Financial Fragility with Rational and Irrational Exuberance

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Abstract

This article formalizes investor rationality and irrationality, exuberance and apprehension, to consider the implications of belief formation for the fragility of an economy's financial structure. The model presented generates a financial structure with portfolio linkages that make it susceptible to contagious financial crises, despite the absence of coordination failures. Investors forecast the likelihood of loss from contagion and may shift preemptively to safer portfolios, breaking portfolio linkages in the process. The entire financial structure collapses when the last group of investors reallocates their portfolios. If some investors are irrationally exuberant, the financial structure remains intact longer. In fact, financial collapse occurs sooner when almost all investors are rationally exuberant than when they are irrationally exuberant. Additionally, a financial crisis initiated by real shocks is indistinguishable from one caused solely by the presence of rationally apprehensive investors in a fundamentally sound economy. Policies that make portfolio linkages more resilient can improve welfare.

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Key Words: financial fragility, contagion, irrational exuberance, financial crises

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> [W]hat does it mean to say that markets are rational? Is it assumed that most markets behave rationally most of the time, or that each and every participant in the market has the same intelligence, the same information, the same purposes, and the same economic model in mind, or that all markets behave rationally all the time? …. Frequently the argument seems to be between two polar positions, one that holds that no market is ever rational, the other that all markets are always so. [Kindleberger, 1996, p. 20]

I. Introduction

Economists from Cantillon to Kindleberger have speculated about whether financial markets are fully rational or with some frequency over- or undervalue assets relative to the assets' fundamentals.[1] This interest in the rationality of financial markets has been fueled by repeated episodes of apparent irrationality, from the Dutch Tulipmania in 1637, to the British South Sea bubble in 1720, to the Japanese real estate bubble in the late 1980s.[2] And it has not been confined to academic circles. In December 1996, with the Dow Jones Industrial Average reaching new heights, Federal Reserve Board Chairman Alan Greenspan suggested that "irrational exuberance" could lead to "unduly escalated asset values, which then become subject to unexpected and prolonged contractions," and questioned how such a situation could be identified and how monetary policy should react. In April 1998, Eisuke Sakakibara, Japan's vice finance minister for international affairs, described the Japanese economy as suffering from "irrational pessimism." Since then, economists worldwide have pondered what it means for financial-market participants to hold irrational beliefs and whether their doing so is bad in some appropriate sense.

This article attempts to address these questions. Given the nature of the questions, the answers necessarily depend on how rationality and exuberance—or the lack thereof—are interpreted. The approach adopted regarding rationality is the one common to much of late-20th-century macroeconomics.[3] Specifically, rational investors are assumed to have rational expectations. That is, they maximize an objective function subject to perceived constraints, and

[1] Eighteenth-century economist Richard Cantillon is likely the first economist to have written on this subject. In his *Essai sur la Nature du Commerce en Général*, published posthumously in 1755, Cantillon writes about the role of expectations and differences in information across investors in foreign exchange speculation, the tendency of people to hold incorrect beliefs about fundamentals, and the South Sea bubble. He describes, albeit indirectly, the circumstances that generated the Mississippi bubble (1719-20), circumstances from which he profited greatly (Bordo 1983, Murphy 1986).

[2] Kindleberger (1996) provides an extensive discussion of the relevant historical experience.

[3] What it means, however, for agents to be rational remains an open question within the economics profession generally. Arrow et al (1996) provide an overview of some of the unresolved issues.

they use all the information available to them, including the true probability distributions of the economy's random variables, when forming expectations. The implication, of course, is that irrational, or boundedly rational, investors do not have rational expectations.[4] In particular, they are assumed to optimize, but to form expectations given their subjective, and incorrect, beliefs about the distributions governing random variables. The subjective priors assumed generate posterior beliefs that are the opposite of the beliefs held by rational investors.

The macroeconomics literature provides little guidance, however, regarding the modeling of exuberance. Dictionaries define "exuberance" as the state of being joyously unrestrained and enthusiastic. This definition is applied here by considering exuberant investors to be those who are very optimistic about the prospects for the economy and thus for their investment portfolios. To better isolate the effect of exuberance in financial markets, the implications of there being a fraction of investors who are not exuberant are also examined. These apprehensive investors, as they are called, perceive the economy's fundamentals as poor and so expect losses on their portfolios.

To model rationality, exuberance, and apprehension under these interpretations, this paper uses a variant of the model in Lagunoff and Schreft (1998). That model consists of many projects that require funding to operate, and many investors who provide the necessary funds. The investors initially hold portfolios that are linked in the sense that an investor's expected and actual portfolio returns depend on the portfolio choices of other investors. Identically and independently distributed shocks to the projects' operations can cause some projects to fail initially. All investors know the probability of project failure and thus know the economy's fundamentals. Realized project failures break some portfolio linkages, which causes some investors to incur losses and reallocate their portfolios, thereby breaking additional linkages. The initial project failures thus spark a contagious financial crisis. Investors who foresee the crisis reducing their portfolio returns can protect themselves if desired by preemptively shifting to a portfolio that is safer in that it reduces their exposure to any ongoing contagion. Since all investors are identical, if one takes such preemptive action, then they all do, causing the instantaneous collapse of all remaining portfolio linkages, despite the absence of coordination failures. An economy is considered more fragile the earlier this total financial collapse occurs.

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[4] This formulation of the distinction between rationality and irrationality is consistent with Sargent (1993).

The model presented below in Section II departs from the Lagunoff and Schreft model in three critical respects. First, here the economy's fundamentals are uncertain in that the probability of project failure, the parameter that characterizes the iid stochastic process, is not known. This requires investors to form expectations about both the fundamentals and their exposure to contagion risk. Second, investors' information about the fundamentals comes from a noisy signal that they observe and use in forming their expectations. Since different investors can observe different signals, heterogeneous beliefs are possible here, in contrast to the Lagunoff and Schreft paper. This heterogeneity can apply to posterior beliefs about fundamentals, about other investors' beliefs, and about the representation of beliefs within the population. Finally, the equilibrium concept used here differs from that in Lagunoff and Schreft in a subtle way. In Lagunoff and Schreft the goal is to look at the inherent fragility of an economy, so the focus is on the set of equilibria that keeps the economy's financial structure intact the longest. As a device for finding such an equilibrium, investors are assumed to make relatively optimistic forecasts. That is, they foresee the possibility of contagion, but not of preemptive behavior. In contrast, in this paper investors make forecasts that account for the possibility of both contagion and total financial collapse.

Two variants of the economy are studied. The first, presented in Section III, takes investors to be rational as defined above and looks at fragility when the fundamentals are strong, so exuberance is justified. The second, which is the subject of Section IV, assumes irrationality and weak fundamentals. This organization is motivated by von Hayek's (1937) belief that "[B]efore we can explain why people commit mistakes, we must first explain why they should ever be right."

Several intriguing findings emerge. First, exuberant investors remain invested at least as long as their apprehensive counterparts, regardless of whether everyone is rational or irrational. As a result, the portfolio choices of the exuberant investors necessarily determine the date of total financial collapse, and thus the economy's fragility. But the presence of apprehensive investors also plays a critical role in determining fragility. An increase in the presence of apprehensive investors in the economy makes the economy at least as fragile. This too is true whether everyone is rational or irrational. And if there is at least one apprehensive investor in the economy, total financial collapse can occur, regardless of the fundamentals and the rationality of investors. This is the case even in an economy where all investors are rational and where the fundamentals are so strong that nothing in the physical environment ever sparks a contagion. The reason is rooted in the factors that generate apprehension. Investors who receive misleading information about the fundamentals, even if they know the true model of the economy (i.e., have the correct prior), perceive a contagious financial crisis as more likely than it actually is. Given this perception, they believe they benefit from reallocating their portfolios to preempt experiencing portfolio losses. This behavior by itself initiates a contagious financial crisis. The rationally exuberant investors in this economy correctly forecast that the fundamentals make financial crises unlikely, but must respond strategically to the portfolio reallocations they expect by the apprehensive investors. They choose to reallocate their own portfolios to protect themselves against losses due to the contagion the apprehensive investors initiate.

Interestingly, rationally exuberant investors who live in a fundamentally strong economy reallocate their portfolios sooner than irrationally exuberant investors who live in a fundamentally weak economy, which makes the strong economy the more fragile one. This finding stems from the effect of irrationality on beliefs. Investors who form expectations irrationally misforecast the behavior of other investors and thus misforecast the fundamentals by a larger margin than they would if all investors were identical. As a result, irrational investors' sentiments about the fundamentals are more extreme than those of rational investors.

These results have some disturbing implications for policymakers concerned about irrationality in financial markets. First, an economy with rationality is indistinguishable from one with irrationality in terms of the types of financial crises experienced. Thus, an observer looking at the realization of crises after the fact cannot tell whether the initial exuberance was rational or irrational. Second, an economy with sound fundamentals and rational investors experiences total financial collapse at the same date as some economy with particular weak fundamentals and identical investors who know those fundamentals. This is discouraging news because it means that a financial crisis that looks as if it were initiated by real shocks to the economy instead could have been caused solely by the presence of rational, but unjustifiably apprehensive, investors.

Section V discusses the implications of these findings for welfare and policy. Since all investors, whether rational or irrational, optimize, they are as well off as possible, conditional on the signals they observe and their beliefs. There will always be some, however, who regret their decisions. They are the ones who do not preemptively reallocate their portfolios in time to avoid incurring losses from contagion. To reduce the likelihood of such regrets, policymakers can try to eliminate contagion. A short-run policy option is for a lender of last resort to make loans to ensure that no project lacks sufficient funding to operate solely because of contagious portfolio reallocations. Section V explains that such a policy is problematic and ineffective because the model, as specified, requires that loans go to investors. At best, the policy can stop contagion for a period or two. Alternatively, if the model allowed for agents who operated the projects, loans could go to them instead. That approach, however, brings with it moral-hazard problems. It remains an open question whether the benefits from such a lending policy exceed the costs.

In the long run, policies aimed at strengthening an economy's financial infrastructure can prove effective. Examples include programs to obtain information about existing portfolio linkages and to encourage diversification. Such policies reduce the likelihood of contagion by increasing the resiliency of portfolio linkages and reducing the cost to investors if links do break. These policies, like the lender-of-last-resort policy, must be implemented by an institution whose authority encompasses all portfolio linkages.

A natural question, given the model's results, is how this paper differs from the papers in the large literature on bubbles. A bubble is said to exist when the price of an asset is inconsistent with market fundamentals. In a narrow sense, then, the model presented here does not generate bubbles because its asset prices are fixed. In a broader sense, however, this model is about bubbles. Bubbles arise when the demands for assets deviate from what is justified by market fundamentals, and such deviations in asset demands do occur in this model. When the demand for a project is sufficiently high, the project operates and pays a positive net return. But when the demand falls sufficiently, the project fails to operate, paying a zero return and having zero value thereafter. When rational exuberance is the dominant sentiment, there is a period during which the demand for projects is lower than what market fundamentals dictate. Likewise, when irrational exuberance is prevalent, the demand for projects is sustained for a period of time beyond that justified by fundamentals. Yet there still exists a date, although possibly infinity, at which the mere anticipation of a sharp decline in the demand for some projects leads investors to dramatically reduce the demand for all projects, driving their values to zero. In this sense, then, this paper is consistent with the bubble literature.

II. The Economy

The model presented below is a variant of that in Lagunoff and Schreft that is more complex in one sense and less so in another. The added complexity derives from the differences in the models' state variables. In Lagunoff and Schreft, the focus was on defining and characterizing fragility and on assessing how fragility changes as an economy increases in size. Consequently, the size of the economy was a key state variable, and the economy's fundamentals were assumed known to all investors. In this paper, in contrast, the objective is to see if investment behavior is consistent with market fundamentals. As a result, the economy's size is fixed, but there is uncertainty about the fundamentals. Additionally, because different agents can observe different signals about the fundamentals, heterogeneous beliefs are possible. This means that agents must forecast not only the risk of loss associated with the various portfolios, but also the forecasts of other investors about investment risks. To offset some of this added complexity, the model presented here takes agents' initial portfolio allocations as exogenous. Lagunoff and Schreft show, however, that there exist economies for which the initial portfolios assumed here are held in equilibria of the type studied in both papers.

A. The Physical Environment

Time is discrete and represented by $t = 0, 1, \ldots$. The economy consists of k investors, each endowed at date 0 with two units of an indivisible object known as dollars and with nothing at later dates. Investors can provide for future consumption by investing their dollars in one of the economy's safe or risky assets. Each investor has access to a safe asset that pays zero interest. He also has the option of investing in the economy's k risky projects, which offer the chance of a higher return.

Specifically, at each date a project yields a random return of $R(I)$ dollars per dollar invested, where I denotes the total number of dollars invested. Each project can be operated only if it has sufficient funding. For simplicity, the critical level of funding—the level at which a project operates and pays the maximum return per dollar, $\overline{R}$ —is taken to be \$2. Projects that have less than two dollars invested in them pay a gross return per dollar of zero. Once a project has been insufficiently funded, it becomes inoperable at all future dates. When a project is overfunded, with more than two dollars invested in it, decreasing returns are realized and the project yields a return per dollar of 2 $\overline{R}$ /*I*.

At date 0 only, there is a second way by which a project can become inoperable: independently and identically distributed shocks can cause projects to fail, pay a zero return, and permanently cease operation. The true probability of a project's failing from an exogenous shock is represented by the random variable $\wp$, which can take one of two values, either zero or $\overline{p}$, where 0 < $\overline{p}$ ≤ 1. Once a project fails, it is forever inoperable. If a project succeeds at date 0, it pays a return that depends on the amount invested, as described above.

In summary, then, a project's return per dollar, assuming the project has not previously ceased operation, satisfies

$$R(I) = \begin{cases} 0 \text{ with probability } \wp \text{ and } \dfrac{2\overline{R}}{I} \text{ with probability } 1-\wp & \text{at } t = 0 \text{ if } I \geq 2, \\ \dfrac{2\overline{R}}{I} & \text{at } t > 0 \text{ if } I \geq 2, \\ 0 & \text{at } t \geq 0 \text{ if } I < 2, \end{cases}$$

where $R^{\max} > \overline{R} > 1$. The upper bound $R^{\max}$ is imposed for simplicity and assumed to be such that no portfolio ever yields a dollar in interest, thus leaving investors with \$3 to invest, and that no investor who sustains a loss of any magnitude ever reinvests in more than one project. Given that investors are initially endowed with \$2 and that dollars are indivisible, the first condition implies that $\overline{R} < 1.5$, while the second, which rules out an unusual situation for notational simplicity, requires $\overline{R} < 1.2$.

The state of the economy depends on which value of $\wp$ is realized. The set of possible states is $\{w_1, w_2\}$, where state w_1 represents the case of $\wp = 0$ and w_2 corresponds to the case of $\wp = \overline{p}$. Each state occurs with probability 0.5. Investors know that iid shocks are possible at date 0, but observe only a noisy signal of the true state, not the state itself. The set of possible signals is $\{x_1, x_2\}$. Investors differ based on which signal they observe: those observing signal x_1 are of type 1, while those observing x_2 are of type 2. The probability that a random investor observes signal x_i in state w_j is $\Pr(x_i|w_j)$. More specifically, $\Pr(x_i|w_i) = 1-e$ and $\Pr(x_j|w_i) = e, j \neq i$, where e is a known constant less than 0.5. That is, signal x_i is more likely to be observed in state w_i than in state w_j, and there is no aggregate uncertainty in the distribution of investor types. It follows that if state w_i occurs, then a fraction $1-e$ of investors is randomly selected to observe x_i, while a fraction e is randomly selected to observe x_j. Investors thus can use the signal they observe in forming expectations of the likelihood of each state and of the fraction of investors of each type in the population. Investors who forecast that the state is w_1 are taken to be exuberant because they think the economy's fundamentals, as measured by the probability of shocks at date 0, are strong. Likewise, those who forecast state w_2 are taken to be apprehensive.

While the true unconditional prior distribution over states is $\Pr(w_i) = 0.5$, much of the analysis of rational and irrational exuberance in subsequent sections is concerned with implications that stem from whether $\Pr(\cdot)$ is known and common knowledge, and thus whether investors are rational. To account for the possibility that investors' subjective beliefs about $\Pr(\cdot)$ are in error, $\Pr^i(\cdot)$ denotes type i's subjective prior. If, indeed, $\Pr(\cdot)$ is common knowledge, then $\Pr^1(\cdot) = \Pr^2(\cdot) = \Pr(\cdot) = 0.5$. It is assumed that type i believes that his model of the world is known to be the true model in the sense that he thinks $\Pr^i(\cdot)$ is the true prior and common knowledge.[5]

Given the information available to him, an investor chooses at each date how to divide his remaining wealth between consumption in the current period and an investment portfolio that he purchases at the beginning of the next date. A portfolio at date $t \geq 0$ is a triple (a_{ht}, a_{jt}, a_{st}), where a_{ht} denotes dollars invested in risky project h at t, a_{jt} denotes dollars invested in risky project j, $j \neq h$, and a_{st} denotes dollars invested in the safe asset. Because dollars are indivisible, there are a limited number of portfolios that can be held at any point in time.[6] An investor who chooses to invest \$2, for example, can hold any of the following portfolios: a diversified loan portfolio ((1,1,0)), an undiversified loan portfolio ((2,0,0) or (0,2,0)), a part-safe-part-risky portfolio ((1,0,1), (0,1,1), (1,0,0), or (0,1,0) since only part of the investor's total wealth is at risk with these portfolios), or a safe portfolio ((0,0,2), (0,0,1), or (0,0,0) since none of the investor's wealth is at risk with these portfolios). An investor who chooses to invest \$1 can hold an undiversified loan portfolio ((1,0,0) or (0,1,0)) or a safe portfolio ((0,0,1) or (0,0,0)). The gross return on portfolio (a_{ht}, a_{jt}, a_{st}), which is realized at t, is $r(a_{ht}, a_{jt}, a_{st})$ and is used to fund future consumption. Without loss of generality, all investments are assumed to be for one period.

[5] Brandenburger and Deckel (1990) discuss the role of common-knowledge assumptions.
[6] The assumption that dollars are indivisible is equivalent to an assumption that investments must be made in \$1 increments.

At each date $t \geq 0$ then, an investor with post-return wealth y_t (or endowment wealth y_{-1}) chooses a sequence $\{(a_{h\,t+1}, a_{j\,t+1}, a_{s\,t+1}), c_t\}$ to solve

$$\max E_t \sum_{h=t}^{\infty} b^{h-t} u(c_h), \quad 0 < b < 1,$$

subject to

$$c_h + a_{h\,h+1} + a_{j\,h+1} + a_{s\,h+1} \leq y_h,$$
$$y_{h+1} = y_h - c_h - (a_{h,h+1} + a_{j,h+1} + a_{s,h+1}) + r(a_{h,h+1}, a_{j,h+1}, a_{s,h+1}),$$
$$a_{h\,h+1}, a_{j\;h+1}, a_{s\,h+1} \in \{0, \$1, \$2\}, \;\; c_h \geq 0, \;\; y_{-1} = \$2,$$

where c_t is wealth consumed at t, and $u(\cdot)$ is an increasing, strictly concave, and time-separable von Neumann-Morgenstern utility function with $u(0) = 0$. The first constraint states that total expenditures on consumption and investment cannot exceed available wealth. The second describes the evolution of post-return wealth and reflects the fact that investors can consume or reinvest their portfolio returns. The final set of conditions states that investments must be made in increments of \$1, that consumption must be nonnegative, and that an investor's initial wealth is \$2.

Investors are assumed to know k, the number of projects available initially, when they choose their first portfolios. These portfolio choices, as well as all later ones, are assumed to be private information: investors know their own portfolio allocations, but not those of others. This assumption is made to capture the notion of large, anonymous economies. Investors also are assumed to have limited ability to communicate and thus to overcome the information restrictions to share risk. Since investors do not know the true state, these assumptions on information and communication imply that they also do not know the realization of shocks at date 0 or how many projects are left at any time after the shocks hit. They are aware, however, of the fate of the projects in which they have invested.

Figure 1 below, which illustrates the timing of economic activity, provides a summary of the physical environment just described. Investors begin date 0 with \$2, which they invest in a portfolio of assets. They then observe a signal regarding the probability of the iid shocks being realized. Next, the shocks are realized, causing some projects to fail. If a project fails, the agents who invested in it lose their entire investment. After returns are realized, both at date 0 and at all later dates, investors choose how to divide their remaining wealth between current consumption and investment at the next date. This completes the description of the physical environment.

$t = 0$
Investors are endowed with \$2 with which they purchase a portfolio of assets.
Investors observe a signal, which determines their type.
Iid shocks to investment projects are realized.
Project returns are realized. Projects that failed become forever inoperable. The consumption-investment decision is made, and consumption occurs.
$t = 1$

$t \geq 1$
Investment occurs.
Project returns are realized. Insufficiently funded projects become forever inoperable. The consumption-investment decision is made, and consumption occurs.
$t + 1$

Figure 1—The Timing of Economic Activity

B. Strategies

It remains to consider investor behavior. The solution to the investor's dynamic choice problem depends on the functional form of utility, on the values of the model's parameters, and on the investor's forecast about the behavior of other investors. Complicating matters is the fact, illustrated by Figure 1, that the investor's problem is recursive after the shocks are realized, but not before. Since the goal here is to study the impact of rational and irrational exuberance on fragility, this paper follows the approach of Lagunoff and Schreft. It considers an economy to exhibit *financial fragility* if the economy possesses a propagation mechanism that allows small exogenous shocks at the initial date to generate financial crises that have large-scale effects on the financial structure and thus on real activity. Lagunoff and Schreft argue that to capture this notion of fragility, a model first must give rise to portfolios that are linked, and second must incorporate some mechanism that breaks linkages. Thus, in what follows attention is restricted to economies with these two features. The first feature arises as long as each investor holds diversified portfolios at the start of date 0; otherwise, an investor has his wealth in at most one risky project. In this model, investors who think that $\wp = 0$ based on the signal they observe are indifferent between holding diversified and undiversified portfolios, but prefer either portfolio to the alternatives. Investors who think that $\wp = \bar{p} > 0$ may prefer to hold a diversified portfolio, depending on their preferences and the values of the economy's parameters. To focus most directly on economies that could possibly be fragile, in what follows investors are assumed to start date 0 with their \$2 endowment invested in the best diversified portfolio—namely a *maximum-return diversified portfolio*. This portfolio consists of two \$1 investments in projects that receive exactly \$2 in funding and thus pay the maximum return of $\overline{R}$ per dollar.[7] An advantage of this assumption is that, in an economy where $\wp = 0$ is common knowledge, investors holding such portfolios never choose to reallocate their portfolios at later dates.

Another advantage of this assumption is that it allows the economy's initial portfolio allocations to be represented by a collection of closed chains. A *closed chain* is a set of projects and investors such that each project is fully funded, each investor is fully invested and diversified, and investor portfolios are all linked. Figure 2 depicts a closed four-link chain, since every investor is linked directly or indirectly to four projects.

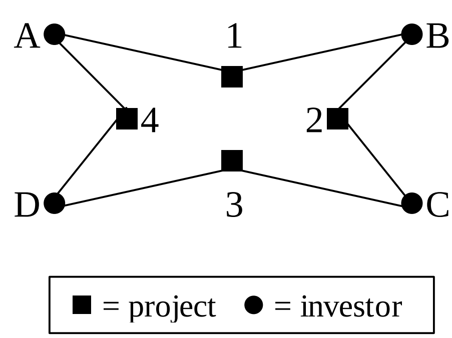

Figure 2—A Closed Chain

When a shock at date 0 causes a project to fail, it turns the closed chain in which the project resides into an open chain. An *open chain* is the same as a closed chain, except that some investors are not diversified because they lost \$1 when a project failed. Figure 3 depicts the open chain that arises when project 3 in Figure 2 fails.

While the shocks at date 0 initiate linkage breaks, fragility requires that there be a propagation mechanism that magnifies the effects of the shocks. In what follows, attention is restricted to economies in which such a propagation mechanism exists. More precisely, attention is restricted to economies in which, at each date $t \geq 0$, an investor who has lost \$1 prefers not to

[7] Lagunoff and Schreft show that there are economies (i.e., specifications of the utility function and ranges of parameter space) for which an equilibrium exists with maximum-return diversified portfolios chosen by investors at the start of date 0.

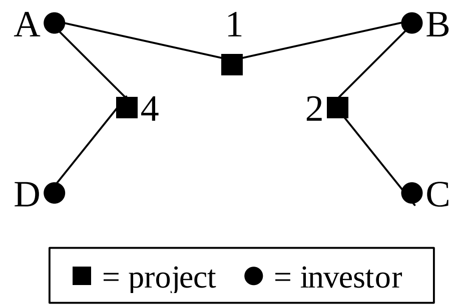

Figure 3—An Open Chain

reinvest in a risky asset.[8] That is, investors who find themselves on the ends of an open chain choose not to reinvest their remaining dollar in the chain at the next date. In Figure 3, for example, this means that investors D and C shift out of risky assets and consume their remaining wealth, which results in projects 4 and 2 receiving insufficient funding and thus investors A and B incurring portfolio losses and subsequently not reinvesting in project 1. The result is that open chains unravel over time.

In this environment, the unraveling of open chains constitutes a financial crisis, a collapse of all or part of the chain structure. It is the result of optimizing behavior by investors in response to portfolio losses sustained. The likelihood of such crises depends on the economy's initial mix of chains and the number and distribution of shocks realized.

The mix of chains that exists at any point in time represents the economy's *chain structure*—or financial structure. Since investors are assumed to hold maximum-return diversified portfolios initially, the date-0 chain structure consists solely of closed chains. Nothing in the model identifies the process that determines the initial chain structure. All investors, however, are assumed to know the true probability distribution over the possible chains in which they can find themselves before shocks hit. At any time after shocks are realized, the chain structure is a mix of closed and open chains, but only those chains that could have evolved from ones possible initially.

An implication of the assumption that portfolios are private information that cannot be communicated is that investors do not know the economy's chain structure, whether they are in an

[8] Lagunoff and Schreft show that there exist economies in which, in an equilibrium of the type studied, investors who have lost $1 rank portfolios in this manner.

open or a closed chain, or their location within their chain. Investors can, however, contemplate the possibilities of their being in the various locations in the various chains and thus can evaluate their exposure to contagious losses and strategically act to protect themselves. It remains to specify investors' behavior in the dynamic recursive game that is played at each decision point after the shocks have been realized.

In addition to observing a signal about the state of the economy, an individual investor observes s_t, the post-return state of his portfolio, that prevailing after returns are realized and when he has to make his consumption-investment decision. That is, he knows at each date, after realizing his returns, whether he has incurred a loss of any or all of his initial \$2 endowment. His state s_t is from the set $\{W_t, H_t, Z_t\}$, where W_t is the state in which he still has his whole initial endowment, H_t is the state in which he has half of his initial endowment because he has incurred a \$1 loss, and Z_t is the state in which he has zero dollars left. Based on the signal observed (x_i) and the state of his portfolio, the investor chooses an action a regarding his portfolio, where a is from the set $\{D, U, P, S\}$. Action D is the choice to continue holding a diversified portfolio at the beginning of the next date. Action U is the choice to deviate to an undiversified portfolio; action P, a part-safe-part-risky portfolio; and action S, a safe portfolio (Section II-A describes these portfolios). With this specification of actions, a *symmetric strategy* for type i is a sequence $f^i \equiv \{f_t^i\}_{t=0}^{\infty}$, where $f_t^i(s_t, x_i)$ denotes the action taken by investor type i in state s_t at time t. Type i's *forecast* of others' behavior is a symmetric strategy profile denoted $\tilde{f}^i \equiv \left(\tilde{f}^i(s_t, x_1), \tilde{f}^i(s_t, x_2)\right)$, with $\tilde{f}_t^i(s_t, x_j)$ representing i's forecast at date t of type j's (j = 1,2) strategy when j is in state s_t. Forecast $\tilde{f}^i$ is a *conditionally correct forecast* if it coincides with strategy f^j that j chooses if j thinks i's prior $\Pr^i(\cdot)$ is the true prior. The notion of equilibrium defined in the next subsection assumes that all investors' forecasts are conditionally correct.

Finally, type i's post-return expected lifetime utility at t from strategy $f_t^i(s_t, x_i)$ and forecast $\tilde{f}^i$ is denoted $V_t\left(f^i, \tilde{f}^i | s_t, x_i\right)$. Appendix A presents the expressions for these expected utilities for each possible state and each possible action an investor can take.

C. Equilibrium and Fragility

The preceding specification of strategies and forecasts implies symmetry in the strategies of investors of the same type. In what follows, then, only symmetric equilibria are studied, and they all share two features. First, as stated previously, to ensure the existence of a propagation mechanism that can permit small shocks to have large-scale effects, the economies studied have the property that investors who lose \$1 do not reinvest in the chain structure. That is, in equilibrium, $f_t^i(H_t, x_i) = S$. Second, since all investors are assumed to hold diversified portfolios before the realization of shocks and to maximize utility, those in state W_t compare the expected lifetime utility at the end of date t from continuing to hold a diversified portfolio at $t + 1$ to that from the best portfolio of each alternative type. If $f^{i,a}$ denotes the strategy of a type-i investor who chooses action a at date t, given state W_t and signal x_i, then the expected utilities associated with the various portfolios can be denoted $V_t\left(f^{i,a}, \tilde{f}^i | W_t, x_i\right)$, for $a = D,U,P,S$. Thus, investors continue to hold diversified portfolios as long as

$$V_t\left(f^{i,D}, \tilde{f}^i | W_t, x_i\right) \geq \max\left\{V_t\left(f^{i,U}, \tilde{f}^i | W_t, x_i\right), V_t\left(f^{i,P}, \tilde{f}^i | W_t, x_i\right), V_t\left(f^{i,S}, \tilde{f}^i | W_t, x_i\right)\right\}.$$

The following definition captures these features of a symmetric equilibrium.

DEFINITION. A *maximal sustainable equilibrium with conditionally correct forecasts* is a symmetric strategy f^{i*} for each investor type i = 1,2, such that

i. For each date t and each state s, $f_t^{i*}(s_t, x_i)$ is chosen to maximize $V_t\left(f^i, \tilde{f}^i | s_t, x_i\right)$, where $\tilde{f}^i$ is a conditionally correct forecast.

ii. For each t, $\tilde{f}_t^i\left(W_t, x_j\right) = D$, provided that for each j = 1,2,
$$V_t\left(f^{jD}, \tilde{f}^i | W_t, x_j\right) \geq \max\left\{V_t\left(f^{jU}, \tilde{f}^i | W_t, x_j\right), V_t\left(f^{jP}, \tilde{f}^i | W_t, x_j\right), V_t\left(f^{jS}, \tilde{f}^i | W_t, x_j\right)\right\}.$$

iii. Investors start date 0 holding maximum-return diversified portfolios.

iv. For each t, $f_t^{i*}(H_t, x_i) = S$.

By construction, then, in a symmetric equilibrium there exists some first date at which investors of type i decide to switch from a diversified portfolio to the next-best alternative, which is necessarily safer in that it involves less exposure to contagion risk (fewer dollars at risk, or dollars at risk from contagion from fewer directions). And by the equilibrium's symmetry, if one investor of type i decides to reallocate his portfolio, then all investors of type i do so. The *exit-decision date* at which type i investors all decide to reallocate and thus exit, or disconnect from, the chain structure at the next opportunity, given *e*, is denoted by $t_i(e)$.

The effect of *e* on $t_i(e)$ is complicated because *e* represents both an investor's uncertainty about the true probability of project failure at date 0 and his belief about the fraction of investors of each type. To sort out the effect of *e* on $t_i(e)$, it proves useful to look at the exit-decision date in an economy in which all investors share the same beliefs and assign probability 1 to the event $\wp = p$ for some $p > 0$. This is exactly the Lagunoff and Schreft economy. In what follows, the exit-decision date for this economy with homogeneous beliefs and certainty about the fundamentals is denoted by $\bar{t}(p)$. By construction, it is the first date *t* for which[9]

$$V_t\left(f^{i,D}, \tilde{f}^{i} | W_t, x_i\right) < \max_{a \neq D} V_t\left(f^{i,a}, \tilde{f}^{i} | W_t, x_i\right).$$

The exit-decision date $\bar{t}(p)$ isolates the effect of *e* on uncertainty about $\wp$. Consequently, the difference between dates $\bar{t}(p)$ and $t_i(e)$ measures the effect of *e* on the existence of heterogeneous beliefs within the population. The smaller is *e*, the farther apart are the beliefs of the type-1 and type-2 investors, as are their exit dates, $t_1(e)$ and $t_2(e)$, respectively. As $e \to 0.5$, the beliefs of the two investor types converge, as do their exit-decision dates: $t_1(0.5) = t_2(0.5)$.

When $t_1(e)$ and $t_2(e)$ differ, the early exiting of some investors has the same effect on the economy as would a second round of shocks to projects. The only difference is each realization of a shock at date 0 causes a single project to fail, whereas each investor in state W_t who exits the chain structure causes two projects to fail, the two that were in his portfolio. The decision at the latter of the $t_i(e)$ of some investors to exit induces a financial crisis involving the simultaneous collapse of all remaining chains, whether open or closed, when the investors actually reallocate at the next date ($\max_i t_i(e) + 1$). The crisis occurs because optimizing investors strategically choose to reallocate their portfolios as protection against possible losses, even if they have not experienced losses themselves and do not know for sure that losses are occurring. Thus, it can occur in addition to the crisis associated with the unraveling of chains, simply because investors act preemptively to protect themselves against the event that their own chain might unravel. And one would expect, as is shown below, that the early exiting of some investors causes it to occur at least as soon as it

[9] The exit-decision date $\bar{t}(p)$ is an equilibrium in pure strategies in the Lagunoff-Schreft economy with homogeneous beliefs of the type specified here. This equilibrium involves a two-action-best-response cycle between strategies *U* and *S* at the exit-decision date. There thus must exist an equilibrium in which investors use mixed strategies at the exit-decision date. It is conjectured, but not proven here, that for some preferences and parameter values, an equilibrium exists in pure strategies as well.

otherwise would have.

It follows that an economy may be characterized as more fragile the earlier the date at which its entire chain structure collapses (i.e., the earlier $\max_i t_i(e)+1$). This does not mean, however, that every economy's financial structure collapses. While $\max_i t_i(e)$ can be zero, in which case the economy's chain structure collapses with certainty when investors reinvest at date 1, it also can be infinity, which means the economy never suffers a complete financial collapse due to strategic investor behavior.

A desirable feature of a maximal sustainable equilibrium is that it gives rise to a date of certain and complete financial collapse later than that of any other equilibrium. Lagunoff and Schreft introduce the concept because their goal is to assess the inherent fragility of an economy. As a modeling device to find such an equilibrium, Lagunoff and Schreft assume that an investor makes an *optimistic forecast*—a forecast $\tilde{f}^i$ such that, for all $t \geq 0$ and each type $i = 1,2$, $\tilde{f}_t^i(W_t, x_j) = D$ and $\tilde{f}_t^i(H_t, x_j) \neq D$. An investor with this forecast expects that other investors continue to hold diversified and linked portfolios until they personally experience losses and then shift to safer portfolios. The optimistic forecast implies that investors foresee the contagious financial crisis initiated by shocks to projects, but not the preemptive behavior that instantaneously induces total collapse. In contrast, the equilibrium concept defined above assumes that investors make conditionally correct forecasts. Such forecasts arise when investors have foresight about both the contagious crisis and the preemptive behavior, and so are less optimistic than optimistic forecasts. These forecasts are necessarily correct conditional on the signal investors observe and on the model they use in forming expectations. If investors observe the better signal, given the true state, and use the true probability distributions in forming expectations, then their forecasts also are unconditionally correct. In subsequent sections, all references to correct or incorrect forecasts refer to *unconditional* accuracy.

Interestingly, the optimistic forecast can be used to construct bounds on the exit-decision date of an investor with conditionally correct forecasts. The reason stems from the difference between an investor with an optimistic forecast and one with a conditionally correct forecast. The latter recognizes that there is a date at which everyone else in the economy exits; the former does not. An investor with an optimistic forecast thus never exits sooner than one with a conditionally correct forecast. And a type-i investor with a conditionally correct forecast remains diversified at all dates $t < t_i(e)$ because, by equilibrium condition ii, he expects everyone of type *i* to do so. It follows that the portfolio reallocations at $t_i(e)+1$ are due solely to beliefs about fundamentals, not to any coordination failure. When *e* = 0, there is only one type of conditionally correct investor, so his exit-decision date is the same as that in an economy with identical investors with optimistic forecasts. That is, for a type-*i* investor who thinks $\wp = p$ with probability 1, $t_i(0) = \bar{t}(p)$ by definition of $\bar{t}(p)$. Thus, in subsequent sections, the optimistic forecast is used to find $t_i(0)$.

This completes the description of the economy. It remains to analyze rational and irrational exuberance. This is done in the remainder of the paper by studying two special cases of the economy: one with strong fundamentals and rational investors, and one with weak fundamentals and irrational investors. For each case, bounds on the exit-decision date for each investor type are found, making use of the case *e* = 0 and its associated exit-decision date $\bar{t}(p)$.

III. Rationality, Exuberance, and Apprehension

A special case of the economy described above is used in this section to study rational exuberance and apprehension. Specifically, the true state of the economy is assumed to be w_1, so that the economy's fundamentals truly are strong ($\wp = 0$), and investors are assumed to have rational expectations as commonly defined. That is, investors are assumed to optimize and to form expectations using the true distributions $\Pr(w_i)$ and $\Pr(x_i|w_j)$, for *i*, *j* = 1,2.

In this rational economy, the investors who observe signal x_1 believe $\Pr(w_1|x_1) = 1-e$ by Bayes' Rule, and thus for $e < 0.5$ believe, correctly, that the true state is most likely w_1 and that shocks are unlikely to be realized at date 0. Given $\Pr(x_i|w_j)$, they believe, also correctly, that the other type-1 investors constitute a fraction $(1-e)^2 + e^2$ of the population and therefore that the type 1s are in the majority.

Similarly, investors who observe signal x_2 believe $\Pr(w_1|x_2) = e$ by Bayes' Rule, and thus believe that the true state is most likely w_2. That is, unlike the exuberant type-1 investors, the type 2s believe that shocks most likely are realized at date 0 and thus are apprehensive about their portfolios' prospects. Given their type and their beliefs about the state, they also believe that the other type-2 investors represent a fraction $(1-e)^2 + e^2$ of the population. They are incorrect, of course, about which state is most likely and about their type's majority status in the economy, but their beliefs are nonetheless rational.

If $e = 0$, this economy is characterized by *pure rational exuberance*: all investors observe signal x_1 and believe correctly that the true state is w_1. No financial crises occur because no shocks initiate the unraveling of chains and no investors anticipate incurring losses because of contagion. The exit-decision date for these rational investors, represented by $t_1^R(0)$, equals $\bar{t}(0)$, which is infinity.

For economies with $e > 0$, no matter how small, the analysis is significantly more difficult because there are two types of investors in the economy, each forecasting the state, the presence of investors of other types, and the beliefs of those other types. Because a total collapse of the economy's financial structure occurs when the last group of investors reallocates their portfolios, it is easiest to start by examining the apprehensive investors—the ones who expect shocks to be realized at date 0. As stated above, the apprehensive investors (type 2) think that there is a fraction e of the population of type 1. To make conditionally correct forecasts of the type 1s' behavior, the apprehensive investors use their own priors, but since their priors are common to all investors, their forecasts are accurate: they believe that the type 1s expect no shocks to be realized and thus to remain diversified at least as long as they themselves do. Each apprehensive investor also forecasts correctly that the other apprehensive investors all make the same forecasts that he makes and find it a best response to hold a diversified portfolio when they expect everyone else to do so. With these beliefs, the environment in which an apprehensive investor pictures himself is the same as that in Lagunoff and Schreft when shocks are realized with probability $\bar{p}$ and investors think everyone else remains diversified at least as long as he does (the optimistic forecast). It follows that an apprehensive (type 2) investor's exit-decision date, denoted $t_2^R(e)$, is simply $\bar{t}(\bar{p}) \geq 0$.

Like their apprehensive counterparts, the exuberant type-1 investors accurately forecast the beliefs of the type 2s because they use the true and common prior $\Pr(w_i)$. As a result, they anticipate the apprehensive investors deciding at date $\bar{t}(\bar{p})$ to exit and initiating a contagion at $\bar{t}(\bar{p}) + 1$. The best response for the exuberant investors is to decide to exit at some date $t_1^R(e)$, where $t_1^R(e) \leq t_1^R(0) = \infty$.

The date $t_1^R(e)$ is a function of how many projects the type-1 investors expect to fail at $\bar{t}(\bar{p})$, which depends, in turn, on which portfolio the type 2s switch to and how many type-2 investors exist. Type-1 investors correctly believe that the type 2s constitute a faction e of the population, and they can accurately forecast the portfolio to which the type 2s switch. If, for example, the apprehensive type 2s decide at date $\bar{t}(\bar{p})$ to switch to a safe portfolio, then at date $\bar{t}(\bar{p})$ + 1 they do not reinvest in any of their previously held projects. The worst-case scenario in terms of fragility occurs when the e type-2 investors have no projects in common in their portfolios. In this case, 2e projects fail when the type 2s exit at $\bar{t}(\bar{p})$ + 1. In the best case, all the type 2s are in adjacent projects in chains, and their portfolio reallocations cause e + 1/k projects to fail. If, instead, the apprehensive investors deviate to a part-safe-part-risky portfolio at date $\bar{t}(\bar{p})+1$, then each immediately consumes a dollar (because the safe asset pays zero interest and investors discount the future) and reinvests a dollar in one project. This causes e projects to fail at $\bar{t}(\bar{p})+1$ in the worst case, $e/2$ to fail in the best case, and type 2s to find themselves with at most $\bar{R}$ dollars to allocate at $\bar{t}(\bar{p})+1$ between current consumption and future investment. By equilibrium condition iv, the type 2s do not reinvest at date $\bar{t}(\bar{p})+2$, causing the failure of another $ek/2$ to ek of all projects.

In choosing strategies, exuberant investors use their forecasts of the type 2s' date-($\bar{t}(\bar{p})+1$) portfolio allocations and of how investors of different types are distributed around chains. Whatever these forecasts, it is clear that the type 1s' best response to the type 2s exiting at $\bar{t}(\bar{p})$ + 1 is to exit themselves at some date $t_1^R(e)$ + 1, where $\infty \geq t_1^R(e) \geq t_2^R(e) = \bar{t}(\bar{p}) \geq 0$, even though they correctly believe that the economy's fundamentals most likely do not warrant it. This result is summarized in the following proposition, which is proven in Appendix B along with all other results.

PROPOSITION 1. If $\bar{p}$ is sufficiently large that $\bar{t}(\bar{p}) < \infty$, then there exists $\bar{e} > 0$ such that $t_1^R(e) \geq t_2^R(e)$ for all $e \in [0, \bar{e})$.

The exiting of the rationally exuberant investors from the economy brings about total financial collapse because they are the last of all investors to exit. The economy could collapse at an earlier date by chance if the right mix of shocks is realized at date 0 and all chains happen to unravel before $t_1^R(e)+1$, but it collapses with certainty at date $t_1^R(e)+1$. And, as the following proposition states, date $t_1^R(e)$ is nonincreasing in e.

PROPOSITION 2. $t_1^R(e_1) \geq t_1^R(e_2)$ if $e_1 \leq e_2$.

Intuitively, when *e* is higher, type-1 investors assign lower probability to state w_1, with $\wp$ = 0, and they believe that a smaller fraction of the population expects that $\wp$ = 0. Thus, an increased presence of apprehensive investors in the economy does not reduce fragility.

A third, and striking, finding is that the financial collapse that occurs because type-1 investors decide at $t_1^R(e)$ to exit is indistinguishable from the collapse that occurs in an economy with homogeneous beliefs and certainty that $\wp = p'$ for some p'. Formally:

PROPOSITION 3. There exists p' such that $\bar{t}(p') = t_1^R(e)$.

That is, a financial crisis that looks as if it had been initiated by real shocks actually could have been caused solely by the presence of a small share of investors who were apprehensive—though rationally so—about the economy's prospects. Without knowledge of the true fundamentals, an observer cannot determine whether a crisis occurred because of less-than-pure rational exuberance in a fundamentally sound economy.

IV. Irrationality, Exuberance, and Apprehension

A second special case of the economy of Section II is suitable for the analysis of irrational exuberance and apprehension. In this variant, the true state of the economy is w_2, the state in which the fundamentals are poor ($\wp = \bar{p}$), and investors' expectations are formed irrationally in the following sense: investors optimize and know the true likelihoods (that is, they know that $\Pr(x_i|w_i)$ = 1–*e* and $\Pr(x_j|w_i)$ = *e* for $j \neq i$), but they do not use the true priors over the w_i. Instead, a type-*i* investor updates his beliefs using the subjective prior

$$\Pr^i(w_i) = \frac{e^2}{(1-e)^2 + e^2},$$

while type $j \neq i$ updates from the prior

$$\Pr^j(w_i) = \frac{(1-e)^2}{(1-e)^2 + e^2}.$$

That is, a type-1 investor uses a *pessimistic prior*, one that assigns more probability weight to the worse state, w_2, while a type-2 investor uses an *optimistic prior*, assigning greater probability weight to state w_1. In addition, investors believe that their own priors are the ones used by all investors, and they do not recognize the dependence of their priors on the state.

Clearly, many subjective priors are consistent with investors forming expectations irrationally in some sense. The priors assumed here are used because they generate beliefs about the probability of the states w_i that are the exact opposite of those held by the rational agents of Section III. Specifically, given these priors, the application of Bayes' Rule yields the posteriors $\Pr^i(w_i|x_i) = e$ and $\Pr^j(w_i|x_j) = 1-e$ after investors observe their private signals. This implies, given that the true state is w_2, that a fraction $1-e$ of investors observes signal x_2 and so is of type 2. The type 2s believe, given their optimistic subjective priors, that the true state is w_1 with probability $1-e$ and w_2 with probability e. Likewise, a fraction e observes x_1 and is of type 1. They pessimistically assign probability e to state w_1 and probability $1-e$ to state w_2. Thus, in this economy with irrational investors, it is the type 2s who end up with posterior beliefs that are unjustifiably exuberant and the type 1s who have posteriors that are apprehensive and relatively more correct, at least about the state.

The use of the subjective priors introduces an interesting complication into the analysis. Since investors believe, incorrectly, that their subjective priors are the priors used by all investors, they make incorrect forecasts of both the distribution of investor types in the economy and the beliefs of investors not of their type. Specifically, in forecasting the beliefs of the type 2s, the pessimistic type-1 investors, who think the state is most likely w_2, use Bayes' Rule with their subjective prior, $\Pr^j(w_i) = (1-e)^2 \big/ \left((1-e)^2 + e^2\right)$, and conclude that the type-2 investors assign probability $e^3 \big/ \left[(1-e)^3 + e^3\right]$, which is less than e for $e < 0.5$, to state w_1. That is, the type 1s believe, incorrectly, that the type 2s think state w_1, with $\wp = 0$, is even less likely than they themselves believe it to be, and thus that the type 2s are the most apprehensive investors in the economy. But given the known likelihoods $\Pr(x_j|w_i) = e$ and $\Pr(x_i|w_i) = 1-e$, the type 1s think their type is a minority of the population and that the type 2s are the majority.[10] In this, at least, they are correct, since the true state is w_2. The optimistic type 2s, who think the true state is w_1, use the same approach to forecasting the beliefs of the type-1 investors. They wrongly conclude

[10] It can readily be verified that a type *i* investor assigns probability $2e(1-e)$ to other investors being of type *i*.

that the type 1s assign probability $(1-e)^3/\left[(1-e)^3+e^3\right]$, which exceeds 1–*e* for $e<0.5$, to state w_1. As a result, the type 2s think the type 1s are the most exuberant investors in the economy. And they wrongly believe that the type 1s are in the majority. Every investor, then, incorrectly believes that his own type is the minority and that everyone else is at least as exuberant or apprehensive as he is.

Investors use these irrational forecasts in determining their strategies. The true apprehensive investors, the type 1s, think that shocks most likely are realized at date 0, but they also think that the type 2s assign even higher probability to such an outcome and believe themselves to be in the majority. The type 1s thus predict that the type 2s believe that the majority of the population thinks a contagious financial crisis to be very likely and decides at some date to exit. If, for example, e = 1/*k*, which approaches zero as *k* approaches infinity, there is one type-1 investor in the economy. This sole type 1 believes first that everyone else (the type 2s) thinks that virtually everyone is of type 2, and second, that the type 2s think $\wp = \overline{p} > 0$ is at least as likely as he thinks it is.[11] The type 1 also perceives type-2 investors as thinking that all investors continue to hold a diversified portfolio as long as their expected lifetime utility from doing so exceeds that from switching to the next-best alternative portfolio. But this is exactly what an investor in the Lagunoff and Schreft economy thinks when the probability of shocks is believed to be $\overline{p}$. Consequently, the type-1 investor forecasts that the type 2s' exit-decision date is $\overline{t}(\overline{p}) \geq 0$. This is an ominous forecast for the type-1 investor because the exiting of the type 2s causes projects to fail at $\overline{t}(\overline{p}) + 1$ in addition to those the type-1 investor expects to fail from the contagion induced by date-0 shocks. The best response for the type-1 investor in this environment is to choose an exit-decision date $t_1^I(e)$, where $t_1^I(e) \geq t_1^I(1/k) = \overline{t}(\overline{p}) \geq 0$. In summary, although the type 1s correctly forecast which state is most likely, they incorrectly forecast the forecasts of the type 2s, and thus perceive the risks to their portfolios as greater than they are in actuality. That is, their prior about the state is pessimistic, but they end up even more pessimistic about the prospects for their portfolios because they incorrectly forecast the beliefs of others. Type-1 investors thus personify Chicken Little, thinking that the sky is falling and reacting accordingly, contributing to the economy's fragility by exiting at least as

[11] It does not make sense to analyze the case of $e = 0$ in this irrational economy because all investors, thinking that their type is a fraction e of the population, would thus think that they did not exist. Such thinking would reflect insanity, not irrationality. Thus, the smallest e can reasonably be is 1/*k*.

early as they would if they recognized the type 2s' true exuberance.

The exuberant type 2s do not fare any better in forecasting the type 1s' behavior. They think it unlikely that any shocks are realized at date 0, and they perceive the type 1s as viewing shock realizations as even less likely and thinking type 1s are in the majority. The type 2s therefore think that the type 1s are likely to continue to hold diversified portfolios at least as long as the type 2s. Again, if, for example, $e = 1/k$, then each type-2 investor believes incorrectly that he is the only type-2 investor, is virtually certain that no shocks hit at date 0, and thinks that everyone else in the economy is at least as certain that no shocks are realized. This is the economy of Lagunoff and Schreft with a true probability of project failure of zero. In the limit, as k approaches ∞, the exit-decision date of the type 2s, denoted $t_2^I(1/k)$, equals $\bar{t}(0)$, which is infinity. Thus, for any e, $t_2^I(e)$ satisfies $0 \leq \bar{t}(\bar{p}) \leq t_1^I(e) \leq t_2^I(e) \leq \bar{t}(0) \leq \infty$. In summary, type-2 investors' prior about the state is optimistic, but they end up even more optimistic about the prospects for their portfolios because of their incorrect forecasts. Type-2 investors, then, are the quintessential Cockeyed Optimists, sustaining the economy's financial structure longer than they would if they recognized the type 1s' true apprehension. The following proposition formalizes these results:

PROPOSITION 4. If $\bar{p}$ is sufficiently large that $\bar{t}(\bar{p}) < \infty$, then there exists $\bar{e} > 0$ such that $t_1^I(e) \leq t_2^I(e)$ for all $e \in [0, \bar{e})$.

Since total financial collapse occurs with certainty when the last investors reallocate their portfolios, the behavior of the type-2 investors determines the date of collapse in the economy with irrationality. Total collapse occurs at date $t_2^I(e) + 1$, where $t_2^I(e) \geq t_1^I(e) \geq 1$. Of course, as e increases, type 2s think all investors assign less probability to state w_1 and believe that there are more type 2s around to exit early. That is,

PROPOSITION 5. $t_2^I(e_1) \geq t_2^I(e_2)$ if $e_1 \leq e_2$.

It follows that the economy is at least as fragile the greater is e.

Furthermore, the irrational type-2 investors are not only irrationally exuberant; they are also more exuberant than the rational type-1 investors are in an economy in which the true state is w_1

(i.e., the type-1 investors studied in Section III). This result arises because the irrationally exuberant type-2 investors misforecast by a large margin both the likelihood of the true state and the beliefs of the minority group (the type-1 investors). Because these type 2s believe the irrationally apprehensive type 1s to be the truly irrationally exuberant investors in the economy, the type 2s are more irrationally exuberant than they would be otherwise. Thus, irrational type-2 investors exit later than rational type-1 investors, which means that the economy with irrational exuberance is less fragile than that with rational exuberance, given ***e***. Similarly, the irrationally apprehensive type-1 investors are even more apprehensive than the rationally apprehensive type-2 investors. The following proposition formally states these results.

PROPOSITION 6. (a) $t_2^I(e) > t_1^R(e) \geq 0$, and (b) $0 \leq t_1^I(e) < t_2^R(e)$.

V. Implications for Policy

The preceding sections examine economies with particular combinations of the state and beliefs for the purpose of characterizing rational and irrational exuberance. Collectively, the results yield a ranking of exit-decision dates:[12]

$$0 \leq \bar{t}(\bar{p}) \leq t_1^I(e) < t_2^R(e) \leq t_1^R(e) < t_2^I(e) \leq \bar{t}(0) = \infty .$$

Since these dates are independent of the true state, they can be applied to more general economies that consist of both rational and irrational, type-1 and type-2 investors.[13]

By using these exit-decision dates, investors maximize their expected lifetime utility, given the signal they observe about the state and their priors. In some sense, then, they are as well off as possible if they use these exit dates. This is true even though some of them exit too early or too late

[12] More specifically, this ranking of exit-decision dates follows from Propositions 1, 4, and 6. Thus, it applies to economies for which those propositions hold. Such economies must have two features. First, they must have the two critical properties discussed in section IIb, namely that portfolios are initially linked and that there is a mechanism for breaking linkages. Lagunoff and Schreft have shown that these properties characterize the equilibrium of some, but not all, economies. Second, they must have the property, required for Proposition 1 to hold (the proof is in Appendix B), that the true probability of a project's failure due to a shock, $\bar{p}$, be large enough that, if all investors believe that $\bar{p}$ is the true probability with certainty, then they have a finite exit-decision date $\bar{t}(\bar{p})$ that is decreasing in $\bar{p}$.

[13] These exit-decision dates are for an economy of a given size *k* (i.e., measured in terms of number of investors and projects). Lagunoff and Schreft show that for sufficiently large economies, the exit-decision date of type *i* investor decreases as the economy becomes even larger. This result applies to each investor type here as well. The ranking of exit-decision dates remains the same as the economy becomes larger since it was derived for any given economy size.

relative to what is appropriate based on the true state. For example, if the true state is w_1, e is approximately zero, and k is very large (approaching infinity), then approximately all investors observe signal x_1 and the exit-decision dates are simply $\bar{t}(\bar{p}) \cong t_1^I < t_1^R \cong \infty$. If investors discover the true state after the collapse is over, the irrational ones realize that no shocks were realized and regret having exited preemptively.

Even if investors never discover the true state, there will always be some who regret their decisions. They are the ones who incur losses from contagion before their scheduled exit date. Depending on the realization of shocks and the initial chain structure, there could be many such investors, or just a few. These investors will look to policymakers for remedies and for assurances that such a crisis will not occur again. Of course, no such assurances can be given because shocks and beliefs are exogenous and the chain structure is unknown, but there are some policy options.

In the short term, policymakers are limited to trying to stem any ongoing contagion. A commonly used approach is for some institution to serve as a lender of last resort who lends funds immediately after shocks are realized to prevent additional losses. In the economy of Section II, such an institution could be modeled as being known to and able to provide information to all investors, but having no greater knowledge of the economy than anyone else (i.e., it does not know the chain structure or realization of shocks). It could announce that it stands ready to extend emergency credit and could raise resources to fund its activities either by creating funds (e.g., issuing a fiat currency) or taxing the return on projects. Agents would reveal themselves to the lender of last resort to obtain credit. Since investors are the only agents in the economy, they are the ones to whom any loans must go. This by itself makes the policy problematic. The reason is that loans must go to investors who actually incur losses and, by equilibrium condition iv, are about to reallocate their portfolios, initiating the first round of contagion. But portfolios are private information, so all investors, whether or not they have incurred losses, have an incentive to request loans under such a policy, and policymakers have no way to identify the legitimate borrowers. The policy also is ineffective because at best it postpones the contagion for two periods.[14] This stems

[14] It is instructive to see why lending to investors does not work here because the reason highlights the role portfolio linkages play in generating contagion. Someone who incurs losses on both investments because of shocks, and thus borrows $2, cannot reinvest in his formerly held projects because they are no longer operative. He instead must put the funds into a safe portfolio, which means he either consumes them immediately or consumes $1 and puts $1 in the safe asset for one period. An investor who only loses $1 from shocks still has a project in which he can reinvest. In the best-case scenario for reducing contagion, he invests the $1 he borrows plus the remaining dollar of his endowment into the

from the simplicity of the model and the nature of the chain structure.

A lender of last resort conceivably could end contagion, at least on average, by making loans to projects. This policy could be used if the model is modified by adding agents who operate projects and can reveal themselves if their projects do not receive sufficient funding to operate. There typically is a moral-hazard problem, however, in making loans to such agents, so it is unclear whether the policy can improve welfare on net.

The model also suggests some longer-term measures that a lender of last resort or government entity can take to reduce the likelihood of contagion. For example, monitoring lenders' portfolios provides information about the economy's linkages that a lender of last resort can use to identify projects at risk of receiving insufficient funding and to coordinate a response among investors. The supervision of banks by governmental regulatory organizations is an example of a policy that achieves this end. A second example is policies that result in greater diversification and thus increase the number of shock-induced project failures needed to initiate a crisis. An implementation of such a policy in the United States is the restriction that allows money-market mutual funds to hold only a limited share of their portfolios in the commercial paper of a single issuer. These types of long-term measures can result in a stronger, more resilient chain structure and thus can reduce the likelihood and severity of contagious financial crises.

For such policies to be effective, however, they must be adopted by an entity with authority regarding all possible portfolio linkages because all linkages contribute to an economy's fragility. Thus, if the model is taken to be one of a small closed economy with strict capital controls, then the entity could be a domestic institution. For open economies with global linkages, fighting contagion requires an organization of international scope.

Appendix A

This appendix presents the expected lifetime utilities associated with an investor's

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one project that is left in his portfolio. But his former coinvestor also reinvests either \$1 or \$2. As a result, his pre-tax return is at most $4\overline{R}/3$. Given the bounds on $\overline{R}$, the tax rate, and equilibrium condition iv, he at best only consumes his interest and reinvests \$1 for one additional period. Thus, a policy of extending emergency credit to investors who incurred losses at most postpones the contagion for a period or two.

continuing to hold diversified portfolios or deviating to another portfolio in the dynamic recursive game that begins after the shocks are realized. As explained in Section II-C, the bounds on the exit-decision dates $t_i(e)$ are the exit-decision dates from an economy with investors who make optimistic forecasts and are homogeneous ($e = 0$) and where the true probability of shocks $\wp$ is believed to be p. That economy is the one described in Lagunoff and Schreft. Hence, the expected utilities for all $t < t_i(e)$ here are those from Lagunoff and Schreft, who provide a thorough derivation in their Appendix A.[15]

In this model, the expected lifetime utility from any portfolio must depend on both the probabilities of being in the various chains and the expected utilities associated with those chains. Calculating the latter for open chains requires considering the utility associated with the various positions in the chain and the likelihood of an investor's being in each position. An open r-link chain has $r - 1$ investors who are in state W_t. Two of them will lose \$1 because their coinvestors are in state H_t and will not reinvest, while the rest will not incur losses if they remain diversified another period but will incur losses eventually as their chain shrinks and their coinvestors end up in state H_t. The expected lifetime utility from remaining diversified if in an open r-link chain, denoted $p^D(r)$, is thus

$$p^D(r) = \begin{cases} \dfrac{2}{r-1}\max\left\{u\left(\overline{R}\right), u\left(\overline{R}-1\right) + bu(1)\right\} + \left(1 - \left(\dfrac{2}{r-1}\right)\right)\left(u\left(2\overline{R}-2\right) + bp(r-2)\right) & \text{if } r \geq 3, \\ 0 & \text{if } r = 2. \end{cases}$$

In contrast, investors in closed chains can remain diversified permanently, enjoying the lifetime utility of $\overline{p} \equiv u\left(2\overline{R}-2\right)/(1-b)$.

With C_{rt} $\left(N_{rt}\right)$ denoting the event that an investor is in a closed (open) r-link chain at the beginning of date t, the probability of an investor's being in a closed (open) chain at the beginning of date t, conditional on his being in state W_t and having observed signal x_i, can be denoted $\Pr\left(C_{rt} \middle| W_{t-1}, x_i\right)$ $\left(\Pr\left(N_{rt} \middle| W_{t-1}, x_i\right)\right)$.[16] A closed r-link chain after returns are realized at $t \geq 0$ must have evolved from a closed r-link chain that existed at the start of date 0 and had no project failures

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[15] Investors' strategies at date $t_i(e)$ in this model may differ from those in Lagunoff and Schreft because here type i investors correctly forecast that at $t_i(e)$ all investors of type i decide to exit, whereas in Lagunoff and Schreft investors forecast that everyone remains diversified who has not incurred a loss. This difference is immaterial, however, because the focus here is only on the date at which investors exit, not on which portfolio they switch to at that date.

[16] The probabilities are unchanged if they are calculated after returns are realized at date $t - 1$, when the investor makes his consumption-investment decision.

due to shocks. If $q(r,k)$ denotes the probability of an investor belonging to a closed r-link chain before the shocks, then $\Pr(C_{r0}|W_{-1},x_i)=q(r,k)$. And, at the start of any $t \geq 1$, the *nonnormalized* conditional probability of a closed r-link chain, denoted $\overline{\Pr}(C_{rt}|W_{t-1},x_i)$, is $q(r,k)(1-p)^{r-2}$ if $2 \leq r \leq k$ and zero otherwise.

Likewise, an open r-link chain at date t had to have evolved from an open chain that existed immediately after the shocks were realized, which itself had to have come from a closed chain that existed initially. The nonnormalized probability of an investor's being in an open r-link chain at the start of date 0, denoted $\overline{\Pr}(N_{r0}|W_{-1},x_i)$, is clearly zero. But

$$\overline{\Pr}(N_{r1}|W_0,x_i)=(r-1)(1-p)^{r-2}\left[p^2\sum_{m=r+2}^{k}q(m,k)+pq(r+1,k)\right]$$

because an open r-link chain at date 1 could have come about by exactly one shock hitting a closed $(r+1)$-link chain at date 0, or two shocks hitting a closed chain with at least $r+2$ links at date 0. This expression makes use of two facts: only the two shocks that create the endpoints of an open chain matter, and an investor knows his own portfolio and thus that, if he is in state W_t, the two projects in his portfolio could not have been hit by shocks. The probability of open chains at later dates takes into consideration that an open chain will have lost two projects in each preceding period. Thus, $\overline{\Pr}(N_{rt}|W_{t-1},x_i)=\overline{\Pr}(N_{n1}|W_{t-1},x_i)$, where $n = r + 2(t-1)$.

These probabilities can be normalized by dividing them by Λ_t, the total probability weight attached to all r-link chains at date t, where

$$\begin{aligned}\Lambda_t &\equiv \sum_{r=2}^{k}\overline{\Pr}(C_{rt}|W_{t-1},x_i)+\sum_{r=2}^{k}\overline{\Pr}(N_{rt}|W_{t-1},x_i)\\ &=\sum_{r=2}^{k}\overline{\Pr}(C_{rt}|W_{t-1},x_i)+\sum_{r=2t}^{k}\overline{\Pr}(N_{r1}|W_0,x_i).\end{aligned}$$

The normalized probabilities are thus

$$\Pr(C_{rt}|W_{t-1},x_i)=\begin{cases}\overline{\Pr}(C_{rt}|W_{t-1},x_i)/\Lambda_t & \text{if } 2\leq r\leq k\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\Pr(N_{rt}|W_{t-1},x_i)=\begin{cases}\overline{\Pr}(N_{r1}|W_0,x_i)/\Lambda_t & \text{if } 2\leq r\leq k-1-2(t-1)\\ 0 & \text{otherwise.}\end{cases}$$

Making use of these expressions, the expected lifetime utilities from an investor's taking various strategies in various states can now be specified:

$$V_t\left(f^i,\tilde{f}^i|Z_t,x_i\right)=0.$$

$$V_t\left(f^i,\tilde{f}^i|H_t,x_i\right)=\begin{cases}\max\left\{u\left(\bar{R}\right),u\left(\bar{R}-1\right)+bu(1)\right\} \text{ if } f_t^i\left(H_t,x_i\right)=S,\\ u\left(\bar{R}-1\right)+b\left\{\left[1-\Pr\left(N_{1\,t+1}|H_t,x_i\right)\right]V_{t+1}\left(f^i,\tilde{f}^i|H_{t+1},x_i\right)\right\}\\ \quad \text{if } f_t^i\left(H_t,x_i\right)=P.\end{cases}$$

$$V_t\left(f^i,\tilde{f}^i|W_t,x_i\right)=\begin{cases}\max\left\{u\left(2\bar{R}\right),u\left(2\bar{R}-1\right)+bu(1),u\left(2\bar{R}-2\right)+bu(2),u\left(2\bar{R}-2\right)+bu(1)+b^2u(1)\right\}\\ \quad \text{if } f_t^i\left(W_t,x_i\right)=S,\\ u\left(2\bar{R}-2\right)+b\left\{\sum_{r=3}^{k}\Pr\left(N_{r\,t+1}|W_t,x_i\right)\left(\frac{2}{r-1}\right)V_{t+1}\left(f^i,\tilde{f}^i|H_{t+1},x_i\right)\right.\\ \quad +\left[\sum_{r=3}^{k}\Pr\left(N_{r\,t+1}|W_t,x_i\right)\left(1-\frac{2}{r-1}\right)+\sum_{r=2}^{k}\Pr\left(C_{r\,t+1}|W_t,x_i\right)\right]V_{t+1}\left(f^i,\tilde{f}^i|W_{t+1},x_i\right)\\ \quad \left.+\Pr\left(N_{2\,t+1}|W_t,x_i\right)V_{t+1}\left(f^i,\tilde{f}^i|Z_{t+1},x_i\right)\right\} \text{ if } f_t^i\left(H_t,x_i\right)=D,\\ u\left(2\bar{R}-1\right)+b\left\{\left[\sum_{r=3}^{k}\left[\Pr\left(N_{r\,t+1}|W_t,x_i\right)\left(1-\frac{2}{r-1}\right)+0.5\Pr\left(N_{r\,t+1}|W_t,x_i\right)\left(\frac{2}{r-1}\right)\right.\right.\right.\\ \quad \left.+\sum_{r=2}^{k}\Pr\left(C_{r\,t+1}|W_t,x_i\right)\right]V_{t+1}\left(f^i,\tilde{f}^i|H_{t+1},x_i\right)\\ \quad \left.+\Pr\left(N_{2\,t+1}|W_t,x_i\right)V_{t+1}\left(f^i,\tilde{f}^i|Z_{t+1},x_i\right)\right\} \text{ if } f_t^i\left(H_t,x_i\right)=P,\\ u\left(2\bar{R}-2\right)+b\left\{\left[\sum_{r=3}^{k}\Pr\left(N_{r\,t+1}|W_t,x_i\right)\left(1-\frac{2}{r-1}\right)+\sum_{r=2}^{k}\Pr\left(C_{r\,t+1}|W_t,x_i\right)\right]u\left(\frac{4\bar{R}}{3}\right)\right.\\ \quad \left.+\sum_{r=2}^{k}\Pr\left(N_{r\,t+1}|W_t,x_i\right)\left(\frac{2}{r-1}\right)\left[\frac{1}{2}\left(\frac{u\left(2\bar{R}-2\right)}{1-b}\right)+\frac{1}{2}u\left(\frac{4\bar{R}}{3}\right)\right]\right\}\\ \quad \text{if } f_t^i\left(H_t,x_i\right)=U.\end{cases}$$

Appendix B

This appendix proves the propositions, making use of the expected utility functions specified in Appendix A. For notational simplicity in what follows, first define

$$\bar{v} \equiv \max\left\{u\left(\bar{R}\right), u\left(\bar{R}-1\right)+bu(1)\right\},$$

$$\bar{\bar{v}} \equiv \max\left\{u\left(2\bar{R}\right), u\left(2\bar{R}-1\right)+bu(1), u\left(2\bar{R}-2\right)+bu(2), u\left(2\bar{R}-2\right)+bu(1)+b^2u(1)\right\}.$$

Further, define probability weights

$$d_t \equiv \sum_{r=3}^{k} \Pr\left(N_{r\,t+1}|W_t, x_i\right)\left(1-\frac{2}{r-1}\right)+\sum_{r=2}^{k} \Pr\left(C_{r\,t+1}|W_t, x_i\right),$$

$$g_t \equiv \sum_{r=3}^{k} \Pr\left(N_{r\,t+1}|W_t, x_i\right)\left(\frac{2}{r-1}\right).$$

It follows that

$$1-d_t-g_t=\Pr\left(N_{2\,t+1}|W_t, x_i\right).$$

Let $f^{i,a}$ denote a strategy in which investor i chooses action a at date t in state W_t. A necessary condition for $f^{i,a}$ to induce a maximal sustainable equilibrium from date $t+1$ on is that equilibrium condition iv—that $f^i_{t+1}\left(H_t, x_i\right)=S$, or equivalently, $V_{t+1}\left(f^{i,S}, \tilde{f}^i|H_{t+1}, x_i\right)=\bar{v}$—holds. Hence, the expected lifetime utility in state W_t from a strategy $f^{i,a}$ that induces a maximal sustainable equilibrium from date $t+1$ on is, for each $a=D,U,P,S$ and for each $t<\bar{t}(p)$

$$\begin{aligned}
&V_t\left(f^{i,S}, \tilde{f}^i|W_t, x_i\right)=\bar{v},\\
&V_t\left(f^{i,D}, \tilde{f}^i|W_t, x_i\right)=u\left(2\bar{R}-2\right)+b\left\{d_tV_{t+1}\left(f^{i,D}, \tilde{f}^i|W_{t+1}, x_i\right)+g_t\bar{v}\right\},\\
&V_t\left(f^{i,P}, \tilde{f}^i|W_t, x_i\right)=u\left(2\bar{R}-1\right)+b\left\{d_t+g_t\right\}\bar{v}, \qquad (1)\\
&V_t\left(f^{i,U}, \tilde{f}^i|W_t, x_i\right)=u\left(2\bar{R}-2\right)+b\left\{d_tu\left(\frac{4\bar{R}}{3}\right)\right.\\
&\qquad\left.+g_t\left(0.5\left(u\left(\frac{4\bar{R}}{3}\right)\right)+0.5\frac{u\left(2\bar{R}-2\right)}{1-b}\right)+\left(1-d_t-g_t\right)\frac{u\left(2\bar{R}-2\right)}{1-b}\right\}.
\end{aligned}$$

The next lemma is used in the proofs of the proposition that follow.

LEMMA. Suppose that $\bar{\bar{v}} \geq 1.5\bar{v}$. Then $\bar{t}(\cdot)$ is decreasing in p.

PROOF. Fix an equilibrium strategy f^{i*} such that investor i chooses action D at date t. Let $f^{i,a*}$ denote the equilibrium strategy identical to f^{i*} except that the investor chooses action a at date t. Let $V_t^p(\;)$ denote the expected lifetime utility from a strategy, so the dependence on p is explicit.

Given p, observe that $\bar{t}(p)$ is the first date t for which

$$V_t^p\left(f^{i*}, \tilde{f}^i|W_t, x_i\right)<\max_{a\neq D} V_t^p\left(f^{i,a*}, \tilde{f}^i|W_t, x_i\right).$$

To show that $\bar{t}(p)$ is decreasing, it suffices to show that if $V_t^p\left(f^{i*}, \tilde{f}^i|W_t, x_i\right) \geq$ $V_t^p\left(f^{i,a*}, \tilde{f}^i|W_t, x_i\right)$ for each $a \neq D$, then $V_t^p\left(f^{i*}, \tilde{f}^i|W_t, x_i\right) - V_t^p\left(f^{i,a*}, \tilde{f}^i|W_t, x_i\right)$ is decreasing in p for each $a \neq D$.

Case (i). $a = S$.

From (1), the difference $V_t^p\left(f^{i*}, \tilde{f}^i|W_t, x_i\right) - V_t^p\left(f^{i,S*}, \tilde{f}^i|W_t, x_i\right)$ may be expressed as

$$\left[u\left(2\bar{R}-2\right)-\bar{\bar{v}}\right]+b\left\{ d_t V_{t+1}^p\left(f^{i*}, \tilde{f}^i|W_{t+1}, x_i\right) + \bar{g}\right\}. \tag{2}$$

By definition, $V_{t+1}^p\left(f^{i*}, \tilde{f}^i|W_{t+1}, x_i\right) \geq \bar{\bar{v}} \geq \bar{v}$ since f^{i*} induces an equilibrium from date t + 1 on such that the investor does no worse than he would from choosing S at time t + 1 and thus getting utility of $\bar{\bar{v}}$. Also, it is clearly the case that $V_t^p\left(f^{i*}, \tilde{f}^i|W_t, x_i\right) \geq V_t^p\left(f^{i*}, \tilde{f}^i|Z_t, x_i\right) = 0$.

To show that (2) is decreasing in p, two facts must be established. First, for $r \geq 3$, it must be shown that the probability weight assigned to $V_{t+1}^p\left(f^{i*}, \tilde{f}^i|W_{t+1}, x_i\right)$ in (2), which is given by $\Pr\left(N_{r\,t+1}|W_t, x_i\right)\left(1-\left(2/r-1\right)\right)+\Pr\left(C_{r\,t+1}|W_t, x_i\right)$, decreases relative to the probability weight assigned to $\bar{v}$, which is $\Pr\left(N_{r\,t+1}|W_t, x_i\right)\left(2/\left(r-1\right)\right)$. That is, it must be shown that

$$\frac{\Pr\left(N_{r\,t+1}|W_t, x_i\right)\left(1-\dfrac{2}{r-1}\right)+\Pr\left(C_{r\,t+1}|W_t, x_i\right)}{\Pr\left(N_{r\,t+1}|W_t, x_i\right)\left(\dfrac{2}{r-1}\right)} \tag{3}$$

is decreasing in p for $r \geq 3$. Second, it must be shown that for $r = 2$, the weight on $V_{t+1}^p\left(f^{i*}, \tilde{f}^i|W_{t+1}, x_i\right)$, which is $\Pr\left(C_{2\,t+1}|W_t, x_i\right)$, decreases relative to the weight on zero. That is,

$$\frac{\Pr\left(C_{2\,t+1}|W_t, x_i\right)}{\Pr\left(N_{2\,t+1}|W_t, x_i\right)} \tag{4}$$

must be shown to be decreasing in p.

Both $\overline{\Pr}\left(N_{r\,t+1}|W_t, x_i\right)$ and $\overline{\Pr}\left(C_{r\,t+1}|W_t, x_i\right)$, defined in Appendix A, are continuous in p.

Using these formulas, expression (3) can be written

$$\frac{r-3}{2}+\frac{q(r,k)}{2\left(p^2 \sum_{m\geq r+2} q(m,k) + pq(r+1,k)\right)},$$

while (4) can be written

$$\frac{q(2,k)}{2\left(p^2 \sum_{m\geq r+2} q(m,k) + pq(3,k)\right)} .$$

Both are clearly decreasing in *p*. By demonstrating term by term in the sum over *r* that the probability weight shifts to smaller expected utilities as *p* increases, it can be established that $V_t^p\left(f^{i*}, \tilde{f}^i | W_t, x_i\right) - V_t^p\left(f^{i,S*}, \tilde{f}^i | W_t, x_i\right)$ decreases in *p*. This concludes the proof of Case (i).

Case (ii). $a = P$.

From (1), the difference $V_t^p\left(f^{i*}, \tilde{f}^i | W_t, x_i\right) - V_t^p\left(f^{i,P*}, \tilde{f}^i | W_t, x_i\right)$ may be expressed as

$$\left[u(2\bar{R}-2) - u(2\bar{R}-1)\right] + b\left\{ d_t\left[V_{t+1}^p\left(f^{i*}, \tilde{f}^i | W_t, x_i\right) - \bar{v}\right] + \frac{g_t}{2}\bar{v}\right\}. \tag{5}$$

Using the same term-by-term arguments given in Case (i), it suffices to show that $V_{t+1}^p\left(f^{i*}, \tilde{f}^i | W_{t+1}, x_i\right) - \bar{v} \geq (\bar{v}/2)$, or $V_{t+1}^p\left(f^{i*}, \tilde{f}^i | W_{t+1}, x_i\right) \geq 1.5\bar{v}$. As stated before, f^{i*} induces an equilibrium from date *t* + 1 on in which the investor can do no worse than to choose *S* at time *t* + 1, getting utility $\bar{\bar{v}}$. Hence, $V_{t+1}^p\left(f^{i*}, \tilde{f}^i | W_{t+1}, x_i\right) \geq \bar{\bar{v}} \geq 1.5\bar{v}$, the latter inequality following from the hypothesis.

Case (iii). $a = U$.

From (1), the difference $V_t^p\left(f^{i*}, \tilde{f}^i | W_t, x_i\right) - V_t^p\left(f^{i,U*}, \tilde{f}^i | W_t, x_i\right)$ may be expressed as

$$b\left\{ d_t\left[V_{t+1}^p\left(f^{i*}, \tilde{f}^i | W_t, x_i\right) - u\left(\frac{4\bar{R}}{3}\right)\right] \right.$$
$$\left. + g_t\left[\bar{v} - 0.5\left(u\left(\frac{4\bar{R}}{3}\right)\right) + 0.5\frac{u(2\bar{R}-2)}{1-b}\right] + (1 - d_t - g_t)\left[-\frac{u(2\bar{R}-2)}{1-b}\right]\right\}. \tag{6}$$

Observe that $\bar{v} \leq 0.5\left(u(4\bar{R}/3)\right) + 0.5\left(u(2\bar{R}-2)/(1-b)\right)$. But since $V_t^p\left(f^{i*}, \tilde{f}^i | W_t, x_i\right) \geq V_t^p\left(f^{i,U*}, \tilde{f}^i | W_t, x_i\right)$ at date *t* by supposition, it must be true that $V_{t+1}^p\left(f^{i*}, \tilde{f}^i | W_{t+1}, x_i\right) \geq u\left(4\bar{R}/3\right)$.

Hence, the term-by-term argument of Case (i) can be used once again to establish the result.

QED. ■

PROPOSITION 1. If $\bar{p}$ is sufficiently large that $\bar{t}(\bar{p}) < \infty$, then there exists $\bar{e} > 0$ such that

$t_1^R(e) \geq t_2^R(e)$ for all $e \in [0, \bar{e})$.

PROOF. Clearly, $t_1^R(0) = \bar{t}(0) = \infty$, and $t_2^R(0) = \bar{t}(\bar{p}) < \infty$, where $\bar{p}$ is again the true probability of a shock in state w_2.

Suppose by contradiction that $t_1^R(e) < t_2^R(e)$. Let $t_i^{R*}(e)$ be the exit-decision date for investors of type i who assign probability 1–e to the value of $\wp$ consistent with state w_i, but assign probability 0 to the existence of state $j \neq i$.

From the supposition that $t_1^R(e) < t_2^R(e)$, it follows that $t_2^R(e) \leq t_2^{R*}(e)$. That is, since type-2 investors exit later than type-1 investors, if type-2 investors think there are no type-1 investors—which is the case when their exit-decision date is $t_2^{R*}(e)$—then type-2 investors need not worry about earlier defaults created by type-1 investors exiting.

Likewise, it follows that $t_1^R(e) \geq t_1^{R*}(e)$. That is, since the later exit of type-2 investors does not affect type-1 investors at the earlier date at which the type-1 investors exit, type-1 investors do not exit any later if they believe that there only exist type-1 investors.

In summary, it has been shown that

$$t_1^{R*}(e) \leq t_1^R(e) < t_2^R(e) \leq t_2^{R*}(e). \tag{7}$$

But it is also the case that, by the Lemma, $t_1^{R*}(e) \to \bar{t}(0) = \infty$ and $t_2^{R*}(e) \to \bar{t}(\bar{p}) < \infty$ as $e \to 0$, which contradicts the string of inequalities in (7). Therefore, $t_1^R(e) \geq t_2^R(e)$ for sufficiently small e. ■

COROLLARY. $\infty = t_1^R(0) \geq t_1^{R*}(e) \geq t_1^R(e) \geq t_2^R(e) \geq t_2^{R*}(e) \geq t_2^R(0) = \bar{t}(\bar{p})$.

PROOF. These inequalities can be constructed from the proof of Proposition 1. ■

PROPOSITION 2. $t_1^R(e_1) \geq t_1^R(e_2)$ if $e_1 \leq e_2$.

PROOF. If $e_1 \leq e_2$, then by Bayes' Rule, type-1 investors believe $\Pr(w_1 | x_1, e_i) = 1 - e$, which is decreasing in e. Hence, when $e = e_1$, type 1s exit no earlier than when $e = e_2$. ■

PROPOSITION 3. There exists p' such that $\bar{t}(p') = t_1^R(e)$.

PROOF. By the proof of Proposition 1 (or its Corollary), $\bar{t}(\bar{p}) \le t_1^R(e) \le \bar{t}(0) \equiv \infty$.

Suppose $\bar{t}(\bar{p}) < t_1^R(e) < \infty$ (otherwise the proof holds for each $\bar{p}$ or $p = 0$). It can now be proven that if $\bar{t}(p_1) = t_1$ and $\bar{t}(p_2) = t_2$, with $t_1 > t_2 + 1$, then there exists $p^* \in (p_1, p_2)$ such that $\bar{t}(p^*) = t_2 + 1$.

By the Lemma, it must be the case that $p_1 < p_2$. Let $V_t^P(a)$ denote $V_t\left(f^{i,a}, \tilde{f}^i | W_t, x_i\right)$, where $f^{i,a}$ is the strategy with $f_t^i(W_t, x_i) = a$ for $a \in \{D, U, P, S\}$. The proof of the Lemma shows that $V_t^p(D) - \max_{a \ne D} V_t^p(a)$ is continuously decreasing in *p*. By the Lemma and the definition of a maximal sustainable equilibrium, $V_{t_2}^{p_1}(D) - \max_{a \ne D} V_{t_2}^{p_1}(a) > 0$, while $V_{t_2}^{p_2}(D) - \max_{a \ne D} V_{t_2}^{p_2}(a) < 0$. By the Intermediate Value Theorem, there exists $p^* \in (p_1, p_2)$ with $V_{t_2}^{p^*}(D) - \max_{a \ne D} V_{t_2}^{p^*}(a) = 0$. Then for some $e > 0$, $V_{t_2+1}^{p^*-e}(D) - \max_{a \ne D} V_{t_2+1}^{p^*-e}(a) < 0$. Therefore, $\bar{t}(p^*) = t_2 + 1$.

Now let $\bar{t}(\bar{p}) = t_2$ and $\bar{t}(0) = \infty = t_1$. Applying the result $\bar{t}(p^*) = t_2 + 1$ iteratively, for each *t* with $\bar{t}(\bar{p}) < t < \infty$, there exists p' such that $\bar{t}(p') = t$. Hence, p' can be chosen such that $\bar{t}(p') = t_1^R(e)$. ■

PROPOSITION 4. There exists $\bar{e} > 0$ such that $\forall e \in [0, \bar{e})$, $t_1^I(e) \le t_2^I(e)$.

PROOF. The proof is analogous to that for Proposition 1. ■

PROPOSITION 5. $t_2^I(e_1) \ge t_2^I(e_2)$ if $e_1 \le e_2$.

PROOF. The proof is analogous to that for Proposition 2. ■

PROPOSITION 6. (a) $t_2^I(e) > t_1^R(e) \ge 0$, and (b) $0 \le t_1^I(e) < t_2^R(e)$.

PROOF OF (a). Label the economy with rational agents (presented in Section III) as economy I and the economy with irrational agents (presented in Section IV) as economy II. Compare the behavior of type-2 investors in state w_2 in economy II to that of type-1 investors in state w_1 in economy I. Both groups constitute a fraction $1 - e$ of the population and assign probability $1 - e$ to the state with favorable fundamentals, w_1. This implies that type-1 investors in economy I exit at date $t_1^{R*}(e)$ (defined in the proof of Proposition 1) if they assign no weight to the existence of investors unlike themselves, while type-2 investors in economy II exit at $t_1^I(e)$ if they give zero weight to the existence of investors like themselves.

In economy I, the smaller group, the type-2 investors, are known to assign probability *e* to state w_1. By contrast, in economy II, if the smaller group of type-1 investors acts according to the prior held by type-2 investors, then they assign probability $(1-e)^3 / \left[(1-e)^3 + e^3\right]$ to w_1. Hence, the smaller group in state w_2 of economy II is believed by the majority to be more optimistic than the smaller group in state w_1 of economy I. Indeed, since $e < 0.5$, $(1-e)^3 / \left[(1-e)^3 + e^3\right] > 1 - e$, which means that type-1 investors are believed to be even more exuberant about state w_1 than type-2 investors. By accounting for type-2 investors' beliefs about the smaller type-1 group in model II, it follows that $t_2^I(e) > t_1^{R*}(e)$. Combining this inequality with the Corollary gives the desired result.

PROOF OF (b). The argument is completely analogous to that for (a), so most of the details are omitted. Observe only that in economy I, type-1 investors believe type 2s have the same prior as their own. Hence, type 1s believe that type 2s have posterior $\Pr(w_2 | x_2) = (1-e)^3 / \left[(1-e)^3 + e^3\right]$ $> 1 - e$. That is, type-2 investors are believed to be even more apprehensive than type 1s. Since type 2s are believed to be the majority by type 1s, the type 1s are more apprehensive than the type 2s in economy I. Reasoning analogous to that for case (a) establishes that $t_1^I(e) < t_2^{R*}(e)$. ■

References

Arrow, Kenneth J., Enrico Colombatto, Mark Perlman, and Christian Schmidt, eds. *The Rational Foundations of Economic Behaviour*. New York: St. Martin's Press, Inc., 1996.

Bordo, Michael David. "Some Aspects of the Monetary Economics of Richard Cantillon," *Journal of Monetary Economics* 12 (1983), 235-58.

Brandenburger, Adam, and Eddie Deckel. "The Role of Common Knowledge Assumptions in Game Theory." In *The Economics of Missing Markets, Information, and Games*, edited by Frank Hahn, pp. 46-61. New York: Oxford University Press, 1990.

Greenspan, Alan. "The Challenge of Central Banking in a Democratic Society." Remarks delivered to The American Enterprise Institute for Public Policy Research, Washington, D.C., December 5, 1996.

Kindleberger, Charles P. *Manias, Panics, and Crashes: A History of Financial Crises*. New York: John Wiley & Sons, Inc., 1996.

Lagunoff, Roger D., and Stacey L. Schreft. "A Model of Financial Fragility," Federal Reserve Bank of Kansas City Research Working Paper No. 98-01, September 1998.

Murphy, Antoin E. *Richard Cantillon: Entrepreneur and Economist*. Oxford: Clarendon Press, 1986.

Sakakibara, Eisuke. Remarks made at a meeting with reporters in Tokyo, as reported in Dow Jones Capital Markets Report via Dow Jones Newswires, April 23, 1998.

Sargent, Thomas J. *Bounded Rationality in Macroeconomics*. Oxford: Clarendon Press, 1993.

von Hayek, F. A. "Economics and Knowledge," *Economica* 4 (1937), 33-54.