)>0$, or what is the same $\delta > n\gamma\frac{2(\alpha-\gamma_0)+\gamma\left(n\left(k-2n_j(\mathrm{g})\right)+2n_j(\mathrm{g})\right)}{(n+1)^2}$. Then, there is a configuration n_j ($\mathbf{g}^*$) such that a leader firm *i* has no incentives of forming any links with the follower firms whatever their connections' structure. So we could have that: $\delta > n\gamma\frac{2(\alpha-\gamma_0)+\gamma\left(n\gamma\left(2\left(1-n^3\right)+7\left(n^2-1\right)\right)\right)}{(n+1)^2}$, that is a contradiction with (2.a).

(3) Finally, we show that network $\mathbf{g}^e$ is an equilibrium network for the leader firm if $\delta > n\gamma\frac{2(\alpha-\gamma_0)+(n-1)n\gamma}{(n+1)^2}$. We prove first that if $\mathbf{g}^e$ is an equilibrium then, in the second part, there's no other equilibrium network.

a) First we establish that any firm has incentive to form links in $\mathbf{g}^e$. Then we have that: $\Pi_i(0,0)-\Pi_i(k,0)>0$, from which we obtain that $\delta > n\gamma\frac{2(\alpha-\gamma_0)+k\gamma n}{(n+1)^2}$. If this result verifies for $k=n-1$ then it is verified for all *k*. We obtain that $\delta > n\gamma\frac{2(\alpha-\gamma_0)+(n-1)n\gamma}{(n+1)^2}$.

b) Now we demonstrate that there's no other equilibrium than the empty network $\mathbf{g}^e$ when it emerges as an optimum topology. We have proved in the first part that a network $\mathbf{g}$ where exists a leader firm i such that $n_i(\mathrm{g}) \notin \{0, n-1\}$ cannot be an equilibrium network. Then we must prove that a network **g**, where exists at least one firm i such that $n_i(\mathrm{g}) = n-1$ is not an equilibrium network. For establishing a contradiction, let suppose that an equilibrium network, $\mathbf{g}^*$, where there is at one firm i such that $n_i(\mathrm{g}^*) = n-1$. We have that $\Pi_i\left(n-1, n_j\left(\mathrm{g}^*\right)\right) - \Pi_i\left(0, n_j\left(\mathrm{g}^*\right)\right) > 0$, which verifies that $\delta < n\gamma \frac{2(\alpha-\gamma_0)+(n-1)\left(n\gamma-\gamma n_j(\mathrm{g})\right)}{(n-1)^2}$, so in this particular case should verifies too: $\delta < n\gamma \frac{2(\alpha-\gamma_0)+(n-1)n\gamma}{(n+1)^2}$. Contradiction. □

Appendix II. Price competition

Proof of Lemma 2.

A. Backward induction first stage

We begin by presenting the following lemma:

> **Lemma AII.1.** *(Billand & Bravard, 2004: 601) In equilibrium, there's at least one follower firma that form links.*

Proof del Lemma AII.1: We establish a contradiction by supposing that there exists an equilibrium network where two (follower) firms j_1 and j_2 has formed links such that $j_1, j_2 \in \{2, \ldots, n\}$. Let suppose that $n_{j_2}(\mathrm{g}) \geq n_{j_1}(\mathrm{g})$ verifies. Given now that $c_{j_2}\left(n_{j_2}(\mathrm{g})\right) \leq c_{j_1}\left(n_{j_1}(\mathrm{g})\right)$, j_1's brute benefit is null given that $p_{j_1} = \gamma_0 - \gamma n_{j_1}$ in equilibrium. For that, we obtain as net benefit's main component is δn_{j_1}:

$$\Pi_{j_1}\left(n_{j_1}(\mathrm{g}), n_{j_2}(\mathrm{g})\right) = -\delta \sum_{j_1 \in N \setminus \{j_2\}} g_{j_1, j_2}$$

Given that firm j_2 has a variable cost $c_{j_2}\left(n_{j_2}(\mathrm{g})\right) < \gamma_0$, firm j_1 should not produce anything if it has no formed any link. Therefore we have

$$\Pi_{j_1}\left(0, n_{j_2}(\mathrm{g})\right) = 0$$

It follows that

$$\Pi_{j_1}\left(0, n_{j_2}(\mathrm{g})\right) - \Pi_{j_1}\left(n_{j_1}(\mathrm{g}), n_{j_2}(\mathrm{g})\right) = \delta \sum_{j_1 \in N \setminus \{j_2\}} g_{j_1, j_2} > 0$$

given that we have supposed that $\sum_{j_1 \in N \setminus \{j_2\}} g_{j_1, j_2} \geq 1$. This is a contradiction. Specifically for own setting, in equilibrium we have that in Stackelberg game's second stage only one follower firm will establish links, while the rest of follower firms will not establish any links. □

Let ξ be the lowest feasible monetary denomination, which we suppose that converges to zero. Then we establish the price this firm would set in the market.

Lemma AII.2. *(Billand & Bravard, 2004: 601) Suppose there is one and only one firm in the market (say firm l) that forms links. Then the Bertrand equilibrium price is given by* $p_l = \gamma_0 - \xi$.

For our case we suppose that *l* is such that $l \in \{2, \dots, n\}$, i.e., it is part of the set of follower firms which behavior leader firm must anticipate. Given that follower firm *l* will face a demand function such that:

$$d_l(p_l)\begin{cases} D(p_l) & \text{if } p_l < p_j, \forall j \in N \\ \frac{D(p_l)}{k} & \text{if } p_l = p_j, \forall j \in N, \text{ with equality for } k \text{ firms} \\ 0 & \text{if } p_l > p_j, \qquad \text{for some } j \neq l \end{cases}$$

which means that if follower firm sets a lower price than any other firm *j* she will supply all the demand alone. If she sets an equal price that any other firm *j*, they will equally shared the demand and if she sets price higher than *j*'s then she will no supply anything. For that if firm *l* is the only one that has formed link then she has the lowest marginal cost then she sets price in $p_l = \gamma_0 - \xi$ so to displace the rest of the firms of the market. So, there will be $n-3$ firms that will not establish any link while one of them, conventionally denoted as firm *l* will form links with all the others follower firms and the leader firm by setting a price a bit lower to fixed cost γ_0 .

How leader firm would react to that? Given that *l* has set a price $p_l = \gamma_0 - \xi$, *i* will play again and would set a price a bit even lower given that there would be only one firm establishing links. Using the same line of reasoning as Lemma AII.1 and AII.2, there would be only one firm forming link and that firm will set the lowest price. Facing the same demand function:

$$d_i(p_i)\begin{cases} D(p_i) & \text{if } p_i < p_l = \gamma_0 - \xi, \\ \frac{D(p_i)}{2} & \text{if } p_i = p_l = \gamma_0 - \xi, \text{ with equality for } i \text{ y } l, \\ 0 & \text{if } p_i > p_l = \gamma_0 - \xi \qquad \text{for some } j \neq l \end{cases}$$

leader firm must decide if she will match firm *l*'s price or if she will cut the price. If she matches p_l given the model's demand rationing rule they will share demand with *l*. If she cuts the price they will earn positive profits. The same would happens if leader firm cut the price by another lowest feasible monetary unit, ξ, being the leader new price set in $p_i = \gamma_0 - 2\xi$. Let find out which decision brings more benefits to leader firm:

$$d(p_i)\left(p_i - c_i(n_i(\mathrm{g}))\right) - \tfrac{d(p_i)}{2}\left(p_l - c_i(n_i(\mathrm{g}))\right) \geq 0$$

$$\underbrace{(\alpha-\gamma_0+2\xi)}_{d(p_i)}\left(\underbrace{\gamma_0-2\xi}_{p_i}-\underbrace{\gamma_0+\gamma n_i(\mathrm{g})}_{c_i(n_i(\mathrm{g}))}\right)-\underbrace{\tfrac{1}{2}(\alpha-\gamma_0+\xi)}_{d(p_i)/2}\left(\underbrace{\gamma_0-\xi}_{p_l}-\underbrace{\gamma_0+\gamma n_i(\mathrm{g})}_{c_i(n_i(\mathrm{g}))}\right)\geq 0$$

$$\frac{1}{2}\left(n_i(\mathrm{g})\gamma(\alpha-3\xi)+\xi(3\alpha+7\xi)-(n_i(\mathrm{g})\gamma+3\xi)\gamma_0\right)\geq 0$$

Given that $\xi \to 0$, then we have that $\frac{1}{2}\gamma n_i(\mathrm{g})(\alpha-\gamma_0)\geq 0$, which verifies for all feasible values of the game. So, leader firm will set $p_i = \gamma_0 - 2\xi$. Under certain functional forms of $D(p)$, price elasticity could play a different role in this interpretation.

B. Backward induction second stage

In the second stage, follower firms will watch the price set by leader firm and they will set their own optimal price. But they will find that $d_j(p_j)=0$ given that $\gamma_0-\xi = p_j > p_i = \gamma_0 - 2\xi$ then for avoiding losses associated with $\Pi_j(n_i(\mathrm{g}), n_j(\mathrm{g})) = -\delta n_j(\mathrm{g})$ they will choose $n_j(\mathrm{g})=0$. □

Proof of Proposition 3. Leader firm's benefit maximization will be determined by

$$\Pi_i(n_i(\mathrm{g}), n_j(\mathrm{g})) = \underbrace{(\alpha-\gamma_0+2\xi)}_{d(p_i)}\left(\underbrace{\gamma_0-2\xi}_{p_i}-\underbrace{\gamma_0+\gamma n_i(\mathrm{g})}_{c_i(n_i(\mathrm{g}))}\right)-\delta n_i(\mathrm{g})$$

which is the same that:

$$\delta = \frac{\gamma n_i(\mathrm{g})(\alpha-\gamma_0+2\xi)-2\xi(\alpha-\gamma_0+2\xi)}{n_i(\mathrm{g})}$$

$$\delta = \gamma(\alpha-\gamma_0+2\xi)-\frac{2\xi(\alpha-\gamma_0+2\xi)}{n_i(\mathrm{g})}$$

and given that $\xi \to 0$, we have that

$$\delta = \gamma(\alpha-\gamma_0).$$

As in Billand & Bravard (2004: 608) here we can distinguish two cases:

1. If $\delta > \gamma(\alpha - \gamma_0)$ benefits function $\Pi_i\left(n_i\left(\mathrm{g}^*\right), n_j\left(\mathrm{g}^*\right)\right)$ would be decreasing in $n_i\left(\mathrm{g}^*\right)$, which implies that leader firm will not have incentives to form links. Leader firm remains isolated and optimal market topology will be the empty network.
2. If $\delta < \gamma(\alpha - \gamma_0)$ then benefits function $\Pi_i\left(n_i\left(\mathrm{g}^*\right), n_j\left(\mathrm{g}^*\right)\right)$ would be increasing in $n_i\left(\mathrm{g}^*\right)$, which implies that leader firm would have higher incentives to form links. Leader firm will connect to all follower firms and optimal market topology will be leader firm sponsored star network. □