ON MONEY AS A SUBSTITUTE FOR PERFECT RECALL*

Ted Temzelides
Department of Economics
University of Iowa
Iowa City, IA 52242
tedt@blue.weeg.uiowa.edu

Jialin Yu
Department of Economics
Princeton University
Princeton, NJ 08544
jialinyu@princeton.edu

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PRELIMINARY - COMMENTS WELCOME

Abstract

In environments with no commitment and with a need for intertemporal trade, bounded recall is shown to be a sufficient friction for a receipt system (fiat money) to lead to improved allocations in an otherwise frictionless Walrasian model. The absence of other frictions makes price determination tractable, thus the model may be used for quantitative monetary policy experiments. Some issues regarding the divisibility of money are also discussed.

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I INTRODUCTION

Recent work by Kocherlakota (1998) identifies "lack of public memory" as a necessary friction for money to be essential in a variety of models. By essential we mean, as is standard, that monetary equilibria support allocations that are Pareto superior to the best supportable allocations in the absence of money. His study includes the overlapping generations and the search models of money.[1] These models typically involve a number of other frictions that are often considered necessary for money to be essential. For example, as it is well known since the work of Kiyotaki and Wright (1989), in addition to lack of public memory, the search model assumes bilateral meetings, random matching, and, of course, lack of commitment. While these frictions are sufficient for money to be essential in this model, there is no formal claim that each of them is indeed necessary. A similar statement can be made for other existing monetary models.

In this paper, we study the question of what is *a minimal* departure from the frictionless Walrasian model that provides an essential role for money in the sense defined above. Of course, one can depart from the Walrasian world in many different ways, and we shall not argue that such a minimal departure can be accomplished in a unique way. Nevertheless, we believe that the question is worth pursuing. From the theoretical point of view, identifying a minimal set of frictions that are sufficient for money to arise is important in order to understand what exactly money is a substitute for. From a more applied point of view, one would always like to dispose, if possible, of frictions that do not resemble what is going on in the actual economy, especially if this can lead to more tractable models.

Our analysis demonstrates that in environments where there is sufficient need for

[1] Standard references on these monetary models include Wallace (1980), Kiyotaki and Wright (1991), and Wallace (1997).

intertemporal trade and no commitment, incomplete recollection of past transactions, which we shall refer to as *imperfect recall,* is by itself *sufficient* for money to be essential. Furthermore, we argue that imperfect recall can be incorporated to the Walrasian paradigm. That lack of commitment is not inconsistent with competitive analysis has been used, for example, by Kehoe and Levine (1993), who define competitive equilibria and demonstrate versions of the welfare theorems in economies with individual rationality constraints.[2] Regarding record keeping, we adopt a benchmark that represents an opposite extreme from the standard Walrasian model. We assume that in any given period, agents or, alternatively, the Walrasian auctioneer, cannot recall past trades. Effectively, this leads to an additional individual rationality constraint in the standard sequential markets model: agents will never agree to trades that make them worse off than their endowment allocation during any given period. In Appendix 1 we discuss how this individual rationality constraint can be derived as a property of any equilibrium outcome of a game between agents with no recall of the past and their future selves.

Kehoe and Levine consider a world of limited liability in which participation constraints ensure that agents are at no time better off reverting to permanent autarky. In our setup, the lack of recall of past trades requires that agents are at no time better off reverting to autarky for that period. Of course, the interesting cases are the ones between the two extremes of perfect recall and complete lack of recall, and ultimately the degree of the incompleteness of record keeping can be formulated as a quantitative question.

We start our analysis by presenting an intertemporal competitive endowment economy in which repeated borrowing and lending is necessary for desirable allocations to

[2] Other, more recent, references include Alvarez and Jermann (1996) and Kocherlakota (1996). As the statement about the first welfare theorem indicates, this setup does not readily provide a role for money.

be supported. With no recall of past trades, however, the only reasonable outcome in this economy will imply no intertemporal trades. On the other hand, useless pieces of paper may be valued playing, in a sense, the role of receipts, indicating that certain transfers of goods took place in the past. For the purposes of our analysis, we restrict ourselves to the case where money consists of intrinsically useless and perfectly storable fiat objects. We show that such objects lead to improved allocations over the best allocations that can be supported in the absence of money; in other words, we show that money is essential. Finally, we discuss the welfare implications of the model from increased divisibility of money.[3]

One might wonder whether lack of commitment alone is sufficient for money to be essential in our setup. The answer, in general, is no. Under perfect recall, there exist simple mechanisms that guarantee "good behavior" by triggering a collective punishment, say, permanent autarky, if anyone has misbehaved in the past. Such punishments might work even if the identity of the deviator is not known. Imperfect recall limits the applicability of such punishments. It is worth mentioning that although here we restrict ourselves to a standard endowment competitive economy, our point is quite general. In the absence of commitment, in *any* setup in which information about the past is relevant in determining allocations, and in which recall is imperfect, the existence of fiat objects may lead to superior outcomes by revealing past trades; i.e., by acting as a receipt system. This would be true, for example, in a dynamic insurance economy like the one in Green (1987), or in a game theoretic setup, such as a repeated market game. One advantage of dealing with a competitive model is that the determination of prices is straightforward.

The paper proceeds as follows. Section II presents the environment without money. Section III studies competitive monetary equilibrium. Section IV deals with monetary injections. In Section V we introduce an extension of the basic model to perfectly

[3]See Kocherlakota (1999) for a discussion of this issue within a search model of money.

divisible money. Finally, Section VI concludes the paper. In Appendix 1 we deal with the derivation of the individual rationality constraint. Appendix 2 contains some proofs.

II THE ENVIRONMENT WITHOUT MONEY

Time is discrete and the horizon is infinite; $t = 1, 2, \ldots$. There is one non-storable, perfectly divisible good per period. We consider an endowment economy populated by a finite number, $i = 1, \ldots, I$, of types of infinitely lived agents. There is one agent (alternatively, a continuum of measure 1 of agents) per type.[4] Each agent receives a positive endowment of the consumption good in each period, e_t^i, and $\{e_t^i\}_t$ denotes agent i's infinite sequence of endowments. Let $u(c_t)$ denote the period utility function from consumption. We assume that u is smooth, monotonically increasing and strictly concave. Agents discount the future at rate $\beta \in (0, 1)$. There is no uncertainty.

If this were a standard competitive economy, Walrasian equilibrium could be characterized in a straightforward fashion. Of course, given that no frictions have been introduced thus far, money could play no welfare-improving role. The approach taken here is quite different. We assume that there is no commitment and that no past trades can be recalled, therefore, allocations in any given period cannot depend on past trades. To guarantee that past trades cannot be reconstructed, we assume that agents cannot recall the utility from their past consumption.

Assumption 1: *For all t, agents do not recall trades or utilities from periods $0, \ldots, t-1$.*

Needless to say, the assumption of no recall of the past is extreme and versions of all our results can be derived under less extreme versions of imperfect recall of the

[4]The basic results throughout the paper generalize to the case where there are many types and to the case where there are many goods.

past. As mentioned earlier, we shall adopt the no recall case as a benchmark.[5]

Remark 1 *Assumption 1 leads us to impose the constraint that* $u(c_t^i) \geq u(e_t^i)$, $\forall i$, $\forall t$.

In the absence of money, what would be a reasonable outcome in such a world? Let l_t^i stand for one-period lending (borrowing if negative) of agent i in period t, and let r_t be the net interest rate.[6] The choice problem of agent i is:

$$\left.\begin{array}{l} \max_{\{c_t^i, l_t^i\}} \sum_{t=1}^{\infty} (\beta^i)^{t-1} u^i(c_t^i) \\ \text{s.t. } c_t^i + l_t^i \square e_t^i + (1 + r_t) l_{t-1}^i, \ \forall t \\ u^i(c_t^i) \geq u^i(e_t^i), \ \forall t \end{array}\right\}. \tag{1}$$

A sequential markets equilibrium is an infinite sequence of period interest rates, lending, and consumptions such that given interest rates, the consumers' choices solve the above problem, and markets clear. Notice that the individual's problem is identical to the standard sequential markets setup, but with the additional individual rationality (IR) constraint that since there is no recall, no individual will lend in exchange for higher future consumption. While the IR constraint seems intuitive, it consists of a reduced way to capture the no recall assumption, and one would like to derive it from primitives. In Appendix 1, we construct a game that agents play against both other agents and their "future selves." In each period, agents choose their lending and their reports on past lending after having formed beliefs about future actions. We argue that all sequential equilibria of that game satisfy the IR constraint in problem (1).

[5] One could argue that some of the frictions in existing monetary models (say, random matching or turnpike assumptions (Townsend (1980)) are made in order to justify the lack of recall of past actions. Our response is that imperfect recall is neither logically implied by these frictions nor does it imply any of them. Using money in our model will be akin to the use of poker chips. Poker players use them to summarize the outcomes of past rounds.

[6] Restricting attention to one period lending is without loss of generality.

While in the case where there are many goods trades could take place within the period as in the standard Walrasian model, the absence of recall will prevent any borrowing or lending. In the case studied here, where there is only one consumption good per period, this will lead to autarky since no intertemporal trades will be realized. This is summarized in the following.

Proposition 1 *In the absence of money, there are no intertemporal trades in equilibrium.*

As mentioned earlier, the assumption of no recall defines one extreme. In general, we could assume that agents have perfect recall of the last T periods, where $T < t$. In that case, intertemporal trades that involve less than T period lending will be executed as in the standard Walrasian model. We introduce money next.

III COMPETITIVE MONETARY EQUILIBRIUM

Here we introduce money into the model. Partly in order to establish a connection between our findings to those of the random matching model, and partly because we view the degree of divisibility of money as an interesting policy variable, we shall assume that money consists of indivisible, perfectly storable fiat objects. Later, we study a version of the model with perfectly divisible money. For convenience, we study a special case with two (types of) agents, $I = \{1, 2\}$, and one consumption good per period.[7] To generate a need for intertemporal trade, we assume that each (type of) agent always receives a low endowment after a high endowment, and a high endowment after a low endowment. Let e_t^1, e_t^2 represent agent (type) 1's and 2's endowment in period t, respectively. Without loss of generality, suppose agent 1 starts out with a high endowment in period 1. When t is odd, $e_t^1 > e_{t+1}^1$ and $e_t^2 < e_{t+1}^2$,

[7]Our analysis easily extends to the case of many types, and the case of many goods under the assumption of CES preferences.

and when t is even, $e_t^1 < e_{t+1}^1$ and $e_t^2 > e_{t+1}^2$.

For concreteness, let the agents' endowment sequences be $e^1 = \{e_h, e_l, e_h, e_l, \dots\}$ and $e^2 = \{e_l, e_h, e_l, e_h, \dots\}$, with $e_h > e_l > 0$. In addition, assume that agent (type) 2 is endowed with one unit of intrinsically useless, perfectly storable, indivisible fiat object in period 1. Informally, since trade corresponds to borrowing and lending, the money offered in exchange for an amount of the good acts as a record, "proving" that an agent offered credit during the previous period. Let q_t denote the price of money at date t in units of the consumption good. Agent i's problem becomes:

$$\left.\begin{array}{l} \max_{\{c_t^i, m_{t+1}^i\}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^i) \\ \text{s.t. } c_t^i + q_t m_{t+1}^i \square e_t^i + q_t m_t^i \\ m_t^i \geq 0, \;\; \forall t \end{array}\right\} . \tag{2}$$

Here, m_t^i is agent i's money holding in period t. A stationary *Competitive Monetary Equilibrium (CME)* consists of an allocation together with a price of money, q, such that the allocation solves each consumer's problem, and the markets for money and the consumption good clear in each period. The following Proposition establishes the existence of a CME when agents are sufficiently patient.

Proposition 2 *Assume that $\beta > \frac{u'(e_h^i)}{u'(e_l^i)}$, $i = 1, 2$. There exists a stationary monetary equilibrium in which an agent who holds money exchanges it for q units of the consumption good in each period.*

The proof of this Proposition is given in Appendix 1. It involves ensuring that there exists a q such that obtaining one unit of money is profitable for the agent with high endowment, while obtaining an additional unit of money is not. These two conditions guarantee that the consumption allocation implied by the monetary equilibrium is individually optimal. We remark that a non-monetary equilibrium in which agents do not accept money anticipating that the price of money in the future will be zero always exists in our setup.

For an example, let $u(\cdot) = \log(\cdot)$, and let $e^1 = \{2, 1, 2, 1, \dots\}$, and $e^2 = \{1, 2, 1, 2, \dots\}$. Assume that agent 2 is endowed with 1 indivisible unit of money at the beginning of period 1. Provided that the discount factor is sufficiently high, $q = \frac{1}{2}$ satisfies the conditions of Proposition 1. Therefore, a stationary CME allocation has each agent consuming $1\frac{1}{2}$ for all t, i.e., one unit of money exchanges for $\frac{1}{2}$ unit of the good in each period. In this CME the agent with high endowment consumes $e_h^i - q$, and the agent with low endowment consumes $e_l^i + q$ in all t. Clearly, this allocation is efficient for the economy under perfect recall. Interestingly, this stationary allocation cannot be supported if money is perfectly divisible.

The above Proposition generalizes to the case where there is aggregate uncertainty under the following two assumptions on the endowment sequences.

Assumption 2: $\alpha \inf_t \min\{|e_t^1 - e_{t+1}^1|, |e_t^2 - e_{t+1}^2|\} \equiv d > 0,$ *where* $\alpha \in (0, \frac{1}{2})$.

Assumption 3: $\beta \in \left(\max\left\{\sup_{t \text{ even}} \frac{u'(e_{t-1}^1 - d)}{u'(e_t^1 + d)}, \sup_{t \text{ odd}} \frac{u'(e_{t-1}^2 - d)}{u'(e_t^2 + d)}\right\}, 1\right)$.

Assumption 2 rules out the case where endowment fluctuations vanish over time and the case where an agent has exactly the same endowment in two adjacent periods. Assumption 3 requires that the discount factor is high enough to ensure the existence of a (possibly not stationary) CME.

IV MONETARY INJECTIONS

In this section we explore the welfare implications of monetary injections. We still assume that there are two (types of) agents. Agent (type) 1 has endowment sequence $\{e_1, e_2, \dots\}$, and agent (type) 2 has endowment sequence $\{e_2, e_1, \dots\}$, with $e_1 > e_2$. We will compare two environments. In environment 1 there is one indivisible unit of fiat money. In environment 2 there are $n > 1$ indivisible units of fiat money. The two environments are otherwise identical. The CME to which we shall restrict our attention is as described previously; i.e., the price of money stays constant over time, and in each period the agent holding money uses his entire money balances to purchase consumption goods. We have the following Proposition. The proof can be found in Appendix 2.

Proposition 3 *Any CME consumption allocation for environment 2 can be supported as a CME consumption allocation for environment 1.*

Intuitively, the environment with less money obtains a greater number of CME. Furthermore, if money is perfectly divisible, the reverse is also true; i.e., any CME allocation for the environment with less money can be supported as a CME allocation for the environment with more money. Though an environment with fewer units of money can support all the CME supportable in an environment with more money, the welfare consequences are difficult to evaluate because there is an equilibrium allocation supportable in both environments that is not dominated by any other equilibrium allocation. We next demonstrate that such an equilibrium allocation exists. We have the following. See Appendix 2 for a proof.

Proposition 4 *There exists a CME allocation that can be supported in both environments, and that is not Pareto dominated by any other CME outcome.*

The above implies that monetary injections cannot be unambiguously evaluated . Although the environment with fewer units of money supports more CME, both environments support a Pareto efficient outcome.

When money is perfectly divisible, the logic behind the above results can be used to show that any CME allocation in environment 1 can be supported as a CME allocation in environment 2. However, when money is not perfectly divisible, a CME allocation in environment 1 is not necessarily supported as a CME outcome in environment 2. To see this, consider an agent's problem in environment 1 with equilibrium price, say $\hat{q}$:

$$\left.\begin{array}{rl} \max & \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t.} & \hat{q} m_{t+1} + c_t \leq \hat{q} m_t + e_t \end{array}\right\}. \tag{A}$$

Here, m_t is assumed to be a nonnegative integer for all t. In order for the equilibrium allocation in the above program to be an equilibrium allocation in environment 2, it has to be that the price in environment 2 is $\frac{1}{n}\hat{q}$ and, therefore, the agent's problem in setup 2 is

$$\left.\begin{array}{rl} \max & \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t.} & \frac{1}{n}\hat{q} m_{t+1} + c_t \leq \frac{1}{n}\hat{q} m_t + e_t \end{array}\right\}. \tag{3}$$

Denoting $\hat{M}_t = \frac{1}{n} m_t$, we can rewrite the problem as

$$\left.\begin{array}{rl} \max & \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t.} & \hat{q} \hat{M}_{t+1} + c_t \leq \hat{q} \hat{M}_t + e_t \end{array}\right\}. \tag{4}$$

This is the same as the earlier problem (A) except that we only need $n\hat{M}_t$ to be a non-negative integer. This allows for more choices than the nonnegative integer constraint for m_t in the original problem (A). Therefore, an optimal consumption choice for the original problem (A) may no longer be optimal here. Intuitively, more money holdings, and the resulting change in prices, allow for finer–tuned trades among agents.

V PERFECTLY DIVISIBLE MONEY

Here, we study a version of the model where money is perfectly divisible. The environment is the same as in the previous section. Assume that agent 2 is endowed with n perfectly divisible units of fiat money in period 1. Although other CME exist, we shall restrict attention to equilibria where agents use their entire money holdings in order to make purchases in each period. This assumption is for simplicity. We have the following Proposition. The proof can be found in Appendix 2.

Proposition 5 *There exists a stationary CME in which consumption satisfies $e_l < c_l^* < c_h^* < e_h$, and the price of money is given by $q^* = \frac{e_h - c_h^*}{n} = \frac{c_l^* - e_l}{n}$. This allocation is the best supportable allocation among those where agents use their entire money holdings in order to make purchases in each period.*

For an example, let $e_h = 3$ and $e_l = 1$, and assume that $u^i(\cdot) = \log(\cdot)$, for all i, and $\beta = .9$. Assume that there is one perfectly divisible unit of fiat money: $n = 1$. We can calculate that $c_h^* = 2\frac{2}{19}$, $c_l^* = 1\frac{17}{19}$, and $q^* = \frac{17}{19}$. Agent 1's equilibrium consumption, c^1, end-of-period money balances, m^1, and discounted marginal utility from consumption (DMU) in each period are given by the following.

t	1	2	3	4	...
c^1	$2\frac{2}{19}$	$1\frac{17}{19}$	$2\frac{2}{19}$	$1\frac{17}{19}$	...
m^1	1	0	1	0	...
DMU	$\frac{19}{40}$	$\frac{19}{40}$	$\frac{81}{100}\frac{19}{40}$	$\frac{81}{100}\frac{19}{40}$	...

(5)

Notice that the agent's marginal utility of consumption is falling over time, so he wants to consume more today. However, at the end of each even period his money balance is zero, which prevents him from consuming more.

VI DISCUSSION

Kocherlakota (1998) formalized the argument that, in a variety of models, incomplete memory is a *necessary* friction for money to be essential. We proposed a model that incorporates no commitment and bounded recall within an otherwise standard Walrasian setup. In the presence of a need for intertemporal trade, these were shown to be *sufficient* for money to be essential.

Our work is related to several existing monetary models. Unlike the prototypical random matching model, here there is an *intertemporal* lack of double coincidence problem. In the presence of bounded recall, this problem cannot be overcome even though agents meet in every period and trade in centralized markets. Our model can be thought of as giving an alternative interpretation to that of Townsend (1980) for the absence of private lending assumption in Bewley (1980). Of course, interpreting the good index as indicating time is one of many possibilities. We could think of the time index as a type index or a location index, etc. In the latter case, bounded recall would restrict memory across locations instead of time periods.

In Appendix 1 we discuss how the individual rationality constraint can be derived as a property of any equilibrium outcome of a game between agents with no recall of the past and their future selves. One interpretation of the model we proposed is that it is a model of *bounded rational* agents. Indeed, record-keeping and computability costs that may result in imperfect recall of the past have been used as justification for restricting the domain of the strategy space.[8] It should also be emphasized that while we concentrate on a Walrasian economy, our main points could be made within a strategic model. Of course, with a large number of agents one would expect the outcome of reasonable specifications of the game to be close to the Walrasian outcome studied here.

[8] See, for example, Cole and Kocherlakota (2000) and references within.

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APPENDIX 1 - THE INDIVIDUAL RATIONALITY CONSTRAINT

Here we discuss how the optimal behavior of agents when there is no recall about past transactions gives rise to the IR constraint imposed in Section 2. We proceed by introducing a suitable game and by arguing that in every equilibrium the individual rationality constraints holds for all agents. Assume that there is one good in each period and a large number of agents. We consider a game which, under perfect recall, has an equilibrium that supports the (assumed to be) unique Walrasian equilibrium allocation of the underlying intertemporal endowment economy. The game is similar in spirit to the pure coordination game in Kocherlakota and Wallace (1998). Here, in addition to the actions of other agents, an agent has to consider his own actions in the future, when he will not recall the past.

Let i_t denote the period t self of agent i. In each period t, player i_t chooses an amount of lending (borrowing, if negative), $\widetilde{l}_t^{i_t}\left(\overline{w}_t^i\right)$. Agent i_t also reports the value (in units of the date t good) of his previous lending, $\widehat{l}_{t-1}^{i_t}$. The two announcements are made simultaneously. Let $\widehat{L}_{t-1} = \sum_{i_t}^{i_t} \widehat{l}_{t-1}^{i_t}$ stand for the date t value of the total reported period $t-1$ lending. Let $\widetilde{L}_t = \sum_{i_t} \widetilde{l}_t^{i_t}$ stand for the total proposed period t lending. Player i_t takes as given the sequence of future endowments $\overline{w}_t^i = \left(w_t^{i_t}, w_{t+1}^{i_{t+1}}, ...\right)$ as well as the actions of the other players for all t, including the actions of his future selves $i_{t+1}, i_{t+2}, \ldots$. Notice that each i_t is "altruistic" in the sense that he values the (appropriately discounted) utility of all his future selves. His payoff can, therefore, be written recursively as $U^{i_t}\left(\overline{w}_t^i\right) = u\left(c_t^{i_t}\right) + \beta U^{i_{t+1}}\left(\overline{w}_{t+1}^{i_{t+1}}\right)$. Each player chooses $(\widetilde{l}_t^{i_t}, \widehat{l}_{t-1}^{i_t})$ in order to maximize $U^{i_t}\left(\overline{w}_t^i\right)$. The consumption allocation is determined by the following rule regarding agents' reports.

$$c_t^{i_t} = w_t^{i_t} - 1_{\widetilde{L}_t}\widetilde{l}_t^{i_t} + 1_{\widehat{L}_{t-1}}\widehat{l}_{t-1}^{i_t} \tag{6}$$

where $1_{\widetilde{L}_t}$, $1_{\widehat{L}_{t-1}}$ are indicator functions defined by

$$1_{\widetilde{L}_t} = \begin{cases} 1, & \text{if } \ \widetilde{L}_t = 0 \text{ and } \sum_{i_t} c_t^{i_t} \square \sum_{i_t} w_t^{i_t}; \\ 0, & \qquad\qquad \text{otherwise}; \end{cases} \tag{7}$$

and

$$1_{\widehat{L}_{t-1}} = \begin{cases} 1, & \text{if } \ \widehat{L}_{t-1} = 0 \text{ and } \sum_{i_t} c_t^{i_t} \square \sum_{i_t} w_t^{i_t}; \\ 0, & \qquad\qquad \text{otherwise.}^{9} \end{cases} \tag{8}$$

In other words, loan repayment takes place in each period only if the total claim from last period's lending matches the total repayment from last period's borrowing. Similarly, current lending takes place only if the current demand for loans matches current supply for loans. In addition, borrowing has to be feasible. We have the following.

Proposition 6 *It is a dominant strategy for each agent to choose* $\widetilde{l}_t\left(\overline{w}_t^i\right) \square 0$ *and* $\widehat{l}_{t-1}^{i_t} \geq 0$, *for all* t. *Thus, the constraint that* $u\left(c_t^i\right) \geq u\left(w_t^i\right)$, *for all* i *and* t *is satisfied. Furthermore, in all outcomes where the actions and beliefs of each agent constitute part of a sequential equilibrium, each agent ends up in autarky.*

The proof is straightforward and we omit it. We remark, however, that the game has many sequential equilibria. For example, having all agents setting borrowing and reported previous lending to infinity constitutes an equilibrium. However, in all equilibria either the current lending and repayment will not realize or the amount of lending and repayment is 0. So agents stay in autarky and $u\left(c_t^i\right) \geq u\left(w_t^i\right)$ is satisfied.

APPENDIX 2 - PROOFS

Proof of Proposition 2: For such an equilibrium to exist, we need that there exists a q satisfying the following two conditions:

$$u(e_h^1) + \beta u(e_l^1) < u(e_h^1 - q) + \beta u(e_l^1 + q), \tag{9}$$

and

$$u(e_h^1 - 2q) + \beta u(e_l^1 + 2q) < u(e_h^1 - q) + \beta u(e_l^1 + q). \tag{10}$$

It suffices to show that given such a q, it is optimal for both agents to follow the scheme described in the Proposition. Then, market clearing follows. In turn, it suffices to show that in a given period, say period 1, agent 1 will exchange q units of the good for a unit of money, and agent 2 will purchase q units of the good. There are four cases to consider.

Case 1: Agent 1 demands less than one unit of money in period 1.
Then in period 2, his consumption, c_2^1, satisfies $c_2^1 \square e_l^1 < e_h^1$. This, together with the first condition above and the concavity of u, implies $u(e_h^1) + \beta u(c_2^1) \square u(e_h^1 - q) + \beta u(c_2^1 + q)$. Therefore, agent 1 can do better by saving one more unit of money in period 1, saving one less unit of money in period 2, and starting period 3 with the same money holdings.

Case 2: Agent 2 offers more than one unit of money in period 1.
This is impossible because agent 2 already uses up all his money holdings.

Case 3: Agent 1 demands more than 1 unit of money in period 1.
Then he saves at least two units of money in period 1. By the second condition above, the cost of obtaining an additional unit of money satisfies

$$\beta[u(e_l^1 + 2q) - u(e_l^1 + q)] < u(e_h^1 - q) - u(e_h^1 - 2q). \tag{11}$$

Therefore, he would be better off saving one less unit of money in period 1, using one less unit of money in period 2, and starting period 3 with the same money holdings. Thus, it has to be the case that $m_3^1 - m_2^1 \geq -1$. In addition, $m_4^1 - m_3^1 \geq 2$ since otherwise,

$$\beta^2[u(e_h^1 - q) - u(e_h^1 - 2q)] < u(e_h^1 - q) - u(e_h^1 - 2q). \tag{12}$$

In that case, agent 1 would be better off saving the extra unit of money in period 3 instead of period 1. Notice that his consumption is not affected because he did not use all his money in period 2. Continuing this way, his sequence of net money savings satisfies $m_2^1 - m_1^1 \geq 2$, $m_3^1 - m_2^1 \geq -1$, $m_4^1 - m_3^1 \geq 2$, $m_5^1 - m_4^1 \geq -1 \ldots$. But this is not optimal since he is accumulating money without ever using it.

Case 4: Agent 2 offers less than one unit of money in period 1.
Similarly to case 3, we can show that he will end up accumulating money without ever using it. We conclude that none of the above four possibilities is optimal. Since each agent's problem is well defined, an optimal choice has to exist, and the scheme described in the Proposition is optimal. ■

Proof of Proposition 3: First, we show that any CME consumption allocation in the environment with more money is supportable in the environment with less money. Consider the agent's problem in environment 2 when the equilibrium price is, say, $q_t = q^*$, for all t.

$$\left.\begin{array}{c} \max \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t. } q^* m_{t+1} + c_t \square q^* m_t + e_t \end{array}\right\}. \tag{13}$$

Let $\{m_t^*\}_t$ be the resulting agent's equilibrium money holdings in environment 2. Note that m_t^* is either 0 or n, for all t. Let $\{c_t^*\}_t$ be the agent's CME consumption sequence.

We will construct an equilibrium for environment 1 in which $\{c_t^*\}_t$ is supported. Let the price be $q_t^{'} = nq^*$, for all t. The agent's problem is

$$\left.\begin{array}{c} \max \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t. } nq^* m_{t+1} + c_t \,\square\, nq^* m_t + e_t \end{array}\right\}. \tag{14}$$

Denoting by $M_t = nm_t$, the agent's problem can be rewritten as

$$\left.\begin{array}{c} \max \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t. } q^* M_{t+1} + c_t \,\square\, q^* M_t + e_t \end{array}\right\}. \tag{15}$$

This is identical to the agent's problem in environment 2, except that when the money is not perfectly divisible, there is an additional constraint that M_t is an integer multiple of n for all t because, by definition, $M_t = nm_t$ and m_t is a nonnegative integer. Since $\{m_t^*\}_t$ solves the agent's problem in environment 2 and satisfies the additional constraint (because m_t^* is either 0 or n), setting $M_t = m_t$ for all t solves the agent's problem in environment 1. It follows that the equilibrium consumption sequence in environment 1 coincides with that in environment 2. ■

Proof of Proposition 4: We demonstrate that there exists an equilibrium allocation that is supportable in both environments and that is not dominated by any other equilibrium consumption allocation. Let $0 < x < e_1$ be such that

$$u^{'}(e_1 - x) = \beta u^{'}(e_2 + x). \tag{16}$$

From our earlier discussion, both environments support the equilibrium in which an agent with high endowment, e_1, consumes $e_1 - x$, and an agent with low endowment, e_2, consumes $e_2 + x$ in each period. This equilibrium outcome is not dominated by any other equilibrium outcome since, for any $x^{'} \neq x$, the agent with endowment $\{e_1, e_2, \dots\}$ is better off consuming $\{e_1 - x, e_2 + x, \dots\}$ rather than $\{e_1 - x^{'}, e_2 + x^{'}, \dots\}$.

The following example gives an equilibrium consumption allocation supportable in environment 1 but not in environment 2. Let $e_1 = 8$, $e_2 = 1$, and assume *log* utility and $\beta = 0.5$. In environment 1 (with one unit of money), let the price of money be 1.5. Since $\log 6.5 - \log 5 = 0.114 \geq 0.5(\log 4 - \log 2.5) = 0.102$, we have that $\{6.5, 2.5, 6.5, 2.5, \dots\}$ can be supported as a CME sequence for the agent beginning with high endowment ($e_1 = 8$). Next, we show that this sequence is not necessarily supportable in environment 2. For this purpose, let $n = 10$ (there are 10 units of money). In order to support the equilibrium where all the money balances change hands in each period, the price of money has to be $\frac{1.5}{10} = 0.15$. However, $\log(8 - 0.15 \times 10) - \log(8 - 0.15 \times 11) = 0.0101$

$< 0.5 \times \log(1 + 0.15 \times 11) - \log(1 + 0.15 \times 10)$. Therefore, $\{6.5, 2.5, 6.5, 2.5, \dots\}$ is not supportable as a CME sequence in environment 2. For the best possible outcome, the first order condition and the market clearing condition give $x = 2$. Therefore, in that outcome an agent with high endowment consumes 6 units, and an agent with low endowment consumes 3 units in each period. ■

Proof of Proposition 5: We consider a planner's problem and demonstrate that the resulting allocation can be supported as a CME. Let $(c_1^1, c_2^1) = (c_h^*, c_l^*)$ solve agent 1's first–best problem restricted to period 1 and 2; i.e.,

$$\left.\begin{array}{l} \max \quad u(c_1^1) + \beta u(c_2^1) \\ \text{s.t. } c_1^1 + c_2^1 = e^h + e^l \end{array}\right\}. \tag{17}$$

The first order conditions for this problem give

$$u'(c_h^*) = \beta u'(c_l^*). \tag{18}$$

Since $\beta > \frac{u'(e_h)}{u'(e_l)}$, we have that $e_l < c_l^* < c_h^* < e_h$. Next, we show that $q^* = \frac{e_h - c_h^*}{n} = \frac{c_l^* - e_l}{n} > 0$ is an equilibrium price and that the equilibrium allocation has agent 1 consume $\{c_h^*, c_l^*, c_h^*, c_l^*, \dots\}$ and for agent 2 to consume $\{c_l^*, c_h^*, c_l^*, c_h^*, \dots\}$. The proof parallels the one for the indivisible case. Again, it suffices to consider period 1. We consider the following four cases, which exhaust all possibilities.

Case 1: Agent 1 saves less.

Then he has less money and has to consume less in period 2, which is not optimal.

Case 2: Agent 2 saves less.

This is impossible because he is already using all his resources.

Case 3: Agent 1 saves more.

Then his marginal utility from consumption in period 1 is higher, which implies that he will consume less than c_l^* in period 2. Then, he has a positive money balance at the end of period 2. This positive balance implies that the agent is not in the corner solution and will consume less than c_l^* in period 3 since, otherwise, he can do better by using more money in period 2 and saving more in period 3. Therefore, agent 1 ends period 3 with a strictly increased money balance. Continuing this way, we can show that agent 1 is accumulating money without using it, which cannot be optimal.

Case 4: Agent 2 saves more.

This means he does not use all his money holdings, which implies that he ends period 1 with a positive money balance. Now he is exactly in the situation of agent 1 in case 3 above. The same argument as the one used there implies that he will end up accumulating money without using it, which is not optimal. Since the agent's problem has a solution, having eliminated all other possibilities, we conclude that consumption sequences $\{c_h^*, c_l^*, \dots\}$ and $\{c_l^*, c_h^*, \dots\}$ are optimal for agent 1 and 2, respectively. The money market clearing follows from the choice of q^*. ■