Mathematical Model of Stock Market Fluctuations in the Absence of Economic Growth

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ABSTRACT

There are shown the ordinary differential equations describing the fluctuations of equities, derivatives, and second derivatives on the stock market.

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1. Introduction

Author described the dynamics of market fluctuations regarding to goods and services, and money based on interrelationships such economic categories as supply, demand, and price in the number of articles [1, 2, 3, 4, 5]. In his latest work [5] author introduced two distinct markets: market of goods and services and market of money, and showed how these markets influence each other.

In the current paper author projects the fluctuations on the market of goods and services, and fluctuations on the monetary market respectively on the fluctuations of the underlying stock.

Then author shows how fluctuations on the stock market induce the fluctuations on the market of derivatives.

All results are based on the mathematical model expressing equilibrium forces acting on markets through the ordinary differential equations, and solutions of these equations are examined.

The problems associated with economic growth are left behind of this article.

2. Primary Market: Goods and Services

Here we describe the imaginary closed market with one only product. At the beginning we assume that market is in equilibrium position, there is no economic growth, and market has product's supply and demand, which both are developing with a constant rate and the product's price is fixed.

When the balance between the product's supply and demand is broken the product's market is experienced the economic forces, which act to bring the market to a new equilibrium position.

These market forces are described by the following ordinary differential equations regarding to the product's supply $V_S(t)$, demand $V_D(t)$, and price $P_C(t)$.

$$\frac{d^2 V_S(t)}{dt^2} = \boldsymbol{l}_1 \cdot \frac{dP_C(t)}{dt} \tag{1}$$

$$\frac{dP_C(t)}{dt} = \boldsymbol{l}_2 \cdot (V_D(t) - V_S(t)) \tag{2}$$

$$\frac{d^2 V_D(t)}{dt^2} = -\boldsymbol{l}_3 \cdot \frac{d^2 P_C(t)}{dt^2} \tag{3}$$

In the equations above the values $l_1,\ l_2,\ l_3 \geq 0$ are constants.

Brief meaning of equations (1) – (3) was described in [5].

3. Primary Market: Money

Similarly here we describe the closed market with one product – money. At the beginning we assume that monetary market is in equilibrium position, there is no growth, and monetary market has money's supply and demand, which both are developing with a constant rate and the money's price is fixed. The price of money is universally represented by an interest rate.

When such balance between the money's supply and demand is broken, the money's market is experienced the economic forces, which act to bring the market to a new equilibrium position.

These market forces are described by the following ordinary differential equations regarding to the money's supply $M_S(t)$, demand $M_D(t)$, and price $P_M(t)$.

$$\frac{d^2 M_S(t)}{dt^2} = m_1 \cdot \frac{dP_M(t)}{dt} \tag{4}$$

$$\frac{dP_M(t)}{dt} = m_2 \cdot (M_D(t) - M_S(t)) \tag{5}$$

$$\frac{d^2 M_D(t)}{dt^2} = -m_3 \cdot \frac{d^2 P_M(t)}{dt^2} \tag{6}$$

In the equations (4) – (6) above the values $m_1, m_2, m_3 \geq 0$ are constants.

Brief meaning of equations (4) – (6) was described in [5].

4. Primary Market: Monetary Economy

Here we describe the imaginary closed market with two commodities – the first one is real product and the second one is money.

At the beginning we assume that market is in equilibrium position. Again there is no economic growth. The money's supply and demand on the market are both developing with a constant rate and the money's price is fixed. The product's supply and demand on the market are both developing with a constant rate and the product's price is fixed. The abovementioned rate, which regards to real product, equals to similar rate, which regards to money.

When either the balance between the product's supply and demand or the balance between the money's supply and demand is broken the market is experienced the economic forces, which act to bring the market to a new equilibrium position.

These market forces are described by the following ordinary differential equations regarding to the product's supply $V_S(t)$, demand $V_D(t)$, and price $P_C(t)$ and the money's supply $M_S(t)$, demand $M_D(t)$, and price $P_M(t)$.

$$\frac{d^2 V_S(t)}{dt^2} = l_1 \cdot \frac{dP_C(t)}{dt} \tag{7}$$

$$\frac{dP_C(t)}{dt} = l_2 \cdot (M_S(t) - V_S(t)) \tag{8}$$

$$\frac{d^2 M_S(t)}{dt^2} = m_1 \cdot \frac{dP_M(t)}{dt} - l_3 \cdot \frac{d^2 P_C(t)}{dt^2} \tag{9}$$

$$\frac{dP_M(t)}{dt} = m_2 \cdot (M_D(t) - M_S(t)) \tag{10}$$

$$\frac{d^2 M_D(t)}{dt^2} = -m_3 \cdot \frac{d^2 P_M(t)}{dt^2} \tag{11}$$

$$V_D(t) = M_S(t) \tag{12}$$

Meaning of equations (7) – (12) was described in [5].

Here I give an illustrative example showing activities of the equilibrium forces.

Let me assume in this example for simplicity that $l_3 = 0$ and $m_3 = 0$.

I assume that at time $t = t_0$ we have the following conditions on the market – with regard to money $\frac{dM_S(t_0)}{dt} = \frac{dM_D(t_0)}{dt} = 0$, $M_D(t_0) = \Delta$, $M_S(t_0) = 0$, $P_M(t_0) = I_0$, and

$\Delta \neq 0$, and with regard to real product $\frac{dV_S(t_0)}{dt} = \frac{dV_D(t_0)}{dt} = 0$, $V_S(t_0) = V_D(t_0) = 0$,

$P_C(t_0) = P_0$.

It was shown in [5] based on [7, 8] that aforesaid change in the money demand $M_D(t_0) = \Delta$ creates the following fluctuations in the product's supply $V_S(t)$,

$$V_S(t - t_0) = \Delta \cdot \left[\frac{m_1 \cdot m_2}{l_1 \cdot l_2 - m_1 \cdot m_2} \cdot \cos\left(\sqrt{l_1 \cdot l_2}(t - t_0)\right) - \frac{l_1 \cdot l_2}{l_1 \cdot l_2 - m_1 \cdot m_2} \cdot \cos\left(\sqrt{m_1 \cdot m_2}(t - t_0)\right) + 1 \right] \tag{13}$$

The solution (13) of this illustrative example represents the sum of two sinusoids.

5. Secondary Market: Equities in Monetary Economy

The markets forces representing the valuation of equities in monetary economy relate the demand, supply, and price of equities to the demand, supply, and price of underlying products through the following ordinary differential equations.

At first I define (in this model) the equities in regard to one of two commodities used in the model (real product and money) as an ownership rights to the part of an entity producing the commodity in question and as consequence an ownership right to the commodity in question itself.

Let me denote $V_S^{II}(t)$ – be supply of equities in regard to the product, $V_D^{II}(t)$ – be demand for equities in regard to the product, and $P_C^{II}(t)$ – be price of equities in regard to the product. Similarly I denote $M_S^{II}(t)$ – be supply of equities in regard to the money, $M_D^{II}(t)$ – be demand for equities in regard to the money, and $P_M^{II}(t)$ – be price of equities in regard to the money.

Then we have the following expressions in regard to the equities of entity supplying the money,

$$\frac{dP_M(t)}{dt} = m_2 \cdot (M_D(t) - M_S(t)) \tag{14}$$

$$\frac{d^2 M_S^{II}(t)}{dt^2} = m_1^{II} \cdot \frac{dP_M^{II}(t)}{dt} \tag{15}$$

$$\frac{dP_M^{II}(t)}{dt} = m_2^{II} \cdot \left(M_D^{II}(t) - M_S^{II}(t)\right) \tag{16}$$

$$\frac{d^2 M_D^{II}(t)}{dt^2} = m_4 \cdot \frac{dP_M(t)}{dt} - m_3^{II} \cdot \frac{d^2 P_M^{II}(t)}{dt^2} \tag{17}$$

In the equations (15) – (17) above the values $m_4, m_1^{II}, m_2^{II}, m_3^{II} \geq 0$ are constants.

Brief meaning of equations (14) – (17) is following. When the money's demand and supply are not equal, the price of money changes. The positive changes in the price of money drive the demand for the underlying equities up, and the negative changes in the price of money drive the demand for the underlying equities down. On the other hand disbalance in the supply and demand for equities as usual affects their prices, which in turn affects their supply and demand.

Analogously we have the following expressions in regard to the equities of entity making the product,

$$\frac{dP_C(t)}{dt} = \lambda_2 \cdot (V_D(t) - V_S(t)) \tag{18}$$

$$\frac{d^2V_S^{II}(t)}{dt^2} = l_1^{II} \cdot \frac{dP_C^{II}(t)}{dt} \tag{19}$$

$$\frac{dP_C^{II}(t)}{dt} = l_2^{II} \cdot \left(V_D^{II}(t) - V_S^{II}(t)\right) \tag{20}$$

$$\frac{d^2V_D^{II}(t)}{dt^2} = l_4 \cdot \frac{dP_C(t)}{dt} - l_3^{II} \cdot \frac{d^2P_C^{II}(t)}{dt^2} \tag{21}$$

In the equations (19) – (21) above the values $l_4,\ l_1^{II},\ l_2^{II},\ l_3^{II} \geq 0$ are constants.

Brief meaning of equations (18) – (21) is following. When the product's demand and supply are not equal, the price of product changes. The positive changes in the product's price drive the demand for the underlying equities up, and the negative changes in the product's price drive the demand for the underlying equities down. On the other hand disbalance in the supply and demand for equities as usual affects their prices, which in turn affects their supply and demand.

Here I give an illustrative example showing activities of the equilibrium forces.

Let me assume in this example for simplicity that $l_3 = 0$, $m_3 = 0$, and $m_3^{II} = 0$.

I assume that at time $t = t_0$ we have the following conditions on the primary and secondary markets – with regard to money $\frac{dM_S(t_0)}{dt} = \frac{dM_D(t_0)}{dt} = 0$, $M_D(t_0) = \Delta$, $M_S(t_0) = 0$, $P_M(t_0) = I_0$, and $\Delta \neq 0$, and with regard to equities of the entity supplying the money $\frac{dM_S^{II}(t_0)}{dt} = \frac{dM_D^{II}(t_0)}{dt} = 0$, $M_S^{II}(t_0) = M_D^{II}(t_0) = 0$, $P_M^{II}(t_0) = I_0^{II}$.

Based on [7, 8] that aforesaid change in the money demand $M_D(t_0) = \Delta$ creates the fluctuating disbalance in the money's supply and demand,

$$M_S(t-t_0)-M_D(t-t_0)=-\Delta\cdot\cos\left(\sqrt{m_1\cdot m_2}(t-t_0)\right) \tag{22}$$

that produces the fluctuating disbalance in the underlying equities' supply and balance,

$$M_S^{II}(t-t_0)-M_D^{II}(t-t_0)=\frac{\Delta\cdot m_2\cdot m_4}{m_1\cdot m_2-m_1^{II}\cdot m_2^{II}}\cdot\cos\left(\sqrt{m_1\cdot m_2}(t-t_0)\right)-\frac{\Delta\cdot m_2\cdot m_4}{m_1\cdot m_2-m_1^{II}\cdot m_2^{II}}\cdot\cos\left(\sqrt{m_1^{II}\cdot m_2^{II}}(t-t_0)\right) \tag{23}$$

Therefore the price of the equities supplying the money will fluctuate as follows,

$$\begin{aligned}P_M^{II}(t-t_0)&=I_0^{II}+\frac{\Delta}{\sqrt{m_1\cdot m_2}}\cdot\frac{m_2\cdot m_4}{m_1\cdot m_2-m_1^{II}\cdot m_2^{II}}\cdot\sin\left(\sqrt{m_1\cdot m_2}(t-t_0)\right)\\&-\frac{\Delta}{\sqrt{m_1^{II}\cdot m_2^{II}}}\cdot\frac{m_2\cdot m_4}{m_1\cdot m_2-m_1^{II}\cdot m_2^{II}}\cdot\sin\left(\sqrt{m_1^{II}\cdot m_2^{II}}(t-t_0)\right)\end{aligned} \tag{24}$$

The solution (24) of this illustrative example represents the sum of two sinusoids.

6. Tertiary Market: Derivatives in Monetary Economy

The markets forces representing the valuation of derivatives in monetary economy relate the demand, supply, and price of derivatives to the demand, supply, and price of both underlying equities and corresponding products through the following ordinary differential equations.

At first I define (in this model) the derivatives of equities in regard to one of two commodities used in the model (real product and money) as an ownership rights to the part of equities for an entity producing the commodity in question, and as consequence an ownership right to corresponding part of the entity itself, and as consequence an ownership rights to the commodity in question itself.

Let me denote $V_S^{III}(t)$ – be supply of derivatives in regard to the product, $V_D^{III}(t)$ – be demand for derivatives in regard to the product, and $P_C^{III}(t)$ – be price of derivatives in regard to the product. Similarly I denote $M_S^{III}(t)$ – be supply of derivatives in regard to the money, $M_D^{III}(t)$ – be demand for derivatives in regard to the money, and $P_M^{III}(t)$ – be price of derivatives in regard to the money.

Then we have the following expressions in regard to the derivatives for the equities of an entity supplying the money,

$$\frac{dP_M(t)}{dt} = m_2 \cdot (M_D(t) - M_S(t)) \tag{25}$$

$$\frac{dP_M^{II}(t)}{dt} = m_2^{II} \cdot \left(M_D^{II}(t) - M_S^{II}(t)\right) \tag{26}$$

$$\frac{d^2 M_S^{III}(t)}{dt^2} = m_1^{III} \cdot \frac{dP_M^{III}(t)}{dt} \tag{27}$$

$$\frac{dP_M^{III}(t)}{dt} = m_2^{III} \cdot \left(M_D^{III}(t) - M_S^{III}(t)\right) \tag{28}$$

$$\frac{d^2 M_D^{III}(t)}{dt^2} = m_5 \cdot \frac{dP_M(t)}{dt} + m_4^{II} \cdot \frac{dP_M^{II}(t)}{dt} - m_3^{III} \cdot \frac{d^2 P_M^{III}(t)}{dt^2} \tag{29}$$

In the equations (27) – (29) above the values $m_5, m_4^{II}, m_1^{III}, m_2^{III}, m_3^{III} \geq 0$ are constants.

Brief meaning of equations (25) – (29) is following. When the money's demand and supply are not equal, the price of money changes. Similarly when the entities' demand and supply are not equal, the price of entities changes. The positive changes both in the prices of money and entities drive the demand for the underlying derivatives up, and the negative changes both in the prices of money and entities drive the demand for the underlying derivatives down. On the other hand disbalance in the supply and demand for derivatives as usual affects their prices, which in turn affects their supply and demand.

Analogously we have the following expressions in regard to the derivatives for the equities of entity making the product,

$$\frac{dP_C(t)}{dt} = l_2 \cdot \left(V_D(t) - V_S(t)\right) \tag{30}$$

$$\frac{dP_C^{II}(t)}{dt} = l_2^{II} \cdot \left(V_D^{II}(t) - V_S^{II}(t)\right) \tag{31}$$

$$\frac{d^2 V_S^{III}(t)}{dt^2} = l_1^{III} \cdot \frac{dP_C^{III}(t)}{dt} \tag{32}$$

$$\frac{dP_C^{III}(t)}{dt} = l_2^{III} \cdot \left(V_D^{III}(t) - V_S^{III}(t)\right) \tag{33}$$

$$\frac{d^2 V_D^{III}(t)}{dt^2} = l_5 \cdot \frac{dP_C(t)}{dt} + l_4^{II} \cdot \frac{dP_C^{II}(t)}{dt} - l_3^{III} \cdot \frac{d^2 P_C^{III}(t)}{dt^2} \tag{34}$$

In the equations (32) – (34) above the values $l_5,\ l_4^{II},\ l_1^{III},\ l_2^{III},\ l_3^{III} \geq 0$ are constants.

Brief meaning of equations (30) – (34) is following. When the product's demand and supply are not equal, the price of product changes. Similarly when the entities' demand and supply are not equal, the price of entities changes. The positive changes both in the prices of product and entities drive the demand for the underlying derivatives up, and the negative changes both in the prices of product and entities drive the demand for the underlying derivatives down. On the other hand disbalance in the supply and demand for derivatives as usual affects their prices, which in turn affects their supply and demand.

7. Quaternary Market: Second Derivatives

The markets forces representing the valuation of second derivatives in monetary economy relate the demand, supply, and price of derivatives to the demand, supply, and price of all underlying derivatives, equities and corresponding products through the following ordinary differential equations.

At first I define (in this model) the second derivatives of derivatives in regard to one of two commodities used in the model (real product and money) as an ownership rights to the part of derivatives for the equities of an entity producing the commodity in question, and as consequence an ownership right to corresponding part of the equities itself, and as consequence an ownership right to corresponding part of the entity itself, and as consequence an ownership rights to the commodity in question itself.

Let me denote $V_S^{IV}(t)$ – be supply of second derivatives in regard to the product, $V_D^{IV}(t)$ – be demand for second derivatives in regard to the product, and $P_C^{IV}(t)$ – be price of second derivatives in regard to the product. Similarly I denote $M_S^{IV}(t)$ – be supply of second derivatives in regard to the money, $M_D^{IV}(t)$ – be demand for second derivatives in regard to the money, and $P_M^{IV}(t)$ – be price of second derivatives in regard to the money.

Then we have the following expressions in regard to the second derivatives for derivatives for the equities of an entity supplying the money,

$$\frac{dP_M(t)}{dt} = m_2 \cdot (M_D(t) - M_S(t)) \qquad (35)$$

$$\frac{dP_M^{II}(t)}{dt} = m_2^{II} \cdot \left(M_D^{II}(t) - M_S^{II}(t)\right) \tag{36}$$

$$\frac{dP_M^{III}(t)}{dt} = m_2^{III} \cdot \left(M_D^{III}(t) - M_S^{III}(t)\right) \tag{37}$$

$$\frac{d^2 M_S^{IV}(t)}{dt^2} = m_1^{IV} \cdot \frac{dP_M^{IV}(t)}{dt} \tag{38}$$

$$\frac{dP_M^{IV}(t)}{dt} = m_2^{IV} \cdot \left(M_D^{IV}(t) - M_S^{IV}(t)\right) \tag{39}$$

$$\frac{d^2 M_D^{IV}(t)}{dt^2} = m_6 \cdot \frac{dP_M(t)}{dt} + m_5^{II} \cdot \frac{dP_M^{II}(t)}{dt} + m_4^{III} \cdot \frac{dP_M^{III}(t)}{dt} - m_3^{IV} \cdot \frac{d^2 P_M^{IV}(t)}{dt^2} \tag{40}$$

In the equations (38) – (40) above the values $m_6, m_5^{II}, m_4^{III}, m_1^{IV}, m_2^{IV}, m_3^{IV} \geq 0$ are constants.

Brief meaning of equations (35) – (40) is following. When the money's demand and supply are not equal, the price of money changes. Similarly when the entities' demand and supply are not equal, the price of entities changes. Likewise when the derivatives' demand and supply are not equal, the price of derivatives changes. The positive changes all in the prices of money, entities, and derivatives drive the demand for the underlying second derivatives up, and the negative changes all in the prices of money, entities, and derivatives drive the demand for the underlying second derivatives down. On the other hand disbalance in the supply and demand for second derivatives as usual affects their prices, which in turn affects their supply and demand.

Analogously we have the following expressions in regard to the second derivatives for derivatives for the equities of an entity making the product,

$$\frac{dP_C(t)}{dt} = l_2 \cdot \left(V_D(t) - V_S(t)\right) \tag{41}$$

$$\frac{dP_C^{II}(t)}{dt} = l_2^{II} \cdot \left(V_D^{II}(t) - V_S^{II}(t)\right) \tag{42}$$

$$\frac{dP_C^{III}(t)}{dt} = l_2^{III} \cdot \left(V_D^{III}(t) - V_S^{III}(t)\right) \tag{43}$$

$$\frac{d^2 V_S^{IV}(t)}{dt^2} = l_1^{IV} \cdot \frac{dP_C^{IV}(t)}{dt} \tag{44}$$

$$\frac{dP_C^{IV}(t)}{dt} = l_2^{IV} \cdot \left(V_D^{IV}(t) - V_S^{IV}(t)\right) \tag{45}$$

$$\frac{d^2 V_D^{IV}(t)}{dt^2} = l_6 \cdot \frac{dP_C(t)}{dt} + l_5^{II} \cdot \frac{dP_C^{II}(t)}{dt} + l_4^{III} \cdot \frac{dP_C^{III}(t)}{dt} - l_3^{IV} \cdot \frac{d^2 P_C^{IV}(t)}{dt^2} \tag{46}$$

In the equations (44) – (46) above the values $l_6,\ l_5^{II},\ l_4^{III},\ l_1^{IV},\ l_2^{IV},\ l_3^{IV} \geq 0$ are constants.

Brief meaning of equations (41) – (46) is following. When the product's demand and supply are not equal, the price of product changes. Similarly when the entities' demand and supply are not equal, the price of entities changes. Likewise when the derivatives' demand and supply are not equal, the price of derivatives changes. The positive changes all in the prices of product, entities, and derivatives drive the demand for the underlying second derivatives up, and the negative changes all in the prices of product, entities, and derivatives drive the demand for the underlying second derivatives down. On the other hand disbalance in the supply and demand for second derivatives as usual affects their prices, which in turn affects their supply and demand.

8. Conclusion

In the current paper I attempted to describe the fluctuations observed on the stock market through ordinary differential equations connecting supply, demand, and prices of the underlying products. Of course this work is just a beginning. If inclusion into the model the multiple products can be possible done by following approach outlined in [5, 6] but questions of insertion there the concept of economic growth represent significant theoretical challenge.

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