A rehabilitation of economic replacement theory

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Abstract

Our objective in this paper is to shed light on the economic forces and the specific way in which they combine to determine the service life, and hence the replacement demand for durables, in the short run and in the long run. For this purpose the received multi-period economic replacement model is extended in the light of more recent theoretical developments and solved for the number and duration of replacements. Owing mainly to the intuition that the latter decisions are inexplicably related to the owner's profit horizon, aside from *steady state replacement*, the model is shown to yield a range of *transitional* and *limiting replacement policies* that have been largely ignored in the literature. In addition, the results indicate that : a) the optimal service life is consistently determined by such conventional forces of short-term variation as utilization, maintenance, operating safety, interest rate, uncertainty due to technological breakthroughs, the price of new and used durables, etc., b) switching among replacement policies produces bursts or slumps in replacement investment much like the "spikes" discovered recently at the plant level, and c) in non-stationary economic environments the error from applying *steady state replacement*, instead of the more appropriate *transitory replacement policies* reported in this paper, may be substantial.

JEL Classification: E22
Keywords: service life, replacement, and scrapping

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I. Introduction

The development of a framework to explain the behavior of business firms and consumers in the replacement of their durables has passed through several phases, not all of which could be considered successful from an evolutionary point of view. Preinreich's (1940) and Terborgh's (1949) seminal contributions, seen through Smith's (1961) perspective a little over ten years later, constituted the culmination of the first significant phase which begun in the early 1920's with the papers of Taylor (1923) and Hotelling (1925).[1] Then, with Jorgenson's (1963) influential article started a new phase during which replacement theory all but eclipsed since the great majority of researchers were content to assume that durables are replaced at a constant rate, proportional to the corresponding stocks. This phase peaked at the beginning of the 1970s when a few researchers initially, and many more afterwards, started to question the theoretical and empirical foundations on which the proportional replacement theory rested.

Wykoff (1970) was the first to raise serious doubts about the validity of the proportionality hypothesis. He found that the resale price of first year automobiles in the post-war period declined consistently far in excess of what was justified by their physical deterioration.[2] Following that contribution there appeared a barrage of additional papers by Feldstein and Foot (1971), Feldstein and Rothchild (1972),[3] Bitros (1972), Wykoff (1973), Bitros and Kelejian (1974) and Smith (1974). These papers showed that the replacement to capital stock ratio varied considerably under the influence of conventional economic forces. So, even though Jorgenson (1974) forcefully defended the proportionality hypothesis, by the middle of 1970's it was clear that the profession was in search of an economic theory of replacement.

Actually, as evidenced by Naslund (1966), Thompson (1968) and Kamien and Schwartz (1971),[4] the efforts to formulate such a theory while the proportionality hypothesis reigned never stopped.[5] But since then the efforts intensified and proceeded in three directions. More specifically, working in the first direction, Malcomson (1975), Nickell (1975), Arnott, Davidson, and Pines (1983), Ye (1990), and Mauer and Ott (1995) presented choice theoretic models highlighting most of the main determinants of replacement. As a result, now it is widely accepted that the decision to replace a durable depends on its age, technological change, maintenance, product and input market conditions, various sources of uncertainty, shocks, etc. The second direction was followed by researchers such as Smith (1975), Westin (1975), Rust (1987), and Rothwell and Rust (1997), who theorized about replacement from certain "stylized facts" characterizing the data from various durables. Finally, several general equilibrium theorists as, for example, Kydland and Prescott (1982), Cooper and Haltiwanger (1993), Cooley, Greenwood and Yorukoglu (1997), and Cooper, Haltiwanger and Power (1999) took the third direction by trying to explain how the decision to replace a piece of equipment is influenced by the state of the economy.

However, in spite of the advances achieved, the quest for a sufficiently general theory of replacement remains unabated. One reason for this is that factors identified in some studies as important determinants of replacement are ignored by others. For example, with the exception of Arnott et. al. (1983), there is no account for the ability of owners to improve their equipment through maintenance. Another reason is that typically the deterioration of durables is attributed merely to the passage of time, thus leading to models that prohibit owners from running their durables down faster (slower) through more (less) intensive utilization. Last, but not least, by assuming that there is no second hand market,[6] that investment is irreversible, and that technical change proceeds at a constant rate, researchers obtain replacement models that are overly restrictive.

In view of the above our objective is to expand on the replacement theory that emanates from the studies in the first tradition so as to derive a model that can explain all important real capital decisions. To this end, in the present paper, aside from laying down the foundations for such a model, we obtain several new results. More specifically, one of them is that, depending on the nature and operating characteristics of the equipment, there emerge several *transitional* and *limiting replacement and scraping policies,* which have been ignored in the relevant literature. Another is that in general *transitional replacement policies* give rise to service lives which are higher than those derived from *equidistant steady state replacements*, thus raising the possibility for significant errors in growth accounting, real business cycle studies and others applications. Still another is that switching among *transitional policies* produces bursts or slumps in replacement investment much like the “spikes” discovered recently at the plant level.[7] And last, but not least, we find that optimal service life is consistently determined by such conventional forces of short-term variation as utilization, maintenance, operating safety, interest rate, uncertainty due to technological breakthroughs, the prices of new and used equipment, etc.

Organizationally the paper is structured as follows. In Section II initially we expand on the received multi-period economic replacement model by allowing explicitly for the processes of *utilization*, *maintenance* and *uncertainty*, due to technological breakthroughs. Then we provide necessary and nearly sufficient conditions for the existence of optimal policies, and lastly we analyze the properties of *transitory*, *limiting* and *steady state replacement and scraping policies* that emerge. Next, in Section III, we use a simple example to highlight the implications of our findings for economic theory and policy and in Section IV we summarize the conclusions. Finally, Appendices A and B provide certain results that supplement the presentation in important respects.

II. A unified theory of replacement and scraping decisions

Preinreich (1940) showed that the economic life of equipment can not be determined in isolation from the economic life of each equipment in a chain of replacements extending as far into the future as the user's profit horizon. However he did not investigate the question of how long the user's profit horizon should be and this led economic theorists writing in his tradition to the simplification that replacements were placed at equally spaced time intervals over an infinite horizon. As a result they restricted attention exclusively on *equidistant steady state replacements*. Instead, according to our rehabilitation of his multi-period model, users of equipment cannot leave the issue of the profit horizon undecided. Subject to technical and market conditions, they are free to plan for any number of replacements and for any sequence of durations, including the one of infinite equidistant replacements. But to realize the benefits of this flexibility at minimum cost to the present value of their profits, users must determine from the beginning the number and the duration of each and every future replacement.

By virtue of this conceptualization we are able to map the complete sequence of *transitory* and *limiting* replacement policies under two qualifications. The first of them concerns the important distinction between owners and lessors. Their difference is that when owners decide to stop operations they scrap their equipment, whereas lessors are obliged to replace it. This difference leads to two distinct sets of replacement policies, i.e. *transitory replacements with terminal scrapping* and *transitory replacements with terminal replacement*. So, even though our model can address the behavior of both users simultaneously, to keep the analysis simple we shall focus only on the problem solved owners. As for the second qualification this stems from the observation that over its lifetime the equipment may change owners through reselling. Whenever such reselling takes place we assume that new owners adopt the optimal policies that were decided by the original owner. Hence, we concentrate on the replacement and scraping decisions only of the last owner.

1. The model

When the multi-period replacement model is based on revenue and cost functions that depend explicitly on time, net profits in any period are inextricably linked to net profits in all future periods, thus leading to solutions that are not manageable from an analytical point of view. The fundamental source of the difficulty arises from the realization that it is utterly impossible to know the technology and the market conditions that will prevail in the distant future. Hence, in order to extract from the model some meaningful properties, over the decades researchers have adopted various restrictions regarding the general and the particular characteristics of the functions involved. Keeping with this tradition below we shall proceed on the assumption that all functions in the model are time invariant.

To highlight the fundamental implication of this assumption, we note that the owner may sell his equipment at any time and stop operations or replace it with a new one. This *replacement process* can be repeated any number of times, ending with *terminal scraping*. Each time a replacement takes place, there starts a new cycle in the life of equipment. We shall refer to these cycles as *operating periods* or *operating intervals.* Thus, if the owner of the equipment chooses to perform v replacements, there will be v operating *periods* with:

$$0 = t_0 < t_1 \cdots < t_v < t_{v+1} = T \ , \qquad \textbf{(1)}$$

where $t_0 = 0$ is the date when the equipment is first put into operation, $t_{v+1} = T$ is the terminal scraping date, and the terminal period may extend to infinity: $T = \infty$. In this framework, had we adopted the property of time dependence, as the number of replacements v increased, we would have to recalculate all revenues and costs from the beginning because both quantities would be dated to absolute time. On the contrary, under time invariance only relative time matters and as v increases we calculate only the extra term corresponding to the replacement that is added each time. This implies that we can solve for each operating period starting at zero time.

With these preliminary remarks in mind, we start now by recalling from previous studies that *maintenance* and *utilization* affect service life through cash flow and salvage value of equipment. With regard to the first we adopt the view that *maintenance* and *utilization* affect cash flow directly, but also indirectly via the quantity and quality of services extracted from the equipment. Thus, ignoring for simplicity the time indexing of variables, the cash flow effects of *maintenance* and *utilization* will be modeled by the expressions:[8]

$$r = r(K, u, m) \tag{2}$$

$$\dot{K} = -w(K, u, m), \tag{3}$$

$K = K(t)$: *Used equipment* measured in efficiency units, reflecting its size and age since first put in operation. New equipment will be denoted by K_0.

$\dot{K} = dK/dt$: *Net deterioration flow,* including aging. Usually negative.

$u = u(t)$: *Utilization intensity* relative to some maximum, with $0 \le u \le 1$.

$m = m(t)$: *Maintenance intensity* expressed as expense relative to some maximum, with $0 \le u \le 1 0 \le m \le 1$.

(u, m): *Operating policy pair,* with u, m piecewise continuous functions of time.

(r, w): *Operating policy functions.*

As for the salvage value effect, the evidence from second-hand markets reveals that, *ceteris paribus*, better maintained and less used pieces of equipment command higher prices than similar pieces of equipment with poor maintenance and more intensive utilization. By **(3)** this effect is incorporated in K, so we adopt the following specification:

$$C(K) = P_0 - S(K) \tag{4}$$

$S(K)$: *Resale value* of used equipment K. We set $S_0 = S(K_0)$

P_0**:** *Purchase cost* of new equipment K_0.

$C(K)$: Cost for replacing used equipment K with new equipment K_0. Usually positive, but it can be also negative, if maintenance is upgrading in the sense of overbalancing the deterioration of equipment. Moreover, If replacement involves scraping with some discount it will hold that $C(K) \le P_0 - S(K)$. To simplify the complications introduced by this possibility, in this paper we shall ignore all incentives granted my manufacturers to buyers of new equipment.[9] Finally, we define $C_0 = P_0 - S_0$ to be the *owner's cost* of holding new equipment. Alternatively, this can be considered as a *transactions cost* or a *second hand discount benefit.*

Turning next to the specification of uncertainty that springs from technological change, we note that the effect of *minor* technological advances, which are more or less predictable, has been accounted for traditionally as a continuous trend incorporated into the time dependence of the functions involved. But in our case consistency with the assumption of time invariance prohibits the adoption of a similar approach. For this reason the effect of such changes will be accounted for only to the extent that maintenance of the *upgrading* type allows their incorporation into the original design of equipment. In addition though we have the possibility of *major* technological breakthroughs which in its simplest form may render the equipment totally obsolete, by eliminating all revenue and scrap value thereafter.[10] This corresponds to a non-repairable equipment breakdown, and following Kamien and Schwartz (1971), we shall model it by:

$$F(t) = \text{Probability of a } \textit{breakthrough} \text{ by time } t, \qquad \textbf{(5)}$$

with $F(0) = 0$ and $F(t) < 1$ for all t.

Given the above specifications, the problem confronted by owners of equipment may be expressed as follows:

Choose the variables $[T, v, \{t_i\}, u, m]$ so as to maximize:

$$\Pi_v = \sum_{i=1}^{v} [Q_i - C_i] + Q_T + R_T - P_0, \qquad \textbf{(6)}$$

Subject to **(3)** and the constraints on the *operating policies* $\{u, m\}$,

with he following quantities denoting expected values:

Π_v : *Total net profit with* v *intermediate replacements,*

$Q_i = \int_{t_{i-1}}^{t_i} r(K,u,m)\varphi(t)dt$ **:** *Net operating revenue in the* $i-th$ *period:* $i = 1, 2, ..., v$,

$C_i = \varphi(t_i)C(K_i)$ **:** *Net cost of the* $i-th$ *replacement*, where $K_i = K(t_i)$,

$Q_T = Q_{v+1}$ **:** *Net operating revenue in the terminal period,*

$R_T = \varphi(T)S(K_T)$: *Revenue from terminal scraping,* and

$\varphi(t) = [1 - F(t)]e^{-\rho t}$: *Effective discount factor.* Assuming a constant discount rate ρ, the discount factor would be $e^{-\rho t}$. To account for **(5)** this should be multiplied by the term $[1 - F(t)]$. In keeping with the specification of time invariance, we will consider only the usual exponential case: $F(t) = 1 - e^{-\theta t}$. By implication, since $\varphi(t) = e^{-(\theta+\rho)t}$, the effect of this type of technological uncertainty is equivalent to the introduction of a revised *effective discount rate*, expressed by the sum: $\sigma = \theta + \rho$

2. Optimality conditions

In solving the problem by optimal control theory, we consider the *present value Hamiltonian*:

$$\tilde{H} = e^{-\sigma t} r(K,u,m) - \tilde{\lambda} w(K,u,m)\,, \tag{7}$$

where $\tilde{\lambda} = \tilde{\lambda}(t)$ is a co-state variable, denoting the *owner' s present unit price* of *K*. Alternatively, we can use the corresponding *current values:*

$$H = e^{\sigma t}\tilde{H} = r(K,u,m) - \lambda w(K,u,m)\,, \quad \textit{with} \quad \lambda = \lambda(t) = e^{\sigma t}\tilde{\lambda}(t)\,. \tag{8}$$

From Leonard and Van Long (1995) and Seierstadt and Sydsaeter (1986) we obtain the following necessary and nearly sufficient conditions for optimality in each period:

For u and m (Maximality Principle):

$$\begin{aligned} &\Rightarrow \max_{u,m}\{\tilde{H} = \varphi(t)r(K,u,m) - \tilde{\lambda}w(K,u,m) \,\|\, 0 \le u \le 1,\;\; 0 \le m \le 1\} \\ &\Rightarrow \max_{u,m}\{H = r(K,u,m) - \lambda w(K,u,m) \,\|\, 0 \le u \le 1,\;\; 0 \le m \le 1\} \end{aligned} \tag{9}$$

For the time evolution of K:

$$\begin{aligned} \dot{K} &= -w(K,u,m) && (i) \\ K(t_{i-1}) &= K_0 && (ii) \end{aligned} \tag{10}$$

For the time evolution of λ *:*

$$\begin{aligned} \dot{\tilde{\lambda}} &= -\partial\tilde{H}/\partial K && \Rightarrow && \dot{\lambda} = -\partial H/\partial K + \lambda(t)\sigma && (i) \\ \tilde{\lambda}(T) &= \partial R_T/\partial K_T && \Rightarrow && \lambda(T) = S'_K(K_T) && (ii) \\ \tilde{\lambda}(t_i^-) &= -C'_K(K_{t_i}) && \Rightarrow && \lambda(t_i^-) = -C'_K(K_{t_i}) && (iii) \end{aligned} \tag{11}$$

The durations T *and* t_i *are determined by the terminal condition on* $H = r - \lambda w$:

$$\tilde{H}(T) + \partial R_T / \partial T = 0 \quad \Rightarrow \quad H(T) = \sigma S(K_T) \qquad (i)$$
$$\tilde{H}(t_i^-) - H(t_i^+) - \partial C / \partial t_i = 0 \quad \Rightarrow \quad H(t_i^-) = H(t_i^+) - \sigma C(K_{t_i}) \qquad (ii) \qquad \textbf{(12)}$$

Moreover, the following remarks complement these conditions in certain important respects:

Remark 1
The above conditions apply if the terminal period satisfies $0 < T < \infty$*. Hence:*
(i)*. If* $T = 0$*, the necessary terminal condition for the Hamiltonian becomes:*

$$\tilde{H} + \partial R_T / \partial T \Big\|_T \leq 0 \Rightarrow H(0) \leq \sigma S(K_0). \qquad \textbf{(13)}$$

Such a solution would occur if it were optimal to scrap the equipment immediately instead of operating it. But if the time and the initial cost of equipment were free, the same conditions with the inequality signs reversed would be necessary for starting operations at $t_0 = 0$ *with* $K(0) = K_0$*.*

(ii)*. If* $T = \infty$ *there is no scrap value to consider and the case needs special treatment. Assuming that* $R_T \to 0$*, the limiting conditions are usually taken to be:*

$$\lambda = \lim \tilde{\lambda} \geq 0 \quad \lim \lambda K = 0 \qquad H: \lim \tilde{H} = 0. \qquad \textbf{(14)}$$

Remark 2
Nothing in our assumptions prevents K *from rising above the original* K_0 *through upgrading maintenance. In this event, replacing* K *would entail net profit, because if* $K > K_0$ *the cost of replacement* C *would be negative. To avoid complications we will assume that in such cases discounting overbalances any possible upgrading so that the operating revenue and the scrap revenue functions are bounded.*

With their help we seek to determine 1) the optimal number of replacements, 2) the optimal duration of each operating period, and 3) the optimal policies u and m as functions of time, during each operating period. However, the last question requires more specifications for the functions involved and will be examined in a separate work. Below then we will concentrate on the first two questions.

3. Optimal replacement and scraping policies

Since under time invariance maximization in each period starts always at zero time, it is convenient to reorder the operating periods to start from the last and going backwards. To this end we introduce the index: $j = v + 1 - i$. Thus the terminal period is denoted by the index $j = 0$, the last replacement period before the last by the index $j = 1$, and so on. With this rearrangement in mind, we will proceed now to highlight the properties of the following three policies: 1) *transitory replacements with terminal scraping, 2) limiting replacement policies,* and 3) *equidistant steady state replacements.*

3.1 Transitory replacements with terminal scraping

The *profit function* for the operating period $[0,T]$ with final scraping is given by:

$$A(T) = Q(T) + e^{-\sigma T} R(K(T)) - P_0 \tag{15}$$

Next, taking the $T-$derivative from **(15),** we obtain

$$A'(T) = \tilde{H}(T) - \sigma e^{-\sigma T} S(K_T) = e^{-\sigma T}[H(T) - \sigma S(K_T)], \tag{16}$$

and define the current value of the final profit rate:

$$\begin{aligned} \alpha(T) &= A'(T)e^{\sigma T} = H(T) - \sigma S(K_T) \\ &= r_T - \lambda_T w_T - \sigma S(K_T) \qquad : \textit{final profit rate,} \end{aligned} \tag{17}$$

where λ_T satisfies conditions **(11)**. In the limit $T \to 0$ we will have also $K(T) \to K_0$. Hence, we obtain the special ***short duration owner's unit price of*** K: $\lambda_0 = S'_K(K_0)$, whereas the *maximality principle* determines the ***short duration optimal policies***:

$$\{u_0, m_0\} \Rightarrow r_0 = r(u_0, m_0, K_0) \Rightarrow w_0 = w(u_0, m_0, K_0) \Rightarrow H_0 = r_0 - \lambda_0 w_0 \tag{18}$$

with the corresponding profit rate:

$$\begin{aligned} \alpha(0) &= H(0) - \sigma S(K_0) = H_0 - \sigma S(K_0) \\ &= r_0 - \lambda_0 w_0 - \sigma S(K_0) \quad : \textit{Short duration profit rate.} \end{aligned} \tag{19}$$

In light of the above, we assume that by solving the optimality conditions we can find for each $T > 0$, the *$T-$optimal solution* for the specified period. We will call *terminal or scraping period optimal duration* T_0 the first local maximum of $A(T)$. The corresponding *maximal profit* will be denoted by:

$$\Pi_0 = A_0 = A(T_0). \tag{20}$$

We will say that the equipment is ***profitable*** if $\Pi_0 > 0$**.**

Now, for any backward sequence of operating durations: $\{T_0, T_1, \cdots\}$, the *total profit with* v *intermediate replacements* will be given by the expression:

$$\Pi_v = A_v + e^{-\sigma T_v} A_{v-1} + \cdots + e^{-\sigma(T_v + \cdots + T_1)} A_0 = A_v + e^{-\sigma T_v} \Pi_{v-1}, \textit{ where } \Pi_0 = A_0. \tag{21}$$

Assuming that $\Pi_0 > 0$, we consider the last replacement period before the scraping period, and the corresponding $1-$*replacement total profit function:*

$$\Pi(T_1) = A(T_1) + e^{-\sigma T_1}\Pi_0 \ . \quad \textbf{(22)}$$

The *optimal* $1-$*replacement duration* T_1 will be defined as the first local maximum of $\Pi(T_1)$ and the corresponding *maximal profit* will be denoted by:

$$\Pi_1 = A_1 + e^{-\sigma T_1}\Pi_0 . \quad \textbf{(23)}$$

We will say that the equipment is $1-$***replaceable,*** if:

$$\Pi_1 > \Pi_0 \Rightarrow A_1 > (1 - e^{-\sigma T_1})\Pi_0 . \quad \textbf{(24)}$$

More generally, assuming that the equipment is $(j-1)-$***replaceable*** with total profit Π_{j-1}, we consider the $j-$*replacement total profit function***:**

$$\Pi_j(T) = A_j(T) + e^{-\sigma T}\Pi_{j-1} . \quad \textbf{(25)}$$

The *optimal* $j-$*replacement duration* T_j will be defined as the first local maximum of $\Pi_j(T)$ and the corresponding *maximal profit* will be denoted by:

$$\Pi_j = A_j + e^{-\sigma T_j}\Pi_{j-1} \quad \textbf{(26)}$$

We will say that the equipment is or $j-$***replaceable,*** if:

$$\Pi_j > \Pi_{j-1} \Rightarrow A_j > (1 - e^{-\sigma T_j})\Pi_{j-1} . \quad \textbf{(27)}$$

Finally, taking the $T-$derivative from the one period profit function **(15),** in conjunction with the current value of the terminal profit rate **(17)**, we note that in general:

$$\Pi_j'(T) = A_j'(T) - \sigma e^{-\sigma T_j}\Pi_{j-1} = e^{-\sigma T}[\alpha(T) - \sigma\Pi_{j-1}] . \quad \textbf{(28)}$$

From the interpretation of the above we conclude the following:

Proposition 1

We assume that for each given $T \geq 0$ we can find the $T-$optimal solution as indicated above and that the function $H(T)$ is continuous.

(i)*.* If $\alpha(0) < 0$*, then we say that the equipment is initially unprofitable. In this case:*

- *If the stationary duration exists then it is* ***unprofitable,*** *but the equipment may be* ***eventually profitable****.*

- *If the stationary duration does not exist then the equipment is* ***unprofitable****.*

(ii)*. If $\alpha(0) > 0 \Rightarrow \sigma < H_0 / S_0$, then we say that the equipment is* ***initially profitable.*** *In this case the optimal scraping duration T_0 is given by the smallest stationary solution:*

$$\alpha(T) = 0 \Rightarrow H(T) = \sigma S(K_T).$$

If it does not exist then the equipment is ***scraping durable:*** *$T_0 = \infty$.*

(iii)*. If the equipment is* ***profitable****, and in addition it satisfies:*

$$\alpha(0) > \sigma\Pi_0 \Rightarrow \sigma < H_0 /(S_0 + \Pi_0),$$

then we say that it is ***initially*** $1-$***replaceable.*** *In this case the optimal $1-$replacement duration T_1 is given by the smallest stationary solution:*

$$\alpha(T) = \sigma\Pi_0 \Rightarrow H(T) = \sigma[S(K_T) + \Pi_0] \ .$$

(iv)*. In general, if the equipment is $j-1$* ***replaceable****, and in addition it satisfies:*

$$\alpha(0) > \sigma\Pi_{j-1} \Rightarrow \sigma < H_0 /(S_0 + \Pi_{j-1}),$$

then we say that it is ***initially*** $j-$***replaceable.*** *In this case the optimal $j-$replacement duration T_j is given by the smallest stationary solution:*

$$\alpha(T) = \sigma\Pi_{j-1} \Rightarrow H(T) = \sigma[S(K_T) + \Pi_{j-1}].$$

(v)*. The replacements, when defined, have successively strictly decreasing durations and strictly decreasing period profits:*

$$T_0 > T_1 > T_2 > \cdots \text{ and } A_0 > A_1 > A_2 > \cdots$$

Remark 3

(i)*. Concerning the continuity of $H(T) = r(T) - \lambda(T)w(T)$, we note that because of the optimization procedure the rate functions $r(T) = r(u(T), m(T), K(T))$ and $w(T) = w(u(T), m(T), K(T))$ can be considered continuous, even though the control functions $u = u(T)$, $m = m(T)$ themselves may exhibit discontinuities if we have linearities. However if, as is often the case, the equipment is subject to operating under safety or other regulatory and more general operating restrictions of the type: $K(T) \geq J$, then the continuity of the inner value $\lambda(T)$ has to be examined as in Leonard and Van Long (1995, pp. 332-342), and Seierstadt & Sydsaeter (1986, pp. 313-356). Also in this case the profit function $A(T)$ may be defined only in a bounded time interval, in which case the maximization procedure will be restricted to this interval.*

(ii)*. The optimal durations are actually given by the first stationary duration provided that stationarity implies optimality, i.e. if the rate function $\alpha(T)$ does not have local minimum at the particular level: $\{\sigma\Pi_0, \sigma\Pi_1, \cdots\}$. This is a reasonable assumption given the discreteness of the values. In any case the optimal duration is given by the first such value that is not local minimum.*

(iii)*. With regard to* ***proposition 1.iv****, we note first that by their definition the successive total profits are increasing and hence the durations are decreasing. Then the period profits are decreasing because $A(T_j)$ is increasing until T_0.*

3.2 Limiting and steady state replacements

In ***Proposition 1*** we introduced the distinction between initial profitability and profitability. This distinction arises only if we have transaction costs: $C_0 = P_0 - S_0 > 0$ and in general if C_0 is very large or if T_j is very small. Under the later circumstances the distinction becomes more crucial, because as the number of replacements increases the replacement durations decrease. For simplicity, below we will start by assuming that $C_0 = 0$ and then consider separately the effect of $C_0 > 0$ if $T_j \to 0$.

Assuming *initial profitability*, we will say that the equipment is ***scraping finite*** $N-$***replaceable*** with *total profit* Π_N, if we have only N profitable replacements, i.e.

$$0 < T_N < \cdots < T_1 < T_0 \leq \infty, \; \textit{with} \; 0 < \Pi_0 < \ldots < \Pi_{N-1} < \alpha(0)/\sigma \quad \& \quad \Pi_N > \alpha(0)/\sigma \,. \quad \textbf{(29)}$$

In particular, if $N = 0$, then we will say that the equipment is ***scraping non-replaceable.*** Otherwise, because of the monotonicities involved, we will have a limit:

$$T_j \downarrow T_\infty \geq 0, \quad A_j \downarrow A_\infty \geq 0, \quad \Pi_j \uparrow \Pi_\infty \leq \alpha(0)/\sigma \,, \qquad \textbf{(30)}$$

where the relations are all either equalities or strict inequalities. If in **(30)** we have equalities we will say that the equipment is ***scraping strongly replaceable,*** in particular $T_\infty = 0$.

Drawing on the above we conclude the following:

Proposition 2 [Ŧ]

With $C_0 = 0$ *a profitable equipment is either* ***scraping finite*** $N-$***replaceable*** *for some* $N \geq 0$*, or else it is* ***scraping strongly replaceable****. Also:*

(i)*. If* $\alpha(0) = \min\{\alpha(T)\}$*, then the equipment is* ***scraping durable:*** $T_0 = \infty$***,*** *and* ***scraping non-replaceable:*** $N = 0$***.***

(ii) *If* $\alpha(0) = \max\{\alpha(T)\}$*, then the equipment is* ***scraping strongly replaceable:*** $T_\infty = 0$.

Independently of the above limiting procedure, steady state solutions can be studied also directly. One such solution that has received overwhelming attention in the literature provides for infinite replacements at equally spaced time intervals.[11] In particular, if this uniform duration is T, the *steady state total profit* would be given by:

$$\Pi^*(T) = \sum_{v=0}^{\infty} e^{-v\sigma T} A(T) = \frac{A(T)}{1-e^{-\sigma T}} \,, \qquad \textbf{(31)}$$

where for each given T, the $T-$***optimal steady state solution*** is the same as the $T-$***optimal solution*** since it involves maximizing $A(T)$ for fixed T. We define the *steady state optimal duration* T^* as the first local maximum of $\Pi^*(T)$ and denote the corresponding *steady state maximal profit by* Π^*. Assuming that the equipment is initially profitable, we will say that it is ***steady state: Durable*** *if* $T^* = \infty$, and ***Replaceable*** *if* $0 \le T^* < \infty$. In the latter case it will be called ***steady state: Weakly replaceable*** *if* $0 < T^* < \infty$, and ***Strongly replaceable*** *if* $T^* = 0$.

In order to characterize the steady state optimal solutions we take the time derivative of the *steady state profit function*:

$$\Pi^{*\prime}(T) = \frac{A'(T)[1-e^{-\sigma T}] - \sigma A(T)e^{-\sigma T}}{[1-e^{-\sigma T}]^2} = \frac{e^{-\sigma T}}{1-e^{-\sigma T}}[\alpha(T) - \sigma\Pi^*(T)] \qquad \textbf{(32)}$$

The *stationary steady state duration* will be defined as the smallest solution $T > 0$ of the stationarity equation:

$$\Pi^{*\prime}(T) = 0 \Rightarrow \alpha(T) = \sigma\Pi^*(T) \Rightarrow H(T) = \frac{\sigma[Q(T) - C(K_T)]}{1-e^{-\sigma T}}. \qquad \textbf{(33)}$$

For small T the properties of the function $\Pi^*(T)$ depend very strongly on the value of C_0. So continuing with the assumption $C_0 = 0$ we observe from **(32)** that $\Pi^*(T)$ is increasing when $\alpha(T)$ is higher, and decreasing when it is lower. In particular a local extremum of $\Pi^*(T)$ is preceded by a similar local extremum of $\alpha(T)$. Also, in view of the assumption $C_0 = 0$ we have $A(0) = -C_0 = 0$ and by applying L' Hopital rule we find:[12]

$$\begin{aligned} &\Pi(0) = \alpha(0)/\sigma, \quad where \ \ \alpha(0) = H(0) - \sigma S_0 \qquad (i) \\ &\Pi'(0) = \alpha'(0)/2\sigma, \ where \ \ \alpha'(0) = H'(0) + \sigma\lambda_0 w_0. \quad (ii) \end{aligned} \qquad \textbf{(34)}$$

Taking into account these relations we distinguish basically three types of equipment as shown in Figure 1 below. We conclude the following:

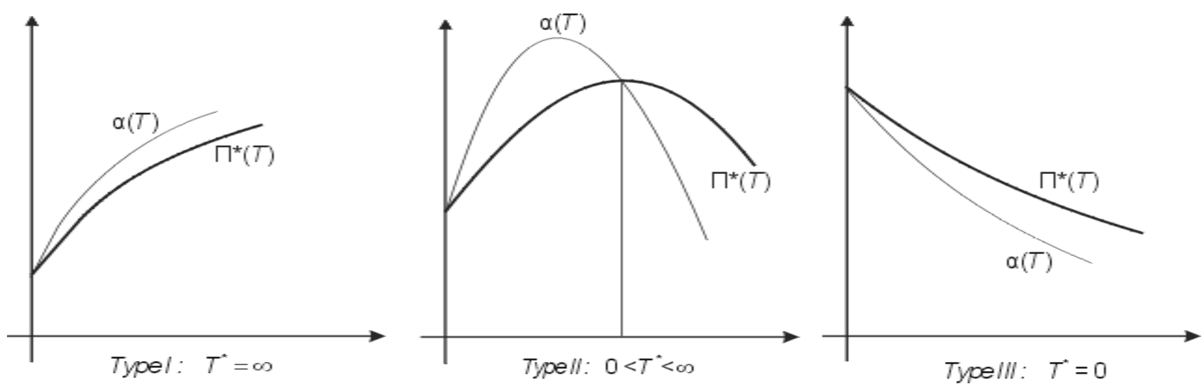

Figure 1

Proposition 3 *Ŧ*

With $C_0 = 0$, and assuming initial profitability: $\alpha(0) > 0$, we distinguish the following types of equipment:

Type I. *If $\alpha'(0) > 0 \Rightarrow H'(0) + \sigma\lambda_0 w_0 > 0$, and the steady state stationary duration does not exist, then the equipment is* ***steady state durable.*** *It is also* ***scraping durable*** *and* ***scraping non-replaceable****, with $T^* = T_0 = \infty$, $\Pi^* = \Pi_0 = A(\infty)$, $N = 0$.*

Type II. *If $\alpha'(0) > 0 \Rightarrow H'(0) + \sigma\lambda_0 w_0 > 0$, and the steady state stationary duration exists, then in general it is equal to the steady state optimal duration, and the equipment is* ***steady state weakly replaceable****. It is also* ***scraping finite $N-$replaceable****, satisfying: $0 < \tau^* < T^* < T_N$ & $\alpha(0)/\sigma < \{\Pi^*, \Pi_N\} < \alpha^*/\sigma$, where τ^* is the first local maximum of $\alpha(T)$ and $\alpha^* = \max\{\alpha(T)\}$ is the global maximum of $\alpha(T)$. We may have $N = 0$, and/or $T_0 = \infty$*

Type III. *If $\alpha'(0) < 0 \Rightarrow H'(0) + \sigma\lambda_0 w_0 < 0$ then the equipment is* ***steady state strongly replaceable.*** *It is also* ***scraping strongly replaceable****, with $T^* = T_\infty = 0$ and $\Pi^* = \Pi_\infty = \alpha(0)/\sigma$.*

Remark 4

The types of equipment defined above are ***local*** *types, in the sense that they depend on the operating properties of the equipment only up to the first maximum of Π_j or Π^*, respectively. In other words, we assume an* ***impatience*** *or* ***caution*** *on the part of the owner the moment he sees his profits falling. However, if $\alpha(T)$ has these characteristics throughout, i.e. if it is* ***quasiconcave*** *being monotonous or having at most two monotonous pieces, then these properties will be global, and we will have* ***global*** *types.*

Finally, turning to the case $C_0 > 0$, in the steady state policy we can separate the part of the profit containing the transaction cost term as follows:

$$\Pi^*(T) = [\frac{A(T) + C_0}{1 - e^{-\sigma T}}] - [\frac{C_0}{1 - e^{-\sigma T}}] = V\Pi^*(T) - C^*(T) \qquad \textbf{(35)}$$

The variable part $V\Pi^*$ corresponds to the case $C_0 = 0$. The correction $C^*(T)$ given by the fixed cost part is monotonically decreasing in T, from $C^*(0) = \infty$ to $C^*(\infty) = 0$. Since $\alpha(T)$ remains the same we find that as C_0 increases the steady state duration increases and the steady state profit falls, as indicated by the broken lines in the diagrams in Figure 2. Equipment of ***Type I*** is not affected except for the reduction in profit, whereas the effect regarding finite replacement policies is similar.

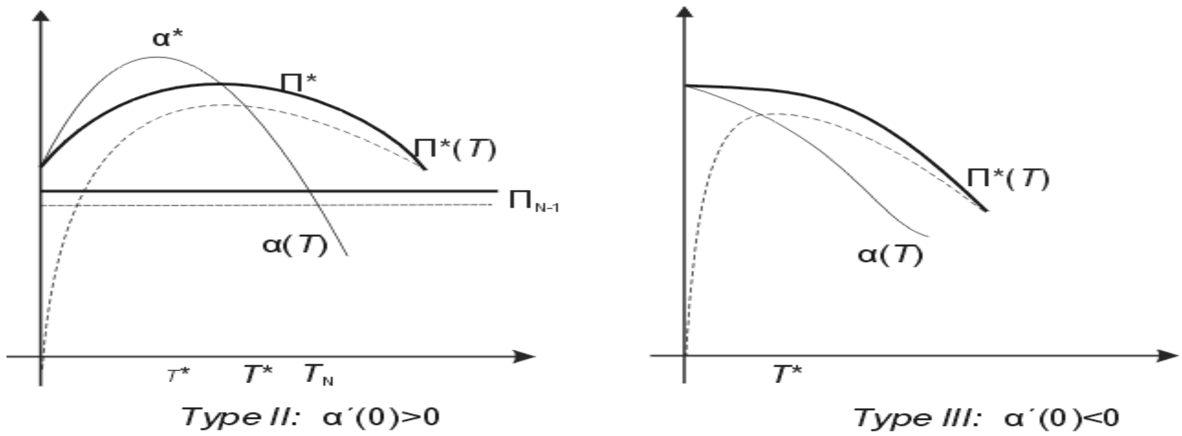

Figure 2

Proposition 4 ŧ

As C_0 increases then all durations increase, except for the scraping duration T_0, which remains invariant. Also all profits decrease, as follows:

(i). *For steady state policy:* $d\Pi^* = -dC^*$, $dT^* = \sigma dC^* / \alpha'(T^*)$, *where*

$$dC^* = dC_0 / (1 - e^{-\sigma T^*}) = dC_0[1 + e^{-\sigma T^*} + e^{-2\sigma T^*} + \cdots].$$

In particular if it is ***steady state strongly replaceable****:* $T^* = 0$, $dC_0 = C_0$, *then*

$$d\Pi^* = -\sqrt{-2C_0\alpha'(0)} / \sigma, \; T^* = dT^* = \sqrt{-2C_0 / \alpha'(0)} \quad \textbf{(36)}$$

(ii)*. For finite replacement policy:*

$$d\Pi_j = -dC_j, \; dT_j = \sigma dC_j / \alpha'(T_j), \quad \textbf{(37)}$$

where $dC_j = dC_0 + e^{-\sigma T_j} dC_{j-1} = dC_0[1 + e^{-\sigma T_j} + \cdots + e^{-\sigma(T_j + \cdots + T_1)}]$

Remark 5

The reduction of the profit is more pronounced in the steady state policy. However the effect on the duration may be counterbalanced by the size of $\alpha'(T)$. We note that by their definition we have: $\alpha'(T^*) < 0$, $\alpha'(T_j) < 0$. Also since $\alpha(T)$ is not affected by C_0, in the finite replacement policy the profit reduction may cause an increase in the number of replacements if C_0 is sufficiently large. Finally we note that if the equipment is strongly replaceable then the replacements with terminal scraping will stop when they cease being profitable: $\Pi_{N-1} < \Pi_N$.

III. Implications for economic theory and policy

Our analysis showed that the multi-period replacement model yields several *transitory* and *limiting replacement policies* that have been ignored in the literature. So the question is why they may be useful. The best way to answer it is by reference to a simple example.

1. Example

Consider a problem in which the various functions involved take the forms:[13]

$$r = qK^{\varepsilon}, \quad w = sK, \quad S = pK, \quad P_0 = pK_0, \quad C_0 = P_0 - S_0 = 0 . \tag{38}$$

The solutions depend crucially on the properties of the operating rate functions: $q = q(u,m)$ and $s = s(u,m)$. In particular, they depend on how flexible is the equipment in allowing varying *utilization* and *maintenance* policies, and on whether it allows *upgrading* policies of the investment or antiques type: $s < 0$. We will examine only the simplest case of equipment, which is:

1. ***Wear regular,*** in the sense that it allows only *downgrading* operating policies: $s > 0$,
2. ***Very rigid***, in the sense that it affords only *fixed* operating policies with constant q *and* s, and
3. ***Characterized by decreasing returns to scale,*** in the sense that $\varepsilon < 1$. Thus while the scrap value decreases at the rate s, the revenue declines at the smaller rate εs.

From the above we compute the relevant quantities as follows:

$$K = K_0 e^{-st}, \quad r = r_0 e^{-\varepsilon st}, \quad Q = \frac{r_0}{\varepsilon s + \sigma}[1 - e^{-(\varepsilon s+\sigma)t}], \quad S = S_0 e^{-st} \qquad (i)$$

$$A(T) = \frac{r_0}{\varepsilon s + \sigma}[1 - e^{-(\varepsilon s+\sigma)T}] + S_0 e^{-(s+\sigma)T} - P_0 \;\Rightarrow A(\infty) = \frac{r_0}{\varepsilon s + \sigma} - P_0 \qquad (ii)$$

$$\alpha(T) = e^{\sigma T} A'(T) = r_0 e^{-\varepsilon sT} - (s+\sigma)\, S_0 e^{-sT} \quad \Rightarrow \alpha(0) = r_0 - (s+\sigma)\, S_0 \qquad (iii)$$

$$\alpha'(T) = -s[\varepsilon r_0 e^{-\varepsilon sT} - (s+\sigma)\, S_0 e^{-st}] \qquad \Rightarrow \alpha'(0) = -s[\varepsilon r_0 - (s+\sigma)\, S_0]. \quad (iv)$$

(39)

Depending on the parameter values we have two possibilities: one for $\alpha'(0) < 0$ and another for $\alpha'(0) > 0$. The profit rate functions to which they correspond are exhibited in Figure 3 below**.** Thus, assuming that $\alpha(0) > 0 : r_0 > (s + \sigma_0) S_0 \Rightarrow \sigma < r_0 / S_0 - s$, we find that the equipment is:

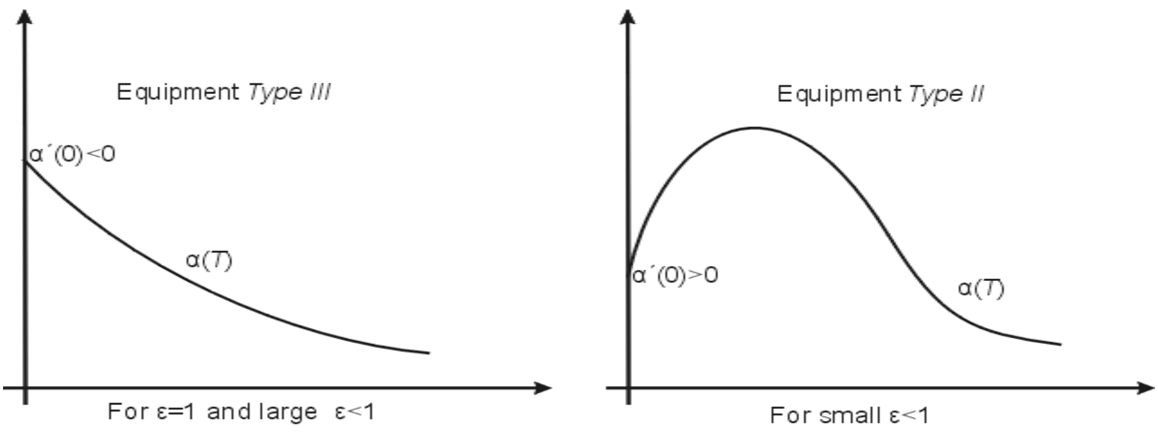

Figure 3

1. ***Scraping durable*** *but* ***steady state replaceable,*** *allways*
2. ***Scraping replaceable*** *if :*
   $\alpha(0) > \sigma\Pi_0 : \varepsilon r_0 > (\varepsilon s + \sigma)S_0 \Rightarrow \sigma < \varepsilon(r_0 / S_0 - s),$ *with*
   $T_1 : \alpha(T) = \sigma\Pi_0 \Rightarrow r_0 e^{-\varepsilon sT} - S_0(s+\sigma)e^{-sT} = \sigma[\frac{r_0}{\varepsilon s + \sigma} - P_0],$
3. ***Steady state weakly replaceable****, and* ***scraping finite N – replaceable*** *if :* **(40)**
   $\alpha'(0) > 0 : \varepsilon r_0 < (s+\sigma)S_0 \Rightarrow \sigma > \varepsilon r_0 / S_0 - s,$ *with*
   $\tau^* : \alpha'(T) = 0 \Rightarrow \tau^* = \frac{1}{(1-\varepsilon)s} \ln \frac{\varepsilon r_0}{(s+\sigma)S_0} < T^* < T_N,$
4. ***Steady state strongly replaceable*** *and* ***scraping strongly replaceable*** *if :*
   $\alpha'(0) < 0 : \varepsilon r_0 > (s+\sigma)S_0 \Rightarrow \sigma < \varepsilon r_0 / S_0 - s.$

2. Nature of the solution

Let $\alpha(0) < 0 \Rightarrow r_0 < (s+\sigma)S_0$. By ***Proposition 1*** the equipment would be ***initially*** and in this case always ***unprofitable***. Consequently, if it is in use its owner ought to scrap it immediately. In this event we would observe scraping without corresponding replacement and the capital stock of the firm or the economy modeled would decline. If the equipment is not in use, it would not be worth purchasing unless some changes in the parameters reversed the inequality, thus rendering the equipment ***initially profitable***, $\alpha(0) > 0$.

Assuming the latter to be the case, then under *steady state replacement* policy we have two options. If $\alpha'(0) < 0$, the owner would be expected to set $T^* = 0$ and treat his equipment as ***steady state strongly replaceable*** (Equipment *Type III* in Figure 3 and expression **(40.4)**). In fact the same solution would be adopted under the terminal scraping policy. Otherwise, if $\alpha'(0) > 0$*,* he would be expected to treat his equipment as ***steady state weakly replaceable*** (Equipment *Type II* in Figure 3 and expression **(40.3)**)*,* and set the duration of replacements equal to T^*. However in this case the scraping solution gives alternative replacement policies. Thus if $\alpha(0) < \sigma\Pi_0$ then by **(40.2)** the appropriate policy is that of treating the equipment as ***scraping durable*** and ***non-replaceable.*** Otherwise, if $\alpha(0) > \sigma\Pi_0$ then the equipment would be treated as ***scraping durable*** and ***replaceable*** with T_1 given by relation **(40.2).** The number of profitable replacements increases as σ decreases. From this we observe that, depending on the particulars of the case under consideration, the policies that emerge under terminal scraping policy may be mixed in the sense that they consist of two phases. A *transitory* one, during which the owner switches from less to more profitable replacement policies, and another *terminal*, during which the owner applies the same policy to the end of the service life of his equipment.

Thus concerning solution **(40.3)** the owner has two choices. Namely, he can apply either *steady state replacements* with durations set equal to T^* or he can apply $N-$*replacements* with service lives set equal to the sequence of T_j. The question is which one he might choose. The answer depends on the preferences of the owner. If he prefers *profitability* to *flexibility* he will choose *steady state replacements* because these give the highest profit. Otherwise, he will choose $N-$*replacements.* In any case the steady state policy binds him to undertaking an infinite number of replacements with small profit in each replacement period which in fact is realized only at the end of each period. On the contrary the terminal scraping policy in general allows him to realize almost the same total profit with few or even without any replacements, as confirmed by the following two numerical examples:

2.1 Numerical solutions

Case 1

$$\{q = 0.2,\ s = 0.1,\ \sigma = 0.06,\ \varepsilon = 0.4\}$$
$$\Rightarrow \alpha(0)/\sigma = 0.67,\ \ \alpha^*/\sigma = 1.25,\ \tau^* = 8$$

$T^*,\ \Pi^*$	$T_0,\ \Pi_0$	$T_1,\ \Pi_1$	$T_2,\ \Pi_2$	$T_3,\ \Pi_3$
21, 1.1	∞, 1	25, 1.1	22, 1.1	21, 1.1

$$\Rightarrow N = 0,\ \ \Pi_0 = 1$$

Case 2

$$\{q = 0.2,\ s = 0.1,\ \sigma = 0.03,\ \varepsilon = 0.4\}$$
$$\Rightarrow \alpha(0)/\sigma = 2.33,\ \ \alpha^*/\sigma = 2.89,\ \ \tau^* = 11.5$$

$T^*,\ \Pi^*$	$T_0,\ \Pi_0$	$T_1,\ \Pi_1$	$T_2,\ \Pi_2$	$T_3,\ \Pi_3$
14, 2.8	∞, 1.8	29, 2.3	22, 2.5	19, 2.6

$$\Rightarrow N = 2,\ \ \Pi_2 = 2.5$$

In the first calculation the steady state policy total profit $\Pi^* = 1.1$ is realized with replacement period $T^* = 21$, while the scraping policy total profit $\Pi_0 = 1$ is realized without any replacements. The equipment is ***scraping durable non-replaceable*** because the scraping period profit $\Pi_0 = 1$ is higher than the critical value $\Rightarrow \alpha(0)/\sigma = 0.67$. In fact, by one more replacement with period $T_1 = 25$, which, however, is not initially profitable, we recover most of the remaining profit $1.1 - 1 = 0.1$.

In the second calculation we half the discount rate and we note that the same equipment becomes ***scraping durable*** $2-$***replaceable***. Now the steady state policy total profit $\Pi^* = 2.8$ is realized with replacement period $T^* = 14$, while the scraping policy total profit $\Pi_2 = 2.5$ is realized with two replacements of duration $T_2 = 29$ and $T_1 = 22$. The 3d replacement is not initially profitable.

3. Implications

Drawing on the above, we may proceed now to demonstrate the usefulness of the proposed rehabilitation of the scraping replacement model. To this effect, assume that the economy under consideration is in long-term equilibrium with capital stock *K**. If the effective discount rate σ increases so that the equilibrium capital stock declines to *K***, the question that would arise is how *K** might adjust to the lower *K***? Under the standard model of *steady state replacements*, where the units of capital are replaced in equal time intervals, the economy would adjust by prolonging the duration of replacement periods. This would occur because the representative firm would find it profitable, on the hand, to postpone replacement at the higher interest cost, and on the other, to apply operating policies with less wear of the equipment. Consequently, along this slow adjustment path we would observe more worn-out capital remaining in operation. On the contrary, under the proposed rehabilitation of the economic replacement model, the representative firm would switch immediately to a pure scraping policy, i.e. scraping without replacement, and at the same time it would stretch out the service life of capital by following suitable utilization and maintenance policies. As a result, this path would be faster and the capital remaining in operation would be less worn-out.

In addition the newly identified replacement and scraping policies raise some questions concerning the validity of research efforts that are based on the assumption of *equidistant steady state replacements*. To highlight the source of the problem, suppose that the equipment under consideration is of *Type II* and that its owner prefers *some flexibility* to *maximum profitability*. Following **(40.3)** he would opt to treat it as ***scraping***

finite N - replaceable by setting its service life equal to $T_N < T_{N-1} < \cdots < T_0$. In standard practice though researchers assume that economic agents decide invariably as profit maximizers by committing to an infinite horizon. Accordingly the owner would be expected to treat his equipment as ***steady state weakly replaceable***, thus equating its service life to T^*. But then, given from ***Proposition 3*** that $T^* < T_N$, the service life of equipment will be consistently underestimated. For this reason the results from past studies of replacement and scrappage should be interpreted with caution as they may involve upward biases of unknown dimensions.

Another feature distinguishing the two policies is the following. As the parameters change the replacement duration also changes. In the steady state policy this change is smooth throughout except at the non-profitability threshold where all activity ceases. However in the case of scraping policy except for this smooth change we have also sudden changes when the parameters cross certain bifurcation values where an additional replacement policy becomes profitable or ceases to be so as indicated by **(40.2)** and **(40.3).** Thus at some changes of parameter values we would observe a burst or a slump in the demand for replacement investment much like the "spikes" discovered in recent years by researchers studying investment at the plant level.

To highlight a particular instance of this feature, assume that the owner of equipment applies the policy of $N = 1 -$ *replacement with scraping.* Also, let σ decline in unison with the interest rate. How would the demand for replacement investment be affected? Taking the derivative from **(40.2)** let the interest rate be such that $\partial T_1 / \partial \sigma < 0$. From this it turns out that a small decline in the interest rate would be expected to increase T_1 and hence to decrease the demand for replacement investment. But if the decline in σ were large so that $\partial T_1 / \partial \sigma > 0$, the owner might be lured to switch to, say, a policy of $N = 2 -$ *replacements with scraping.* Then he would apply service life T_2, which by ***Proposition 1*** is sharply lower than T_1. Thus a large decline in the interest rate would be expected to increase the demand for replacement investment. Consequently, by exploiting the asymmetry of these two effects we may be able to discriminate among regular periods and exceptional ones that coincide with switches in replacement policies.

Still another notable feature emanates from the structure of *replacement* and *scraping* policies themselves. Looking for example at **(40.2)** we see that the policy of $N = 1 -$ *replacement with scraping* is expressed in terms of such variables as: the rates of interest and technological uncertainty, the rates of utilization and maintenance, and the prices of new and used equipment. In addition, under more general specifications of the functions r, w, S, P_0, and C_0 we could allow for product and input prices, the nature of safety constraints, disposal costs, etc. But these are exactly the determinants of short-term variation that a researcher would need for the study of *replacement* investment and *scrappage* over the business cycle. Therefore, by confronting the criticism levelled by Feldstein and Foot (1971, p.50) that: "…previous studies of optimal steady state service life were of limited use because they lacked certain crucial determinants of short-term variation of replacement," the proposed rehabilitation of the established replacement model gains significantly in empirical relevance.

Next, consider the implications regarding second-hand markets. The literature dealing with the questions why such markets exist and what factors determine their breath and depth has developed along two strands. More specifically, the one (e.g. Bond (1983)) stresses the heterogeneity of firms in terms of their input prices and various operating conditions, whereas the other emphasizes the presence of transaction costs (e.g. Sandfort (1999)). In the note in Appendix B we establish that both these strands can be analyzed in a unified framework and in the process gain valuable insights as to the relative magnitude and direction of influences exercised by the determinants of replacement durations. For as we argue, the shorter the replacement durations are, the more intensive the economic activity that takes place in both the primary and the secondary equipment markets.

Finally, the proposed rehabilitation of the established replacement model provides a framework of analysis, which is amenable to several generalizations and extensions. For example, one would be to allow explicitly for the structure of the market in which the services of the equipment are sold so as to trace the impact of output prices on service life decisions. Another would be to introduce investment and thus turn the model into a general framework for the analysis of all real capital related decisions. Still another would be to address the safety of equipment and other regulatory concerns. In particular, depending on the objectives the respective authorities pursue, this could be accomplished by making the parameter J in ***Remark 3.(i)*** a function of, say, the previous accident records, or the pollutants emitted in the environment.[14] Moreover, while it would be a rather straightforward task to incorporate taxes and tax incentives, the specification of **(3)** as a stochastic differential equation would be more demanding.

IV. Summary and conclusions

Despite the advances made during the last six decades, the quest for a sufficiently general theory of replacement investment remains unabated. So, in order to contribute towards this objective, drawing on Preinreich's (1940) theorem, we proposed an improved specification of the established multi-period replacement model and solved it for the owner's profit horizon. By virtue of this new conceptualization, apart from traditional *steady state replacement*, we were able to derive a whole range of *transitory replacement policies*.

Since these policies have been ignored in the literature, we then proceeded to highlight their implications for economic theory and policy by reference to a simple example. From this analysis we obtained the following main results. The replacement policies reported in the paper explain the transition from any initial position of a firm or an economy up to *steady state replacement.* Switching among replacement policies produces bursts or slumps in replacement investment much like the "spikes" discovered recently at the plant level. Applying *steady state replacement* in non-stationary economic environments leads to underestimation of service lives, and hence to overestimation of replacement demand for investment. The optimal service life of equipment is consistently determined by such conventional forces of short-term variation as utilization, maintenance, interest rate, uncertainty due to technological breakthroughs, the price of new and used equipment, transaction costs, etc. And last, but not least, the estimates from econometric studies of *replacement* and *scrappage*, particularly when they use time series data covering periods with unknown switches in replacement policies, may be seriously biased.

In addition, we highlighted the reasons for the existence of second-hand markets for equipment and suggested several generalizations and extensions of the model. But before embarking on any of these endeavors, the immediate task is to analyze *utilization* and *maintenance* policies. To this we shall turn in our next paper.

Appendix A

Proofs

Proof of *of equation (34)*:

$$\alpha(T) = H(T) - \sigma S(K_T),\ \ \alpha'(T) = H'(T) - \sigma S'(K_T)\dot{K}_T = H'(T) + \sigma\lambda(T)w(T)$$

1. $\Pi^*(0) = \dfrac{0}{0} \sim \dfrac{A'(T)}{(1-e^{-\sigma T})'}\Big|_{T=0} = \dfrac{\alpha(0)}{\sigma}$

2. $\Pi^{*\prime}(0) = \dfrac{\alpha'(T) - \sigma\Pi^{*\prime}(T)}{(1-e^{-\sigma T})'}\Big|_{T=0} = \dfrac{\alpha'(0) - \sigma\Pi^{*\prime}(0)}{\sigma} \quad \Rightarrow \quad \Pi^{*\prime}(0) = \dfrac{\alpha'(0)}{2\sigma}$

Proof of *Proposition 2*:

Assume $T_j \downarrow T_\infty > 0$. Then

$A_j \downarrow A_\infty = A(T_\infty),\ \alpha(T_j) \uparrow \alpha_\infty = \alpha(T_\infty),\ \Pi_j \uparrow \Pi_\infty = A_\infty + e^{-\sigma T_\infty}\Pi_\infty$, *with* $\alpha_\infty = \sigma\Pi_\infty$.

From the definition of the optimal durations it follows that $\alpha(T)$ can not have strictly smaller values to the left of T_∞, i.e. we have: $\alpha(T) \ge \alpha(T_\infty)$ for $T \le T_\infty$. Hence:

$$A_\infty = \int_0^{T_\infty} A'(T)dT = \int_0^{T_\infty} e^{-\sigma T}\alpha(T)dT \ge \alpha_\infty(1-e^{\sigma T_\infty})/\sigma,$$

with equality only if $\alpha(T)$ is constant for $T \le T_\infty$. Substituting we obtain:

$$\Pi_\infty = A_\infty + e^{-\sigma T_\infty}\Pi_\infty > \alpha_\infty(1-e^{\sigma T_\infty})/\sigma + e^{-\sigma T_\infty}\alpha_\infty/\sigma \quad \Rightarrow \Pi_\infty > \alpha_\infty/\sigma$$

a contradiction, unless $\alpha(T)$ is constant for $T \le T_\infty$ in which case we can apply any smaller replacement duration without affecting the total profit, i.e. we have again $T_j \to 0$.

(i). We have $A'(T) = e^{-\sigma T}\alpha(T) \ge e^{-\sigma T}\alpha(0) > 0$ and therefore the equipment is ***scraping durable***. Also the assumption implies $\Pi_0 = A(\infty) \ge \alpha(0)/\sigma$, with equality only if $\alpha(T)$ is constant in which case the replacement durations are indeterminate without increasing the total profit, and hence the equipment can again be considered as ***non-replaceable***.

(ii). As above the assumption gives $\Pi_j \le \alpha(0)/\sigma$, with equality only if we have $\alpha(T) = \alpha(0)$ for $T \le T_j$. As above we conclude that all replacements are profitable, with $T_j \to 0$.

Proof of *Proposition 3*:

(I). If $\Pi^*(T)$ is strictly increasing then so is $A(T) = (1-e^{-\sigma T})\Pi^*(T)$. Hence it is ***scraping durable***. Also by **(37)** we will have $\alpha(T) > \sigma\Pi^*(T) > \sigma\Pi^*(0) = \alpha(0)$, and by ***Proposition 2*** it is ***scraping durable non-replaceable***.

(II). The equipment is not ***strongly scraping replaceable*** because by assumption

$\alpha(T)$ is initially rising and hence $\alpha(0)$ can not be approached by lower values from the right. Hence it is finite $N-$**replaceable** for some $N \geq 0$. Also $\alpha(T)$ is higher then $\sigma\Pi^*(T)$ until T^*, because in this interval $\sigma\Pi^*(T)$ is rising. Also we have $\sigma\Pi_{N-1} < \alpha(0) < \sigma\Pi^*$, and therefore $\alpha(T)$ meets first the level $\sigma\Pi^*$ and later on the level $\sigma\Pi_{N-1}$. Hence, we have $T^* < T_N$. Also $\alpha(T)$ has a local maximum at some τ^* before T^*, because it is initially rising but then falling when it crosses the level $\sigma\Pi^*$. Hence $\tau^* < T^*$ and $\sigma\Pi^* < \alpha^*$. Concerning Π_N, we have:

$$\Pi_N{}'(T) = e^{-\sigma T}[\alpha(T) - \sigma\Pi_{N-1}] \leq e^{-\sigma T}(\alpha^* - \sigma\Pi_{N-1})$$

$$\Rightarrow \Pi_N \leq \Pi_N(0) + \int_0^{T_N} e^{-\sigma T}(\alpha^* - \sigma\Pi_{N-1})dT = A(0) + \Pi_{N-1} + (1 - e^{-\sigma T_N})(\alpha^* - \sigma\Pi_{N-1})/\sigma$$

$$= \alpha^*/\sigma - e^{-\sigma T_N}(\alpha^* - \Pi_{N-1})/\sigma < \alpha^*/\sigma$$

(III). It follows from the definition.

<u>Proof of *Proposition 4*</u>:

Concerning the profits it is a direct consequence of the ***envelope theorem*** applied on the functions:

$$\Pi^*(T) = A(T)/(1 - e^{-\sigma T}),\ \Pi_j(T) = A(T) + e^{-\sigma T_j}\Pi_{j-1}.$$

With regard to the durations we use the relations:

$$\alpha(T) = \sigma\Pi^*(T),\quad \alpha(T) = \sigma\Pi_j.$$

For the case of strong replaceability, we can use the linear approximation at $T^* = 0$:

$$d\alpha(0) = \sigma[dV\Pi^*(0) - dC^*] \Rightarrow \alpha'(0)T^* = \frac{1}{2}\alpha'(0)T^* - \frac{C_0}{T^*} \Rightarrow T^* = \sqrt{-2C_0/\alpha'(0)}.$$

Appendix B

A note on second-hand markets

Using a simple example in which equipment was assumed to be ***wear regular*** and ***totally rigid*** with ***decreasing returns to scale***, we investigated two types of replacement policies:

A. *Infinite steady state replacements of equal durations*, and

B. *Finite replacements of varying durations with terminal scraping*.

Replacement durations T were analyzed with respect to the following fixed parameters: r : net revenue flow, *s*: rate of change of scrap value, ε : coefficient of revenue change to scrap value change, σ : effective discount rate, S *:* scrap or second hand price, P_0: purchase price of new equipment, and C_0: owner's fixed cost or second hand discount benefit.

We can associate the scrap price with a second-hand or secondary market, where of course the conditions of operating the equipment are different than those pertaining in the new equipment or primary market. Short replacement durations indicate strong primary and secondary market activity. In the simple model presented in the text, we distinguished for both replacement policies ***two regimes***, which are characterized by the following relations:

1. $\sigma > \sigma_0 = \dfrac{r_0}{S_0} - s$: *high discount, low net revenue/scrap price, high wear,*

2. $\sigma < \sigma_0 = \dfrac{r_0}{S_0} - s$: *low discount, high net revenue/scrap price, low wear,* **(B.1)**

where σ_0 stands for a threshold effective discount rate defining the switching value of profitability.

If $\sigma > \sigma_0$, the equipment is not profitable, and hence, we have economic activity neither in the primary nor in the secondary equipment market. On the contrary, if $\sigma < \sigma_0$, we observe economic activity. We note now the following distinctive difference between the two types of policies. In the steady state policy model both markets, primary and secondary appear simultaneously. On the contrary in the case of scraping policy model and for the case examined, at first we have only primary activity since the equipment becomes ***scraping durable non-replaceable***. However, as the interest rate declines, there appears another interest threshold:

$$\sigma_1 = \varepsilon(r_0 / S_0 - s), \qquad \textbf{(B.2)}$$

below which the equipment becomes ***replaceable***, indicating the emergence of second hand activity. In fact, there follows a sequence of such interest ***bifurcation values*** $\{\sigma_2 > \sigma_3 > \cdots\}$ leading to abrupt changes in the replacement durations and a consequent sharp increase in primary and secondary market activity.

Finally, we should like to point out that the above results could be generalized by altering the specification of the functions involved. For example, by setting $s < 0$ we can analyze the case of antiques or other special equipment that upgrades in time. By setting $r = qK^{\varepsilon}$ for $\varepsilon \geq 1$ in addition to ***decreasing returns to scale*** $(\varepsilon < 1)$**,** we can analyze the cases of ***constant*** or ***increasing returns to scale*** $(\varepsilon \geq 1)$. And, of course, by allowing the operating policies u and m to vary we can analyze ***more flexible*** types of equipment than the one considered in the example in the text.

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Endnotes

[*] The Centre for Economic Research of the Athens University of Economics and Business Science, the Greek Ministry of Science and Technology, and several business concerns supported our research from which the present paper derives. To all of them, as well as to Omar Licandro, Giuseppe Bertola and the participants in the Macroeconomics Seminar at the European University Institute, we extend our sincere appreciation.

[1] Where the state-of-the-art was back in the late 1950's may be easily ascertained from Dean's (1962) extensive survey of the literature.

[2] In fairness to historical accuracy it should be stressed that the first researcher who reported this finding was O'Herlihy (1965). Nevertheless, as his results had appeared in a rather obscure journal, they did not get the attention they deserved.

[3] It may be of some interest to note that this paper is the same as the one that appeared in 1974 under the same title in *Econometrica*, Vol. 42, pp. 393-423.

[4] As well as the work of a few isolated researchers, such as, for example, Eilon, King and Hutchinson (1966), who, defying the new trend, continued to contribute in the earlier tradition.

[5] It may be of some interest to mention that during the same period Jorgenson, McCall, and Radner (1967) contributed significantly in the earlier tradition. They did so by helping us understand how the replacement decision is related to the processes of *regular maintenance,* as prescribed in the service manuals that accompany all durables, and *preventive periodic* and *opportunistic maintenance,* in order to forestall the costs of a serious, and sometimes catastrophic, breakdown.

[6] Admittedly in this paper we are making only a modest effort to include the impact of a second hand market on the replacement decision. But in the light of the significant contributions by Rust (1983) and Sandfort (1999) this issue cannot be ignored, as most of the literature has done so far.

[7] Dunne (1994), Abel and Eberly (1996), Caballero, Engel and Haltiwanger (1995), Cooper, Haltiwanger and Power (1999), and others, have discovered in recent years that "spiked" patterns of investment at the firm level are more likely to occur the older is the existing capital stock. Overhauling of plants may be as liable for this finding as technological progress.

[8] For an alternative approach to modelling the implications of utilization in the replacement of durables, see Jin and Kite-Powell (1999).

[9] To drive a wedge between new and used equipment prices that manufacturers offer to initial buyers of their equipment various extra enticements. Such enticements take the form of warranties, guarantees, free service and repairs for so many years, free optional accessories, low or even zero interest loans, etc.

[10] Clearly, this is an extreme assumption because in the great majority of actual cases technological breakthroughs do not render older equipment worthless at the time of their appearance. However, since modelling the case where the breakthrough reduces the earning power and the salvage value of the equipment by a certain percentage would be quite straightforward, we chose this specification in order to keep the analysis as simple as possible.

[11] For necessary and sufficient conditions establishing the optimality of *equidistant steady state replacements* in the presence of linear technological change, see Elton and Gruber (1976) and Van Halten (1991).

[12] The proof for the following equation as well as the propositions marked by the symbol **Ŧ** are furnished in Appendix A

[13] To drive a wedge between new and used equipment prices without further complications, the assump-

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tion that $p_0>p$ should be construed to reflect the extra enticements that manufacturers offer to initial buyers of their equipment. Such enticements take the form of warranties, guarantees, free service and repairs for so many years, free optional accessories, low or even zero interest loans, etc.

[14] Extensions along these lines would provide ample theoretical support for models testing the effectiveness of incentives to accelerate automobile scrappage rates for emissions control.