I – Q Cycles

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Abstract

We develop a model of "intrinsic" business cycles, driven by the decentralized behaviour of entrepreneurs and firms making continuous, divisible improvements in their productivity. We show how equilibrium cycles, associated with strategic delays in implementation and endogenous innovation, arise even in the presence of reversible investment. We derive the implications for the cyclical evolution of both tangible (physical) and intangible (knowledge) capital. In particular, our framework is consistent with key aspects of the somewhat puzzling relationship between fixed capital formation and the stockmarket at business cycle frequencies.

1 Introduction

Fluctuations in the aggregate investment rate are a central feature of the business cycle. As Figure 1 illustrates, the rate of U.S. investment in fixed, non–residential assets displays regular, and recurring patterns of activity over time.[1] In recent decades, economists have attempted to understand movements in aggregate investment as an optimal response to measurable incentives. In the RBC model, for example, fluctuations in aggregate investment are driven by exogenous disturbances that change the incentives to produce investment goods relative to consumption goods. However, the short–term empirical relationship between aggregate investment and contemporaneous measures of investment incentives is extremely weak. In particular, while there is some evidence of a long run relationship, neither micro nor macro level empirical work has generally found a significant short–run relationship between investment and Tobin's Q — the ratio of the equity value of firms, to the book value of the capital stock.

As is well known, one cannot necessarily infer from this that investment is sub–optimal because Tobin's Q need not reflect the *marginal* incentives to invest (see Abel, 1979 and Hayashi, 1982). Moreover, equity values are likely to include the values of intangible, as well as tangible capital, as emphasized by Hall (2001), so that measured investment and capital stocks may understate true levels. But then the question arises as to what kind of relationship we should expect to observe between investment and *measurable* proxies of financial incentives and financial values over the business cycle. Figure 1 shows the investment rate and Tobin's Q for the US between 1953 and 2003. Figure 2 shows the rate of change in the four–quarter moving average of each time series.

Because the investment rate is most highly correlated with the value of Tobin's Q lagged about four quarters, one interpretation of this data is that there is a lead–lag relationship between the two variables. In principle, such a relationship might occur for essentially mechanical reasons: a rise in Tobin's Q signals profitable investment opportunities, but actual investment can only respond with a considerable delay. Note that this is not the same as slow adjustment of the capital stock due, say, to a time–to–build constraint. RBC models require explicitly built–in "time to plan" constraints in order to explain the observed pattern in this way.[2] However, such an interpretation leaves unanswered what we view as the critical puzzle of investment cycles — why should these apparent investment opportunities arise simultaneously across diverse sectors of

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[1] The shaded regions in Figure 1 are NBER–dated recessions. Investment data is constructed from NIPA. Data for Tobin's Q is constructed using Federal Reserve quarterly data.

[2] Christiano and Todd (1996), Bernanke, Gertler and Gilchrist (1999) and Christiano and Vigfusson (2001) introduce "time to plan" as a fixed time period between the date when the decision to invest more (less) is made and the date when the actual funds are allocated. Wen (1998) suggests perhaps one exception to this comment. He shows that a time to build framework with short lags in the production of capital can generate endogenously long lags in capital formation.

the economy? Moreover, why do many of the sharp upturns in the stockmarket appear to occur precisely during periods when physical investment rates are crashing (often during NBER–dated recessions)?

An alternative interpretation of Figures 1 and 2 is that a large component of stockmarket movements do not directly reflect new information about investment opportunities. Instead they are the result of costly bursts of investment in intangible capital that are anticipated to raise future productivity. According to this view, production units bundle together two distinct types of capital; physical and knowledge capital. Recently, Hall (2001), Hobijn and Jovanovic (2001) and Laitner and Stolyarov (2003) interpret the large equity adjustments since 1974 as the result of changes in the value of intangible capital associated with GPTs. [3] However, in order to assess the role of intangible capital at business cycle frequencies, one must consider the more modest and widely dispersed productivity improvements that characterize a typical business cycle. Fluctuations at this frequency exhibit striking sectoral co–movement in productivity, investment, output, and factor usage through the typical business cycle (see Christiano and Fitzgerald, 1998). To get at this sort of fluctuation it is necessary to understand the aggregate implications of the actions of "small" actors in multiple sectors making independent choices over the timing and level of investments in both physical and knowledge capital.

This is the goal of the present paper. We take as our starting point a recent body of theoretical work incorporating intangible capital accumulation into the analysis of business cycle fluctuations. This literature consists of two distinct strands. In one class of models (e.g. Bental and Peled, 1996, Freeman Hong and Peled, 1999, Li, 2001, Walde, 2002) cycles come about through innovation booms in a single sector.[4] While in such models it is relatively straightforward to accommodate capital accumulation, the single sector and (often) large, indivisible nature of innovations render them ill–suited to the analysis of high frequency business cycle fluctuation. As has often been remarked, these models are perhaps more suited to analysis of long–waves, or GPT type innovations. A second class of models emphasizes the bunching of productivity improvements in multiple disparate sectors due to optimal delay in implementation (e.g. Shleifer, 1986, Gale, 1996, Francois and Lloyd–Ellis, 2003). These models feature multiple disparate sectors (and firms) each following privately optimal investment decisions, and thus better correspond to NBER length business cycles which, by definition, involve cross–sectoral co–movement. However, the delay central to generating aggregate fluctuations in these models seems, at first blush, to be

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[3]Speficially the latter two are concerned with the IT negative shock of the early 70's and Hall focuses on the dramatic run-up in equities of the late 90's, which he also attributes to IT. There is little evidence supporting the arrival of GPTs at business cycle frequencies (see Jovanovic and Lach, 1998, and Andolfatto and Macdonald, 1998). Indeed, Laitner and Stolyarov (2003) cite evidence suggesting there have only been seven major technological innovations of this kind identified in the last 200 years.

[4]Aghion and Howitt (1992) also consider the existence of cycles, but in a model without capital.

undermined by the consumption–smoothing possibilities afforded by the accumulation of physical capital.[5] Households will partially consume any anticipated increases in productivity in advance of implementation thereby offsetting the gains to delay.

A key theoretical contribution of this paper is to demonstrate that, in fact, although capital accumulation does indeed smooth out consumption in this class of model, endogenous cycles still persist because fluctuations in the rate at which the capital stock itself is optimally adjusted provide incentives for disparate innovators to cluster implementation. We show that, when productivity improvements are costly to obtain, a robust cycle with endogenous delay in multiple sectors exists, even in the presence of smoothly accumulable and reversible physical capital. Moreover, within this class of cycling equilibria, there exist relatively weak sufficient conditions under which any such cyclical equilibrium is unique.[6] We use this framework to explore the interaction between tangible and intangible capital accumulation along the economy's cycling path and explore incentives for these as reflected by the value of the economy's financial assets.

Along the equilibrium growth path that we study, expansions are triggered by the implementation of accumulated productivity improvements. These improvements arrive stochastically across sectors during the recession, gradually increasing firm values, so that Tobin's Q starts to rise prior to the boom. However, since firms optimally choose to delay implementation, investment lags behind the increase in Q. Physical capital accumulation peaks at the start of the expansion and capital is accumulated continuously and smoothly, though at a declining rate, until its end. At this point, the economy enters a recessionary phase where output falls and capital accumulation declines more precipitously, though still remaining positive. The anticipated fall in demand causes Tobin's Q to fall even while investment is above its historical average so that Q leads investment into the recession too.

The framework we build endogenously generates relatively volatile investment (both physical and intangible) in the presence of smooth consumption behavior, and shows that these will naturally tend to display an asynchronous pattern. It also explains crucial aspects of observed timing at business cycle frequencies: the lead-lag relationship between Tobin's Q and observed investment; investment declines that precede recessions; an asynchronous pattern of cycles in

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[5]Matsuyama's (1999, 2001) is one approach that does not neatly fit this scheme. The cycles that arise in his model do not depend on delay, and are thus robust to capital accumulation through the cycle. However, Matsuyama's framework is more suited to understanding longer–term movements in the *nature* of growth (e.g. productivity slowdown), rather than business cycle fluctuations. In particular, there is no phase of his cycle that could be called a recession: production and consumption never decline. Moreover, productivity rises immediately when Q rises, reflecting the fact that innovations, which raise Q, are immediately implemented. Tobin's Q would be 1 throughout his cycle.

[6]Both the existence and uniqueness of the cyclical equilibrium are a result of endogenous innovation as in Francois and Lloyd-Ellis (2003). These results would not hold with exogenous innovation as in Shleifer (1986) and Gale (1996).

tangible and intangible capital formation. The process of investment in intangible capital over the business cycle generated by our model compares well with the counter–cyclical pattern of intangible capital formation implied by Hall's (2001) recently constructed estimates.

The paper proceeds as follows: Section 2 sets up the basic model and Section 3 posits the cyclical behavior of entrepreneurs and capital owners describes the cyclical growth path. Section 4 characterizes the implied movement of key aggregates and prices through the posited cycle and derives necessary conditions for implied behavior to be optimal. Section 5 develops sufficient conditions for the uniqueness of a stationary cyclical equilibrium and demonstrates existence. Section 6 explores the model's implications for key aggregates. Section 8 concludes. An appendix provides all proofs.

2 The Model

2.1 Assumptions

There is no aggregate uncertainty. Time is continuous and indexed by $t \geq 0$. The economy is closed and there is no government sector. The representative household has isoelastic preferences

$$U(t) = \int_t^\infty e^{-\rho(\tau-t)} \frac{C(\tau)^{1-\sigma} - 1}{1-\sigma} d\tau \tag{1}$$

where ρ denotes the rate of time preference and σ represents the inverse of the elasticity of intertemporal substitution. The household maximizes (1) subject to the intertemporal budget constraint

$$\int_t^\infty e^{-[R(\tau)-R(t)]} C(\tau) d\tau \leq S(t) + \int_t^\infty e^{-[R(\tau)-R(t)]} \left[w(\tau) + \psi(\tau) \right] d\tau \tag{2}$$

where $w(t)$ denotes wage income, $S(t)$ denotes the household's stock of assets (firm shares and capital) at time t and $R(t)$ denotes the discount factor from time zero to t. The term $\psi(\tau)$ represents lump–sum transfers from the government (see below). The population is normalized to unity and each household is endowed with one unit of labor hours, which it supplies inelastically.

Final output is produced according to a Cobb–Douglas production function utilizing physical capital, $K(t)$, and a continuum of intermediates, x_i, indexed by $i \in [0,1]$:

$$Y(t) = K(t)^\alpha X(t)^{1-\alpha}, \tag{3}$$

where

$$X(t) = \exp\left(\phi t + \int_0^1 \ln x_i(t) di \right) \tag{4}$$

Note that production is subject to exogenous total factor productivity growth at rate ϕ.[7] We assume that $\alpha < \min\left[\frac{1}{2}, \sigma\right]$ and that $\rho + (\sigma-1)\phi > 0$. Final output can be used for consumption,

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[7]Here we have expressed this as augmenting intermediate goods, but with Cobb-Douglas technology it could equally well be capital-augmenting or Harrod neutral.

$C(t)$, investment, $\dot{K}(t)$, or (potentially) stored:

$$C(t) + \dot{K}(t) + \delta K(t) \leq Y(t), \tag{5}$$

where δ denotes the rate of physical depreciation. Although we allow physical capital to be reversible in principle, in the equilibrium we study negative investment never actually occurs.

Output of intermediate i depends upon the state of technology in sector i, $A_i(t)$, and labor hours, $L_i(t)$, according to a simple linear technology:

$$x_i(t) = A_i(t)L_i(t) \tag{6}$$

Intermediates are completely used up in production, but can be produced and stored for later use. Incumbent intermediate producers must therefore decide whether to sell now, or store and sell later. We assume that any profits earned from intermediate production are taxed at a constant rate, ω. Revenue from this tax is redistributed back to households in a lump–sum fashion, so that the government's budget is balanced at every date t:

$$\psi(t) = \omega\pi(t). \tag{7}$$

Profit taxes are not a necessary part of a cyclical equilibrium. However, including them helps in terms of generating cycles with realistic features. Moreover, these tax effects can be thought of as a convenient representation of various other realistic factors, including implementation costs, imitation and labor market imperfections.[8]

The innovation process is exactly as in the quality–ladder model of Grossman and Helpman (1991). Competitive entrepreneurs in each sector allocate labor effort to innovation, and finance this by selling equity shares to households. The rate of success in instant t is $\mu h_i(t)$, where μ is a parameter, and h_i represents the labor hours allocated to innovation in sector i. At each date, entrepreneurs decide whether or not to allocate labor hours to innovation, and if they do so, how much. The aggregate labor hours allocated to innovation is given by

$$H(t) = \int_0^1 h_i(t)dt. \tag{8}$$

New ideas and innovations dominate old ones by a factor e^γ, where $\gamma > 0$. Successful entrepreneurs must choose whether or not to implement their innovation immediately or delay implementation until a later date.[9] Once they implement, the associated knowledge becomes

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[8] For example, in Francois and Lloyd–Ellis (2005), the deadweight loss due to a worker–firm contracting problem acts very much like a tax on profits.

[9] We adopt a broad interpretation of innovation. Recently, Comin (2002) has estimated that the contribution of measured R&D to productivity growth in the US is less that 1/2 of 1%. As he notes, a larger contribution is likely to come from unpatented managerial and organizational innovations.

publicly available, and can be built upon by rivals. However, prior to implementation, the knowledge is privately held by the entrepreneur.[10] We let the indicator function $Z_i(t)$ take on the value 1 if there exists a successful innovation in sector i which has not yet been implemented, and 0 otherwise. The set of instants in which new ideas are implemented in sector i is denoted by Ω_i. We let $V_i^I(t)$ denote the expected present value of profits from implementing a success at time t, and $V_i^D(t)$ denote that of delaying implementation from time t until the most profitable future date.

2.2 Definition of Equilibrium

Given initial state variables $\{A_i(0),\ Z_i(0)\}_{i=0}^1$, $K(0)$ an equilibrium for this economy consists of:
(1) sequences $\left\{\hat{p}_i(t),\ \hat{x}_i(t).\ \hat{L}_i(t),\ \hat{H}_i(t),\ \hat{A}_i(t),\ \hat{Z}_i(t),\ \hat{V}_i^I(t),\ \hat{V}_i^D(t)\right\}_{t\in[0,\infty)}$ for each intermediate sector i, and

(2) economy wide sequences $\left\{\hat{Y}(t),\ \hat{K}(t),\ \hat{R}(t),\ \hat{w}(t),\ \hat{q}(t),\ \hat{C}(t),\ \hat{S}(t)\right\}_{t\in[0,\infty)}$
which satisfy the following conditions:

• Households allocate consumption over time to maximize (1) subject to the budget constraint, (2). The first–order conditions of the household's optimization imply that

$$\hat{C}(t)^\sigma = \hat{C}(\tau)^\sigma e^{\widehat{R}(t)-\widehat{R}(\tau)-\rho(t-\tau)} \qquad \forall\ t,\tau, \tag{9}$$

and that the transversality condition holds

$$\lim_{\tau\to\infty} e^{-\hat{R}(\tau)}\hat{S}(\tau) = 0 \tag{10}$$

• Final goods producers choose capital and intermediates, x_i, to minimize costs given prices p_i, subject to (3). The derived demand for intermediate i is

$$x_i^d(t) = (1-\alpha)\frac{Y(t)}{p_i(t)}. \tag{11}$$

The conditional demand for capital is given by

$$K(t) = \frac{\alpha Y(t)}{q(t)} \tag{12}$$

• The unit elasticity of demand for intermediates implies that limit pricing at the unit cost of the previous incumbent is optimal. It follows that

$$p_i(t) = \frac{w(t)}{e^{-\gamma}A_i(t)} \qquad \forall t \tag{13}$$

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[10]Even for the case of intellectual property, Cohen, Nelson and Walsh (2000) show that firms make extensive use of secrecy in protecting productivity improvements. Secrecy likely plays a more prominent role for entrepreneurial innovations, which are the key here.

The resulting instantaneous profit (before any taxes) earned in each sector is given by

$$\pi(t) = (1 - e^{-\gamma})(1 - \alpha)Y(t). \tag{14}$$

• Labor markets clear:

$$\int_0^1 \hat{L}_i(t)di + \hat{H}(t) = 1 \tag{15}$$

• Arbitrage trading in financial markets implies that, for all assets that are held in strictly positive amounts by households, the rate of return between time t and time s must equal $\frac{\hat{R}(s) - \hat{R}(t)}{s - t}$.

• Free entry into innovation — entrepreneurs select the sector in which they innovate so as to maximize the expected present value of the innovation, and

$$\mu \max[\hat{V}_i^D(t), \hat{V}_i^I(t)] \leq \hat{w}(t), \quad \hat{H}_i(t) \geq 0 \qquad \text{with at least one equality.} \tag{16}$$

• At instants where there is implementation, entrepreneurs with innovations must prefer to implement rather than delay until a later date

$$\hat{V}_i^I(t) \geq \hat{V}_i^D(t) \quad \forall\, t \in \hat{\Omega}_i. \tag{17}$$

• At instants where there is no implementation, either there must be no innovations available to implement, or entrepreneurs with innovations must prefer to delay rather than implement:

$$\begin{aligned}
\text{Either } \hat{Z}_i(t) &= 0, \\
\text{or if } \hat{Z}_i(t) &= 1,\ \hat{V}_i^I(t) \leq \hat{V}_i^D(t) \quad \forall\, t \notin \hat{\Omega}_i.
\end{aligned} \tag{18}$$

• Free entry of replacement capital.

3   The Cyclical Equilibrium Growth Path

Although there exists an acyclical equilibrium growth path that satisfies the conditions stated above, our focus here is on a cyclical equilibrium growth path. In this section, we posit a temporal pattern of innovation and implementation behavior by entrepreneurs. Section 4 then derives the implications of this for the evolution of aggregate variables, and a set of sufficient conditions under which the implied evolution of aggregate variables, and market clearing, yield optimal entrepreneurial behavior corresponding with the originally posited behavior.

Suppose then that implementation occurs at discrete dates denoted by T_ν where $v \in \{1, 2, ..., \infty\}$. We adopt the convention that the vth cycle starts at time T_{v-1} and ends at time T_ν. The posited behavior of innovators and investors over this cycle is illustrated in Figure 3. After implementation at date T_{v-1} an **expansion** is triggered by a productivity boom and continues through

subsequent fixed capital formation. During this phase there is no innovation and consequently all workers are used in production. At some time T_v^*, innovation commences and labor starts to be withdrawn from production. Innovative successes are not implemented immediately but are withheld until time T_v. During this second **contraction** phase, fixed capital formation continues, but the rate of investment declines rapidly. As aggregate demand falls, labor continues to be released from production, so that innovation grows in anticipation of the subsequent implementation boom.

Let $P_i(s)$ denote the probability that, since time T_{v-1}, no entrepreneurial success has been made in sector i by time s. It follows that the probability of there being no entrepreneurial success by time T_v conditional on there having been none by time t, is given by $P_i(T_v)/P_i(t)$. Hence, the value of an incumbent firm in a sector where no entrepreneurial success has occurred by time t during the vth cycle can be expressed as

$$V_{0,i}^I(t) = (1 - \omega) \int_t^{T_v} e^{-[R(\tau)-R(t)]} \pi_i(\tau) d\tau + \frac{P_i(T_v)}{P_i(t)} e^{-\beta(t)} V_{0,i}^I(T_v). \tag{19}$$

where

$$\beta(t) = R_0(T_v) - R(t) \tag{20}$$

denotes the discount factor used to discount from time t during the cycle to the beginning of the next cycle.[11] The first term in (19) represents the discounted profit stream that accrues to the entrepreneur with certainty during the current cycle, and the second term is the expected discounted value of being an incumbent at the beginning of the next cycle.

Lemma 1 : *In a cyclical equilibrium, successful entrepreneurs can credibly signal a success immediately and all innovation in their sector stops until the next round of implementation.*

Unsuccessful entrepreneurs have no incentive to falsely announce success. As a result, an entrepreneur's signal is credible, and other entrepreneurs will exert their efforts in sectors where they have a better chance of becoming the dominant entrepreneur.

In the cyclical equilibrium, entrepreneurs' conjectures ensure no more entrepreneurship in a sector once a signal of success has been received, until after the next implementation. The expected value of an entrepreneurial success occurring at some time $t \in (T_v^*, T_v)$ but whose implementation is delayed until time T_v is thus:

$$V_i^D(t) = e^{-\beta(t)} V_{0,i}^I(T_v). \tag{21}$$

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[11]Throughout, we use the subscript 0 to denote the value of a variable immediately after the boom. Formally, for any variable $X(\cdot)$, we define $X(t) = \lim_{\tau \to t^-} X(\tau)$ and $X_0(t) = \lim_{\tau \to t^+} X(\tau)$.

In the cyclical equilibrium, such delay is optimal; i.e. $V_i^D(t) > V_i^I(t)$ throughout the contraction. Successful entrepreneurs are happier to forego immediate profits and delay implementation until the boom in order to ensure a longer reign of incumbency. Since no implementation occurs during the cycle, by delaying, the entrepreneur is assured of incumbency until at least T_{v+1}. Incumbency beyond that time depends on the probability of another entrepreneurial success.[12]

The symmetry of sectors implies that entrepreneurial effort is allocated evenly over all sectors that have not yet experienced a success within the cycle. In the posited cyclical equilibrium, the probability of not being displaced at the next implementation is

$$P_i(T_v) = \exp\left(-\mu \int_{T_v^*}^{T_v} h_i(\tau)d\tau\right). \tag{22}$$

4 Within–Cycle Dynamics

In equilibrium, factor prices are proportional to their marginal products. Consequently, standard aggregation results hold and aggregate output can be expressed as

$$Y(t) = e^{(1-\alpha)\phi t}\bar{A}_{v-1}^{1-\alpha}K(t)^\alpha L(t)^{1-\alpha}, \tag{23}$$

where

$$\bar{A}_{v-1} = \exp\left(\int_0^1 \ln A_i(T_{v-1})di\right). \tag{24}$$

Note that this endogenous component of TFP is fixed through the cycle. In order to afford a stationary representation of the economy it is convenient to normalize aggregates by dividing by total factor productivity using lower–case letters to denotes these deflated variables:

$$k(t) = \frac{K(t)}{\bar{A}_{v-1}e^{\phi t}}, \quad c(t) = \frac{C(t)}{\bar{A}_{v-1}e^{\phi t}}, \quad y(t) = \frac{Y(t)}{\bar{A}_{v-1}e^{\phi t}}. \tag{25}$$

Consequently, the intensive form production function is given by

$$y(t) = k(t)^\alpha L(t)^{1-\alpha}. \tag{26}$$

The household's Euler equation during the cycle can be expressed as

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\sigma} - \phi, \tag{27}$$

where $r(t) = \dot{R}(t)$. The economy's aggregate resource constraint is

$$\dot{k}(t) = y(t) - c(t) - (\phi + \delta)k(t) \tag{28}$$

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[12] A signal of further entrepreneurial success submitted by an incumbent is not credible in equilibrium because incumbents have incentive to lie to protect their profit stream. No such incentive exists for entrants since, without a success, profits are zero. Note also that the reason for delay here differs from Shleifer (1986) where the length of incumbency is exogenously given.

Finally, factor prices can be expressed as

$$q(t) = \alpha k(t)^{\alpha-1} L(t)^{1-\alpha} \tag{29}$$

$$w(t) = e^{-\gamma}(1-\alpha)e^{\phi t}\bar{A}_{v-1}k(t)^{\alpha}L(t)^{-\alpha}. \tag{30}$$

Note that the wage rate is less than its marginal product by a factor $e^{-\gamma}$, reflecting the fact that a fraction $1 - e^{-\gamma}$ goes in the form of profits to intermediate producers. Moreover, free entry of replacement capital implies that

Lemma 2 :

$$r(t) = q(t) - \delta. \tag{31}$$

4.1 Phase 1: The Expansion $(T_{v-1} \to T_v^*)$

We now trace out the evolution of the economy implied by the behavior posited above. We start immediately following an implementation boom, when capital, consumption and output take on the initial values $k_0(T_{v-1})$, $c_0(T_{v-1})$ and $y_0(T_{v-1})$, respectively. During the expansion all labor is used in production so that

$$L(t) = 1. \tag{32}$$

Combining this condition with (26), (27), (28), (29) and (31) yields transitional dynamics that are identical to those of the Ramsey model:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha-1} - \delta - \rho}{\sigma} - \phi \tag{33}$$

$$\frac{\dot{k}(t)}{k(t)} = k(t)^{\alpha-1} - \frac{c(t)}{k(t)} - \phi - \delta. \tag{34}$$

These dynamics are illustrated using a phase diagram in Figure 4.

During this expansionary phase, both consumption and capital grow, so we restrict attention to the lower left quadrant of the phase diagram. As capital accumulates, the wage grows and the interest rate declines. However, for the dynamic path to be consistent with the cyclical equilibrium more stringent conditions must be met.

Proposition 1 : *If, during an expansion, the dynamic paths of consumption and capital satisfy*

$$\left(1 - \frac{\alpha}{\sigma}\right)k(t)^{\alpha} + \frac{\rho + (1-\sigma)\delta}{\sigma}k(t) > c(t) > \left(\frac{1-\alpha}{\alpha}\right)(\phi+\delta)k(t), \tag{35}$$

then there exists a T_v^* such that for the first time

$$\mu V^D(T_v^*) = w(T_v^*), \tag{36}$$

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The left hand inequality in (35) is depicted in Figure 4 by points below the curve OA. This curve is a concave function passing through the origin and the intersection of the $\dot{k} = 0$ and $\dot{c} = 0$ locii. As long as the path of the economy lies below this curve during this phase, the consumption–capital ratio declines though time. This is consistent with the fact that the marginal product of capital is relatively high, inducing rapid investment and a capital stock that is growing relative to consumption.

The right hand inequality in (35) is depicted in Figure 4 by points above the line OB. This line is a ray from the origin that intersects the $\dot{k} = 0$ locus at its peak. During the first phase of the cycle it cannot be optimal to engage in innovation and then delay implementation until the subsequent boom. That is

$$\mu V^D(t) < w(t). \tag{37}$$

As the capital stock accumulates and TFP grows, $w(t)$ rises through time. Moreover, as the subsequent boom approaches $V^D(t)$ grows at the rate of interest. As long as the path of the economy lies between OA and OB it must be true that

$$r(t) > \frac{\dot{w}(t)}{w(t)}. \tag{38}$$

Consequently, the first phase of the cycle comes to an end in finite time.

After the date, T_v^*, if all workers were to remain in production, returns to entrepreneurship would strictly dominate those in production. As a result, labor hours are re–allocated from production and into innovation and this triggers the next phase of the cycle. The following Lemma demonstrates that during the transition from one phase to the next, all aggregate variables evolve smoothly.

Lemma 3 : *At time T_v^*, when entrepreneurship first commences in a cycle $L(T_v^*) = 1$, and output, investment and consumption must evolve continuously.*

4.2   Phase 2: The Downturn $(T_v^* \rightarrow T_v)$

During this phase, capital continues to be accumulated so that (31) must still hold. However, now there is innovation, so that $L(t) < 1$. Free entry into innovation implies $\mu V^D(t) = w(t)$, so that it must be the case that

$$r(t) = \frac{\dot{V}^D(t)}{V^D(t)} = \frac{\dot{w}(t)}{w(t)}. \tag{39}$$

Combining these conditions with (26), (27), (28), (29) and (30) implies that during the slowdown, consumption, capital and the labor force in production evolve according to the following dynamical system:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha-1}L(t)^{1-\alpha} - \delta - \rho}{\sigma} - \phi \tag{40}$$

$$\frac{\dot{k}(t)}{k(t)} = k(t)^{\alpha-1}L(t)^{1-\alpha} - \frac{c(t)}{k(t)} - \phi - \delta \tag{41}$$

$$\frac{\dot{L}(t)}{L(t)} = -\frac{c(t)}{k(t)} + \left(\frac{1-\alpha}{\alpha}\right)(\phi+\delta) < 0 \tag{42}$$

The negative value of $\dot{L}(t)/L(t)$ at the beginning of this phase is ensured by condition (35). Initially, the consumption–capital ratio $c(t)/k(t)$ also continues to decline. However, as $L(t)$ declines, the marginal product of capital falls and investment starts to fall. Eventually, in the hypothesized cycle, $c(t)/k(t)$ starts to rise again.

Note that the implied path for $L(t)$ during this phase implies a path for the fraction of the labor force used in innovation, $H(t) = 1 - L(t)$. This, in turn, determines the measure of sectors that innovate at each date:

$$-\dot{P}(t) = \mu\left[1 - L(t)\right], \tag{43}$$

where $P(T_v^*) = 1$.[13] At the end of the cycle, the fraction of sectors that have experienced an innovative success is therefore

$$1 - P(T_v) = \int_{T_v^*}^{T_v} \mu\left[1 - L(\tau)\right] d\tau. \tag{44}$$

4.3 The Implementation Boom

We denote the improvement in total factor productivity during implementation, $e^{(1-\alpha)\Gamma_v}$, where $\Gamma_v = \ln\left[\bar{A}_v/\bar{A}_{v-1}\right]$. Productivity growth at the boom is given by

$$\Gamma_v = \gamma\left(1 - P(T_v)\right). \tag{45}$$

A key implication of the assumption that investment is (at least partially) reversible is that household consumption must evolve smoothly over the period T_v — it cannot jump discontinuously. Intuitively, if households anticipated a sharp rise in consumption in the future they could raise their utility by converting some of the capital stock into consumption goods immediately. As a result, the household's Euler equation implies that the rate of return on any asset held over the

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[13]The rate of change in P is given by $\frac{\dot{P}}{P} = -\mu h_i$. But since labor is allocated symmetrically to innovation only in the measure P of sectors where no innovation has occurred, $h_i = \frac{H}{P}$, so that $\dot{P} = -\mu H$.

boom must equal zero.[14] In particular, the returns to storing intermediate goods until after the boom must be zero. A positive return would exist if the wage rose discontinuously upon implementation because it would be cheaper to produce extra intermediates at the low wage just before the boom and substitute them for production at the high wage afterwards. The fact that, in equilibrium, the wage must therefore evolve smoothly across the boom pins down a tight relationship between the growth in productivity and the labor effort allocated back into production:

Proposition 2 *Asset market clearing under reversible investment at the boom requires that*

$$(1 - \alpha)\Gamma_v = -\alpha \ln L(T_v) \tag{46}$$

During the boom, firm values and wages grow in proportion to labor productivity. Since, just before the boom $\mu V^I(T_v) = w(T_v)$, an immediate corollary is that

$$\mu V_0^I(T_v) = w_0(T_v) = (1 - \alpha)e^{-\gamma}e^{\phi t}\bar{A}_v k_0(T_v)^\alpha. \tag{47}$$

Output growth through the boom is given by

$$\Delta \ln Y(T_v) = (1 - \alpha)\Gamma_v - (1 - \alpha)\ln L(T_v) = \left(\frac{1 - \alpha}{\alpha}\right)\Gamma_v \tag{48}$$

It follows directly from Proposition 2 that growth in output exceeds the discount factor across the boom. Since profits are proportional to output, this explains why firms are willing to delay implementation during the downturn. Because investment is reversible, consumption cannot jump at the boom, and so all of the increase in output must be associated with a sharp rise in investment.

4.4   Optimal Entrepreneurial Behavior During the Cycle

Optimal entrepreneurial behavior imposes the following requirements on our hypothesized equilibrium cycle:

• Successful entrepreneurs at time $t = T_v$ must prefer to implement immediately, rather than delay implementation until later in the cycle or the beginning of the next cycle:

$$V_0^I(T_v) > V_0^D(T_v). \tag{E1}$$

• Entrepreneurs who successfully innovate during the slowdown and downturn must prefer to wait until the beginning of the next cycle rather than implement earlier and sell at the limit price:

$$V^I(t) < V^D(t) \qquad \forall \, t \in (T_v^*, T_v) \tag{E2}$$

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[14]This is in stark contrast to Francois and Lloyd-Ellis (2003), where consumption jumps at the boom.

• No entrepreneur wants to innovate during the expansion of the cycle. Since in this phase of the cycle $\mu V^D(t) < w(t)$, this condition requires that

$$\mu V^I(t) < w(t) \qquad \forall\, t \in (T_{v-1}, T_v^*) \tag{E3}$$

• Finally, in constructing the equilibrium above, we have implicitly imposed the requirement that the downturn is not long enough that all sectors innovate:

$$P(T_v) > 0. \tag{E4}$$

Taken together conditions (E1) through (E4) are restrictions on entrepreneurial behavior that must be satisfied for the cyclical growth path we have posited to be an equilibrium.

5   The Stationary Cyclical Growth Path

We focus on a stationary cyclical equilibrium growth path in which the boom size is constant at Γ every cycle and the cycle length is given by

$$\Delta = T_v - T_{v-1} \quad \forall\, v. \tag{49}$$

In addition, we denote the length of the stationary expansion phase as

$$\Delta^* = T_v^* - T_{v-1} \quad \forall\, v. \tag{50}$$

Along this path, $\bar{A}$ rises by e^Γ at each implementation boom, but consumption and capital evolve continuously so that their normalized values at the beginning of each cycle are given by

$$c_0(T_v) \;=\; e^{-\Gamma}c(T_v) = \hat{c} \tag{51}$$
$$k_0(T_v) \;=\; e^{-\Gamma}k(T_v) = \hat{k}. \tag{52}$$

In Appendix B, we demonstrate that, for a given stationary cycle length and boom size, a stationary equilibrium is equivalent to that which would be chosen by a social planner who is constrained to follow the specific innovation and implementation path associated with (Γ, Δ). This implication holds despite the presence of imperfect competition in the intermediate goods market for two reasons: (1) all intermediate sectors consist of monopolists who behave symmetrically and (2) within the cycle, labor and entrepreneurship are supplied inelastically. Consequently, relative prices are the same as they would be under perfect competition. The only implication of monopoly *within the cycle* is for the distribution of household income between profits and wages. The existence and uniqueness of a stationary solution to the constrained planner's problem,

14

$\hat{k}(\Gamma, \Delta)$, is demonstrated using a standard contraction mapping theorem, and requires only that utility is bounded:[15]

$$\rho + (\sigma - 1)\left(\phi + \frac{\Gamma}{\Delta}\right) > 0. \tag{53}$$

Thus, for each pair (Γ, Δ) satisfying (53), there is a unique stationary path for the endogenous variables $\{c(t), k(t), L(t)\}_{T_{v-1}}^{T_v}$ which repeats itself every cycle. We can summarize these within–cycle dynamics in terms of the consumption–capital ratio and the capital–labor ratio:

$$\frac{(c\dot{/}k)}{(c/k)} = \left(\frac{\alpha}{\sigma} - 1\right)\left(\frac{k(t)}{L(t)}\right)^{\alpha-1} + \frac{c(t)}{k(t)} + \frac{\delta(\sigma-1) - \rho}{\sigma} \tag{54}$$

$$\frac{(k\dot{/}L)}{(k/L)} = \left(\frac{k(t)}{L(t)}\right)^{\alpha-1} - \left(\frac{\phi + \delta}{\alpha}\right), \tag{55}$$

where $L(t) = 1$ in phase 1 and $L(t)$ is determined by (42) in phase 2. The associated phase diagram is shown in Figure 5. In a stationary cycle, the consumption–capital ratio must be the same at the end as at the beginning Also the capital–labor ratio must be higher at the end than at the beginning which implies the economy must be to the left of the $(k\dot{/}L) = 0$ locus. During phase 1 $(T_{v-1} \to T_v^*)$, condition (35) implies that the economy lies below the $(c\dot{/}k) = 0$ locus, so that c/k falls and (since $L(t) = 1$) $k(t)$ rises. During phase 2 $(T_v^* \to T_v)$, c/k initially continues to fall, but eventually the economy crosses the $(c\dot{/}k) = 0$ locus, so that it starts to rise again.

For any arbitrary pair of values (Γ, Δ) such a stationary cyclical path need not, however, satisfy either the free entry into innovation condition (Proposition 1) or the asset market clearing condition at the boom (Proposition 2). In Appendix B we show that under fairly weak parametric restrictions, only one pair of values $\left(\hat{\Gamma}, \hat{\Delta}\right)$ can satisfy these conditions along a stationary cyclical path.[16]

Proposition 3 : *There exists a $\sigma^* \in [0, 1)$, such that if $\sigma > \sigma^*$, and if*

$$(\rho + (\sigma - 1)\phi)e^{-\gamma} < \mu(1 - e^{-\gamma})(1 - \omega) \tag{56}$$

then a stationary cyclical equilibrium $(\hat{\Gamma}, \hat{\Delta}, \hat{k})$ satisfying all the conditions above is unique within this class of cycling equilibria.

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[15]The fact that household utility is bounded in equilibrium is equivalent to (E1). To see this observe that (E1) can be expressed as $V_0^I(T_v) > e^{-[R_0(T_{v+1}) - R_0(T_v)]}e^{\Gamma + \Delta\phi}V_0^I(T_v)$, which holds only if $R_0(T_{v+1}) - R_0(T_v) = \sigma(\Gamma + \Delta\phi) + \rho\Delta > \Gamma + \Delta\phi$. Re-arranging yields (53).

[16]Obviously this does not imply that the equilibrium is globally unique, only that it is so within the class of stationary cyclical paths described above. In particular, we know that there exists at least one other equilibrium growth path — the standard acyclical one.

5.1 Baseline Example

We numerically solve the model for various combinations of parameters that satisfy (56) and check the existence conditions (E1)–(E4). The parameters for our baseline example are given in Table 1.[17] The parameters α and γ imply a capital share of 0.3, a labor share of about 0.6 and a profit share of 0.1. The value of γ corresponds to a markup rate of around 25%. The unit–valued intertemporal elasticity of substitution implies logarithmic preferences. Given these values, we chose μ, ρ and ω so as to match a long–run annual growth rate of about 2%, an average risk–free real interest rate of roughly 4%, and a cycle length of approximately 8 years. These values roughly correspond to average data for the post–war US. The implied value of ω is admittedly rather high if we interpret it purely as a tax on profits.[18] However, as noted earlier, we view ω as representing a number factors that may affect the ratio of profits to wages (e.g. labor market frictions, implementation costs, imitation).

Figure 6 depicts the evolution of key aggregates over the cycle for this baseline case. In this example, the capital stock grows monotonically through the first phase and into the second, before starting to decline towards the end of the cycle. Note that this decline is purely due to lack of maintenance, not because of negative investment. As shown in Figure 7, although the investment rate falls rapidly in phase 2 it never goes below zero. Consumption evolves much more smoothly than investment, rising through the first phase and slowing down somewhat in the second, before accelerating at the subsequent implementation boom.

After rising gradually during the expansion, output — defined here as the sum of consumption goods and fixed capital formation — falls dramatically in phase 2. However, correctly measured GDP should include the payments made to labour used in producing intangibles. As illustrated in Figure 6, GDP also falls during phase 2, but much less dramatically. The reason GDP falls is that labor used in production is being paid below its marginal product. As labour effort is transferred into innovative activities, the marginal cost in terms of lost output exceeds the marginal benefit of innovation. In effect, the transfer of labour imposes a negative externality on the profits of incumbent producers. Because it is offset by the fall in intangible investment, the rise in GDP at the boom is also less dramatic.

Figure 8 depicts the evolution of $w(t)/\mu$, the value of incumbent intermediate producers that have not been displaced, $V^I(t)$, and the value of innovations whose implementation is delayed until the subsequent boom. As can be seen the value functions conform with conditions (E1)–(E3). $V^I(t)$ falls through the expansion as dividends are paid out. During the contraction the likelihood

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[17] The Gauss program used to generate the numerical simulations and the diagrams contained here is downloadable from the following URL: http://qed.econ.queensu.ca/pub/faculty/lloyd-ellis/research.html

[18] McGrattan (1994) estimates taxes on capital income in the US to be approximately 50%.

16

of being displaced at the boom declines and towards the end of the cycle this factor dominates, driving $V^I(t)$ sharply upward at the end. Interestingly, the wage does not vary much relative to trend over the cycle. In this baseline example it is mildly countercyclical, but in general it could rise more or less rapidly in the contraction. The growth of the wage over the cycle depends on the relative effects of falling capital versus falling labour in production on the rate at which the capital–labour ratio rises. The effect of the growth in TFP at the boom is exactly offset by the reallocation of labor effort back into production.[19]

5.2   Comparative Stationary Cycles

Table 2 documents several statistics from the model for various deviations in parameter values from the baseline example. In most cases we raised or lowered the individual parameter by 10% from its baseline value. In all of the cases considered in the table, the investment is positive throughout the cycle. The implications for the average annualized rates of growth, $\bar{g}$, and interest, $\bar{r}$, are qualitatively similar to what one would expect along an acyclical growth path. Perhaps more interesting are the model's implications for capital formation and intangible accumulation which we consider in the next section.

6   Implications for Investment and the Stockmarket

6.1   Tobin's Q and Investment

Tobin's Q is typically measured as the ratio of the value of firms to the book value of their capital stock. In our model Tobin's Q is given by

$$Q(t) = 1 + \frac{\Pi(t)}{K(t)}, \tag{57}$$

where $\Pi(t)$ denotes the stock market value of the intangible capital tied up in firms. Figure 9 illustrates the evolution of Tobin's Q and the aggregate investment rate over the cycle in the baseline example.

During an expansion the value of intangible capital is equal to the value of incumbent firms: $\Pi(t) = V^I(t)$. Since $K(t)$ rises and $V^I(t)$ declines, Tobin's Q falls monotonically throughout this phase. During a contraction, some sectors experience innovations, so there exist production methods that are certain to be made obsolete at the next round of innovation. At time t the measure of such sectors is $1 - P(t)$, and so

$$\Pi(t) = (1 - P(t))[V^T(t) + V^D(t)] + P(t)V^I(t), \tag{58}$$

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[19]Francois and Lloyd-Ellis (2005) explore in more detail the implications of intrinsic cycles for wages and employment when labor supply is endogenous.

where

$$V^T(t) = V^I(t) - \frac{P(T_v)}{P(t)} V^D(t) \tag{59}$$

is the value of these "terminal" production methods. During the contraction $Q(t)$ initially declines as $K(t)$ continues to grow. However, eventually the growth in the value of intangible capital starts to dominate as we approach the boom, so that $Q(t)$ rises in anticipation.

The predicted evolution of Q and the investment rate during contractions matches well the relationship their counterparts in US data. During every post–war U.S. recession (except perhaps the last one), as depicted by the shaded regions in Figure 1, I/K falls. But more importantly Tobin's Q reaches a turning point either at the start, or early in the recession, after which it heads upwards.[20] The model's dramatic boom makes matching investment behavior in expansions more difficult, but here there are similarities too. In US expansions, I/K peaks well ahead of the economy's peak (though typically not at the boom as our model would have it), and then begins a gradual decline through the remainder of the expansion before beginning the steeper decline characteristic of recessions. As in our model, the rate at which I/K tapers off at the end of expansions in the data is generally more gradual than its steeper decline through the recessions.

6.2 The Relationship between Tangible and Intangible Capital Formation

A key feature of our results is the asynchronous cyclical movement in the accumulation of tangible and intangible capital. In our model, the fluctuations in Tobin's Q reflect the changing value of intangibles relative to the capital stock. However, in the real world the ratio of the market value of firm to the replacement cost of their capital stock (i.e. Q in Figure 2) may fluctuate over the business cycle for reasons other than the value of intangibles. In particular, even in a purely competitive economy with no profits from innovation, this ratio might fluctuate purely because of changes in adjustment costs or changes in the stock of inventories held by firms. Indeed there are good reasons to believe that these factors may induce counter–cyclical variation in the value of firms. In order to obtain a true measure of the value of intangibles, we should net out the effects of these factors on the ratio.

It turns out that exactly this calculation has recently been undertaken by Hall (2001) for the US over the period 1952–1999. Using an estimated quadratic adjustment cost model of investment and data on inventories, Hall backs out a series for the component of aggregate firm values that is not associated with these other factors.[21] He argues that this series can be viewed as a measure of the value of the stock of intangibles in the US economy. Although his focus is on the run–up

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[20]The long 1974 recession is somewhat different from the others because since Q rises much later, but it is still qualitatively consistent with the model's predictions.

[21]Downloadable from Hall's web-site http://www.stanford.edu/~rehall/index_files/Page1379.htm

in the stock of intangibles during the 1990s, the data is quarterly and is arguably well–suited to an analysis of the cyclical properties of the value of intangible capital. Figure 10 plots the cyclical components of the investment rate (I/K) and the ratio of Hall's intangibles to the capital stock, computed using a Hodrick–Prescott filter. As may be seen, intangible capital generally rises relative to trend when I/K is low and falls in expansions when I/K is high, thus exhibiting a strikingly counter–cyclical pattern, as predicted by our model.

The asynchronous movements in I and Q in our model are the result of the endogenous delay at the centre of our cyclical equilibrium. In contractions, though potential productivity is higher (i.e. knowledge capital is being built) and, hence, equities are rising, the anticipated boom induces innovators to hold–off on implementation. Since investment will not pick up again until after the innovations are implemented, investment lags the movement of equity values (or Tobin's Q). In contrast, in the models of Bental and Peled (1996), Freeman et al. (1999), Matsuyama (1999, 2001) and Walde (2002) the market value of intangible capital is perfectly correlated with productivity, suggesting that its value should rise contemporaneously with the increase in incentives for physical capital accumulation.

6.3 Tobin's Q and Investment at the Sectoral Level

To understand the distinction between the mechanical lead–lag view of the relationship between the stockmarket and investment, and the mechanism emphasized here, it is useful to think about the implications at the sectoral level. There is no unique way to disaggregate the model in order to consider its cross–sectoral implications. Most sectors produce final goods and intermediate goods which are then used by other sectors. Here we consider the implications of one method of disaggregation which seems reasonable. Specifically, we think of each sector as producing final goods using capital and intermediates, just as in the aggregate economy, and also producing an intermediate good, some of which it uses itself and the rest it sells to other sectors. We then ask what the observed relationship will be between the observed relative values of Q across sectors and their investment levels.

At some date t during the contraction, a measure $1 - P(t)$ of sectors will have seen the arrival of innovations which render the current methods soon to be obsolete. As noted above, the value of each of these sectors is $V^T(t) + V^D(t)$. The value of sectors where no innovation has occurred is $V^I(t) < V^T(t) + V^D(t)$. Since during the boom all sectors in our model increase investment symmetrically, the predictive content of the relative stock market values of each sector at any date $t < T$ is therefore zero. In other words, in our framework, the stockmarket is not an aggregate index of "news" about sectoral investment opportunities. Rather it is forecasting

the boom in aggregate demand that will affect all sectors symmetrically, even though only some experience productivity increments. Although news about future investment opportunities in different sectors is also undoubtedly important, this is not the mechanism at work here.

Interestingly, most studies of the relationship between investment and Tobin's Q at the sectoral level have generally found at best weak evidence of a relationship (contemporaneous or lagged), and certainly much weaker than suggested by the correlation between de–trended investment and lagged Q at the aggregate level.[22] Our model offers one interpretation of this difference — part of the aggregate relationship may be driven by general equilibrium effects reflecting endogenous delay.

7 Conclusion

Firms are repositories of both tangible and intangible capital. This paper develops a general equilibrium theory of investment in both types of capital at business cycle frequencies. Because of the dynamic externalities inherent in the process of innovation, indepedently acting firms have incentives to undertake both types of investment in a clustered and seemingly coordinated fashion. Consequently, the aggregate economy oscillates between periods of intensive physical capital accumulation, rapid productivity growth and expanding output, and periods of slowing fixed capital formation and declining output, but intangible capital accumulation. The behaviour of aggregates, investment, prices, consumption, interest rates, and the relationship between Tobin's Q and tangible investment conform reasonably well with key features of the patterns observed at business cycle frequencies in US data. Moreover, the evidence regarding intangible capital accumulation presented by Hall (2001) and others is also consistent with the asynchronous $I - Q$ cycles generated here.

There are two main directions that we plan to pursue in future research. Firstly, the cycle that we study here is deterministic and stationary. Variations in the aggregate growth rate and the length and amplitude of the cycle could be introduced into the present framework by adding a noise component to the exogenous component of productivity growth. A more ambitious and interesting extension would be to model the interaction between elements of exogenous aggregate variation and the model's cyclical components. We have in mind a framework in which shocks trigger phases that exhibit recurring and stable patterns of aggregate behavior similar to that

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[22]There is a large literature here, most of which has found these effects to be weak – Blundell, Bond, Devereux, Schiantarelli (1992), Abel and Blanchard (1986), Hayashi and Innoue (1991) and Gilchrist and Himmelberg (1995) are a few. However recent research constructing an alternative measure of Q based on securities analysts' earnings forecasts: Cummins, Hassett and Oliner (1999) and Bond and Cummins (2001), do much better suggesting that perhaps basic Q theory, with the correct measure of firm value, can provide guidance to investment behavior at the micro level.

modelled here. While we are some way from producing this more ambitious extension, we believe that such a framework would generate non–linear output dynamics akin to a duration-dependent, Markov switching process.

A second direction is to study the welfare implications of the endogenous two–way interaction between growth and cycles present in our framework. Lucas (1987) argued that consumption volatility is small and likely to be of second order importance to welfare in comparison with factors affecting secular growth rates. In contrast Barlevy (2004) argues that the volatility of investment induced by the business cycle contributes adversely to long run growth and hence welfare. In our model growth rates are effected by changes in factors that smooth cycles, but changes in factors effecting growth rates also effect the volatility of aggregates at cyclical frequencies. In contrast to our earlier work (Francois and Lloyd-Ellis (2003)) consumption here evolves smoothly, providing a more realistic environment in which to consider the welfare implications of counter-cyclical and secular policy initiatives.

Appendix A – Proofs

Proof of Lemma 1 We show: (1) that if a signal of success from a potential entrepreneur is credible, other entrepreneurs stop innovation in that sector; (2) given (1), entrepreneurs have no incentive to falsely claim success.

Part (1): If entrepreneur $i's$ signal of success is credible then all other entrepreneurs believe that i has a productivity advantage which is e^γ times better than the existing incumbent. If continuing to innovate in that sector, another entrepreneur will, with positive probability, also develop a productive advantage of e^γ. Such an innovation yields expected profit of 0, since, in developing their improvement, they do not observe the non-implemented improvements of others, so that both firms Bertrand compete with the same technology. Returns to attempting innovation in another sector where there has been no signal of success, or from simply working in production, $w(t) > 0$, are thus strictly higher.

Part (2): If success signals are credible, entrepreneurs know that upon success, further innovation in their sector will cease, from Part (1), by their sending of a costless signal. They are thus indifferent between falsely signalling success when it has not arrived, and sending no signal. Thus, there exists a signalling equilibrium in which only successful entrepreneurs send a signal of success.∎

Proof of Lemma 2: The present value of the capital owners' net income in sector i is:

$$V^K(t) = \int_t^\infty e^{-[R(\tau)-R(t)]}\left[q(\tau)K(\tau) - \dot{K}(\tau) - \delta K(\tau)\right]d\tau. \tag{60}$$

Differentiating (60) with respect to time yields

$$\dot{V}^K(t) = r(t)V^K(t) - q(t)K(t) + \dot{K}(t) + \delta K(t). \tag{61}$$

Free entry implies that if $\dot{K}(t) > 0$, then $V^K(t) = K(t)$ and $\dot{V}^K(t) = \dot{K}(t)$. The result follows immediately.∎

Proof of Proposition 1: Subtracting (34) from (33) we get

$$\frac{\dot{c}(t)}{c(t)} - \frac{\dot{k}(t)}{k(t)} = \frac{c(t)}{k(t)} - \left(1 - \frac{\alpha}{\sigma}\right)k(t)^{\alpha-1} - \frac{\rho + (1-\sigma)\delta}{\sigma} \tag{62}$$

It follows that in order for $c(t)/k(t)$ to be declining in the first phase

$$\frac{c(t)}{k(t)} < \left(1 - \frac{\alpha}{\sigma}\right)k(t)^{\alpha-1} + \frac{\rho + (1-\sigma)\delta}{\sigma}. \tag{63}$$

which is the left hand inequality in (35).

For $r > \dot{w}/w$, we require that

$$\alpha k(t)^{\alpha-1} - \delta > \phi + \alpha \frac{\dot{k}(t)}{k(t)} \tag{64}$$

Substitution using (34)yields

$$\alpha k(t)^{\alpha-1} > \phi + \delta + \alpha \left(k(t)^{\alpha-1} - \frac{c(t)}{k(t)} - \phi - \delta \right), \tag{65}$$

which rearranges to the right hand inequality in (35).■

Proof of Lemma 3: Since capital depreciation rates are independent of utilization, and its marginal product is always positive, installed capital is always fully utilized. At T_v^*, since the discount factor does not jump, neither does V^D. With non–variable capital utilization, the wage must jump up if $L(t)$ jumps down. Since just before T_v^*, $\mu V^D(t) < w(t)$, this is not possible. It follows that $L(T_v^*) = 1$ and falls smoothly from that point on.

Since L adjusts smoothly, and capital utilization is non–variable, output cannot jump down at T_v^*. Since the discount factor does not jump, consumption cannot jump at T_v^* either. Consequently, investment, $\dot{K}$, cannot jump at T_v^*. Note further that $r(t) = q(t) - \delta$ cannot jump down instantaneously with non-variable capital utilization. It follows that wage growth $\frac{\dot{w}(t)}{w(t)}$ must jump up at T_v^* and employment growth in production $\frac{\dot{L}(t)}{L(t)}$ must jump to a negative level.■

Proof of Proposition 2: During the boom, for entrepreneurs to prefer to implement immediately, it must be the case that $V_0^I(T_v) > V_0^D(T_v)$. Just prior to the boom, when the probability of displacement is negligible, the value of implementing immediately must equal that of delaying until the boom: $\delta V^I(T_v) = \delta V^D(T_v) = w(T_v)$. Free entry into entrepreneurship at the boom requires that $\delta V_0^I(T_v) \le w_0(T_v)$. The opportunity cost of financing entrepreneurship is the rate of return on shares in incumbent firms in sectors where no innovation has occurred. Since this return across the boom must equal zero, it must be the case that $V_0^I(T_v) = V^I(T_v)$. It follows that asset market clearing at the boom requires

$$\log \left(\frac{w_0(T_v)}{w(T_v)} \right) = (1 - \alpha)\Gamma_v - \ln L(T_v) \ge 0. \tag{66}$$

In sectors with no entrepreneurial success, incumbent firms could sell claims to stored output, use them to finance greater current production and then store the good to sell at the beginning of the next boom. Free entry into storage implies that the rate of return (the growth in the wage) to this activity must satisfy

$$\log \left(\frac{w_0(T_v)}{w(T_v)} \right) = (1 - \alpha)\Gamma_v - \ln L(T_v) \le 0. \tag{67}$$

Combining (66) and (67) yields (46).■

Appendix B – Uniqueness of the Stationary Equilibrium Cycle

We first show that the within–cycle decentralized equilibrium is equivalent in its aggregate implications to that which would be chosen by a social planner who is constrained to follow the innovation and implementation cycle assumed above.

Lemma B1: *For a given cycle length, target value of $P(T_v)$, and boundary values for the capital stock, k_0 and k_T, the within–cycle dynamics are equivalent to that which would be chosen by a social planner that is constrained to attain $P(T_v)$ and k_T.*

Proof: Fix the value of Γ and let $P^* = 1 - \Gamma/\gamma$. Consider first the within–cycle problem, taking the cycle length Δ, and the boundary values k_0 and k_T as given

$$V^P(k_0, k_T; \Gamma, \Delta) = \max_{P(.),c(.),L(.),k(.)} \left\{ \int_0^\Delta e^{-(\rho+(\sigma-1)\phi)t} \left(\frac{c(t)^{1-\sigma} - 1}{1 - \sigma} \right) dt \right\} \tag{68}$$

subject to

$$\dot{k}(t) = k(t)^\alpha L(t)^{1-\alpha} - (\phi + \delta)k(t) - c(t) \tag{69}$$

$$-\dot{P}(t) = \mu [1 - L(t)] \tag{70}$$

$$k(0) = k_0, \ k(\Delta) = k_T, \ P(0) = 1, \ P(\Delta) = P^* = 1 - \Gamma/\gamma \tag{71}$$

$$L(t) \leq 1 \tag{72}$$

where T_{v-1} and T_v are normalized to 0 and Δ respectively.

The Hamiltonian associated with the within–cycle planning problem is

$$
\begin{aligned}
&H(c(t), k(t), L(t), P(t), \lambda_1(t), \lambda_2(t), \psi(t), t) \\
&= e^{-\rho t + (1-\sigma)\phi t} \left(\frac{c(t)^{1-\sigma} - 1}{1 - \sigma} \right) + \lambda_1(t) \left[k(t)^\alpha L(t)^{1-\alpha} - (\phi + \delta)k(t) - c(t) \right] \\
&\quad + \lambda_2(t)\mu [1 - L(t)] + \psi(t) [1 - L(t)]
\end{aligned}
$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the costate variables and $\psi(t)$ is the Lagrange multiplier on the labor constraint such that $\psi(t) [1 - L(t)] = 0$. The Hamiltonian conditions are (69), (70) and

$$\frac{\partial H}{\partial c} = e^{-[\rho-(1-\sigma)\phi]t} c(t)^{-\sigma} - \lambda_1 = 0 \tag{73}$$

$$\frac{\partial H}{\partial L} = \lambda_1(t)(1-\alpha)k(t)^\alpha L(t)^{-\alpha} - \lambda_2(t)\mu - \psi(t) = 0 \tag{74}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial k} = -\lambda_1(t) \left[\alpha k(t)^{\alpha-1} L(t)^{1-\alpha} - (\phi + \delta) \right] \tag{75}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial P} = 0 \tag{76}$$

24

Case 1: If $L(t) = 1$ then $\psi(t) > 0$. Then differentiating (73) w.r.t. time and using (75) to substitute out $\dot{\lambda}_1/\lambda_1$ we get

$$\rho + (\sigma - 1)\phi + \sigma\frac{\dot{c}}{c} = \alpha k(t)^{\alpha-1} - (\phi + \delta) \tag{77}$$

This condition combined with (69) (with $L(t) = 1$) are, of course, the Ramsey conditions and are the same as those from the first phase of the cycle. Differentiating (74) w.r.t. time and using $\dot{\lambda}_2 = 0$, we can write

$$\frac{\dot{\psi}(t)}{\lambda_2\mu + \psi} = \frac{\dot{\lambda}_1}{\lambda_1} + \alpha\frac{\dot{k}(t)}{k(t)} = -\alpha k(t)^{\alpha-1} + (\phi + \delta) + \alpha\left(k(t)^{\alpha-1} - (\phi + \delta) - \frac{c(t)}{k(t)}\right) \tag{78}$$

$$= -\alpha\frac{c(t)}{k(t)} + (1 - \alpha)(\phi + \delta) \tag{79}$$

This condition implies that in order for the constraint on labour used in production to eventually become non–binding, the initial consumption level for a given k_0 must be in a range so that during the first phase

$$\frac{c(t)}{k(t)} > \left(\frac{1 - \alpha}{\alpha}\right)(\phi + \delta) \tag{80}$$

If this is the case, then along the optimal path, if $\psi > 0$ then $\dot{\psi} < 0$ and eventually hits zero. In this case there exists a T^* such that $L(t) = 1$ if $t < T^*$ and $L(t) < 1$ if $t > T^*$.

Case 2: If $L(t) < 1$ then $\psi(t) = 0$. In this case totally differentiating (73) and (74) w.r.t. time and noting that $\dot{\lambda}_2 = 0$ yields

$$-\frac{\dot{\lambda}_1}{\lambda_1} = \rho + (\sigma - 1)\phi + \sigma\frac{\dot{c}}{c} \tag{81}$$

$$-\frac{\dot{\lambda}_1}{\lambda_1} = \alpha\left(\frac{\dot{k}}{k} - \frac{\dot{L}}{L}\right) \tag{82}$$

Substituting out $\dot{\lambda}_1/\lambda_1$ using (75) and $\dot{k}/k$ using (69) we get

$$\rho + (\sigma - 1)\phi + \sigma\frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1}L(t)^{1-\alpha} - \phi - \delta \tag{83}$$

$$\frac{\dot{L}(t)}{L(t)} = -\frac{c(t)}{k(t)} + \left(\frac{1 - \alpha}{\alpha}\right)(\phi + \delta) \tag{84}$$

But these two conditions combined with (69) are identical to those from the decentralized equilibrium in phase 2 of the cycle.∎

Lemma B2: *Given* (Γ, Δ) *satisfying*

$$\rho + (\sigma - 1)\left(\phi + \frac{\Gamma}{\Delta}\right) > 0$$

the stationary cyclical path implies that the normalized capital stock at the start of each cycle takes on a unique stationary value, $k_0(T_v) = \hat{k} \; \forall \; v$.

Proof: We can express the social planner's problem as a Bellman equation given by

$$W(k; (\Gamma, \Delta)) = \max_{k'} \left\{ V^P(k, e^\Gamma k'; (\Gamma, \Delta)) + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} W(k'; (\Gamma, \Delta)) \right\} \qquad (85)$$

The optimal choice for k' must satisfy

$$
\begin{aligned}
\lambda_1(T) &= e^{-(\rho + (\sigma-1)\phi)\Delta} e^{(1-\sigma)\Gamma} \frac{\partial W(k(T))}{\partial k} \\
c(T)^{-\sigma} &= e^{-(\sigma-1)\Gamma} \frac{\partial W(k(T))}{\partial k} = e^{-\sigma\Gamma} c_0^{-\sigma}(T) \\
c(T) &= e^\Gamma c_0(T)
\end{aligned}
$$

Note that this implies $C(T) = C_0(T)$, (i.e. consumption cannot jump).

We show that the right–hand side of (85) is a contraction mapping in the space of relevant bounded functions so that it has a fixed point. Let $\Xi = [k_{\min}, k_{\max}]$. and let $f(\cdot)$ and $g(\cdot)$ be any two continuous functions from Ξ to Ξ. The maximized within cycle utility function can be expressed as

$$
\begin{aligned}
V^P(k_0, k_T; (\Gamma, \Delta)) &= \int_0^\Delta \left[H(\hat{c}(t), \hat{k}(t), \hat{L}(t), \hat{\lambda}_1(t), \hat{\lambda}_2(t), \psi(t), t) + \dot{\hat{\lambda}}_1(t)\hat{k}(t) \right] dt \qquad (86) \\
&\quad + \hat{\lambda}_1(0)k_0 - \hat{\lambda}_1(T)k_T - \hat{\lambda}_2(T)(1 - P(\Gamma))
\end{aligned}
$$

Observe that the value function is increasing and concave in k_0 and decreasing in k_T:

$$
\begin{aligned}
V_1^P &= \frac{dV^P}{dk_0} = \hat{\lambda}_1(0) = c(0)^{-\sigma} > 0 \\
V_{11}^P &= \frac{d^2V^P}{dk_0^2} = -\sigma c(0)^{-\sigma-1} \frac{dc(0)}{dk_0} < 0 \\
V_2^P &= \frac{dV^P}{dk_T} = -\hat{\lambda}_1(T) = -e^{-(\rho + (\sigma-1)\phi)\Delta} c(T)^{-\sigma} < 0
\end{aligned}
$$

Define the operator Ψ by

$$\Psi \circ f(k) = \max_{k'} \left\{ V^P(k, e^\Gamma k') + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} f(k') \right\}$$

subject to $k' \in \Xi$. To show that Ψ is a contraction mapping we must show that it satisfies two sufficient conditions (Blackwell's conditions):

26

(a) Monotonicity: Suppose $f(k) \geq g(k)$ $\forall k \in \Xi$. Let k_f and k_g attain $\Psi \circ f$ and $\Psi \circ g$, respectively for some arbitrary given $k \in \Xi$. Then

$$
\begin{aligned}
\Psi \circ g(k) &= V^P(k, e^\Gamma k_g) + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} g(k_g) \\
&\leq V^P(k, e^\Gamma k_g) + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} f(k_g) \quad \text{(since } f(k_g) \geq g(k_g)) \\
&\leq V^P(k, e^\Gamma k_f) + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} f(k_f) \quad \text{(by definition)} \\
&= \Psi \circ f(k)
\end{aligned}
$$

(b) Discounting: We have to show that there exists a $\beta \in (0,1)$ such that for any constant $x \geq 0$,

$$
\begin{aligned}
&\max_{k'} \left\{ V^P(k, e^\Gamma k') + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} \left[f(k') + x \right] \right\} \\
&\leq \max_{k'} \left\{ V^P(k, e^\Gamma k') + e^{-(\rho + (\sigma-1)\phi)\Delta} e^{-(\sigma-1)\Gamma} f(k') \right\} + \beta x
\end{aligned}
$$

Clearly, setting $\beta = e^{-(\rho - (1-\sigma)\phi)\Delta} e^{(1-\sigma)\Gamma}$ will satisfy this condition as long as

$$
\rho + (\sigma - 1)\left(\phi + \frac{\Gamma}{\Delta}\right) > 0
$$

which must be true in the cyclical equilibrium.

Let $k' = F(k)$ be the optimal policy attaining $W(k)$. Since $F(k) : \Xi \to \Xi$, and is continuous, it has a fixed point k^* in Ξ, so that $k^* = F(k^*)$ and

$$
W(k^*, \Gamma, \Delta) = V^P(k^*, e^\Gamma k^*, \Gamma, \Delta) + e^{-(\rho - (1-\sigma)\phi)\Delta} e^{-(\sigma-1)\Gamma} W(k^*, \Gamma, \Delta). \blacksquare
$$

Lemma B3: *There exists a $\sigma_0 \in [0,1)$ such that if $\sigma > \sigma_0$ then*
(a) an increase in Γ decreases the steady state capital stock, $\hat{k}$.
(b) an increase in Δ increases the steady state capital stock, $\hat{k}$.
Proof: The proof uses the following first and second derivatives of the maximized within-cycle value function, $V^P(\hat{k}, e^\Gamma \hat{k}, \Gamma, \Delta)$:

$$
\begin{aligned}
V_1^P &= c_0^{-\sigma} > 0, & V_2^P &= -e^{-\hat{\rho}\Delta} c_T^{-\sigma} < 0 \\
V_{11}^P &= -\sigma \frac{V_1^P}{c_0} \frac{\partial c_0}{\partial k_0} < 0, & V_{21}^P &= -\sigma \frac{V_2^P}{c_T} \frac{\partial c_T}{\partial k_0} > 0 \\
V_{12}^P &= -\sigma \frac{V_1^P}{c_0} \frac{\partial c_0}{\partial k_T} > 0, & V_{22}^P &= -\sigma \frac{V_2^P}{c_T} \frac{\partial c_T}{\partial k_T} > 0 \\
V_{1\Gamma}^P &= -\sigma \frac{V_1^P}{c_0} \frac{\partial c_0}{\partial \Gamma} > 0, & V_{2\Gamma}^P &= -\sigma \frac{V_2^P}{c_T} \frac{\partial c_T}{\partial \Gamma} < 0 \\
V_{1\Delta}^P &= -\sigma \frac{V_1^P}{c_0} \frac{\partial c_0}{\partial \Delta} < 0, & V_{2\Delta}^P &= -\hat{\rho} V_2^P - \sigma \frac{V_2^P}{c_T} \frac{\partial c_T}{\partial \Delta} > 0
\end{aligned}
$$

Along an optimal stationary cyclical path for a given Γ and Δ, the planner's problem implies the Euler condition that

$$Z(\hat{k}, e^{\Gamma}\hat{k}, \Gamma, \Delta) = V_2^P(\hat{k}, e^{\Gamma}\hat{k}, \Gamma, \Delta) + e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P(\hat{k}, e^{\Gamma}\hat{k}, \Gamma, \Delta) = 0.$$

Differentiating Z with respect to $\hat{k}$ we get

$$
\begin{aligned}
\frac{\partial Z}{\partial \hat{k}} &= V_{21}^P + e^{\Gamma}V_{22}^P + e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}\left(V_{11}^P + e^{\Gamma}V_{12}^P\right) \\
&= -\sigma\frac{V_2^P}{c_T}\frac{\partial c_T}{\partial k_0} - e^{\Gamma}\sigma\frac{V_2^P}{c_T}\frac{\partial c_T}{\partial k_T} - e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}\left(\sigma\frac{V_1^P}{c_0}\frac{\partial c_0}{\partial k_0} + e^{\Gamma}\sigma\frac{V_1^P}{c_0}\frac{\partial c_0}{\partial k_T}\right)
\end{aligned}
$$

Using the Euler condition, we can write this as

$$
\begin{aligned}
\frac{\partial Z}{\partial \hat{k}} &= \sigma e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P\left(\frac{1}{c_T}\frac{\partial c_T}{\partial k_0} + \frac{e^{\Gamma}}{c_T}\frac{\partial c_T}{\partial k_T} - \frac{1}{c_0}\frac{\partial c_0}{\partial k_0} - \frac{e^{\Gamma}}{c_0}\frac{\partial c_0}{\partial k_T}\right) \\
\frac{\partial Z}{\partial \hat{k}} &= \sigma e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P\left(\frac{\partial \ln(c_T/c_0)}{\partial k_0} + e^{\Gamma}\frac{\partial \ln(c_T/c_0)}{\partial k_T}\right) < 0
\end{aligned}
$$

The negative sign follows from the fact that both derivatives in brackets must be negative within a cycle. The first because, $\frac{\partial r(t)}{\partial k_0} < 0$ for all t in the cycle, and $\frac{\partial \ln(c_T/c_0)}{\partial R} > 0$, where R is the interest rate discounting from 0 to T. The second holds similarly because $\frac{\partial r(t)}{\partial k_T} < 0$ for all t in the cycle up to T, and again $\frac{\partial \ln(c_T/c_0)}{\partial R} > 0$, where R is the interest rate discounting from 0 to T.

(a) Differentiating the Euler condition with respect to Γ yields

$$
\begin{aligned}
\frac{\partial Z}{\partial \Gamma} &= V_{2\Gamma}^P + e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_{1\Gamma}^P - (\sigma-1)e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P \\
&= -\sigma\frac{V_2^P}{c_T}\frac{\partial c_T}{\partial \Gamma} - e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}\sigma\frac{V_1^P}{c_0}\frac{\partial c_0}{\partial \Gamma} - (\sigma-1)e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P \\
&= \sigma e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P\left(\frac{\partial \ln(c_T/c_0)}{\partial \Gamma}\right) - (\sigma-1)e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P
\end{aligned}
$$

When Γ increases more labor must be allocated towards innovation in the second phase. Consequently, consumption must grow less during the cycle so that the first term above is negative. Clearly if $\sigma \geq 1$ the second term is negative. Define $\sigma_0 < 1$: the two terms are equal. For $\sigma > \sigma_0$ the first term dominates and $\frac{\partial Z}{\partial \Gamma} < 0$. It follows that at the stationary optimum:

$$\left.\frac{d\hat{k}}{d\Gamma}\right|_{Z=0} = -\frac{\left(\frac{\partial Z}{\partial \Gamma}\right)}{\left(\frac{\partial Z}{\partial \hat{k}}\right)} < 0. \tag{87}$$

(b) Differentiating the Euler condition with respect to Δ yields

$$
\begin{aligned}
\frac{\partial Z}{\partial \Gamma} &= V_{2\Delta}^P + e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_{1\Delta}^P - \hat{\rho}e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P \\
&= -\hat{\rho}V_2^P - \sigma\frac{V_2^P}{c_T}\frac{\partial c_T}{\partial \Delta} - e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}\sigma\frac{V_1^P}{c_0}\frac{\partial c_0}{\partial \Delta} - \hat{\rho}e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P \\
&= \sigma e^{-\hat{\rho}\Delta - (\sigma-1)\Gamma}V_1^P\left(\frac{\partial \ln(c_T/c_0)}{\partial \Delta}\right) > 0
\end{aligned}
$$

28

When Δ increases, relatively more output can be allocated towards consumption in the second phase so that the derivative in brackets must be positive. It follows that

$$\left.\frac{d\hat{k}}{d\Delta}\right|_{Z=0} = -\frac{\left(\frac{\partial Z}{\partial \Delta}\right)}{\left(\frac{\partial Z}{\partial \hat{k}}\right)} > 0. \tag{88}$$

Proof of Proposition 3: The proof shows that given the social planning solution for each pair (Γ, T) only one of these pairs $(\hat{\Gamma}, \hat{T})$ is consistent with both

(1) no–arbitrage in the presence of intermediate storage (with an ε storage cost)

$$e^{-[R(t)-R(s)]}\frac{w(t)}{w(s)} \leq 1 \ \forall t, s$$

(2) free entry into innovation

$$\left[\mu V^D(t) - w(t)\right] L(t) = 0 \ \forall t.$$

Essentially, we show that (1) can be represented by a negative relationship between Γ and Δ like that labelled AMC in Figure 11 and (2) can be represented by a positive relationship like that labelled FEI.

(1) Since the social planner's solution implies that

$$r(t) \geq \frac{\dot{w}(t)}{w(t)} \ \forall t$$

at all points within the cycle, condition (1) boils down to the equilibrium requirement that the (Γ, Δ) must satisfy

$$w(T) = w_0(T)$$

which implies that the expected return from storage across the boom must be zero:

$$Z = \Gamma + \left(\frac{\alpha}{1-\alpha}\right) \ln L_T(\Gamma, \Delta, \hat{k}(\Gamma, \Delta)) = 0$$

where $L_T(\Gamma, \Delta, \hat{k}(\Gamma, \Delta)) = L(T; \Gamma, \Delta, \hat{k}(\Gamma, \Delta))$. Totally differentiating yields

$$\frac{\partial Z}{\partial \Delta} = \left(\frac{\alpha}{1-\alpha}\right)\left[\frac{\partial \ln L_T}{\partial \Delta} + \frac{\partial \ln L_T}{\partial \hat{k}}\cdot\frac{\partial \hat{k}}{\partial \Delta}\right]$$

$$\frac{\partial Z}{\partial \Gamma} = 1 + \left(\frac{\alpha}{1-\alpha}\right)\left[\frac{\partial \ln L_T}{\partial \Gamma} + \frac{\partial \ln L_T}{\partial \hat{k}}\cdot\frac{\partial \hat{k}}{\partial \Gamma}\right].$$

It follows that along the AMC curve where $Z = 0$:

$$\left.\frac{d\Gamma}{d\Delta}\right|_{AMC} = \frac{-\left(\frac{\alpha}{1-\alpha}\right)\left[\frac{\partial \ln L_T}{\partial \Delta} + \frac{\partial \ln L_T}{\partial \hat{k}}\cdot\frac{\partial \hat{k}}{\partial \Delta}\right]}{1 + \left(\frac{\alpha}{1-\alpha}\right)\left[\frac{\partial \ln L_T}{\partial \Gamma} + \frac{\partial \ln L_T}{\partial \hat{k}}\cdot\frac{\partial \hat{k}}{\partial \Gamma}\right]} < 0$$

29

To see that this slope is indeed negative, note that from Lemma B3 we have that $\frac{\partial \hat{k}}{\partial \Gamma} < 0$ and $\frac{\partial \hat{k}}{\partial \Delta} > 0$. Also note that $\frac{\partial \ln L_T}{\partial \hat{k}} > 0$ since the higher is the capital stock the more costly it is to divert a marginal unit of labour from production at any date (since its marginal product is higher). Since innovative efforts are cumulative, increasing Δ makes it optimal to reduce innovation $(1 - L(t))$ for all t, ceteris parabus, consequently $\frac{\partial \ln L_T}{\partial \Delta} > 0$. It follows that the numerator is positive. Similarly increasing Γ, holding Δ fixed, requires more labour diverted from production at any t, so that $\frac{\partial \ln L_T}{\partial \Gamma} < 0$. Although the term in square brackets in the denominator is therefore negative, it must be less than -1. Since $\alpha < 1/2$, the negative slope of AMC follows.∎

(2) Since the social planners solution implies that whenever $L(t) < 1$,

$$r(t) = \frac{\dot{w}(t)}{w(t)},$$

condition (2) reduces to the equilibrium requirement that

$$\mu V^D(\Delta^*; \Gamma, \Delta) = w(\Delta^*; \Gamma, \Delta)$$

$$\mu e^{-[R(T) - R(T^*)]} e^{\Gamma} V_0^I(0) = w(\Delta^*; \Gamma, \Delta)$$

$$\frac{\mu e^{-[R(\Delta) - R(\Delta^*)]} e^{\Gamma}}{1 - P(\Gamma) e^{-(\rho + (\sigma-1)\phi)\Delta} e^{(1-\sigma)\Gamma}} \int_0^{\Delta} e^{-\int_0^{\tau} r(s)ds} \pi(\tau) d\tau = w(\Delta^*; \Gamma, \Delta)$$

$$\frac{\mu(1-\omega)(\frac{1-e^{-\gamma}}{e^{-\gamma}}) e^{-[R(\Delta) - R(\Delta^*)]} e^{\Gamma}}{1 - P(\Gamma) e^{-(\rho + (\sigma-1)\phi)\Delta} e^{(1-\sigma)\Gamma}} \int_0^{\Delta} e^{-\int_0^{\tau} r(s)ds} w(\tau) L(\tau) d\tau = w(\Delta^*; \Gamma, \Delta)$$

Re-arranging we have

$$\frac{e^{-[R(\Delta) - R(\Delta^*)]} e^{\Gamma}}{1 - P(\Gamma) e^{-(\rho + (\sigma-1)\phi)\Delta} e^{(1-\sigma)\Gamma}} \int_0^{\Delta} e^{-\int_0^{\tau} r(s)ds} \frac{w(\tau)L(\tau)}{w(\Delta^*)} d\tau = \frac{e^{-\gamma}}{\mu(1 - e^{-\gamma})(1 - \omega)}$$

$$\frac{e^{-[R(\Delta) - R(\Delta^*)]} e^{\Gamma}}{1 - P(\Gamma) e^{-(\rho + (\sigma-1)\phi)\Delta} e^{(1-\sigma)\Gamma}} \left[\int_0^{\Delta^*} e^{-\int_0^{\tau} r(s)ds} \frac{w(\tau)}{w(\Delta^*)} d\tau + e^{-R(\Delta^*)} \int_{\Delta^*}^{\Delta} L(\tau) d\tau \right] = \frac{e^{-\gamma}}{\mu(1 - e^{-\gamma})(1 - \omega)}$$

$$\frac{e^{-\rho\Delta} e^{(1-\sigma)\Gamma}}{1 - P(\Gamma) e^{-(\rho + (\sigma-1)\phi)\Delta} e^{(1-\sigma)\Gamma}} \left[\int_0^{\Delta^*} \frac{e^{-R(\tau)} w(\tau)}{e^{-R(\Delta^*)} w(\Delta^*)} d\tau + (\Delta - \Delta^*) - \frac{\Gamma}{\mu\gamma} \right] = \frac{e^{-\gamma}}{\mu(1 - e^{-\gamma})(1 - \omega)} = X$$

Now observe that for $\tau < \Delta^*$,

$$\frac{e^{-R(\tau)} w(\tau)}{e^{-R(\Delta^*)} w(\Delta^*)} = \exp\left(\int_{\tau}^{\Delta^*} \left(r(s) - \frac{\dot{w}(s)}{w(s)} \right) ds \right)$$

$$= \exp\left(\int_{\tau}^{\Delta^*} \left(\alpha \frac{c(s)}{k(s)} - (1-\alpha)(\phi + \delta) \right) ds \right)$$

For all $\tau < \Delta^*$, $\frac{c(\tau)}{k(\tau)}$ must be increasing in the steady–state capital stock at the start of each cycle — an increase in k^* reduces the marginal product of capital, so that the social planner allocates

30

marginally less income to investment and more to consumption. It follows from Lemma B3 that at each date $\tau < \Delta^*$

$$\frac{d}{d\Gamma}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right) < 0$$

$$\frac{d}{d\Delta}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right) > 0.$$

Differentiating X with respect to Γ yields

$$\frac{\partial X}{\partial \Gamma} = -\frac{e^{-(\rho+(\sigma-1)\phi)\Delta}e^{-(\sigma-1)\Gamma}}{1-P(\Gamma)e^{-(\rho+(\sigma-1)\phi)\Delta}e^{-(\sigma-1)\Gamma}}\left[\frac{1}{\mu\gamma}+\frac{X}{\gamma}+\frac{(\sigma-1)X}{e^{-\rho\Delta}e^{-(\sigma-1)\Gamma}}-\int_0^{\Delta^*}\frac{d}{d\Gamma}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right)d\tau.\right]$$

Since $\int_0^{\Delta^*}\frac{d}{d\Gamma}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right)d\tau \leq 0$, a sufficient condition for $\frac{\partial X}{\partial \Gamma} < 0$ is that $\frac{1}{\mu\gamma}+\frac{X}{\gamma}+\frac{(\sigma-1)X}{e^{-(\rho+(\sigma-1)\phi)\Delta}e^{-(\sigma-1)\Gamma}} > 0$. Clearly this must hold for $\sigma \geq 1$, and will hold for $\sigma < 1$ if sufficiently large.

Differentiating X w.r.t. Δ we get

$$\frac{\partial X}{\partial \Delta} = \frac{e^{-(\rho+(\sigma-1)\phi)\Delta}e^{-(\sigma-1)\Gamma}}{1-P(\Gamma)e^{-(\rho+(\sigma-1)\phi)\Delta}e^{-(\sigma-1)\Gamma}}\left[1-(\rho+(\sigma-1)\phi)X+\int_0^{\Delta^*}\frac{d}{d\Delta}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right)d\tau\right]$$

Since $\int_0^{\Delta^*}\frac{d}{d\Delta}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right)d\tau \geq 0$, a sufficient condition for $\frac{\partial X}{\partial \Delta} > 0$ is

$$\frac{(\rho+(\sigma-1)\phi)e^{-\gamma}}{\mu(1-e^{-\gamma})(1-\omega)} < 1$$

Thus, the slope of the FEI curve is given by

$$\left.\frac{d\Gamma}{d\Delta}\right|_{FEI} = -\frac{\left(\frac{\partial X}{\partial T}\right)}{\left(\frac{\partial X}{\partial \Gamma}\right)} = \frac{1-(\rho+(\sigma-1)\phi)X+\int_0^{\Delta^*}\frac{d}{d\Delta}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right)d\tau}{\frac{1}{\mu\gamma}+\frac{X}{\gamma}+\frac{(\sigma-1)X}{e^{-(\rho+(\sigma-1)\phi)\Delta}e^{-(\sigma-1)\Gamma}}-\int_0^{\Delta^*}\frac{d}{d\Gamma}\left(\frac{e^{-R(\tau)}w(\tau)}{e^{-R(\Delta^*)}w(\Delta^*)}\right)d\tau} > 0$$

If $\sigma \geq 1$ and $(\rho+(\sigma-1)\phi)e^{-\gamma} < \mu(1-e^{-\gamma})(1-\omega)$ (so that $(\rho+(\sigma-1)\phi)X < 1$), this is clearly positive. For lower values of σ it may still be positive.∎

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Table 1: Baseline Parameters

Parameter	Value
α	0.30
γ	0.20
σ	1.00
ρ	0.025
μ	1.80
δ	0.10
ϕ	0.00
ω	0.70

Table 2: Comparative Stationary Cycles

Parameters		Γ	Δ	Δ^*	k^*	$\bar{g}$ (%)	$\bar{r}$ (%)	$P(T)$
Baseline		0.15	7.83	5.06	2.30	1.90	4.40	0.26
$\alpha =$	0.27	0.12	6.53	4.15	1.94	1.90	4.40	0.38
	0.33	0.18	9.33	6.12	2.70	1.92	4.42	0.11
$\gamma =$	0.18	0.16	10.26	7.12	2.33	1.54	4.04	0.12
	0.22	0.14	6.19	3.70	2.22	2.33	4.83	0.34
$\delta =$	0.08	0.15	7.86	5.12	2.89	1.88	4.38	0.26
	0.12	0.15	7.96	5.15	2.06	1.89	4.38	0.25
$\phi =$	0.005	0.15	7.90	5.12	2.19	2.38	4.88	0.26
	0.010	0.15	7.75	4.96	2.06	2.94	5.44	0.25
$\mu =$	1.62	0.16	9.58	6.43	2.27	1.70	4.20	0.19
	1.98	0.14	6.51	4.03	2.29	2.14	4.64	0.30
$\rho =$	0.0225	0.14	7.06	4.32	2.37	1.97	4.23	0.30
	0.0275	0.16	8.40	5.59	2.19	1.93	4.68	0.19
$\sigma =$	0.9	0.14	7.39	4.65	2.37	1.89	4.20	0.30
	1.1	0.16	8.29	5.48	2.21	1.92	4.61	0.20
$\omega =$	0.6	0.17	6.56	3.75	2.08	2.57	5.07	0.16
	0.8	0.13	10.82	8.08	2.54	1.20	3.71	0.35

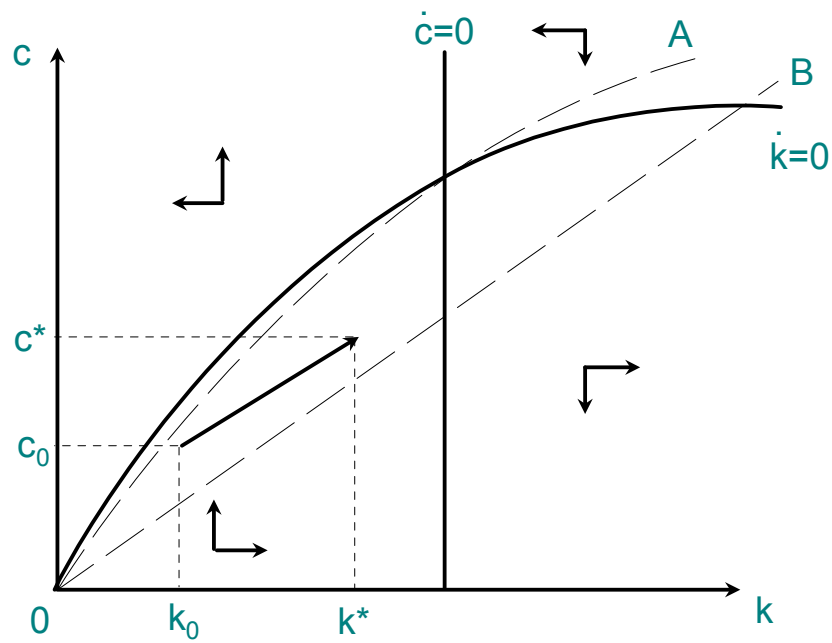Figure 3: Innovation and Investment in the vth cycle
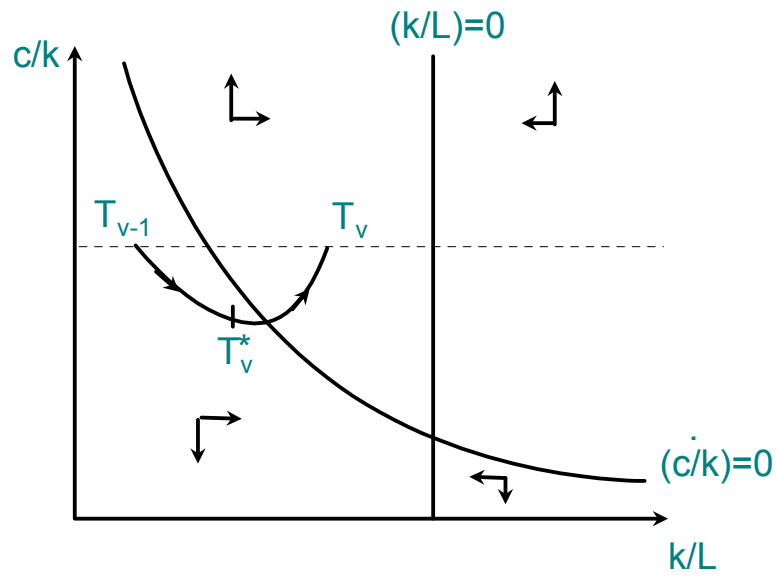

Figure 4: Dynamics in Phase 1

Figure 5: Within-cycle Dynamics in the Stationary Cyclical Equilibrium
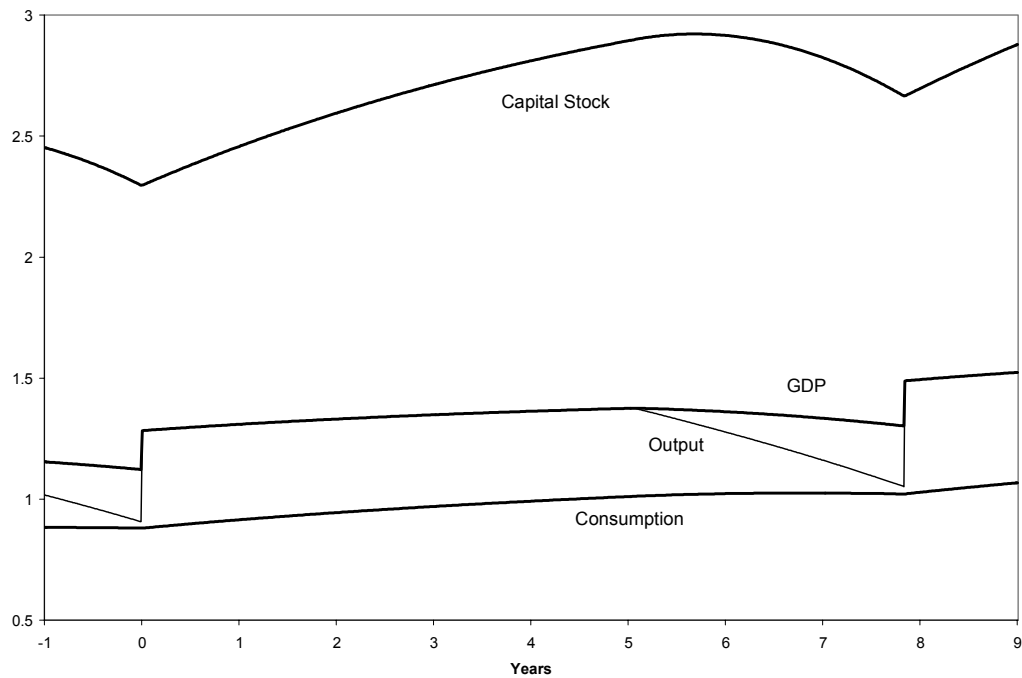

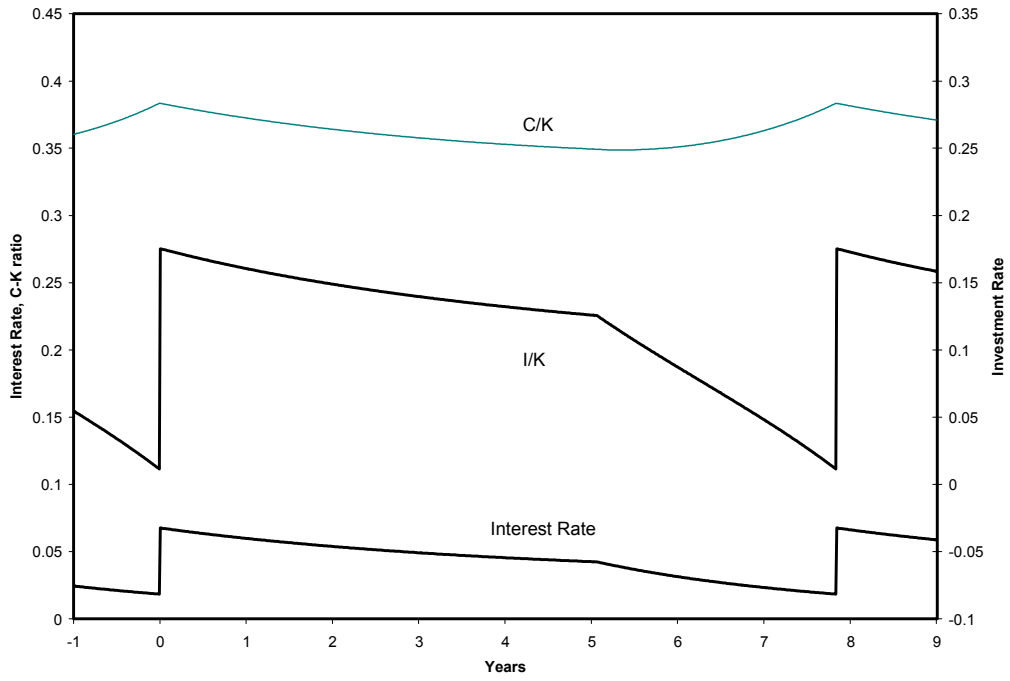Figure 6: Evolution of Key Aggregates during the Cycle
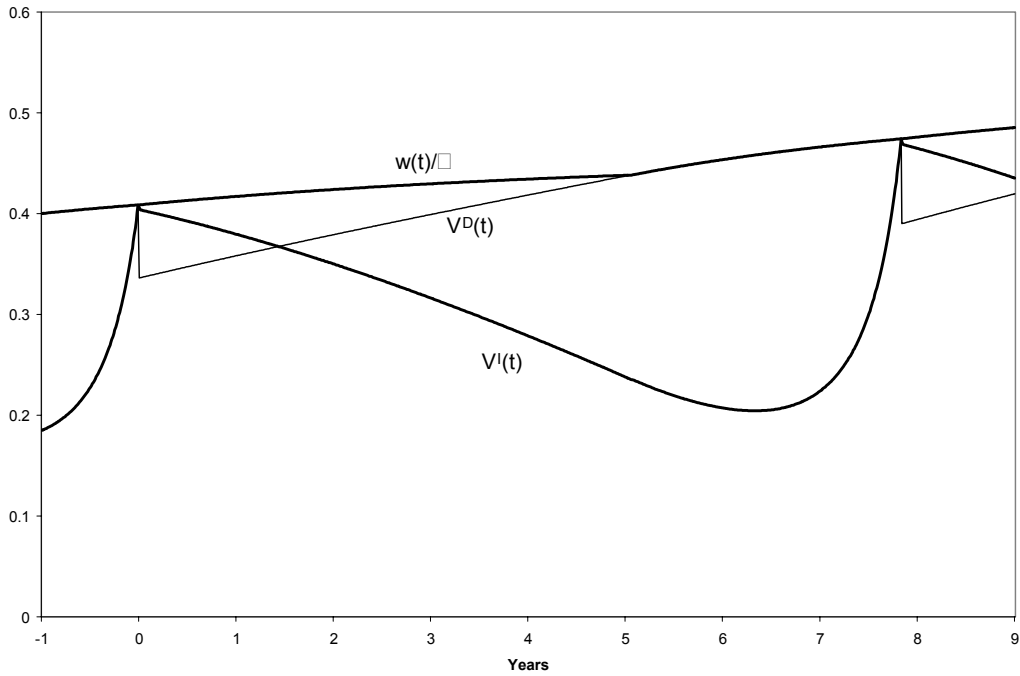
Figure 7: Rates and Ratios during the Cycle

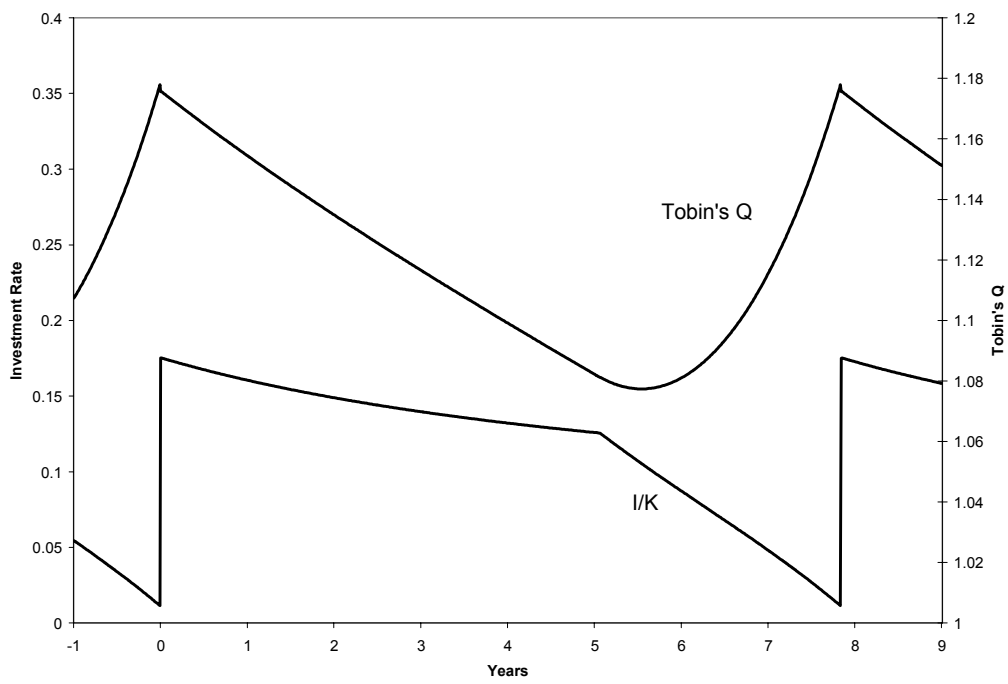

Figure 8: Evolution of Firm Values and the Wage

38

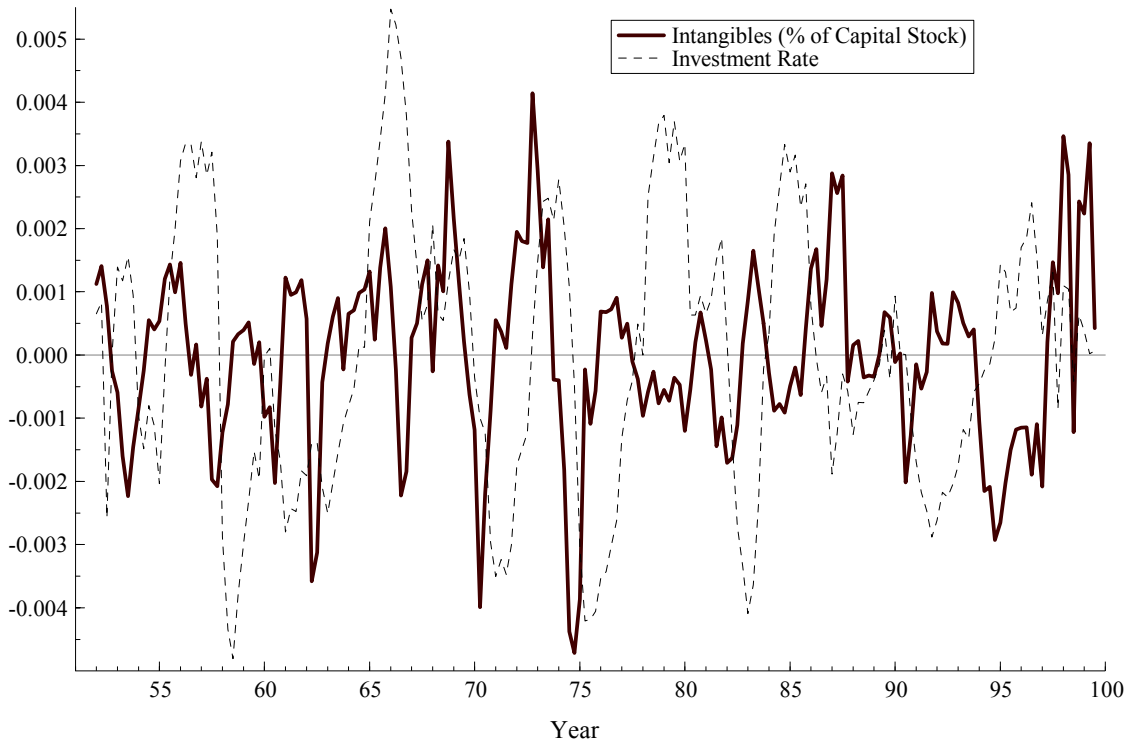Figure 9: Tobin's Q and the Rate of Investment

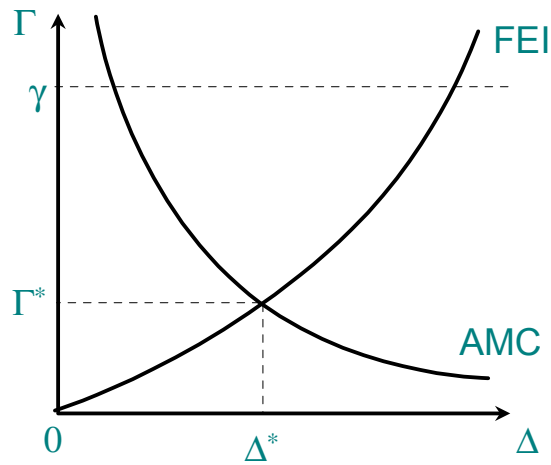Figure 10: Cyclical Component of Fixed Investment and Intangibles using Hall's (2001) data

Figure 11: Uniqueness of the Stationary Equilibrium Cycle