Utilization and maintenance in a model with terminal scrapping

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Abstract

We draw on three strands of literature dealing with *utilization*, *maintenance*, and *scrapping* in order to analyze the properties of the respective policies and their interactions. We do so by focusing on the last period of the received multi-period service life model and extending it in three directions: first, by associating the physical deterioration of equipment to the intensity of its *utilization* and *maintenance*; second, by expanding on the range of explainable *operating policies* to allow for *idling*, *mothballing*, *capacity depleting*, *capacity preserving, full capacity, upgrading*, and *downgrading*; and, third, by linking the *operating policies* to the *capital policy* of *scrapping*. Owing to these enhancements, the analysis leads to several important findings. One among them is that optimal *operating policies* depend on the properties of the *operating function*. If it exhibits linearities, then the operating policies jump to policies involving more *utilization* and more *maintenance* or vice versa. If not, then the policies develop monotonously, proceeding in time from *harder* to *softer* or vice versa, depending on the net revenue earning capability of the equipment. Another is that profit (loss) making equipment is *scrappable* iff on the average the *operating capital* deteriorates faster (slower), or equivalently improves slower (faster), than the *scrapping capital*. And still another result is that *operating policies* are determined jointly with *capital policies*, thus suggesting that empirical investigations of their determinants should allow for this simultaneity.

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1. Introduction

Owners' decisions with respect to their durables may be classified into two categories.[2] The first concerns the decisions that are primarily directed at changing the condition of durables themselves and includes *replacement, scrapping, expansionary investment, upgrading, downgrading, overhauling* and *stripping.* Below we shall refer to them as *capital policies*. The second category comprises the decisions that are associated with *utilization* and *maintenance* and we shall refer to them as *operating policies*.

In Bitros and Flytzanis (2002a) we extended the multi-period service life model and laid down the groundwork to derive all these policies from a unified analytical framework based on rational economic behavior. But partly because of the overwhelming attention they have received in the literature and partly because the presentation had to be kept within reasonable limits, in that paper we placed the emphasis on *replacement* and *scrapping* and kept all other policies in the background. As a result this left for us the tasks, on the one hand, to expand the model so as to incorporate the remaining *real capital policies,* and on the other, to investigate the properties of *operating policies* and their interactions with *capital policies.* Thus, having accomplished recently in Bitros and Flytzanis (2002b) the former of these two tasks, our goal in this paper is to pursue the latter.

The long and arduous endeavor to integrate operating with capital policies has evolved along three paths. Following the trail of thoughts by Keynes (1935), the objective in the first path was to allow for the depreciation of durables due to the intensity of their utilization. This started with the contribution by Taubman and Wilkinson (1970); Nadiri and Rosen (1974), Shapiro (1986), Bischoff and Kokkelenberg (1987), and Johnson (1994) developed it further; and progress peaked with the papers by Prucha and Nadiri (1996) and Jin and Kite-Powell (1999).[3] In the second path the goal was to introduce maintenance. This began with Masse (1962); it continued with Naslund (1966), Jorgenson, McCall and Radner (1967), and Thompson (1968); and while it culminated with Kamien and Schwartz (1971), the interest in it has not subsided because of the wide implications and significant relative size of maintenance expenditures.[4] Finally, working in the third path, Bitros (1972, 1976a, 1976b) and Parks (1977,1979) in the 1970's, Epstein and Denny (1980), Everson (1982) and Kim (1988) in the 1980's, and Licandro and Puch (2000), Collard and Kollintzas (2000), and Boucekkine and Tamarit (2003), more recently, have pushed for a model of capital services with endogenous utilization, maintenance and gross investment.

From the proceeding it follows that the present paper falls in the last group of studies. But it differs from them in that they fail to characterize the properties of *operating policies* and their interactions with *capital policies*. To substantiate this claim, suppose that we would like to obtain advice on the following questions. When should the representative firm stop operations and proceed to *idle, mothball* or even *scrap* its equipment? Under what conditions is it profitable to *upgrade* or *downgrade* the equipment? Do the analytic forms of the functions relating *utilization* and *maintenance* to cash flow and equipment deterioration matter, and if so, in what way? If one searched for enlightenment in the literature cited above, one would not find much. And the same is true with the literature from such fields as *operations research* and *operations management*. To the best of our knowledge then, this constitutes the first attempt to shed light on these questions.

Owing to the new setting, the results that emerge are quite illuminating. Unlike previous studies that led to indeterminate *utilization*, *maintenance* and *service life* policies, the ones obtained here are determinate and computable to any desired approximation. At his own discretion the owner may run down his equipment through more intensive *utilization* and *downgrading.* Technological improvements permitted under the original design of equipment may be incorporated gradually through *upgrading*. Technological breakthroughs generate uncertainty, which raises the effective rate of discount. If either of the two flow functions relating *utilization* and *maintenance* to cash flow and equipment wear is strictly concave, the optimal path of operating policies is in fact unique and continuous. Otherwise there may be jumps to *operating policies* of lower intensity, i.e. both lower *utilization* and *maintenance*, and vice versa. Last but not least, the owner may stop using his equipment and decide to: a) *scrap* it, b) *idle* it temporarily in order to weather unfavorable market conditions or even *mothball* it for use much later.

Section 2 describes the model, the optimality conditions, and the policies. Since the building blocks of the model have been elaborated extensively in Bitros and Flytzanis (2002a, 2002b), the presentation here is meant to serve only as a vehicle to introduce certain clarifications and to identify the totality of operating and capital policies. In Section 3 we obtain the general solution of the model and analyze the dependence of optimal operating and capital policies on the parameters. In Section 4 we construct an example by adopting separable specifications for the flow rate functions r and w. In Section 5 we highlight the implications of our results for economic theory and policy. In Section 6 we summarize our findings and conclusions, and, finally, in the Appendix we supply some technical material, which supplements the presentation significantly.

2. The model

2.1 Model specification

In Bitros and Flytzanis (2002a), we examined the problem of optimal service life of equipment in the framework of the multi-period replacement model, allowing for any number of consecutive replacements to be followed by terminal scrapping. In particular, we examined the relation between the time durations of the consecutive replacement periods and the terminal scrapping period. Furthermore, we related the above to the case of steady state replacements at equal time intervals. Here we concentrate only in one period of operations, which leads to scrapping. In fact in our previous work we showed that very often the optimal policy is that of scrapping without replacement, and further that even when it is optimal to replace, the last scrapping period is where most of the profit is made.[5] In this scrapping period the objective for the owner of the equipment may be stated as follows:

$$
\begin{aligned}
&\text{Choose } [T, u(t), m(t)] \text{ so as to maximize}: \\
&A = Q + S = \int_0^T q(u,m,K)\varphi(t)dt + \varphi(T)S(K_T, T) \\
&\text{s.t.} \quad K = -s(u,m,K), \text{ with } K(t_0) = K_0, \text{ and} \\
&\qquad 0 \le u \le 1,\ 0 \le m \le 1,
\end{aligned} \tag{1}
$$

where the various symbols are defined as follows:

$Q = \int_0^T q(u,m,K)\varphi(t)dt$: *Expected net operating revenue for operating horizon T*.

$K \quad K(t)$: *Used equipment* measured in efficiency units, reflecting its size and age since first put in operation. New or unused equipment will be denoted by $K_0 = K(0)$.

$u \quad u(t)$: *Utilization intensity* relative to some extremal values, with $0 \quad u \quad 1$.

$m \quad m(t)$: *Maintenance intensity* expressed as expense relative to some extremal values, with $0 \quad m \quad 1$.

(u,m) : *Operating policy factors*.

$q(u,m,K)$: *Flow of net operating revenue*.

$s(u,m,K)$: *Flow of net capital wear*

(q,s): *Operating policy flows.*

$S = S(K_T, T)$: *Scrap value* of used equipment at T. For the scrap value of unused equipment we set $S_0 = S(K_0, 0)$.[6]

$\varphi(t) = e^{-\sigma t}$: *Effective discount factor.* Let $F(t)$ denote the probability of a *technological breakthrough* by time *t,* with $F(0) = 0$ and $F(t) < 1$ for all t. Assuming a constant discount rate ρ, the discount factor would be $e^{\rho t}$. To account for technological uncertainty this is multiplied by $[1 - F(t)]$. In keeping with the specification of time invariance, we consider only the usual exponential case: $F(t) = 1 - e^{-\theta t}$. Then, since $\varphi(t) = e^{-(\theta+\rho)t}$, the effect of uncertainty is equivalent to introducing a revised *effective discount rate*, expressed by $\sigma \quad \theta \quad \rho$.

Expression **(1)** describes the general setting of an optimal control problem. We will proceed with a more specific model by assuming q and s of the following type:

$q = rK^{\varepsilon}$: Where $r = r(u,m)$ is the *operating net revenue rate*. Usually positive, but it can also be negative. Increasing in *u,* decreasing in *m,* concave in *(u, m)*.

$s = wK$: Where $w = w(u,m)$ is the *capital stock wear rate*. Increasing in *u,* decreasing in *m,* convex in (u,m). It expresses the effect on equipment of maintenance and usage, including aging. Usually positive but it can also be negative, if aging causes upgrading or if investment type of maintenance overbalances the wear of equipment, allowing K to even rise above the original K_0.

(w,r)*: Operating policy rates*

These rate functions characterize the operating features of the equipment. They have been taken to be time invariant. However, we will allow time variations for the prices, of the constant percentage type, by setting:

$S = pe^{\eta T}K$: *Scrap value* of equipment at time T, where:

η: *Relative rate of price change*. It is the difference between equipment price change and operating revenue price change, because any common part can be subtracted from the discount rate σ. It can have either sign, or be zero.

With the help of these specifications, we will investigate the dependence on the parameters $\{\varepsilon,\sigma,\eta,p,K_0\}$, of: a) the *operating policies* defined by the optimal rates of utilization and maintenance as functions of time: $\{u=u(t),m=m(t)\}$, and b) the *scrapping policy* defined by the optimal duration T^*.

2.2 Policy types.

Concerning *scrapping policy*, we will say that the equipment is *nonprofitable,* if $T^*=0$, *scrappable,* if $0<T^*<\infty$, and *durable,* if $T^*=\infty$.

As for the operating policies we refer to Figure 1(a) below. We will say that a policy pair $v:(u,m)$ is of:

Higher intensity, if both utilization and maintenance are higher,
Lower intensity, if both utilization and maintenance are lower.

In this ordering, we distinguish the two extremal policies, of lowest and highest intensity:

$$v_0:(u=0,\ m=0)\ \ \&\ v_1:(u=1,\ m=1).$$

More important is their ordering according to the resultant wear-revenue rates: (w,r). We will say that a policy pair is:

Harder, if it gives higher rates both for wear and revenue,
Softer, if it gives lower rates both for wear and revenue.

In this ordering, we distinguish the two extremal policies: [7]

$\underline{v}:(u=0,m=1)$: *Softest,* with the lowest rates: $(\underline{w},\underline{r})$
$\overline{v}:(u=1,m=0)$: *Hardest* with the highest rates: $(\overline{w},\overline{r})$

Moreover, we will say that a policy pair (w,r) is:

Profit making, if $r>0$, *loss making,* if $r<0$,
Downgrading, if $w>0$, *upgrading,* if $w<0$
Break even of zero revenue, if $r=0$,
Capacity preserving of zero wear, if $w=0$.

For particular equipment any of the above policy types may or may not be available. Classifying equipment according to the totality of the available policies, we say that it is:

Profit making, if $r \geq 0$, *loss making* if $r \leq 0$,
Revenue-flexible, if both profit making and loss making policies are available,
Downgrading, if $w \geq 0$, *upgrading,* if $w \leq 0$,
Wear- flexible, if both upgrading and downgrading policies are available.

Finally, we will say that the equipment is:

Special, if it has policies that are both profit making and upgrading at the same time,
Common, if it is not special.

We will find that the equipment behaves differently depending mainly on its revenue type.

Remark 1
Referring to various policy types, in practice we often use the following terminology:
1. Among the minimal utilization policies: $u = 0$, we distinguish the following:
   (i) *Closedown,* with no maintenance. It is the policy of lowest intensity:
   $$v_0 : (u = 0,\ m = 0)$$
   (ii) *Idling,* with some maintenance $(u = 0,\ m > 0)$.[8]
   (iii) *Mothballing,* with full maintenance. It is the *softest policy*:
   $$\underline{v} : (u = 0,\ m = 1),$$
   with the lowest rates: $(\underline{w}, \underline{r})$.[9]
2. Among the maximal utilization policies: $u = 1$, we distinguish the following:
   (i) *Capacity depleting,* with no maintenance. It is the *hardest policy*:
   $$\bar{v} : (u = 1,\ m = 0),$$
   with the highest rates: $(\bar{w}, \bar{r})$.[10]
   (ii) *Full capacity,* with maximal maintenance. It is the policy of highest intensity
   $$v_1 : (u = 1,\ m = 1).$$[11]

2.3 Two revenue based measures of capital

We examine first some preliminary notions that will help us interpret the results. We start by distinguishing the two sources of revenue, the operating revenue and the scrap revenue. The capacity of capital to produce these two revenues is affected by operations and also by time discounting. But their effects are exercised in different ways, as follows:

Remark 2
1. Concerning the effect of operations on the two revenues, we have:
   $-q / q = \varepsilon w$: *Deterioration rate of operating revenue*, of either sign.
   $-S / S = w - \eta$: *Deterioration rate of scrapping revenue*, of either sign.
   We note that if $\varepsilon > 1$, then operations affect the services more than the equipment, after we account for price changes due to η . The opposite is the case if $\varepsilon < 1$.
2. Concerning the effect of time discounting, we note that for the same operating policies, K units of capital at time T are equivalent presently to:
   $rK_{oc}^{\varepsilon} = rK^{\varepsilon}e^{-\sigma T} \Rightarrow K_{oc} = Ke^{-\sigma T/\varepsilon}$ capital units for operating revenue.

$pK_{sc} = pKe^{\eta T}e^{-\sigma T} \Rightarrow K_{sc} = Ke^{-(\sigma-\eta)T}$ capital units for scrapping revenue.

Thus, we have two discounting rates for future capital:

σ / ε: *Discounting rate for operating revenue*, positive

$\sigma - \eta$: *Discounting rate for scrapping revenue*, of either sign.

We note that if $\varepsilon > 1$, then future capital is more heavily discounted for its scrap value than for its services, after accounting for the price changes due to η. The opposite is the case if $\varepsilon < 1$.

3. We can summarize these differences by considering two measures of capital:

   K : *Scrapping capital,* determining the scrap revenue.

   K^{ε} : *Operating capital,* determining the operating revenue,

   where as noted above ε is the *deterioration (improvement) coefficient* for the services rendered by the equipment relative to the downgrading (upgrading) of the equipment itself.

4. At the beginning of the operating period the unit prices of the two capital measures are defined respectively by:

   $\lambda_0 = pK_0 / K_0 = p$: *Owner's unit logistic value for new scrapping capital.*

   $\mu_0 = pK_0 / K_0^{\varepsilon} = pK_0^{1-\varepsilon}$: *Owner's unit logistic value for new operating capital.*

   If K_0 is fixed, i.e. if it is not a parameter, then we can choose capital units and also adjust r, so that the two initial values are equal: $K_0 = 1 \Rightarrow \lambda_0 = \mu_0 = p$

The main results so far can be summarized as follows:

Remark 3.

1. Scrapping policy is determined mainly by the deterioration rates: $\{\varepsilon w, w - \eta\}$
2. Operating policies are determined mainly by the discount rates: $\{\sigma / \varepsilon, \sigma - \eta\}$
3. In all cases the policies depend on whether the price p is "low" or "high".

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2.4 Optimality conditions

Examining the problem in the setting of optimal control theory, we consider the *total profit flow* given by the current value *Hamiltonian:*

$$H = [q(u,m)K^{\varepsilon} - \lambda s(u,m)K] = [q(u,m) - \lambda K^{1-\varepsilon}s(u,m)]K^{\varepsilon},$$

with co-state variable

$\lambda = \lambda(t)$: *Owner's unit logistic value for scrapping capital.*

In place of $\{H, \lambda\}$, we have also the pair $\{h, \mu\}$, where

$h = H / K^{\varepsilon} = r(u,m) - \mu w(u,m)$: *Total profit flow rate per unit of operating capital*

$\mu = \lambda K / K^{\varepsilon}$: *Owner's unit logistic value for operating capital*

From Leonard & Van Long (1995) or Seierstadt & Sydsaeter (1986), we obtain the following necessary conditions for optimality:[12]

(i). For the operating policies (u,m) the maximality principle:

$$\max_{u,m} H \Rightarrow \max_{u,m}\{h=[q(u,m)-\mu s(u,m)]\| \ \ 0\le u\le 1, \ \ 0\le m\le 1\}$$

(ii). For the capital stock:

$$\dot{K}=s(u,m)K\,, \ \ with\ K-initial\ condition\ K(0)=K_0 \qquad \textbf{(2)}$$

(iii). For the logistic value:

$$\dot{\lambda}=-H'_K+\sigma\lambda \Rightarrow \dot{\mu}=-\varepsilon h+\sigma\mu=\varepsilon\mu(\sigma/\varepsilon-h/\mu)$$
$$\text{with } T-\text{final condition}: \ \lambda_T=S'_K(K_T,T) \ \ \Rightarrow \mu_T=pe^{\eta T}K_T^{1-\varepsilon}$$

(iv). For the duration, the scrapping $H-$terminal condition:

$$H=\sigma S-S'_T \ \Rightarrow h/\mu=\sigma-\eta$$

The solution will be obtained by the following procedure. First we solve the maximality principle 3(i), to express $\{u,m\}$ as functions of μ. Then for given duration T we solve the autonomous dynamical equation 3(iii) for μ. This gives the optimal solution for given $T: 0<T<\infty$. For $T\to 0$ we find $h_0=r_0-\mu_0 w_0$, and we consider the initial condition:

$$H(0)>\sigma S(K_0,0)-S'_T(K_0,0)\Rightarrow h_0/\mu_0>(\sigma-\eta): Profitability\ condition \qquad \textbf{(3)}$$

If it is not satisfied then the optimal duration is zero: $T^*=0$, and the equipment is *non-profitable*.[13] If it is satisfied then it is profitable, and we consider two possibilities. If the terminal condition 1(iv) does not have solution, then the optimal duration is un-bounded: $T^*=\infty$, and the equipment is *durable*. If it has solution, then the equipment is *scrappable*,[14] and we take the first such solution as the scrapping duration $T^*=T_s$.[15] In this case we examine also the operating policies.

As can be seen from the conditions, pivotal role is played by the quantity:

$$i=H/\lambda K=h/\mu: Total\ profit\ index \qquad \textbf{(4)}$$

It expresses the *total profit flow per unit logistic value of capital,* expressed either in terms of the operating capital or in terms of the scrapping capital.[16]

3. Equipment characteristics

3.1 Optimal path

The maximality principle **3(i)** determines for given μ the optimal $(u,m)-$policies. By convex programming and by the monotonicity properties of the functions involved, the totality of available optimal policies can be obtained also as solutions of either of the following constrained optimization problems, where the Lagrange multiplier of the first problem coincides with μ:

$$\begin{aligned} &\textbf{(i)}\ \max_{u,m}\{r(u,m) \| w(u,m)=w,\ 0\le u\le 1,\ 0\le m\le 1\}, \text{ for any } w \\ &\textbf{(ii)}\ \min_{u,m}\{w(u,m) \| r(u,m)=r,\ 0\le u\le 1,\ 0\le m\le 1\}, \text{ for any } r \end{aligned} \qquad \textbf{(5)}$$

They can be characterized as follows:

> ***Remark 4***
> 1. Among the policies that give the same rate of capital wear, optimal are those that maximize the rate of operating revenue, and
> 2. Among the policies that give the same rate of operating revenue, optimal are those that minimize the rate of capital wear.

Actually the above constrained optimization problems determine pairs of optimal rates: (w,r). As indicated in **Figure 1(a)**, for each such pair the contact points of their isorate curves in the (u,m) plane give the corresponding policies, or else they are boundary. These points form a path in the (u,m) plane, which we will call *optimal path.* In general, each contact consists of a single point and then the optimal path is uniquely determined and continuous. In special cases, it may be only upper-semicontinuous, with portions where the policies are not uniquely determined, in the sense that they give the same (w,r) values. In practice, these appear as discontinuity jumps to policies of higher or lower intensity, like part AB in Figure **1(a)**. As μ increases, the optimal path moves from harder to softer policies. On the average this will lead also to lower utilization and higher maintenance. However we may have portions of the path where it leads to policies of higher or lower intensity.

3.2 Operating function

The constrained maximization problem determines the maximal revenue rate that can be obtained for given wear rate, and defines a maximal value function, which we will call:

$$r = r(w): \textit{Operating function.}$$

It is concave increasing. In Figure 1(b) we give the graphs for two such functions, corresponding to revenue flexible and wear flexible equipment, of the special type and of the common type, respectively.

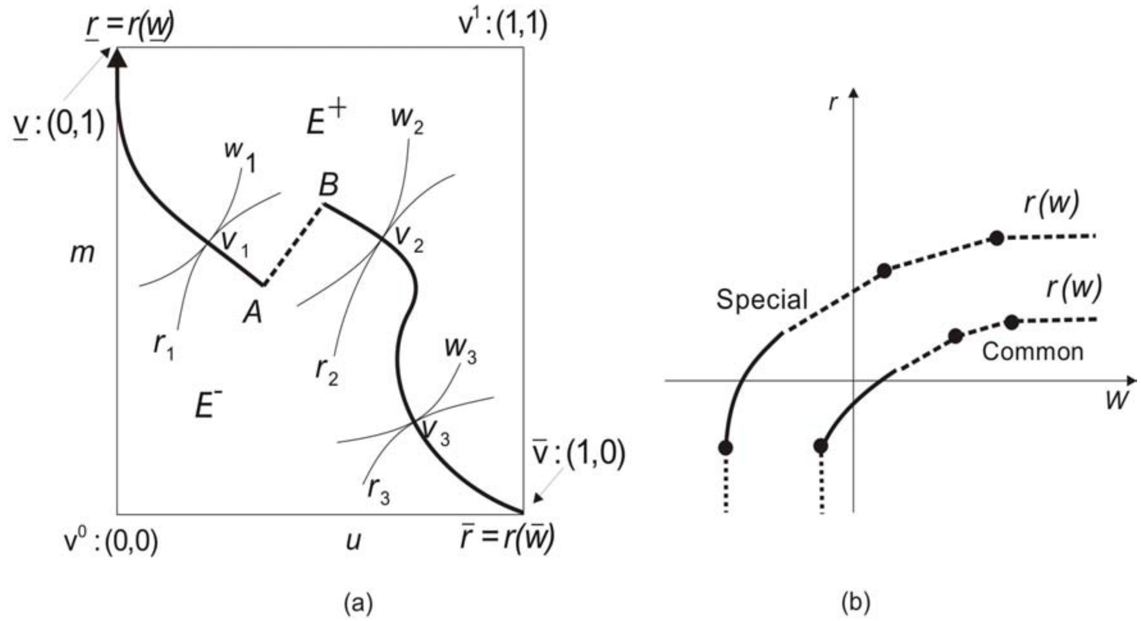

Figure 1: Derivation of the operating function: $r = r(w)$

The generalized derivative of the operating function measures the extra revenue that can be obtained per unit of increase in the wear rate. It will be called

$$r'(w) = dr / dw : \textit{Substitution rate}$$

It is positive decreasing in w. It defines the Lagrange multiplier of the constrained maximization problem and it coincides with μ. Considering the ordering of policies according to their hardness, we have the extremal values:[17]

$\bar{r}' = r'(\underline{w}) \leq +\infty$: Highest substitution rate at the softest policy: $(\underline{w}, \underline{r})$
$\underline{r}' = r'(\bar{w}) \geq 0$: Lowest substitution rate at the hardest policy: $(\bar{w}, \bar{r})$

Thus for given μ in this range, the corresponding optimal policies are determined by the relation:

$$\mu = r'(w): \textit{Operating policy function.}$$

The operating function $r = r(w)$ may have corners with discontinuous derivative, and linear parts with constant derivative. As μ changes continuously along the curve of the operating function, it will stay for a finite duration at the corners of the operating function waiting to cover the derivative discontinuity. They will be called *stable* or *persistent* policies. The nonstable policies can be called *transient.* Also it will skip the linear parts that have constant substitution rate, jumping to milder or harder policies, depending on the direction, and occasionally to policies of lower or higher intensity. They will be called *skipped* policies*.* The extremal policies are usually stable policies. In actual practice operating functions are expected to be piecewise linear, consisting of a few alternative stable policies, the rest being skipped. Summarizing, we can phrase the maximality principle in our case, as follows:

Lemma 1. Logistic value

1. For given $\mu = \lambda K^{1-\varepsilon}$, the optimal wear-revenue pair (w,r) is the one for which the substitution rate coincides with the unit operating capital logistic value μ:
$$\{r = r(w), r'(w) = \mu\} \Rightarrow \{w = w(\mu), r = r(\mu)\},$$
if this rate is attained, i.e. if $\underline{r}' \leq \mu \leq \overline{r}'$. If it is not attained, i.e. if the value of μ is outside the range of available substitution rates then it is the corresponding most extremal rates policy, i.e. the hardest or the softest.
2. As μ changes the policies move along the curve of the operating function. They move to softer policies if μ increases, to harder policies if μ decreases, persisting at the stable policies, and skipping the skipped policies.

From now on we will be working only with optimal policies, as determined by the operating function $r = r(w)$, since it summarizes all the relevant characteristics of the equipment. In this context, an optimal policy can be determined by any of the three quantities:

$$\mu : (w,r), \text{ where } -\infty \leq \mu \leq +\infty, \underline{w} \leq w \leq \overline{w}, \underline{r} \leq r \leq \overline{r}.$$

We note that all negative $\mu-$values correspond to the hardest policy:

$$\mu \leq 0 : (\overline{w}, \overline{r}) \Rightarrow (u = 1, m = 0).$$

If the equipment is wear flexible, then it has a capacity preserving policy of zero wear with the corresponding revenue and substitution rates:

$$\mu^w : (w = 0, r^w).$$

Similarly, if the equipment is revenue flexible then it has a break-even policy of zero revenue, with corresponding wear and substitution rates:

$$\mu^r : (w^r, r = 0).$$

If these policies are stable then we will actually have a whole interval of corresponding $\mu -$values. We note the following:

Remark 5
If the equipment is revenue flexible, then:
1. A policy is profit making if $\mu < \mu^r$, loss making if $\mu > \mu^r$.
2. The equipment is special if the break-even policy is upgrading: $w^r < 0$, common if it is not upgrading: $w^r \geq 0$ (as in **Figure 1(b)**).

The above refer to the operating characteristics of the equipment. What part of the operating function will actually be covered and in what direction is determined by the parameters $\{\varepsilon, \sigma, \eta, p, K_0\}$. The solution depends on the properties of the total profit index, which we examine next.

3.3 Total profit index

Substituting from the maximality principle we can express the total profit index as function of μ:

$$h = r(\mu) - \mu w(\mu) \Rightarrow i = \frac{h(\mu)}{\mu} = \frac{r(\mu)}{\mu} - w(\mu)$$

For $\mu \leq 0$ it is always the profit index of the hardest policy and is given by the hyperbola:

$$i = \overline{r} / \mu - \overline{w}.$$

Note that for $\mu < 0$, negative index indicates positive total profit flow. For $\mu \geq 0$, it depends on the revenue type of the equipment, as indicated in the Appendix. In **Figure 2** below we give the graph of the index function for equipment of various revenue types. In each case the position of the μ axis where $i = 0$, depends on the signs of $\{\underline{w}, \overline{w}, w^r\}$, i.e. on the wear type. We have placed it for the case of wear flexible equipment. The main characteristics can be summarized as follows:

Lemma 2. Total profit index.
As the policies soften with increasing μ, the total profit index decreases if the policies are profit making, it increases if they are loss making. In particular, for equipment of the revenue flexible type we distinguish a critical $\mu -$value given by

the substitution rate at the break even optimal policy:

$$\mu^r : (w^r, r = 0)$$

where the policies change from profit making to loss making and the total profit index attains its smallest value:

$$i^r = -w^r$$

as it changes from decreasing to increasing.

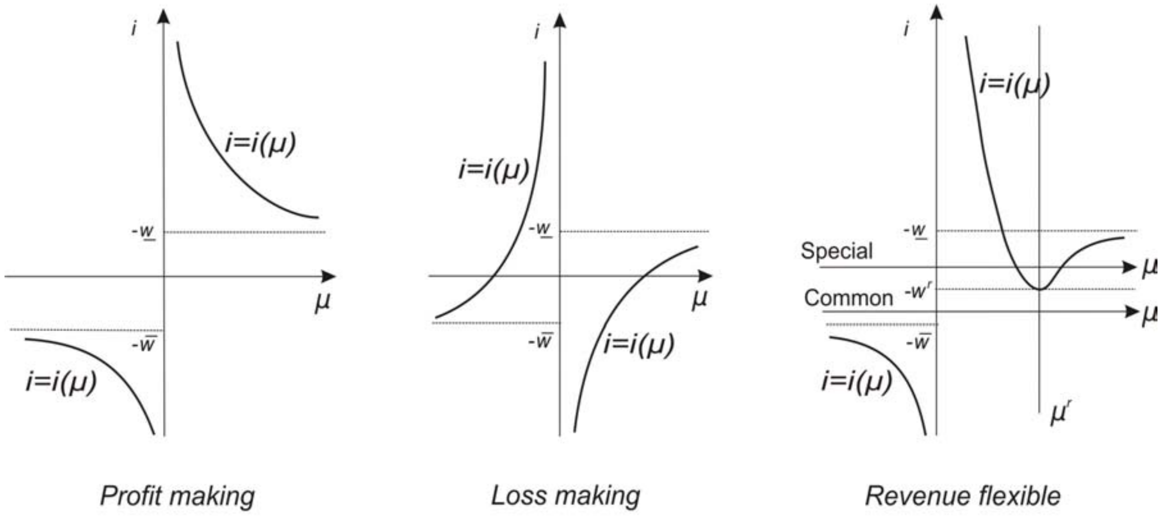

Figure 2. Total profit index function $i=i(\mu)$

4. Solution

Using the total profit index function we can rephrase the optimality conditions **(1)** and **(2)** as follows:

- **(i)**. For operating policies: $\{r = r(w),\, r'(w) = \mu\} \Rightarrow \{w = w(\mu),\, r = r(\mu)\}$
- **(ii)**. For capital stock: $K = w(\mu)K$, with initial condition $K(0) = K_0$
- **(iii)**. For logistic value: $\mu = \varepsilon\mu[\sigma/\varepsilon - i(\mu)]$, with final condition: $\mu_T = pe^{\eta T}K_T^{1-\varepsilon}$ **(6)**
- **(iv)**. For duration, the terminal scrapping condition: $i(\mu_T) = \sigma - \eta$.
- **(v)**. For profitability: $\sigma - \eta < i(\mu_0)$ where $\mu_0 = pK_0^{1-\varepsilon}$.

We will proceed now to the solution, investigating first the question of profitability, then the operating policies and finally the scrapping policy.

4.1 Profitability.

For given duration T , the T – optimal solution is determined by the T – final condition:

$$\lambda_T = pe^{\eta T} \Rightarrow \mu_T = pe^{\eta T}K_T^{1-\varepsilon}$$

In particular, for $T \to 0$ the price of new equipment determines a μ value and corresponding policy, by:

$$\lambda_0 = p \Rightarrow \mu_0 = pK_0^{1-\varepsilon} : \ (w_0, r_0)$$

It is the optimal policy to be applied if the equipment is to be operated for very short time duration. The initial profitability condition **(3)** is written as:

$$\sigma - \eta < i_0(p) \ \text{ where } \ i_0(p) = i(\mu_0)$$

Graphically, it is given by the region below the positive part: $\mu > 0$ of the graph of the index function $i = i(\mu)$, with $\sigma - \eta$ on the vertical axis and μ_0, or equivalently $p = \mu_0 K_0^{\varepsilon-1}$, on the horizontal. The position of the $p-$axis where $\sigma - \eta = 0$ depends on the sign of the terms $\{\underline{w}, \overline{w}, w^r\}$, i.e. on the wear properties of the equipment. In **Figure 3**, we have placed it for the case of wear flexible equipment. We can summarize the main properties as follows:

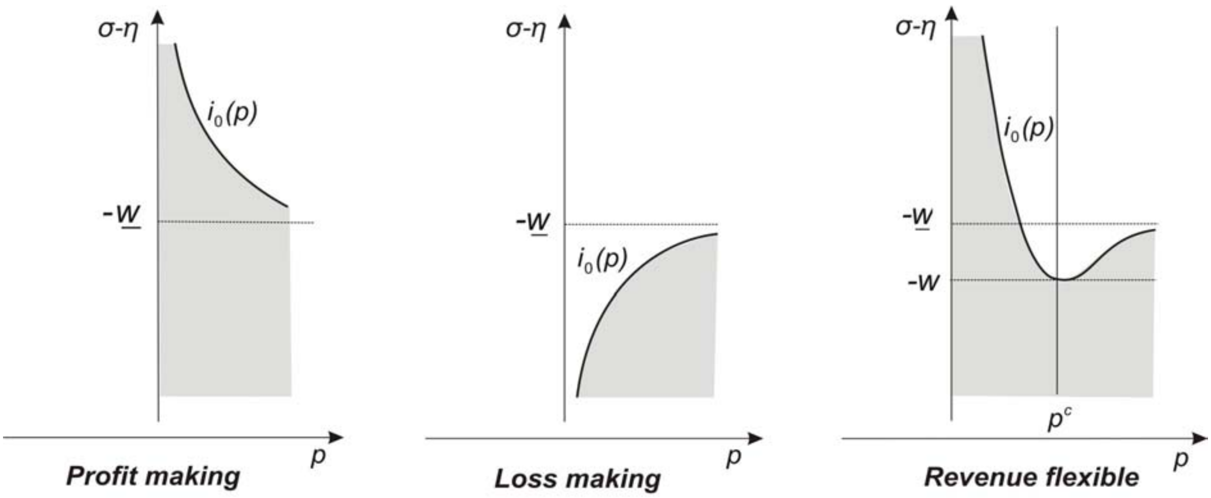

Figure 3: Profitability regions: $\sigma - \eta < i(p)$

Proposition 1. Profitability
For given price p, the equipment is profitable if the scrapping capital discount rate is smaller than the total profit index of new capital:

$$\sigma - \eta < i_0(p), \text{ where } i_0(p) = i(\mu_0) = r_0 / \mu_0 - w_0 \quad \Rightarrow \frac{q_0}{S_0} > w_0 + \sigma - \eta .$$

We note in particular, that if the equipment is of the revenue flexible type, then there is a critical price, given by: $p^c = \mu^r K_0^{\varepsilon-1}$, such that the equipment behaves like the profit making type if its price is lower, like the loss making type if its price is higher. We will say that the price is: low if $p < p^c$, *high* if $p > p^c$

Remark 6
For revenue flexible equipment, low prices are characterized by the property that short duration optimal policy is profit making, while high prices are characterized by the fact that short duration optimal policy is loss making. Extending this notion we can say, by convention, that for profit making equipment all prices are "low", while for loss making equipment all prices are "high".

4.2 Operating policies.

During the operating period the $\mu-$value develops according to equation **6(iii)**. This determines the operating policies on the operating function $r = r(w)$ according to equation **6(i)**. Recalling that larger $\mu-$values correspond to softer policies, we will investigate, for *scrappable* equipment: a) the shift to softer or harder operating policies as time passes, and b) the range of applied operating policies: $\mu \geq \mu(t) \geq \mu \Rightarrow w \leq w(t) \leq w,\ r \leq r(t) \leq r$.

Concerning time development, we note that equation **6(iii)** is autonomous, and since μ is continuous it will move in time monotonously. The sign of the derivative μ determines the direction of monotonicity at any time, in particular at the terminal time T , if the equipment is *scrappable.* Substituting from **6(iv)**, we find:

$$\mu_T = \varepsilon[\sigma / \varepsilon - (\sigma - \eta)]\mu_T , \quad \text{where } \mu_T = pe^{\eta T} K_T^{1-\varepsilon} > 0 .$$

Hence the monotonicity property depends on the relative magnitude of the two discount rates: $\{\sigma / \varepsilon, \sigma - \eta\}$. We have:

Proposition 2a. Time shift of operating policies
If the equipment is *scrappable*, then we distinguish the following cases:
1. If $\sigma / \varepsilon > \sigma - \eta \Rightarrow \eta > (1 - 1/\varepsilon)\sigma$, i.e. if the operating discount is higher than the scrapping discount, then $\mu(t)$ increases in time from harder to softer policies
2. If $\sigma / \varepsilon < \sigma - \eta \Rightarrow \eta < (1 - 1/\varepsilon)\sigma$, i.e. if the operating discount is lower than the scrapping discount, then $\mu(t)$ decreases in time from softer to harder policies.
3. If $\sigma / \varepsilon = \sigma - \eta \Rightarrow \eta = (1 - 1/\varepsilon)\sigma$, i.e. if the operating discount and the scrapping discount are equal, then $\mu(t)$ stays fixed in time at the equilibrium policy.

Thus the equipment is worked harder at the beginning if the operating discount is larger than the scrapping discount, and conversely. We note however that the wear-revenue flows: $q = rK^{\varepsilon}$, $s = wK$, depend not only on how hard the equipment is worked but also on K itself.

Regarding the range of applied policies we consider the dynamics of the equation **6(iii)** and we locate:

1. The fixed $\mu -$values, if they exist, given by the policies with profit index equal to the operating capital discount rate:

   $$i(\mu) = \sigma / \varepsilon \Rightarrow \mu_e : (w_e, r_e): \textit{ Equilibrium solutions}$$

2. The scrapping $\mu -$values, if they exist, given by the policies with profit index equal to the scrapping capital discount rate:

   $$i(\mu) = \sigma - \eta \Rightarrow \mu_s > 0 : (w_s, r_s): \textit{ Scrapping solutions}$$

   We have restricted the scrapping solutions to be positive because they satisfy also the $T -$final condition **2(iii)**: $\mu_T = pe^{\eta T} K_T^{1-\varepsilon} > 0$.

We note now that as we increase the operating duration T, μ_T moves about along the positive part of the total profit index curve: $\mu > 0$, starting from μ_0. The equipment will be *scrappable* if it meets μ_s, *durable* if it does not. In particular it is durable if there is no scrapping solution. From the graph of the index function we find that if the equipment is of the profit making type or of the loss making type then there is at most one scrapping solution. In these cases there is also at most one positive equilibrium solution which is unstable repelling if the equipment is of the profit making type, stable attractive if it is of the loss making type. Since the flow is always from the nearest repelling equilibrium or repelling extremal value towards the scrapping value, we conclude the following:

Proposition 2b. Bounds on the operating policies

For *scrappable* equipment the applied policies are bounded at the terminal scrapping time by the scrapping value. At the initial time they are bounded as follows:

1. If the equipment is of the profit making type, then initially they are bounded by the equilibrium value, unless there is no equilibrium value in which case they are initially bounded by the extremal softest policy. This happens if the extremal softest policy is upgrading and the operating capital discount is sufficiently low:

   $$\underline{w} < 0 \text{ and } \sigma / \varepsilon < -\underline{w}$$

2. If the equipment is of the loss making type, then initially they are bounded by an extremal policy, as follows:

   (i) The extremal softest policy if $\sigma / \varepsilon < \sigma - \eta \Rightarrow \eta < (1 - 1/\varepsilon)\sigma$.

   (ii) The extremal hardest policy if $\sigma / \varepsilon > \sigma - \eta \Rightarrow \eta > (1 - 1/\varepsilon)\sigma$

3. If the equipment is of the revenue flexible type then:

   (i) If its price is low then it behaves like the profit making type with w^r replacing $\underline{w}$

(ii) If its price is high then it behaves like the loss making type in case 2(i) above, while in case 2(ii) it is bounded initially by the profit making equilibrium policy instead of the hardest.

Proof.
Parts 1 and 2 are direct consequences of the dynamics and the properties of the index function. In these cases depending on the value of σ / ε we obtain the flows in the diagrams of **Figure 4** below. Concerning part 3, we note that if the equipment is of the revenue flexible type, and $\sigma - \eta$ is in the range:

$$-w^r < \sigma - \eta < -\underline{w},$$

then there will be two scrapping solutions, one profit making the other loss making as follows:

$$\textit{profit making}: \mu_s^1 < \mu^r < \mu_s^2 : \textit{loss making}$$

We note however that in this case, the profitability assumption gives:

$$i(\mu_0) > \sigma - \eta \Rightarrow \begin{cases} \mu_0 < \mu_s^1 < \mu_s^2 \text{ if the price is low}: \mu_0 < \mu^r \\ \mu_s^1 < \mu_s^2 < \mu_0 \text{ if the price is high}: \mu^r < \mu_0 \end{cases}$$

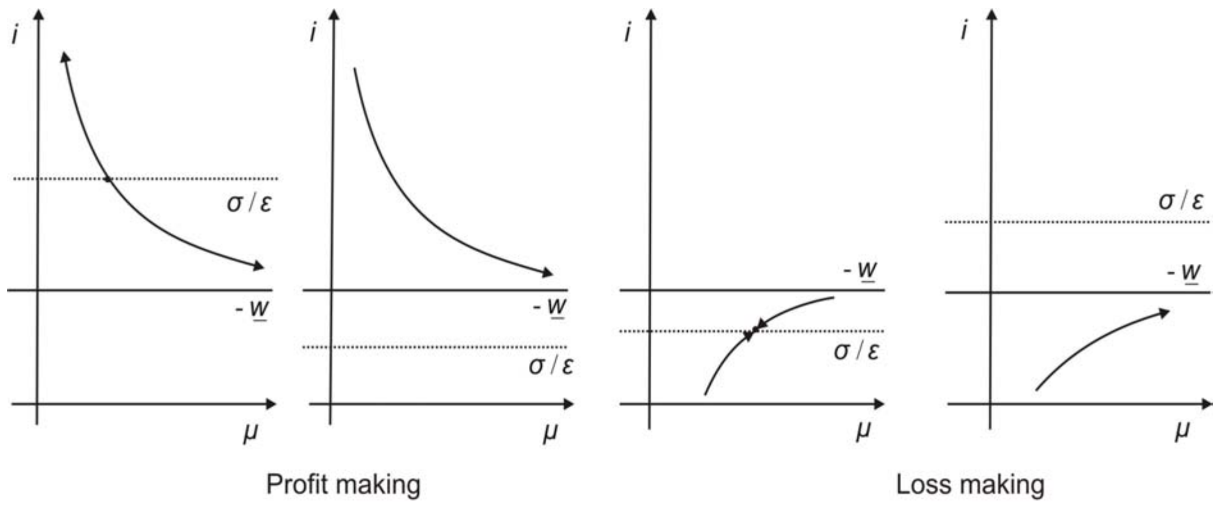

Figure 4: Flow diagrams for $\mu \geq 0$

Hence as μ_T moves about starting from μ_0, it will hit first, if any, the profit making value μ_s^1 in the first case, the loss making value μ_s^2 in the second case. Thus, the terminal scrapping operating policy is uniquely defined in this case also. It is hard profit making if the price is low, soft loss making if the price is high.[18] The rest follows as in 1 and 2 with some minor modifications, as indicated.

The above are the best possible bounds on the operating policies for scrappable equipment. We note also that the bounds given by the terminal and the initial policy have well defined wear and revenue properties. Hence, the corresponding properties for the operating policies are determined by noting that for small durations they stay close to the terminal policy, and as the duration increases they spend increasing amount of time close to the initial policy before ending at the terminal policy, in accordance with the general turnpike property.

4.3 Scrapping policy

For the above results to be relevant we proceed now to determine the scrapping policy, i.e. under what conditions we have scrapping and what is the scrapping duration. For this we will need the bounds on operating policies for scrappable equipment obtained above. We note first that a profitable equipment is durable if the scrapping condition **6(iv)** does not have solutions. If it has solutions, then we can determine uniquely the terminal μ_s value as indicated above. If the equipment is *scrappable*, then the operating policies will be bounded by: $w \le w \le w$, as indicated above.

In this case, the solution is determined by the terminal value μ_s and the scrapping duration T. We define the quantity:

$$\omega = \frac{1}{T}\int_0^T w(t)dt : \textit{Average wear rate, } w \le \omega \le w \,.$$

Proposition 3. Scrapping policy

We consider equipment K_0 of unit price p, with coefficient ε and operating function $r = r(w)$, with $\underline{w} \le w \le \overline{w}$, and we compute:

$r'(w) = \mu$: Operating policy function,

$i(\mu) = r(\mu)/\mu - w(\mu)$: Total profit index function.

$i_0(p) = i(\mu_0)$: Profit index of new capital, where $\mu_0 = pK_0^{1-\varepsilon}$

The equipment is profitable if

$$\sigma - \eta < i_0(p) .$$

In this case the scrapping policy depends on the revenue type as follows:

1. If the equipment is of the profit making type, we distinguish two cases:
   (i). If $\sigma - \eta < -\underline{w}$, then it ıs durable for any price p
   (ii). If $-\underline{w} < \sigma - \eta < i_0(p)$, then it is scrappable iff on the average the operating capital deteriorates strictly faster or equivalently improves strictly slower than the scrapping capital. Thus, it is:
      (a). Durable if $\varepsilon\omega \le \omega - \eta \Rightarrow \eta \le (1-\varepsilon)\omega$.
      (b). Scrappable if $\varepsilon\omega > \omega - \eta \Rightarrow \eta > (1-\varepsilon)\omega$.
      In the last case we compute also the scrapping value: $i(\mu) = \sigma - \eta \Rightarrow \mu_s$, and then the scrapping duration is given by:
      $$T_s = \frac{1}{\varepsilon\omega - (\omega - \eta)} \ln \frac{\mu_s}{\mu_0}, \text{ where } \varepsilon\omega > \omega - \eta,\ \mu_s > \mu_0, \text{ and } w \le \omega \le w .$$

2. If the equipment is of the loss making type with $\sigma - \eta < i_0(p)$, then it is scrappable iff on the average the scrapping capital deteriorates strictly faster or

equivalently improves strictly slower than the operating capital. Thus, it is:
(i). Durable if $\omega - \eta \le \varepsilon\omega \Rightarrow \eta \ge (1-\varepsilon)\omega$.
(ii). Scrappable if $\omega - \eta > \varepsilon\omega \Rightarrow \eta < (1-\varepsilon)\omega$.
In the last case we compute also the scrapping value: $i(\mu) = \sigma - \eta \Rightarrow \mu_s$,
and then the scrapping duration is given by:

$$T_s = \frac{1}{(\omega-\eta)-\varepsilon\omega}\ln\frac{\mu_0}{\mu_s},\ \text{where } \omega-\eta>\varepsilon\omega,\ \mu_s<\mu_0,\ \text{and } w \le \omega \le w .$$

3. If the equipment is of the revenue flexible type, we compute also the following:
w^r : Break even wear rate,
$p^c = \mu^r K_0^{\varepsilon-1}$: Critical price, where $\mu^r = r'(w^r)$

Then:
(i) If $\sigma - \eta < -w^r$ then it is durable for any price p
(ii) If $-w^r < \sigma - \eta < i_0(p)$, then it behaves like the profit making equipment if its price is lower than the critical price, like the loss making equipment if its price is higher than the critical price.

Proof.
For parts **3.1(i)**, **3.3(i)** we simply note that under these conditions there is no scrapping solution. For the rest we note that if the equipment is scrappable then the solution will be determined by the terminal scrapping value μ_s and the scrapping duration T . We note now that a solution of duration T will give final capital stock:

$$K(T) = K_0 e^{-\omega T},\ \text{where } \omega = \frac{1}{T}\int_0^T w(t)dt : \textit{average wear rate, } w \le \omega \le w$$

The solution will be optimal if the final values $\{\mu_T, K_T\}$ satisfy relations **6(iii)** and **6(iv)**

$$\mu_T = pe^{\eta T}K_T^{1-\varepsilon} \text{ and } \mu_T = \mu_s,\ \text{where } \mu_0 = pK_0^{1-\varepsilon} \Rightarrow K_T = (\mu_s/\mu_0)^{1/(1-\varepsilon)}K_0 e^{-\eta T/(1-\varepsilon)}$$

Hence it will be scrappable iff the two functions $K(T)$ and K_T coincide for some T , which will also be the scrapping duration. We will proceed only with case of **Proposition 3.1**. In this case the index function is decreasing and the profitability condition can be written:

$$\sigma - \eta < i(\mu_0) \Rightarrow \mu_s > \mu_0,\ \text{because } i(\mu_s) = \sigma - \eta$$

We consider the two capital functions:
$K_1(T) = K(T) = K_0 e^{-\omega T}$ and $K_2(T) = K_T = (\mu_s/\mu_0)^{1/(1-\varepsilon)}K_0 e^{-\eta T/(1-\varepsilon)}$, with $\mu_s > \mu_0$.
We will examine separately the three cases depending on the $\varepsilon-$value:

1. If $\varepsilon < 1$, then $K_1(0) < K_2(0)$, and hence $K_1(T)$ meets $K_2(T)$ iff
$-\omega > -\eta/(1-\varepsilon) \Rightarrow \eta > (1-\varepsilon)\omega$
2. If $\varepsilon > 1$, then $K_1(0) > K_2(0)$, and hence $K_1(T)$ meets $K_2(T)$ iff
$-\omega < -\eta/(1-\varepsilon) \Rightarrow \eta > (1-\varepsilon)\omega$
3. If $\varepsilon = 1$, then it is scrappable iff $\mu_T = \mu_0 e^{\eta T} = \mu_s$ for some T , which happens iff $\eta > 0$.

The conditions and the formulas obtained can be verified in the simple case where the applied policy is fixed: $w - w = 0$. In the general case they can be used to provide easily applicable sufficient conditions, as follows:

Corollary 1

1. If it is profit making with $-\underline{w} < \sigma - \eta < i_0(p)$ or if it is revenue flexible with low price and $-w^r < \sigma - \eta < i_0(p)$, then it is profitable and:
   (i). It is durable if (a). $\varepsilon \le 1$ and $\varepsilon w \le w - \eta$ or (b). $\varepsilon \ge 1$ and $\varepsilon w \le w - \eta$.
   (II). It is scrappable if (a). $\varepsilon \le 1$ and $\varepsilon w > w - \eta$ or (b). $\varepsilon \ge 1$ and $\varepsilon w > w - \eta$.
2. If it is loss making with $\sigma - \eta < i_0(p)$ or if it is revenue flexible with high price and $-w^r < \sigma - \eta < i_0(p)$, then it is profitable, and:
   (i). It is durable if (a). $\varepsilon \le 1$ and $\varepsilon w \ge w - \eta$ or (b). $\varepsilon \ge 1$ and $\varepsilon w \ge w - \eta$.
   (II). It is scrappable if (a). $\varepsilon \le 1$ and $\varepsilon w < w - \eta$ or (b). $\varepsilon \ge 1$ and $\varepsilon w < w - \eta$.

4.4 Parameter dependence

We have ascertained that the operating and scrapping policies depend on the quantities:

$$\mu_0 = pK_0^{1-\varepsilon},\ i(\mu) = \sigma - \eta \Rightarrow \mu_s : (w_s, r_s),\ i(\mu) = \sigma / \varepsilon \Rightarrow \mu_e : (w_e, r_e)$$

From the monotonicity properties of the profit index function we obtain the following general results. More specific results can be obtained in special cases.

Proposition 4. Parameter dependence

1. If the equipment is profit making or revenue flexible with low price, then:
   (i). If scrappable, then the operating policies harden if $\sigma - \eta$ or σ / ε increase.
   (ii). The scrapping duration is $p-$decreasing. Also it is $\sigma-$decreasing if $\varepsilon \ge 1$.
2. If the equipment is loss making or revenue flexible with high price, then:
   (i). If scrappable, then the operating policies soften if $\sigma - \eta$ increases.
   (ii). The scrapping duration is $p-$increasing. Also it is $\sigma-$decreasing if $\varepsilon \le 1$.

5. Special cases

5.1 The case $\varepsilon = 1$

The conditions and the bounds obtained for the scrapping policy become more determinate as

$$(1-\varepsilon)(w - w) \to 0\text{, i.e. as } \varepsilon \to 1\text{, or as } w - w \to 0.$$

In fact for $\varepsilon = 1$, they become completely determinate, as follows:

Corollary 2.

We consider profitable equipment for which the scrapping capital and the operating capital coincide: $\varepsilon = 1$.

1. If it is profit making with $-\underline{w} < \sigma - \eta < i_0(p)$ or if it is revenue flexible with low price and $-w^r < \sigma - \eta < i_0(p)$, then it is scrappable iff $\eta > 0$, and then the scrapping duration is:

$$T_s = \frac{1}{\eta}\ln\frac{\mu_s}{\mu_0}\text{, where } \eta > 0 \text{ and } \mu_s > \mu_0.$$

2. If it is loss making with $\sigma-\eta<i_0(p)$ or if it is revenue flexible with high price and $-w^r<\sigma-\eta<i_0(p)$, then it is scrappable iff $\eta<0$, and then the scrapping duration is:

$$T_s=\frac{1}{-\eta}\ln\frac{\mu_0}{\mu_s}=\frac{1}{\eta}\ln\frac{\mu_s}{\mu_0}\text{, where } \eta<0 \text{ and } \mu_s<\mu_0 .$$

Remark 5

Consider **Figure 5** below. In all cases, for given price p, we distinguish a nonprofitable region of high $\sigma-\eta$ values, an intermediate profitable region with mixed policies, both scrappable and durable depending on the sign of η, and a profitable region of low $\sigma-\eta$ values with durable policy. We note that as $\sigma-\eta$ decreases crossing these critical values there may appear discontinuities in the duration. Thus as it crosses the value $i_0(p)$ it may jump from 0 to ∞. We note also that for each fixed η the revenue flexible type has the maximal scrapping duration:

$$\max T_s=\max\frac{1}{\eta}\ln\frac{\mu_s}{\mu_0}=\frac{1}{\eta}\ln\frac{\mu^r}{\mu_0}.$$

But when $\sigma-\eta$ crosses the critical value $-w^r$, it jumps to $T=\infty$. As remarked previously these discontinuities are caused by our specification that due to the effect of obsolescence risk the operator does not accept revenue reduction and stops at the first maximum.

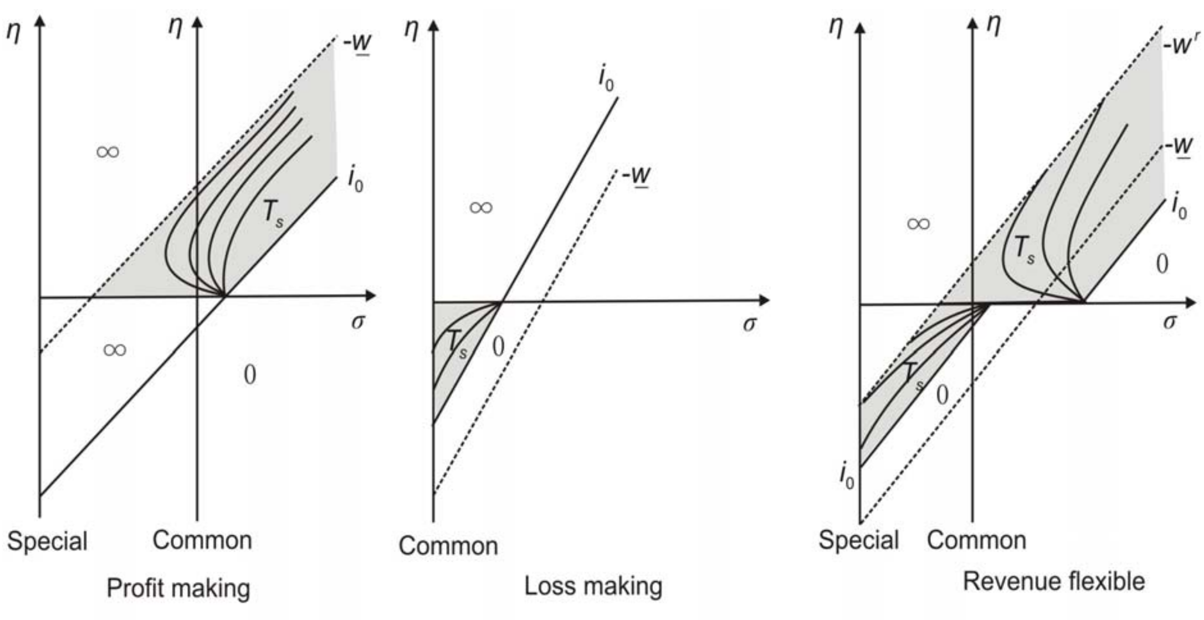

Figure 5: Scrapping policies for ε=1

5.2 An example: linear operating function

We will compute the solution quantities for equipment with only two stable policies, a soft upgrading and loss making: $(\underline{w}<0,\underline{r}<0)$, and a hard downgrading and profit making: $(\overline{w}>0,\overline{r}>0)$. The remaining are skipped policies. In this case the revenue rate will be a linear increasing function of the wear rate:

$$r = \alpha w + \beta .$$

The equipment is revenue flexible and wear flexible. In **Figure 6** we give the diagrams of the operating function and of the profit index function for $\mu \geq 0$, separately for the special type and for the common type.

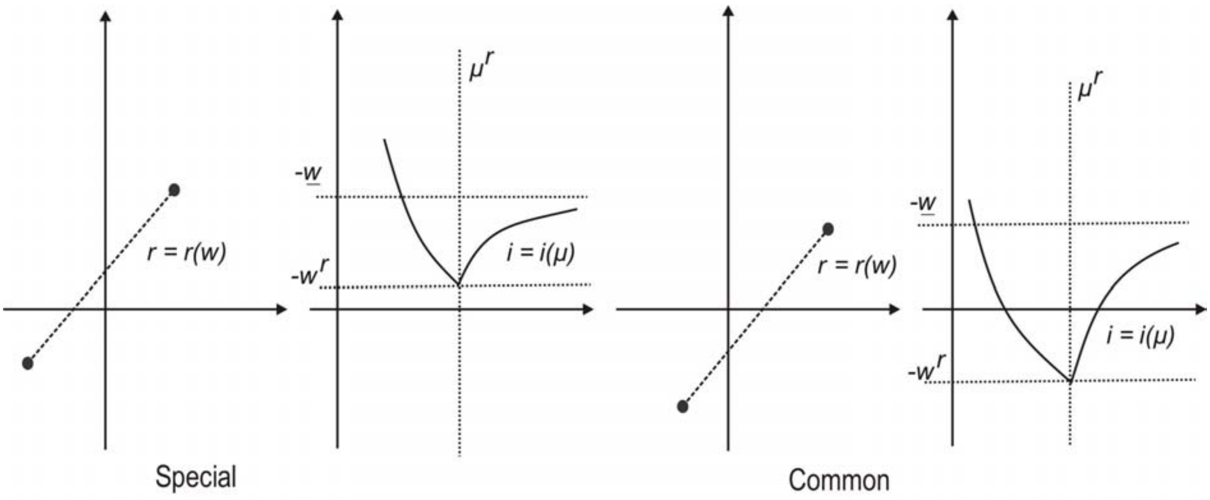

Figure 6: Linear operating function

For convenience we take $K_0 = 1 \Rightarrow \lambda_0 = \mu_0 = p$, and we find:

1. Operating function: $r = \mu^r (w - \underline{w}) + \underline{r}$ for $\underline{w} \leq w \leq \bar{w}$, with $\mu^r = \dfrac{\bar{r} - \underline{r}}{\bar{w} - \underline{w}}$
2. Break even policy: $r = 0 \Rightarrow w^r = \dfrac{\bar{r}\underline{w} - \underline{r}\bar{w}}{\bar{r} - \underline{r}}$, $\mu^r = \dfrac{\bar{r} - \underline{r}}{\bar{w} - \underline{w}}$,
3. Total profit index: $i = \{\bar{r} / \mu - \bar{w} \;\; if \;\; \mu \leq \mu^r\}, \{\, \underline{r} / \mu - \underline{w} \;\; if \;\; \mu \geq \mu^r\}$
4. Lowest total profit index: $i^r = -w^r = \dfrac{\underline{r}\bar{w} - \bar{r}\underline{w}}{\bar{r} - \underline{r}}$, negative if common, positive if special.
5. Critical price: $p^c = \mu^r = \dfrac{\bar{r} - \underline{r}}{\bar{w} - \underline{w}} \Rightarrow \begin{cases} p \leq p^c : \textit{low prices} \\ p \geq p^c : \textit{high prices} \end{cases}$.

Concerning applied optimal policies, we note the following:

6. It is profitable $\Leftrightarrow \sigma - \eta < i_0(p) \Rightarrow \begin{cases} \bar{r} / p > \bar{w} + \sigma - \eta \;\; \textit{if the price is low} \;\; p \leq p^c \\ -\underline{r} / p < -\underline{w} + \eta - \sigma \;\; \textit{if the price is high} \;\; p \geq p^c \end{cases}$

7. If $\sigma-\eta \le -w^r = \dfrac{\underline{r}\overline{w}-\overline{r}\underline{w}}{\overline{r}-\underline{r}}$ then it is profitable and durable for all p.

The remaining policies depend in more particular ways on the parameters. We will examine some special cases.

8. We assume $\varepsilon=1$. Then:

(i) If the price is low $p<p^c$ and if $-w^r<\sigma-\eta<i_0(p) \Rightarrow \dfrac{\underline{r}\overline{w}-\overline{r}\underline{w}}{\overline{r}-\underline{r}}<\sigma-\eta<\dfrac{\overline{r}}{p}-\overline{w}$,

then it is scrappable iff $\eta>0$. In this case we apply the hard policy $(\overline{w},\overline{r})$, with scrapping duration:

$$T_s=\frac{1}{\eta}\ln\frac{\mu_s}{\mu_0}=\frac{1}{\eta}\ln\frac{\overline{r}}{p(\overline{w}+\sigma-\eta)}$$

(ii) If the price is high: $p>p^c$, and if $-w^r<\sigma-\eta<i_0(p) \Rightarrow \dfrac{\underline{r}\overline{w}-\overline{r}\underline{w}}{\overline{r}-\underline{r}}<\sigma-\eta<\dfrac{\underline{r}}{p}-\underline{w}$,

then it is scrappable $\Leftrightarrow \eta<0$. In this case we apply the soft policy $(\underline{w},\underline{r})$, with scrapping duration:

$$T_s=\frac{1}{\eta}\ln\frac{\underline{r}}{p(\underline{w}+\sigma-\eta)}=\frac{1}{-\eta}\ln\frac{p(-\underline{w}+\eta-\sigma)}{-\underline{r}}.$$

9. Under certain conditions we can have shifts between the two policies, the hard and the soft. In particular:

(i). If the price is low, then the terminal policy will be the hard policy. However as indicated in part 3a of **Proposition 3a**, if

$$w^r<0 \text{ and } \sigma/\varepsilon<-w^r$$

i.e. if it is special and σ/ε is sufficiently low, then as the price p decreases the scrapping duration will increase according to proposition 4 and hence it will spend initially increasing amount of time close to the initial soft policy before switching to the hard policy at the end.

(ii). Similarly if the price is high, then the terminal policy will be necessarily the soft policy. However as indicated in part 3b of **Proposition 3a**, if

$$\sigma/\varepsilon>\sigma-\eta,$$

then by Proposition 4 as the price p increases the scrapping duration will also increase and it will spend increasing amount of time close to the initial hard policy before switching to the soft policy at the end.

6 Implications for economic theory and policy

Recent efforts to integrate *operating* and *capital policies* in a unified analytical framework leave much to be desired. The reason being that the proposed models are founded invariably on the presumption that the representative firm adopts a capital policy of perpetual replacements at equal time intervals, even though market conditions during certain periods may recommend *terminal scrapping.* So what we want to do in this section is to highlight the implications of our results by reference to three areas of contemporary research.

6.1 New theories of economic growth and real business cycles

Consider, for example, the model presented by Boucekkine and Tamarit (2003).[19] The view taken by these authors is that the representative firm chooses *utilization* and *maintenance*, along with *gross investment* and *labor*, in order to maximize the present value of an infinite stream of profits, subject to the usual accumulation constraint in which capital depreciates at a positive rate. In this framework it can be shown that maximization over an infinite horizon is equivalent to assuming that the representative firm plans for a perpetual series of equidistant investments having infinite service lives. But when capital goods are durable, the policies of *utilization* and *maintenance* may not be well defined, particularly in the multi-period setting of the model. Consequently their specifications may be seriously flawed.

That this is most likely the case can be ascertained by reference to the difficulties that the authors encountered in obtaining coherent solutions. One such difficulty concerns the direction of influence that *utilization* and *maintenance* exercise on the rate of depreciation. In this respect Boucekkine and Tamarit (2003, pp 10-16) find that when *utilization* worsens depreciation, *maintenance* does too. So faced with this counter-intuitive result they are forced to impose an extra constraint, which guarantees that the influence of the two operating policies runs in opposite directions.

Another difficulty stems from the observation that their model does not yield explicit characterizations of *utilization* and *maintenance*. For this reason, such significant questions as: how do operating policies develop in time; when and under what technical and market conditions do they develop from harder to softer, and vice versa; and, how and to what extent do they depend on the various parameters, are left unanswered. Lastly, one additional difficulty springs from the realization that the service life of capital, which constitutes a crucial determinant of operating policies, is ignored.

In light of these shortcomings, the models of growth and real business cycles under consideration stand to gain significantly from our results. More specifically, improvements may be pursued in at least three directions. First, by bringing the service life of capital to the forefront of the analysis. Focusing on *terminal scrapping* can accomplish this easily. Second, by extending the analysis to obtain determinate *utilization* and *maintenance* policies even in the case of durable capital. This may be hard to accomplish, if at all possible, but it is desirable because depending on market conditions the representative firm may be obliged at times to treat capital as durable; and thirdly, by generalizing the model to allow for a finite number of *replacements* before *terminal scrapping*.

6.2 Capital budgeting

The process of capital budgeting assists decision makers in business firms and other organizations to resolve questions associated with investment. It deals with such issues as the selection of projects, the timing and the duration of investment, the determination of the amount to be invested within any given period of time, the arrangement of financial means necessary for the completion of projects, etc. Some of the decision variables involved in the process are determined endogenously, some other are considered exogenous, and still some other are taken as fixed. **Table 1** below lists the most common decision variables in each of these three categories under the current state in the theory and practice of capital budgeting.

Table 1: Endogenous, exogenous and fixed variables in capital budgeting under the current state of theory and practice.

Endogenous	Exogenous	Fixed
Amount of investment	Service life	Discount rate
Quantity of output	Duration of opportunity[1]	Initial capital stock
Quantity of inputs	Utilization rate	
Technology	Maintenance	
	Output and input prices	
	Scrap value	

1. Duration of opportunity is the overall period over which it may be profitable to replace the equipment several times.

From this we observe that presently a large number of decision variables are treated as exogenous, which implies that the generality of the models employed in capital budgeting is severely limited. On the contrary our model renders most of the variables in the second group endogenous. In particular, their classification in the framework of our model becomes as shown in **Table 2**.

Table 2: Endogenous, exogenous and fixed variables in capital budgeting in the framework of our model.

Endogenous	Exogenous	Fixed
Amount of investment	Horizon of opportunity	Discount rate
Quantity of output	Output and input prices	Initial capital stock
Quantity of inputs		
Technology		
Service life		
Utilization rate		
Maintenance		
Scrap value		

Consequently, the gains achieved in the generalization of the standard capital budgeting model are substantial. Actually the generalization could be stretch even further by allowing the horizon of opportunity to be determined endogenously. But then the problem would become quite cumbersome to solve.

6.3 Retirement acceleration programs

Government policies, for example, to reduce car emissions, to increase the safety of passengers in coastal shipping, and to encourage switching to energy efficient technologies take the form of restrictions on the one hand to lower their service lives and on the other to influence the modes of their operation. Historically in all these cases the instruments of intervention have been of the command and control type. But in recent years interested researchers and policy makers have turned their attention to models based on economic incentives, a classic example being the programs of accelerated vehicle replacement that have been adopted in many countries.

The standard model driving these programs is described, for example, in Alberini, Harrington and McConnell (1995). It postulates that automobiles are replaced at equal time intervals over an infinite horizon and the question asked is what subsidy would the government have to offer in order to achieve the expected participation in the program. However, as it ignores the effects of *utilization* and *maintenance on* the rate of depreciation, the model is least appealing. For if the resale value of a car exceeds the bounty because it has a better than average *utilization* and *maintenance* record, contrary to the prediction of the model, its owner will be unwilling to participate in the program.

Another limitation is the presumption embedded in the model that cars depreciate physically and economically at a constant rate. Yet as we stressed in Bitros and Flytzanis (2002a) this hypothesis has come under attack on theoretical and empirical grounds in the last three decades and the dominant view now is that at least it should be tested as in Prucha and Nadiri (1996). Hence, due to the inappropriate specification of the depreciation rate, the model does not yield an explicit characterization of the determinants of *terminal scrapping*.

In conclusion, the models that drive retirement acceleration programs are in need of extension in two directions: first, by focusing on *terminal scrapping,* and, secondly, by introducing *utilization* and *maintenance* as endogenous determinants of the depreciation rate. Our results contribute towards this objective by offering a consistent framework of analysis with a wide range of specification choices.

7. Summary and conclusions.

In this paper we pursued several objectives. One was to introduce *utilization* and *maintenance* in a model with *terminal scrapping* in order to study the properties of the respective policies and their interactions. Another was to gauge the nature of implications that emerge for economic theory when operating and capital policies are integrated in an analytical framework based on rational entrepreneurial behavior. And still a third objective was to come up with a model, which is adequately flexible to facilitate further extensions, but also tractable enough to provide meaningful assistance in economic applications. Below we summarize the main features of the approach we adopted to attain these objectives as well as the conclusions derived from our results.

7.1 Main features of the model

Drawing on our earlier research we adopted a model of *utilization* and *maintenance* in which the investment opportunity lasts for a single operating period, at the end of which equipment is terminally scrapped. Moreover, the following features characterize our approach:

1. We introduced two measures of capital corresponding to its dual function as equipment and as services rendered. This allowed us to measure the different effect of the operations on the equipment and on the services by the use of a single coefficient ε.
2. We introduced a single index, the *total profit index,* i, which summarizes

the technical characteristics of the equipment and allows us to compare the various policies.

3. We distinguished two equipment price domains. Low prices, where the short duration policy is profit making, and high prices, where the short duration policy is loss making. The critical price p^c separating them is determined by the properties of the break-even policy.
4. We have ascertained how the applied wear, revenue, and scrapping policies depend on the above characteristics and on the market parameters, particularly the discount rate σ and the difference η in the rate of market price changes between the equipment and the services rendered. In the general case, we distinguished three domains of $\sigma - \eta$ values:
   (i) Low values, where the equipment is profitable and durable for all prices.
   (ii) Intermediate values, where the equipment is profitable if the price is sufficiently low or sufficiently high, i.e. there is an intermediate range of equipment prices that are not profitable, which widens as $\sigma - \eta$ increases. In this domain we have mixed scrapping policies, durable or scrappable, depending on the wear properties of the two capital measures.
   (iii) High values, where it is profitable only for low prices, becoming lower as $\sigma - \eta$ increases.
5. As $\sigma - \eta$ changes crossing these critical values, we may have sudden changes in the scrapping policies. These are caused mainly by the effect of equipment obsolescence risk on the owner.
6. Beside the above jumps in scrapping policy, we may have also jumps in operating policies to higher or lower wear/revenue rates caused by linearities in the operating functions.
7. We can determine conditions under which we apply extremal policies, of closedown, mothballing, idling, capacity depleting, or full capacity.

Remark 6
In this context It would be interesting to examine the above policies in the areas of real estate policies or shipping.

7.2 Conclusions

Our results differ significantly from the scanty evidence on record. Foremost among the differences is that *utilization*, *maintenance* and *service life* depend critically on the revenue earning properties of the equipment under consideration. By contrast the view propagated in the literature is that the corresponding policies are invariant with respect to the type of equipment. Another distinct difference lies in that all three policies are determined jointly. This implies in turn that empirical models investigating their determinants should allow for this simultaneity. And still another difference has to do with the development of operating policies in time. In particular, *optimal operating policies* provide for decreasing (increasing) *utilization* and increasing (decreasing) *maintenance* according as the discount rate for operating capital is higher (lower) than the discount rate of scrapping capital, but not necessarily throughout. The reason being that, if at least one of the functions defining the *operating factors* is not strictly concave, there will be jumps to lower (higher) intensity policies i.e. both lower (higher) *utilization* and at the same time lower (higher) *maintenance*. So the puzzle in the literature that has been associated with the optimal combination of these two policy instruments, and which was encountered recently by Boucekkine and Tamarit (2003), is resolved without having to impose extra restrictions.

Appendix A

A1. Total profit index function

We consider equipment with increasing concave operating function: $r = r(w)$, and we define the adjusted Hamiltonian flow rate function:

$$h(\mu) = \max_{w}\{r(w) - \mu w\} \Rightarrow \quad r'(w) = \mu \;\; or \;\; extremal \, .$$

It can be interpreted as the usual profit function, where the revenue r, the input w, and the input unit cost μ, may have any sign. Hence it is the dual of $r = r(w)$ in the context of convex analysis, and as such it is convex, with linear segments at the corners of $r(w)$, and corners at the linear segments of $r(w)$. In fact differentiating we find:

$$h'(\mu) = h'(w)w'(\mu) = [r'(w) - \mu'(w)w - \mu(w)] \, / \, \mu'(w) = -w(\mu) \, ,$$

where $-w(\mu)$ is an increasing function of μ, constant at the corners and discontinuous at the linear parts. It has minimum at the zero wear value: $w = 0 \Rightarrow \mu^{w} = r'(0)$. So depending on the wear properties, the graph of $h(\mu)$ will have one of the forms in **Figure A1**, with the asymptotes:

$$h(\mu) \quad \underline{r} - \mu\underline{w} \;\; as \;\; \mu \to +\infty \;\; \text{and} \;\; h(\mu) = \overline{r} - \mu\overline{w} \;\; \text{for} \;\; \mu \le 0 \, .$$

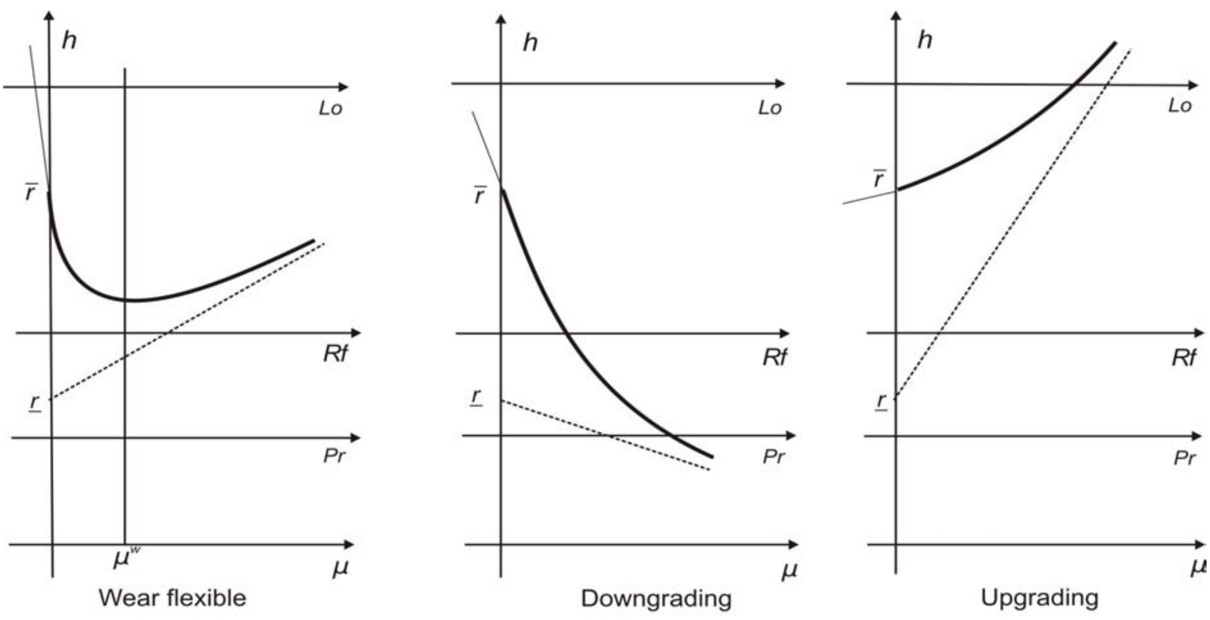

Figure A1: Adjusted Hamiltonian function: $h = h(\mu)$

In each case, the position of the μ axis where $h=0$, depends on the revenue properties. It can be in any one of the three positions indicated, according as the equipment is:

***Pr**ofit making*: $\underline{r} \geq 0$, ***Lo**ss making*: $\bar{r} \leq 0$, ***R**evenue **f**lexible*: $\underline{r} < 0 < \bar{r}$.

A2. Profit index

The slope of the radius to the graph of the function $h = h(\mu)$ gives the total profit index:

$$i = h(\mu)/\mu$$

For $\mu \leq 0$, it is always the hyperbola:

$$i = \bar{r}/\mu - \bar{w}$$

For $\mu \geq 0$, it depends on the revenue type of equipment. If it is revenue flexible: $\underline{r} < 0 < \bar{r}$, then it decreases from $+\infty$ past the value of the asymptote with $i = -\underline{w}$, reaching a minimum at the unit elasticity point, given by the zero revenue value:

$$h(\mu)/\mu = h'(\mu) \Rightarrow r(\mu) = 0 \Rightarrow \mu^r, \text{ with minimum value } i(\mu^r) = -w^r.$$

If it is not revenue flexible, then it decreases monotonically to $i = -\underline{w}$ if it is profit making: $\underline{r} \geq 0$, it increases monotonically to $i = -\underline{w}$ if it is loss making: $\bar{r} \leq 0$. We obtain the graphs in **Figure 2** in the text. In each case the position of the μ axis where $i = 0$, depends on the wear properties, i.e. the signs of $\{\underline{w}, \bar{w}, w^r\}$.

For given μ, the total profit index i can also be obtained directly from the operating function $r = r(w)$ as indicated in the first diagram of **Figure A2** below.

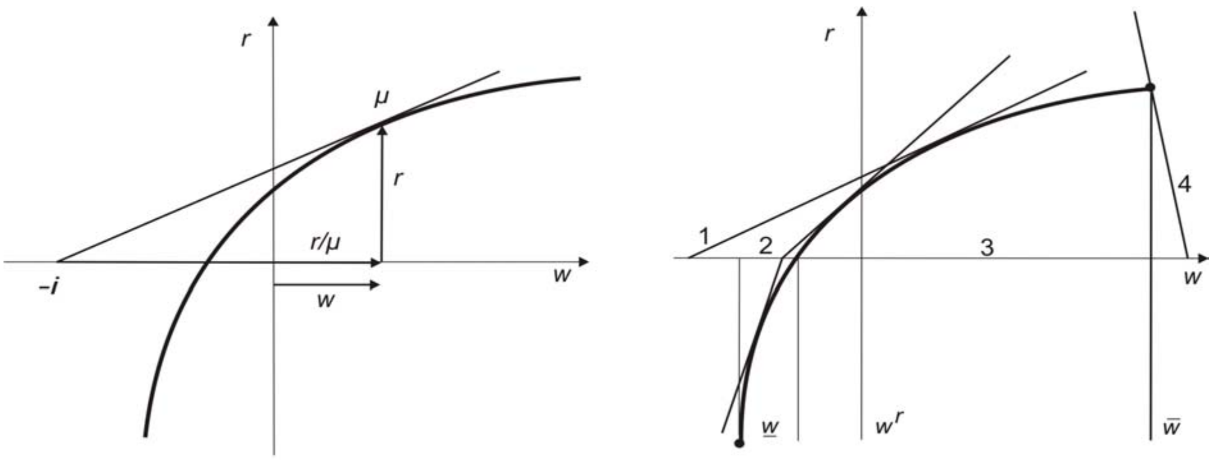

Figure A2: Total profit function: $i = \frac{r}{\mu} - w$

Conversely, from the same diagram, for given i we can find the $\mu-$values and the policies (w,r) with this profit index by drawing the appropriate tangent lines from the position $-i$ on the $w-$axis. Thus, in the second diagram of **Figure A2**, we consider revenue flexible equipment of the special type and we find the $\mu-$values and the policies (w,r) corresponding to four index values, as follows:

1. $-i < \underline{w} \Rightarrow i > -\underline{w}$: Unique profit making
2. $\underline{w} < -i < w^r \Rightarrow -w^r < i < -\underline{w}$: Two policies, one profit and one loss making
3. $w^r < -i < \overline{w} \Rightarrow -\overline{w} < i < -w^r$: No policy
4. $\overline{w} < -i \Rightarrow i < -\overline{w}$: Extremal policy, the hardest.

The results are in accord with the third graph of **Figure 2** in the text.

A3. Extremal policies

As noted in Lemma 1, if μ becomes negative then the applied policy is the extremal hardest. Since we don't have disposal costs, this cannot happen at the terminal scrapping time. However, if $\underline{r}' > 0$ then, even if μ is positive, it can move below $\underline{r}'$ and the corresponding policy will be the extremal hardest: $\overline{w}$. Similarly if $\overline{r}' < \infty$ then μ can move above $\overline{r}'$ and the corresponding policy will be the extremal mildest $\underline{w}$. In these cases the extremal policies are stable policies, and the scrapping and equilibrium policies can be extremal under certain market conditions. Then it will be optimal to apply the corresponding extremal policy throughout the operating period. We will examine one such case.

Corollary A1. Extremal policies

We consider scrappable equipment of the profit making type.

1. If the extremal hardest policy is stable:

$$\underline{r}' > 0 .$$

and both discount rates are higher than the profit index of the extremal hardest policy:

$$\sigma - \eta \geq i(\overline{w}) = \overline{r} / \underline{r}' - \overline{w} \quad \text{and} \quad \sigma / \varepsilon \geq i(\overline{w}) = \overline{r} / \underline{r}' - \overline{w} ,$$

then the extremal hardest policy will be applied throughout the operating period.

2. If the extremal softest policy is stable:

$$\overline{r}' < \infty .$$

and both discount rates are lower than the profit index of the extremal softest policy:

$$\sigma - \eta \leq i(\underline{w}) = \underline{r} / \overline{r}' - \underline{w} \quad \text{and} \quad \sigma / \varepsilon \leq i(\underline{w}) = \underline{r} / \overline{r}' - \underline{w} ,$$

then the extremal softest policy will be applied throughout the operating period.

A4. Substitution rates

We consider an equipment with operating rate functions $w(u,m)$, $r(u,m)$, and we define two *substitution rates*, one for each policy factor:

$$r^u = r'_u / w'_u \text{, decreasing in } u \text{ ,} \quad r^m = r'_m / w'_m \text{, increasing in } m \text{.}$$

Referring to **Figure 1(a)** in the text, we obtain two policy domains, the *high intensity* policies in the upper right side and the *low intensity* policies in the lower left side, characterized by:

$$E^+ : r^u < r^m \quad \text{and} \quad E^- : r^u > r^m$$

Their common boundary is the optimal path determined by the relation:

$$r^u = r^m$$

This common ratio is the substitution rate $r'(w)$. Thus the optimal policies are of intermediate intensity. However for some or even all w values one of these domains may be empty and then the optimal policies are on the boundary consisting of extremal utilization or maintenance policies: $u = \{0,1\}$ or $m = \{0,1\}$. In this case the substitution rate is given by the substitution rate of the non-extremal policy factor. In

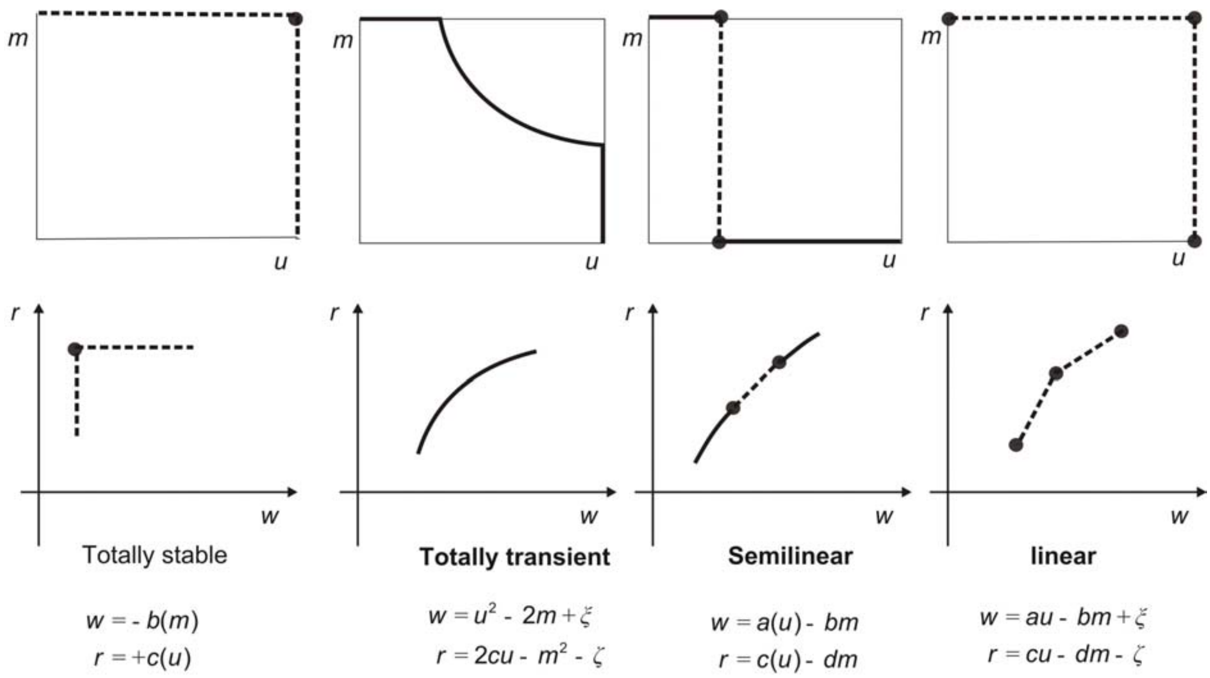

Figure A3: Optimal paths and operating functions

Figure A3 we give four examples of optimal paths and corresponding operating function. They are all of the separable type to be explained bellow. We can adjust the constants to obtain any revenue or wear type.

A5. Separable policies

We will consider operating rate functions with separable utilization and maintenance controls:

$$w = a(u) - b(m), \quad r = c(u) - d(m)$$

The substitution rates are also separable, each depending only on the respective policy factor, and monotonic by the convexity assumptions:

$$r^u = c'(u) / a'(u) \ \downarrow, \quad r^m = d'(m) / b'(m) \ \uparrow$$

The high and low intensity regions are determined by the inequalities:

$$E^+ : r^u(u) \leq r^m(m) \quad \text{and} \quad E^- : r^u(u) \geq r^m(m),$$

respectively. The substitution rate and the optimal path are determined by:

1. $r' = r^u(u) = r^m(m)$, in the interior of the control region.
2. $u = \{0,1\},\ r' = r^m(m)$, on the vertical boundaries of extremal utilization.
3. $m = \{0,1\},\ r' = r^u(u)$ on the horizontal boundaries of extremal maintenance.

From the monotonicity properties of the two substitution rates, we conclude the following:

Remark 6
If the controls are separable, then in the direction of softer policies with decreasing wear-revenue rates, we have also decreasing utilization and increasing maintenance, i.e. we do not have policy movements in time, to higher or lower intensity policies.

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Endnotes

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[2] For the reasons cited in Bitros and Flytzanis (2002b) we will continue to deal with the problems faced by *owners* of equipment and relegate the analysis of those confronted by *lessors* to future research endeavors.

[3] An excellent survey of the literature on *utilization* that developed during the decades that proceeded these efforts can be found in Winston (1974).

[4] For a recent account of the implications and the relative size of maintenance and repair expenditures see McGrattan and Schmitz (1999).

[5] As it can be ascertained from, say, Howe and McCabe (1983), in the theory of finance, and particularly in its segment dealing with issues in capital budgeting, the relevant literature refers to *scrapping* as asset *abandonment*. So the implications of the analysis undertaken below regarding *scrapping* may transcend the core area of economic theory.

[6] This concerns the owner of the capital stock. If he were actually leasing it, then assuming that he has to return it in the original state, we would have a cost term at the end of the operating period, and assuming no transaction costs, the net scrap value would be:

$$S - C = S(K_T, T) - S(K_0, T)$$

[7] In general, softest is the policy pair: $(u = 0, m = 1)$, and hardest is the policy pair: $(u = 1, m = 0)$. However, if the rate functions have flat sections then the operating function being increasing concave will have an initial vertical or a final horizontal segment in the direction of increasing w. These we can ignore, as non-applied. The extremal values are determined by the remaining part.

[8] In practice *idling* takes place for the purpose of adjusting the available productive capacity to temporary shortfalls in demand. For an excellent analysis of this policy see Das (1991).

[9] *Mothballing* of equipment lasts for extended periods and is decided either because market conditions are not expected to improve any time soon or because of strategic reasons (e.g. *mothballing* by Exxon Corporation of newly built plant for producing crude from shale after the first oil crisis).

[10] Under this policy the owner of the equipment stops completely its maintenance but continues to operate it at full capacity until it comes to a standstill, naturally or otherwise. This implies that $w > 0$ and explains the capacity depleting nature of the policy.

[11] Maintenance is most intensive at times of carrying out the *surveys* that are mandated by regulatory authorities (e.g. ships and airplanes) as well as while repairing damages from major accidents. However, the latter are unexpected events and as such cannot be accounted for when scheduling the utilization of equipment. For this reason, this policy should be interpreted to imply that full maintenance $(m = 1)$ takes place during periods that the equipment is normally off-duty, because then it does not interfere with full utilization $(u = 1)$.

[12] We note that these conditions are also valid if the quantities: $\{w, r, \sigma, \eta\}$, are time dependent, denoting external influences that are predictable. Actually η itself corresponds to predictable changes in the price of the equipment and the services offered.

[13] The above applies to the case where the equipment is owned or equivalently if the decision for its acquisition must be made at time $\tau = 0$. If this is not the case then we must put the additional condition that it is not profitable to delay the acquisition. If we neglect transaction costs so that new and unused

stock have the same value, then this restriction is given by the $H-$ *initial condition*:

$$-H(0)+S_T'(K_0,0)<0 \quad \textit{with} \ \lambda(0)=0 \ \Rightarrow \max r>(\sigma-\eta)pK_0^{1-\varepsilon}$$

14 As it was shown in our previous work, the notions: "scrappable, durable", are not equivalent to the notions: "replaceable, non-replaceable". In fact the two are independent. A scrappable stock may or may not be replaceable, the same for a durable stock.

15 As in the case of profitability we take the first maximum as optimal. This could reflect a reluctance on the part of the operator to accept even temporary reduction in the over all net revenue, e.g. as a precautionary measure to the risk of capital obsolescence . Anyways we will indicate situations where in fact there may appear subsequent maxima. However their comparison would require additional specifications.

16 We consider the two financial indices:

P/E : Price per total earnings, and P/BV : price per logistic value

then the profit index is given by their ratio

$i=E/BV$: Total earnings (including stock adjustment) per logistic value.

17 In general, softest is the policy pair: $(u=0, m=1)$, and hardest is the policy pair: $(u=1, m=0)$. However if the rate functions have flat sections then the operating function being increasing concave will have an initial vertical or a final horizontal segment in the direction of increasing w. These we can ignore, as non-applied. The extremal values are determined by the remaining part.

18 We noted that for revenue flexible equipment some $\sigma-\eta$ give two scrapping solutions, and then the scrapping duration is determined by the one closest to μ_0, in accordance with the position that the operator does not accept revenue reduction, as in endnote 14 above. In this case as T increases further it may meet the second scrapping solution giving a minimum, and then we will have a second maximum at infinite duration. Also, as we see from the graph of the profit index function, as $\sigma-\eta$ decreases the two $\mu-$values come closer together and then it is expected that at some point the second maximum will dominate the first. For this investigation we need more specifications.

19 We address our comments to their model because it is of the latest vintage and we can avoid detailed references to older models by, say, Licandro and Puch (2000) and Collard and Kollintzas (2000), which belong in the same class.