Collateral Constraints in a Monetary Economy

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Abstract

This paper studies the role of collateral constraints in transforming small monetary shocks into large persistent output fluctuations. We do this by introducing money in the heterogeneous-agent real economy of Kiyotaki and Moore (1997). Money enters in a cash-in-advance constraint and is injected via open-market operations. We find that a one-time exogenous monetary shock generates persistent movements in aggregate output, whose amplitude depends on whether or not debt contracts are contingent. If contingent contracts cannot be written, money shocks can trigger large output fluctuations. In this case a one time money expansion triggers a boom, while money contractions generate recessions. In contrast, if contracts are contingent amplification is not only smaller, but it can generate the reverse results. When the possibility of default and renegociation is considered, the model can generate asymmetric business cycles with recessions milder than booms. Finally, one-time shocks monetary shocks generate a highly persistent dampening *cycle* rather than a smoothly declining deviation.

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1 Introduction

The extent and mechanism through which monetary policy affects real economic activity over the business cycles has been a long-standing question in macroeconomics. Different mechanisms that explain the propagation of money shocks have been proposed. These include sticky prices, wage contracting, monetary misperceptions, and limited participation.[1] Another mechanism that has received special attention in recent years is credit-market imperfections. In particular, the agency-cost model of Bernanke and Gertler (1989) has been extended to monetary environments in order to analyze how fluctuations in borrowers' net worth can contribute to the amplification and persistence of exogenous money shocks to the economy- see Fuerst (1995), Bernanke, Gertler and Gilchrist (1999), and Carlstrom and Fuerst (2000).[2]

In contrast with these agency-costs models, little attention has been devoted to analyzing *monetary* economies in which agents face endogenous credit limits determined by the value of collateralized assets. The environment we have in mind is one in which lenders cannot force borrowers to repay their debts unless debts are secured. Using *real*-economy models, Kiyotaki and Moore (1997), Kiyotaki (1998) and Kocherlakota (2000) among others, have shown that collateral constraints may be a powerful mechanism of amplification and persistence of real shocks.[3] These papers show that when debts need to be fully secured by collateral, say land, and the collateral is also an input in production, then a small shock to the economy can be largely amplified. For instance, a small negative shock that reduces the net worth of credit-constrained firms forces them to curtail their investment in land. Land prices and output fall because credit-constrained firms are by nature more productive in the use of land. The fall in the value of the collateral reduces even more the

[1] See Cooley and Hansen (1998) for an illustration of the role of monetary shocks in the equilibrium business cycle theory.

[2] Credit-market imperfections in these models emerge from asymmetric information and costly-state verification. In this framework, entrepreneurs borrow to pay the amount of the factor bill that is not covered by their net worth. Lenders must pay a monitoring cost in order to observe the entrepreneur's project outcome. If an entrepreneur has little net worth invested in the project, monitoring costs increase because there is larger divergence between the interests of the entrepreneur and the lender, and so the premium for external financing is larger. With procyclical net worth, periods of low output are associated with higher monitoring costs and a higher external finance premium. This mechanism amplifies the effects of external shocks on production and investment.

[3] Scheinkman and Weiss (1986) also study the effects of *borrowing* contraints in the presence of uninsurable risk. They simulate a lump-sum monetary injection that changes the distribution of assets across agents.

debt capacity of constrained firms, causing additional falls in investment, land prices, and output.

This paper studies the potential role of collateral constraints as a transmission mechanism of monetary shocks. We do this by introducing a cash-in-advance constraint for consumption and investment in the real-economy model of Kiyotaki and Moore (1997). We exploit the simplicity of this framework to study monetary injections carried out via open-market operations, as opposed to the less realistic but simpler helicopter drops employed by many monetary models. Due to the presence of credit-market imperfections, the exact path of the money supply is crucial to determine the real effects of open-market operations. We choose a parsimonious type of monetary paths which avoid changes in long-run inflation and fiscal variables. Thus, current monetary expansions need to be offset by future monetary contractions to avoid changes in inflation or unstable government-bond paths. As in the real-economy models, the price of the collateral plays a central role in generating large and persistent effects of exogenous shocks. Moreover, the response of the nominal interest rate becomes also critical in determining the effects of shocks.[4]

The main finding of this paper is that a one-time monetary shock can generate persistent movements in aggregate output, whose amplitude depends on whether or not debt contracts are contingent. In particular, if contingent contracts cannot be written, then unanticipated money shocks can trigger large output fluctuations. In this case a one-time unanticipated money expansion triggers an economic boom, while unanticipated money contractions generate depressions. The basic mechanism at work is the Fisher effect, according to which unexpected debt-deflation (even if small) redistributes resources from borrowers to lenders. In our model, due to the existence of collateral constraints, this Fisher effect can significantly amplify output fluctuations. In contrast, if contracts are contingent upon the monetary shock, output amplification is not only smaller, but it is possible for money expansions to generate output downturns. This occurs because even though contingent contracts prevent any redistribution of resources from lenders to borrowers, the inflationary tax reduces borrowers' net worth.

Unanticipated shocks may induce default and renegotiation if *expost*, the value of debts exceed

[4]In Kiyotaki and Moore (1997) and Kocherlakota (2000) the equilibrium interest rate is constant.

the collateral value. The possibility of renegotiation can generate asymmetric business cycles as default may occur during depressions but not during booms. We find that if contracts are non-contingent, an unanticipated monetary contraction reduces the value of the collateral and therefore may induce default and renegotiation, depending on the timing of the shock. Renegotiation benefits borrowers and prevents a larger output downturn, i.e. renegotiation can substantially reduce output amplification.

A third property of the model is that monetary shocks trigger highly persistent dampening *cycles* rather than smoothly declining deviations. This occurs due to the interplay between cash-in-advance and collateral constraints. In particular, the full impact of a shock that increases net worth is delayed in this model because with a binding cash-in-advance constraint, collateral can only be accumulated gradually. The cyclical dynamics of the model is consistent with the hump-shaped pattern of output response to monetary shocks that has been observed in the data.[5]

Finally, the model also generates endogenous limited participation in the government-bonds market due to the fact that in equilibrium, collateral constraints are binding only for a set of agents. This implies that only unconstrained agents hold government bonds and can participate in open-market operations. In this context, the propagation of the money shock is nontrivial because agents differ not only in whether they are or not credit constrained, but also in their productivity.

The reminder of the paper is organized as follows. Section 2 presents the model and characterizes the steady state. In Section 3 we discuss the dynamics of the model in response to a monetary shock. The dynamic structure of the model can be summarized by a nonhomogeneous second-order difference equation in the distribution of capital across agents. We parameterize the model and provide a numerical illustration of the dynamics in Section 4. Finally, Section 5 concludes. Technical details omitted in the text are presented in the Appendix.

[5] See Bernanke, Gertler and Gilchrist (1999), and Carlstrom and Fuerst (2000).

2 The model

The model for this heterogeneous-agent economy is an extension of the framework of Kiyotaki and Moore (1997). We keep the main features of their model and introduce money using a cash-in-advance (CIA) constraint. There are two goods in this economy: a durable asset (capital), and a nondurable commodity (output). We focus on the effects of monetary shocks on the distribution of capital across agents and abstract from capital accumulation. Capital is available in an aggregate fixed amount $\bar{K}$.

There are two types of private agents in this economy. They are both risk neutral, but operate different technologies and have distinct discount factors. As will become clear below, around the steady state the more patient agents become lenders, while the impatient agents become borrowers. To abbreviate, let us refer to the two types of agents as borrowers and lenders. Both types of agents face a cash-in-advance (CIA) constraint and a collateral constraint. Finally, the government in this economy has the only role of controlling money supply through open-market operations.

Events in this model occur as follows. Assume that there are two identical members per household who carry out different activities. Households enter each period with money balances stored from the previous period. Production takes place overnight. Early in the morning households observe the money shock and borrowers repay their outstanding debts in output.[6] During the day, all markets are opened simultaneously. The first member of the household uses the money balances to make transactions in both the capital and goods markets. He can buy or sell capital, and buy goods. The second member stays at home selling the goods the household has produced, making transactions in the money market and contracting new debt. Financial transactions must satisfy a standard budget constraint for the household, as well as a collateral constraint.

[6] Borrowers repay their outstanding debts at the beginning of the period to ensure that if the debt is repudiated, lenders can appropriate the collateral. As in other CIA models, we assume that households value the different "types" of output produced by other households. This implies that when lenders get paid in output, they will sell it in exchange for money, and buy other varieties of output.

2.1 Lenders

The mass of lenders in the economy is n. Lenders differ from borrowers in their production technology and preferences. Lenders use a strictly concave technology, and they are more patient than borrowers. Their production function is given by $y_t = G(k'_{t-1})$, where $G' > 0$, $G'' < 0$, $G'(0) = \infty$, and k'_{t-1} is their capital stock at the end of last period. Lenders choose sequences of consumption $\{x'_t\}$, capital holdings $\{k'_t\}$, nominal money balances $\{m'_t\}$, bonds holdings $\{b'_t\}$, and government-bonds purchases $\{h'_t\}$, to solve the following problem for given sequences of output prices $\{p_t\}$, nominal interest rates $\{R_t\}$, and nominal capital prices $\{q^n_t\}$.

$$\max \sum_{t=0}^{\infty} \beta'^t x'_t$$

subject to

$$q^n_t(k'_t - k'_{t-1}) + p_t x'_t \leq m^{d\prime}_{t-1}, \tag{1}$$

$$m'_t + R_t b'_{t-1} + h'_t \leq p_t G(k'_{t-1}) + b'_t + R_t h'_{t-1}, \tag{2}$$

where the prime denotes a lender's decision variable. We define the nominal rate R_t as the interest rate paid at t on loans made at $t-1$. Equation (1) is the CIA constraint. Money is required for both consumption $p_t x'_t$ and investment $q^n_t(k'_t - k'_{t-1})$. Equation (2) is the budget constraint. The revenues collected through output sales $p_t G(k'_{t-1})$, new bonds issued b'_t, and the proceeds from government-bond holdings $R_t h'_{t-1}$ must be enough to accumulate new money balances m'_t, pay outstanding debt obligations $R_t b'_{t-1}$, and purchase government bonds h'_t.

Let $\beta'^t \Omega_t$ be the Lagrange multiplier associated to the CIA constraint and $\beta'^t \Lambda_t$ the one for the budget constraint. Then, the first order optimality conditions for the problem above are given by[7]

$$x'_t : 1 = \Omega_t p_t,$$

$$m'_t : \Lambda_t = \beta' \Omega_{t+1},$$

[7] These are the first order conditions for interior solutions. Assumptions above guarantee such result.

$$b_t', h_t' : \Lambda_t = \beta' R_{t+1} \Lambda_{t+1},$$

$$k_t' : q_t^n \Omega_t - \beta' q_{t+1}^n \Omega_{t+1} = \beta' \Lambda_{t+1} p_{t+1} G'(k_t').$$

From the optimality conditions above is easy to obtain expressions for the nominal interest rate and the users cost of capital for the lenders $u' \equiv q_t - \beta' q_{t+1}$

$$R_t = \frac{1+\pi_t}{\beta'}, \tag{3}$$

$$u_t' = \frac{\beta'^2}{1+\pi_{t+1}} G'(k_t'), \tag{4}$$

where $\pi_t = \frac{p_{t+1}-p_t}{p_t}$ is the inflation between t and $t+1$. Equation (4) states that lenders equate their users cost of capital with the present value of its marginal product. Since in equilibrium these agents are not credit constrained, the users cost is simply the difference between the cost of buying capital today q_t and the discounted value of selling capital tomorrow $\beta' q_{t+1}$. Notice that the lenders' users cost is not affected by inflation since the proceeds of selling the capital can be consumed or invested immediately, without requiring previous accumulation of cash. In contrast, the marginal product of capital has to be discounted by $\frac{\beta'^2}{1+\pi_{t+1}}$because output needs to be exchanged for money before it can be consumed. This means that the investor has to wait two periods and pay the inflation tax before he can consume the returns of the investment.

2.2 Borrowers

The measure of borrowers is normalized to one. Their technology is given by the production function $y_t = (a+c)k_{t-1}$. They choose sequences of consumption $\{x_t\}$, capital holdings $\{k_t\}$, nominal money balances $\{m_t^d\}$, private issued bonds $\{b_t\}$, and government-bonds purchases $\{h_t\}$ to solve the following problem for given sequences of output prices, nominal capital prices, nominal interest rates, and government-bonds nominal rates

$$\max \sum_{t=0}^{\infty} \beta^t x_t$$

subject to

$$q_t^n(k_t - k_{t-1}) + p_t x_t \leq m_{t-1}^d, \tag{5}$$

$$m_t^d + R_t b_{t-1} + h_t \leq (a + c) p_t k_{t-1} + b_t + R_t h_{t-1}, \tag{6}$$

$$R_{t+1} b_t \leq q_{t+1}^n k_t, \tag{7}$$

In addition to the CIA and budget constraints, borrowers face a collateral constraint, given by equation (7). Borrowing can only take place up to the point where the principal plus interest $R_{t+1}b_t$ is secured by the market value of the capital owned by the household $q_{t+1}^n k_t$. Lenders also face a collateral constraint but we did not explicitly write it. Around the steady state this constraint is not binding under the following assumption:

Assumption 1. $\beta' > \beta$.

It is also assumed that only the fraction a of the output is tradable between borrowers and lenders. The fraction c can be traded only among borrowers, and it can be interpreted as a subsistence minimum consumption. We refer to this fraction as the nontradable output. The purpose of the assumption is to avoid the situation in which borrowers continuously postpone consumption.[8]

In Appendix A we prove that around the steady state of the model the borrower's optimal plan is to consume only the nontradable fraction of output, i.e. $x_t = ck_{t-1}$, to borrow up to the limit imposed by the collateral constraint, and to invest all remaining resources. This implies that borrowers do not purchase government bonds, i.e. $h_t = 0$, and that the CIA constraint is binding. Notice that since borrowers do not hold government bonds while lenders do, this model generates endogenous limited participation in this market. These results hold under the following additional assumption

[8]Kiyotaki and Moore (1997) introduce a similar assumption. As will be explained later on, due to the linearity of preferences borrowers would like to continuosly postpone consumption in exchange for investment. This is avoided by introducing a nontradable fraction of output, which we think of as subsistence minimum consumption. Notice that money is required to buy nontradable output because this type of output can be traded among borrowers. One can think that households can only produce say fruit of a particular color, but they value fruits of all colors.

Assumption 2.

$$\frac{c}{a} > \frac{(1-\beta)}{\beta^2}\frac{(2-\beta-\beta')}{(1-\beta')}.$$

This condition is easy to satisfy if the discount factors are close to 1.[9] We can use equations (5), (6) and (7) to obtain

$$k_t = \frac{1}{u_t}\left[(a+q_t)k_{t-1} + \frac{1}{1+\pi_{t-1}}\frac{m^d_{t-1}}{p_{t-1}} - \frac{R_t}{1+\pi_{t-1}}\frac{b_{t-1}}{p_{t-1}} - \frac{m^d_t}{p_t}\right], \tag{8}$$

where $q_t \equiv \frac{q^n_t}{p_t}$ is the real price of capital. The term in brackets corresponds to the real net worth of borrowers, which consists of the value of tradable output ak_{t-1}, plus the value of capital held from the previous period $q_t k_{t-1}$, plus the real money balances brought from the previous period $\frac{1}{1+\pi_{t-1}}\frac{m^d_{t-1}}{p_{t-1}}$, minus the real value of debt repayments $\frac{R_t}{1+\pi_{t-1}}\frac{b_{t-1}}{p_{t-1}}$, minus money balances reserved for next period's purchases $\frac{m^d_t}{p_t}$. Finally, the users cost of capital for borrowers, u_t, is given by

$$u_t \equiv q_t - \frac{1+\pi_t}{R_{t+1}}q_{t+1}. \tag{9}$$

Thus, equation (8) says that borrowers use all their net worth to finance the difference between the value of their capital $q_t k_t$ and the amount they can borrow against each unit of capital $\frac{q_{t+1}}{R_{t+1}}(1+\pi_t)\,k_t$ in real terms. Notice that borrowers discount the future value of the capital at the nominal interest rate. This is the case, as will become clear below, because in equilibrium borrowers need to borrow in order to buy capital.

2.3 Government

The government controls money supply in this economy through open-market operations (OMOs), which take place in the bonds market. Let H^s_t be the nominal supply of government-issued bonds.

[9] In this case, $\frac{(2-\beta-\beta')}{(1-\beta')}$ is some constant near to 2, and $\frac{(1-\beta)}{\beta^2}$ is close to zero. Further, in the proposed equilibrium $\frac{c}{a}$ is the ratio between the marginal propensity to consume and the marginal propensity to save for borrowers, which can be assumed to be bounded away from zero.

The stock of money supply, M_t^s, in this economy is given by

$$M_t^s = M_{t-1}^s - H_t^s + R_t H_{t-1}^s, \tag{10}$$

where

$$H_t^s = \tau H_{t-1}^s, \ \tau \geq 0 \tag{11}$$

so that at time t the government withdraws an amount τH_{t-1}^s of money and injects $R_t H_{t-1}^s$ back into the economy. There are two comments in order. First, we choose a simple law of motion for government bonds H_t^s. This simplicity is convenient for our purpose of analyzing the effects of a one-time money shock. Notice that following this shock, unless $\tau < 1$ for all t, government bonds may exhibit an explosive path. To avoid this, any one-time money expansion through OMOs must be eventually followed by a "policy reversal" or "sterilization" that guarantees convergence back to the steady state. In particular, the size of τ determines the speed at which such monetary contraction takes place. In the analysis below we consider τ very close to 1 in order to simulate a very slow policy reversal. Since credit markets are imperfect in this economy, real effects of monetary shocks may depend on the path of government debt. Although we choose a parsimonious law of motion for H_t^s, we will discuss below the role of the size of τ in our results, as well as other paths for government debt.

Second, notice that we do not consider a rebate of the inflationary tax. Since some agents face corner solutions, such rebate cannot be lump-sum in general. For example, simple helicopter drops redistribute wealth, and affect agents decisions. Since here we want to focus on the effects of the "pure monetary shock", we do not include any rebates in the model. Tax rebates in fact may reinforce the results of the paper.[10]

[10]The intuition for this result is simple. Suppose the economy starts off at the steady state and there is a one-time money expansion. Assume that borrowers were to receive a money transfer that compensates them for the inflationary tax in an amount higher than their optimal consumption. This may happen, for example, with helicopter drops. In this case, borrowers will buy capital with the extra resources, and next period output would increase. This reinforces our results because, as will be shown below, in this economy monetary expansions generate booms. More details on this are available from the authors upon request.

2.4 Aggregate resource constraints

Let K_t, K'_t, B_t, B'_t, H_t, H'_t, M^d_t, $M^{d\prime}_t$ be the aggregate variables corresponding to the lowercase individual variables. There are five markets in the model: consumption goods, capital, money, private bonds, and public bonds. By Walras' Law one needs only to consider four of them. The equilibrium conditions to clear the last four markets are

$$M^s_t = M^d_t + M^{d\prime}_t = m^d_t + nm^{d'}_t,$$

$$B_t = b_t = -B'_t = -nb'_t,$$

$$\bar{K} = K_t + K'_t = k_t + nk'_t,$$

and since $H_t = 0$,

$$H'_t = nh'_t = H^s_t.$$

Using the market clearing conditions above along with equations (2) and (6) we obtain

$$M^s_t + H^s_t - R_t H^s_{t-1} \equiv M^s_{t-1} = p_t \left[(a+c)K_{t-1} + nG\left(\frac{\overline{K} - K_{t-1}}{n} \right) \right], \tag{12}$$

which is just the quantity equation.

2.5 Steady state

Define a steady state where all real variables are constant, and all nominal variables grow at the constant rate π, which is the steady-state growth rate of money supply. From equations (10) and (11) it follows that $\pi = \tau - 1$ if $\tau \geq 1$, and $\pi = 0$ if $\tau < 1$.

Next, it is easy to see that the steady-state users cost of capital for lenders and borrowers is the same: $u = u' = q(1 - \beta')$. Further, since under the proposed equilibrium the collateral constraint (7) binds for the borrowers, we can use R, u' and the budget constraint of these agents (6) to get: $u = a + c - \frac{M^d}{pK^*}$, where K^* is the borrowers' steady-state capital level. Next, using the CIA constraint (5) one obtains: $\frac{M^d}{p} = cK^* (1 + \pi)$, i.e. borrowers' real money balances exactly cover their consumption adjusted by inflation.

Combining the last two expressions we obtain: $u = a - \pi c$. Notice that if $\pi = 0$, we obtain the intuitive results that $\frac{M^d}{p} = cK^*$, and $u' = u = a$. This last equation means in a steady state with no money growth, the users cost equals the tradable marginal product of capital.

Finally, using equation (4), we obtain an implicit solution K^*

$$G'\left(\frac{\overline{K} - K^*}{n}\right) = \frac{1+\pi}{(\beta')^2}(a - \pi c) \tag{13}$$

The equation above, along with Assumption 2 imply that in equilibrium borrowers have higher marginal product of capital than lenders. The following proposition summarizes the main features of the steady state.

Propostion 1. *Under Assumptions 1 and 2,*

(i) if $G'\left(\overline{K}/n\right) < \frac{1+\pi}{(\beta')^2}(a - \pi c)$ *there exists a unique steady state;*

(ii) $\frac{\partial K^*}{\partial \pi} \neq 0$ for $(1 + 2\pi)c \neq a$, *so that inflation affects the steady-state output* Y^*.

Proof: The existence of a unique steady state level K^* is guaranteed from the properties of the production function $G(.)$. It is easy to see that the left-hand side of equation (13) is continuous and strictly increasing in K, while the right-hand side is a constant. If $G'\left(\overline{K}/n\right) < \frac{1+\pi}{(\beta')^2}(a - \pi c)$ the left and right-hand side cross only once. Figure 1 illustrates the determination of the steady state. The second property follows easily. ■

It is interesting that in the long run money is not superneutral as indicated by Proposition 1, *(ii)* even though aggregate capital is constant. It is well known (Abel 1984) that money is not superneutral if investment enters in the CIA constraint, because inflation acts as a tax on capital accumulation. However, aggregate capital is constant in our model so that the standard result does not apply. The non-supernetrality arises because inflation acts as a tax for all agents, but in the margin it affects differently borrowers and lenders. Higher inflation decreases the marginal cost of investing for both types, i.e. it decreases the users cost of capital. For a given K^*, borrowers net worth decreases with higher inflation because they must demand more money to sustain their consumption, cK^*. Further, since borrowers are credit constrained, they are in a corner solution. In contrast, lenders have an interior solution and since u has decreased, their marginal benefit of investing needs to decrease, which can only happen if lenders' capital holdings, $\overline{K} - K^*$, increase. Thus, money is not superneutral due to the asymmetric effect of inflation on constrained and unconstrained agents.[11]

3 Dynamics

To simplify the analysis, we only present the dynamics of the model around the steady state, and assume zero steady-state inflation, $\pi = 0$. The solution for the case $\pi > 0$ is summarized in Appendix D. We also assume that β' is close to 1. This occurs, for example, if the length of the periods is small. This assumption allows as to obtain some sharp analytical results, but numerical simulations confirm that the main results hold even if β' is far from 1. Let $g_t \equiv \frac{M_t^s}{M_{t-1}^s}$, i.e. g_t is one plus growth rate of money supply, $v_t \equiv \frac{p_{t+1}}{p_t}$, i.e. v_t is one plus the inflation rate. Thus, in the steady state, $g = v = 1 + \pi$. In general, let $\widehat{x}_t = \frac{x_t - x^*}{x^*}$ denote the rate of deviation of a variable x from its steady state value.

Assume that the economy starts off at the steady state, and that an unexpected one-time increase in the growth rate of money $\varepsilon > 0$ occurs at $t = 0$, i.e. $\widehat{g}_0 = \frac{\varepsilon}{1+\pi}$. Since the monetary expansion occurs through OMOs, H_0 decreases below its steady state level ($H_0 < 0$). According to the law of motion for government bonds, $H_t^s = \tau H_{t-1}^s$, H_t gradually returns to zero to avoid changes in the long term inflation rate. Thus, the one-time money expansion at $t = 0$ is followed by a monetary contraction, i.e. by a "sterilization policy". In particular, the size of $\tau < 1$ determines

[11] If the borrowers' propensity to consume $\frac{c}{a+c}$ is larger than 0.5 then higher inflation reduces output, a result consistent with Abel (1984). However, if money is injected via helicopter drops rather than via OMOs, higher steady-state inflation may have the opposite results, i.e. higher π implies larger K^* and larger Y^*. This occurs if borrowers receive a fraction of the transfer higher than their steady-state consumption share, $\alpha \equiv \frac{cK^*}{Y^*}$. In this case, borrowers are overcompensated for the inflationary tax and, as a result, they can afford to buy additional capital with the extra resources. In addition, inflation increases the marginal cost of investing, u, but lenders are particularly hurt because they face and interior solution.

the speed at which such monetary contraction takes place.

Using the law of motion of money supply and bonds, one can obtain the following path of money growth[12]

$$\widehat{g}_0 = -\partial d_0,$$

and

$$\widehat{g}_t = -(R - \tau)\tau^{t-1}\widehat{g}_0.$$

Notice that this path is fully determined by the exogenous initial shock, and converges to zero at a rate determined by the size of τ. In particular, a larger τ implies a smoother sterilization of the monetary contraction. In what follows we will assume a τ very close to 1 in order to simulate a very slow monetary contraction.[13]

To complete the characterization of the dynamics of the model, we need to solve for the paths of $\widehat{v}_t$, $\widehat{q}_t$ and $\widehat{K}_t$. Linearizing equation (12) yields

$$\widehat{v}_t = \widehat{g}_t - \rho\left(\widehat{K}_t - \widehat{K}_{t-1}\right),$$

where $\rho = (a + c - G')\frac{K^*}{M^s/p}$. Notice that $\widehat{v}_{-1} = 0$ because both output and the money supply used for transactions in the goods market are predetermined. Next, linearizing equation (4) we obtain

$$\widehat{q}_t - \beta'\widehat{q}_{t+1} = \left(1 - \beta'\right)\left(\frac{1}{\eta} - \rho\right)\widehat{K}_t + \left(1 - \beta'\right)\rho\widehat{K}_{t+1} - (1 - \beta')\widehat{g}_{t+1}, \tag{14}$$

where $\frac{1}{\eta} = -\frac{G''K^*}{nG'} > 0$.[14] The equation above describes the forward-looking nature of capital

[12]Here we compute the absolute deviations of the government-debt to money ratio ∂d_0 instead of the percentage deviations from the steady state because $d = 0$. The first expresion follows from linearizing the stationary version of the equation $M_0^s = M_{-1}^s - H_0^s$. For $t \geqslant 1$, combine the law of motion for H_t^s and M_t^s, transform variables to render them stationary, and linearize to obtain $\widehat{g}_t = (R - \tau)\partial d_{t-1}$. Next, use the law of motion of government debt to obtain $\partial d_t = \tau\partial d_{t-1} = \tau^t\partial d_0$, which together with the previous expresion implies that $\widehat{g}_t = (R-\tau)\tau^{t-1}\partial d_0 = -(R-\tau)\tau^{t-1}\widehat{g}_0$.

[13]When there is a monetary expansion at $t = 0$ and τ is very close to one, then the money contraction at $t = 1$ is very small. Ideally, for a more "realistic" scenario, one could have a more persistent money expansion, eventually followed by a contraction. In the context of our simple model, since we analyze a one-time money expansion, by using τ very close to one most of the subsequent money contraction occurs several period after the money expansion.

[14]The term $\frac{1}{\eta}$ can be rewritten as: $\frac{1}{\eta} = -\frac{G''(\overline{K}-K^*)/n}{G'}\frac{K^*}{\overline{K}-K^*}$, and so it can be interpreted as a measure of the elasticity of the marginal product of borrowers' capital, weighted by the ratio of borrowers to lenders' capital in the

prices, i.e. the price of capital at $t = 0$ depends on the full path of capital distributions across types.

Finally, using the three expressions above, as well as the linearized versions of equations (5), (6) and (7), it is easy to show that $\widehat{K}_t$ satisfies the following non-homogeneous second order difference equation for $t \geqslant 2$

$$\theta_0 \widehat{K}_t = \theta_1 \widehat{K}_{t-1} + \theta_2 \widehat{K}_{t-2} + \mu_0 \tau^{t-2} \widehat{g}_0, \tag{15}$$

where θ_0, θ_1, θ_2,and μ_0 are constants that depend on steady-state variables (see Appendix B). It can be shown that for β' close to 1, these constants are given by: $\theta_0 = 1 - \rho > 0$, $\theta_1 \approx 2\theta_0$, $\theta_2 \approx -\theta_0$, and $\mu_0 \approx -(1-\tau)^2$. This last term reflects that a money injection at $t = 0$ generates a negative trend in K_t as a result of the sterilization that takes place after the injection.

The previous equation summarizes the equilibrium dynamics of the model. It can be shown that $\widehat{K}_t$ exhibits persistent and dampening cycles, as summarized in the following proposition:

Proposition 2. *For β' sufficiently close to 1 and $\pi = 0$,*

(i) the general solution to (15) is

$$\widehat{K}_t = Ar^t \cos(\omega t - \phi) + A_\tau \tau^t \widehat{g}_0 \tag{16}$$

where A and ϕ are constants, $r = \sqrt{-\theta_2/\theta_0}$, $\omega = \cos^{-1}\left(\frac{\theta_1/\theta_0}{2r}\right)$, and $A_\tau = \frac{\mu_0}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}$.[15]

(ii) r is close to, but less than, 1, and ω is close to, but larger than, zero.

Proof: See Appendix B. ■

Corollary. *$\widehat{K}_t$ exhibits persistent and dampening cycles.*

The existence of dampening cycles rather than monotonic dynamics occurs in this model due to the interplay between the CIA and collateral constraints. In particular, the full impact of a shock

steady state.

[15] Note that $\lim_{\beta' \to 1} A_\tau = -\frac{1}{(1-\rho)}$.

that increases net worth is delayed in this model because with a binding CIA constraint, collateral can only be accumulated gradually. Recall that investment enters in the CIA constraint. This cyclical dynamics of the model is consistent with the hump-shaped pattern of output response to shocks that has been observed in the data.

To fully characterize the equilibrium solution, we require two additional conditions on the trajectory of $\widehat{K}_t$. For reasons that we explain briefly, monetary injections via OMOs imply $\widehat{K}_0 = 0$. Thus, the equilibrium path of the distribution of capital can be completely characterized in terms of $\widehat{K}_1$. Using these two conditions, we obtain

$$A = \frac{\widehat{K}_1 - A_\tau \tau \widehat{g}_0}{\cos(\omega - \phi)} \text{ and } \cos(\phi) = -\frac{A_\tau \widehat{g}_0}{A}.$$

The result that $\widehat{K}_0 = 0$ follows from the following four facts: *i)* the money contraction occurs in the bonds market; *ii)* the shopper's only resources are the money balances accumulated during the previous period and the land holdings; *iii)* near the steady state, borrowers' consumption is equal to the nontradable output which is predetermined. As a consequence, lenders consumption is also predetermined; *iv)* the nominal price of consumption at the moment of the shock does not change. These facts together imply that at the moment of the shock households cannot change their investment level.

We now solve for $\widehat{K}_1$ following a monetary shock at $t = 0$. For this purpose, combine (5), (6), and (7) to obtain

$$q_1(K_1 - K_0) + cK_0 = \frac{a+c}{1+\pi_0}K_{-1} + \frac{1}{1+\pi_0}\frac{B_0}{p_0} - \frac{R_0}{(1+\pi_0)(1+\pi_{-1})}\frac{B_{-1}}{p_{-1}}, \tag{17}$$

where $\frac{B_{-1}}{p_{-1}}$ is the aggregate steady-state level of debt in real terms, and K_{-1} corresponds to the borrowers' steady-state capital level. We consider two relevant cases at this point: non-contingent and contingent debt contracts. The difference between these two is that under contingent contracts, borrowers must compensate lenders for any unexpected inflation. Thus, debt repayments at time zero are immune to period one's inflation π_0, i.e., $R_0 B_{-1} = \frac{1+\pi_0}{\beta'} B_{-1}$.

3.1 Non-contingent contracts

When debt contracts are non-contingent, R_0 in (17) is simply the steady-state nominal interest rate $R = \frac{1}{\beta'}$. Linearizing (17) in this case yields

$$\widehat{K}_1 = \frac{1}{1-\beta'\rho}\left[\beta'\widehat{q}_1 + \left(2 - r^h - \beta'\tau\right)\varepsilon\right]. \tag{18}$$

To solve for $\widehat{K}_1$ we first need to solve for $\widehat{q}_1$, which in turn depends on the full sequence $\{\widehat{K}_t\}$. Solving equation (14) forward we can obtain a solution for $\widehat{q}_1$, as shown in Appendix C. The expression that relates $\widehat{q}_1$ with $\widehat{K}_1$ is algebraically complicated.

We can use equation (18) to gain some intuition on the real effects of a monetary expansion $\varepsilon > 0$ under non-contingent contracts. Suppose initially that the real price of capital remains unchanged after the monetary shock so that $\widehat{q}_1 = 0$. In this case, the real effects of the shock depend on the size of τ. Specifically, if $\tau < \overline{\tau} \equiv \frac{1}{\beta'}\left(2 - r^h\right)$, then K_1, and also Y_2, move in the same direction as the monetary shock. This is always the case because when β' is close to 1, then $\overline{\tau}$ is close to 2. Finally, this change in the distribution of capital toward the more productive agents induces an increase in the price of capital, $\widehat{q}_1 > 0$, which reinforces the initial effect of the shock. Therefore, a one-time monetary expansion under non-contingent debt contracts induces a redistribution of capital towards borrowers, and increases output.

3.2 Contingent contracts

As indicated above, if debt contracts are contingent, then $R_0 = \frac{1+\pi_0}{\beta'}$. In this case, linearization of equation (17) when $\pi = 0$ yields

$$\widehat{K}_1 = \frac{1}{1-\beta'\rho}\left[\beta'\widehat{q}_1 + \left(1 - r^h - \beta'\tau\right)\varepsilon\right]. \tag{19}$$

The equation above is very similar to (18), except that now we have a smaller coefficient on ε. This smaller coefficient has two important implications. First, the real effects of the one-time money shock will be smaller than in the case of non-indexed debt contracts, i.e. the redistribution of capital as well as the output amplification are smaller. Second, different from the case of non-contingent debt contracts, a monetary expansion may now produce an output *downturn* rather than a boom. To see why, assume for a moment that $\widehat{q}_1 = 0$. In this case, the real effects of the shock again depend on the size of τ. However, different from the non-contingent case, now $\overline{\tau}$ is much smaller: specifically, if $\tau < \overline{\tau} \equiv \frac{1}{\beta'}\left(1 - r^h\right)$, then the monetary expansion will increase output. But for β' close to 1, $\overline{\tau}$ is close to 1. Thus it is possible to find $\tau > \overline{\tau}$ such that the monetary expansion generates a downturn.

To better highlight the mechanisms behind this result, it is useful to rewrite equation (17) as

$$q_1(K_1 - K_0) + cK_0 = \frac{a+c}{1+\pi_0}K_{-1} + \frac{\beta' q_1 K_0}{1+\pi_1} - \frac{R_0}{(1+\pi_0)(1+\pi_{-1})}\frac{B_{-1}}{p_{-1}}, \tag{20}$$

where the left-hand side represents consumption and investment in $t = 1$, and the right-hand side are the real balances brought from period $t = 0$. In particular, the first term on the right-hand side are output sales; the second term is new debt contracted in $t = 0$; and the third term is the repayment for debt contracted in $t = -1$. Notice that consumption in $t = 1$ is fixed because K_0 remains at the steady-state level. Thus, the only way borrowers increase their investment in capital K_1, is if the right-hand side of the equation is large than its steady-state value.

First, notice that under contingent debt contracts, since $R_0 = \frac{1+\pi_0}{\beta'}$, then the last term on the right-hand side does not change with the money expansion, i.e. it remains at its steady-state value. This implies that lenders do not transfer any wealth to borrowers in the period of the shock via interest rate repayments.

Second, due to the monetary expansion $\widehat{g}_0 > 0$, the level of prices in $t = 1$ increases, so that π_0 increases ($\widehat{v}_0 > 0$). This hurts borrowers because the first term on the right-hand side $\frac{a+c}{1+\pi_0}K_{-1}$ decreases. Thus, the only way borrowers could increase K_1 is if the second term on the right-hand side $\frac{\beta' q_1 K_0}{1+\pi_1}$ increases by more than the decrease in $\frac{a+c}{1+\pi_0}K_{-1}$. In general, π_1 decreases ($\widehat{v}_1 < 0$) due to the policy reversal, i.e. the money contraction that follows the one-time expansion. However, for τ sufficiently close to 1, case in which the money contraction in $t = 1$ is very small, the drop in π_1 is so small, that $\frac{\beta' q_1 K_0}{1+\pi_1}$ increases very little. In this case, borrowers decrease K_1 and the money expansion causes a downturn. In summary, the only way a monetary expansion can generate an increase in output when debt contracts are indexed is when this expansion is quickly reverted by a monetary contraction (i.e., small τ).

Finally, notice that when debt is non-contingent, the third term on the right-hand side of (20) decreases with respect to the steady state because π_0 increases due to the money expansion. What this means is that there is a transfer of wealth from creditors to debtors, which ultimately allows borrowers to increase K_1 and generate an output expansion.[16]

3.3 Default and asymmetric business cycles

Up to now we have considered the same model of debt as in Kiyotaki and Moore (1997). There are two important assumptions in their model. The first is that once a borrower has started to produce with capital K_t, he is the only one with the skill to complete production in period $t+1$. In other words, if the borrower were to withdraw his labor between t and $t+1$, there will be no output in $t+1$. A second assumption is that when the shock arrives, the borrower has already input his labor into the production project, and so it is too late for him to threaten the creditors by withdrawing his labor. The borrower thus never has incentives to repudiate his debt contract in the face of a shock.

Consider now what would happen if the shock arrived before the borrower had input his labor. In this case, if the value of the collateral falls with the shock below debt value, the borrower can threaten his creditors by withdrawing his labor and defaulting on his debt. Since there is no output without the borrower's labor, then the borrower could be able to renegotiate his debt down to the market value of the collateral. It turns out that if such renegotiation is possible, then our model can generate asymmetric business cycles. In particular, if contracts are non-contingent and there is a monetary *contraction*, borrowers have incentives to renegotiate their debt and the output downturn is smoother.

The intuition for this result is as follows. If contracts are non-contingent, the interest rate

[16] The redistribution of wealth between debtors and creditors following a money shock has been emphasized by Fisher (1933).

borrowers pay is the steady-state rate. If there is a money contraction, this steady-state rate will be higher than the equilibrium rate, due to deflation. Thus, borrower's net worth is reduced, as well as their capital holdings. This in turn triggers a decrease in the price of capital. In this case, borrowers have incentives to repudiate their debt and pay back just the market value of the collateral. The fact that borrowers end up paying back less, makes the output downturn smoother. Notice that in contrast, if there is a monetary expansion, borrowers will not have incentives to repudiate and renegotiate their debt. This is so because in this case the price of capital increases, while the nominal interest rate remains at its steady state.[17]

Analytically, when debt is renegotiated, R_0 in (17) is not the equilibrium value. Rather, the term $R_0 \frac{B_{-1}}{p_{-1}}$ is replaced by the market value of the collateral $q_0 K_{-1}$. Linearizing equation (17) in this case yields

$$\widehat{K}_1 = \frac{1}{1-\beta' \rho} \left[\beta' \widehat{q}_1 - \widehat{q}_0 + \left(2 - r^h - \beta' \tau \right) \varepsilon \right]$$

which holds for both the contingent and non-contingent debt cases. It turns out that when debt is renegotiated, the solution for $\widehat{K}_1$ can be simply written as[18]

$$\widehat{K}_1 = \frac{1}{\theta_0} \left[\left(1 - r^h\right) - \left(1 - 2\beta'\right) (R - \tau) \right] \widehat{g}_0. \tag{21}$$

Since the expression above is algebraically simple, we can use it to analyze whether following the one-time monetary contraction in period $t = 0$, it is the case that $\widehat{K}_1 < 0$ and so $\widehat{Y}_2 < 0$. Further, if $\widehat{K}_2 < \widehat{K}_1$, since the model exhibits persistent dampening cycles, we should observe a downturn in the economic activity as borrowers' capital level decreases. Proposition 3 summarizes the conditions under which these results hold. Let $\alpha \equiv \frac{cK^*}{Y^*} < 1$ be the fraction of steady-state output consumed by the borrowers.

[17]If debt contracts are contingent, renegociation occurs depending on how interest payments change compared to the change in the price of the collateral. For instance, suppose τ is small enough so that a monetary expansion triggers an increase in output. Then, if the nominal interest rate increases by more than the increase in the price of capital, borrowers will have incentives to renegociate.

[18]Under renegociation, the solution for $\widehat{K}_1$ is the same as that implied by the non-homogeneous second order differential equation for $t = 1$ and $\widehat{K}_0 = 0$.

Proposition 3. *For β' sufficiently close to 1 and $\pi = 0$,*

(i) following a one-time decrease in the money growth rate $\varepsilon < 0$ at $t = 0$, borrowers decrease their capital holdings in period $t = 1$, i.e. $\widehat{K}_1 < 0$. Further, the lower τ, the larger $\left|\widehat{K}_1\right|$ is.

(ii) if τ is sufficiently close to 1 then $\widehat{K}_2 < \widehat{K}_1$, while if $\tau \to 0$ then a sufficient condition for $\widehat{K}_2 < \widehat{K}_1$ is that $\alpha < \frac{1}{3}$.

Proof: (i) When $\beta' \to 1$ it is the case that $\rho \to \alpha$ and that $r^h \to 0$. Then, $\theta_0 \to (1-\alpha)$. Thus, when $\beta' \to 1$ from equation (21) we have that: $\widehat{K}_1 \to \frac{2-\tau}{1-\alpha}\widehat{g}_0$, and since $\tau < 1$ and $\widehat{g}_0 < 0$ it follows that $\widehat{K}_1 < 0$. Notice that the more slowly government debt returns to the steady state, i.e. the larger τ, the lower the multiplier of monetary policy in the first period.

(ii) From equation (15) we have: $\widehat{K}_2 = \frac{\theta_1}{\theta_0}\widehat{K}_1 + \frac{\mu_0}{\theta_0}\widehat{g}_0$, and since when $\beta' \to 1$ we have that $R \to 1$ and so $\mu_0 \to -(1-\tau)^2$, then: $\widehat{K}_2 \to \frac{1}{1-\alpha}\left[\frac{2-\tau}{1-\alpha} - (1-\tau)^2\right]\widehat{g}_0$. If $\tau \to 1$, then $\widehat{K}_2 \to \frac{1}{1-\alpha}\widehat{K}_1$ and so $\widehat{K}_2 < \widehat{K}_1$. On the other hand, if $\tau \to 0$, then $\widehat{K}_2 \to \left[\frac{1}{1-\alpha} - \frac{1}{2}\right]\widehat{K}_1$, so that $\widehat{K}_2 < \widehat{K}_1$ if $\alpha < \frac{1}{3}$. ■

Part *(ii)* in Proposition 3 indicates the role of τ in strengthening the real effects of a monetary contraction. In fact, when the sterilization policy is smooth, i.e. when τ is large, borrowers further decrease their capital stock in $t = 2$. This implies that they would be able to borrow less against their collateral, and their capital holdings will decrease for a number of periods after the shock. This occurs because when the sterilization is smooth, then the government expands the money supply in small amounts during several periods, and so the nominal interest rate remains above the steady state, i.e. $\widehat{R}_t > 0$ for a longer time. In contrast, when the monetary contraction is reverted quickly, i.e. when $\tau \to 0$, this dynamic pattern for capital may not necessarily hold, unless further conditions are imposed.

4 Simulations

To illustrate the magnitude and persistence of monetary shocks in this economy, we assign values to the parameters of the economy and simulate the effects of a one-time 1% change in the growth rate of money. The only purpose here is to illustrate the dynamics generated by our model, not to calibrate our stylized model. As such, the quantitative results presented here are not to be taken literally. We choose the parameters of the model to satisfy the assumptions imposed. We set $\beta' = 0.995$ to simulate a time period equal to a month. Note that β' is close enough to 1, in line with many of the proofs presented above.

We normalize to unity the total stock of capital, i.e. $\bar{K} = 1$, as well as the nontradable fraction of output, i.e. $c = 1$. The production technology for lenders is: $G(K) = B(\bar{K} - K)^{\gamma}$, where B is also normalized to unity and set $\gamma = 0.3$. We set $n = 3$, which implies that in this economy only 25% of the agents are constrained. Finally, we choose a steady-state capital distribution of $K = 0.25$, i.e. lenders hold 25% of the total capital.

Figure 2 displays the effects of a one-time increase of 1% in the growth rate of money when $\pi = 0$, $\tau = 0.9$, and debt contracts are non-contingent. The figure shows percentage deviations from steady-state values. Since τ is large, the subsequent money contraction is smooth and government bonds go back gradually to their steady-state $H^s = 0$. As is shown in the graph, this policy generates ample and persistent dampening cycles. The cycle starts with an increase in borrowers' capital holdings, as well as an increase in output. The peak of the cycle is reached about 50 months after the shock, when output is around 40% above the steady state. Of course, this quantitative result is unrealistically large and due to non-standard assumptions of the model such as linear utility. What lies behind such large amplification is the redistribution of wealth from lenders to borrowers due to the non-contingent nature of the debt contract. In the figure, since the collateral constraint binds, real borrowers' debt mimics the behavior of capital.

These results emerge from the combination of two mechanisms that affect both sides of the collateral constraint: one is the asset-price effect, and the other is the interest-rate effect. First, there is an increase in real price of capital that increases the value of the collateral for a number of periods. This increase in the asset price comes from the fact that to clear the capital market, the users cost for lenders has to increase. Notice that the real price of capital is above the steady state for 50 months, which is exactly the time at which capital and bonds reach their peaks. Second, the nominal interest rate is at its steady-state value in the period of the shock, but it then decreases below the steady state.

Figure 3 displays the effects of a one-time increase of 1% in the growth rate of money when $\pi = 0$, but now debt contracts are contingent. In this case we choose $\tau = 0.999$ in order to illustrate the case in which a monetary expansion can generate an output downturn. The most striking feature of Figure 3 is that even though we observe an output downturn, the real effects of the money shock under contingent contracts are very small when compared to those obtained in Figure 2. This clearly highlights the difference between the non-contingent and the contingent cases: when debt is contingent, there is no redistribution of wealth from less productive lenders to more productive borrowers.

Finally, Figure 4 illustrates the case in which debt renegotiation takes place. It displays the effects of a one-time decrease of 1% in the growth rate of money when $\pi = 0$, $\tau = 0.9$, and debt is non-contingent. In this case, the model still exhibits persistence, and the amplitude of the effects is much smaller than the ones observed in Figure 2. Notice that output reaches a trough of about -2.5%. Renegotiation avoids a deep economy downturn because it partially protects borrowers from deflation.

5 Concluding comments

This paper analyzes the propagation of monetary shocks by combining collateral and cash-in-advance constraints in a world where changes in money supply occur via open-market operations. We find that a one-time unanticipated monetary injection generate persistent movements in aggregate output, whose amplitude depends on whether debt contracts are contingent or not. In general, output fluctuations are larger if contingent contracts cannot be written. Due to the interaction between the cash-in-advance and collateral constraints, monetary shocks trigger a highly persistent dampening *cycle* rather than a smoothly declining deviation.

The model analyzed here is simple enough to provide insights on how collateral constraints work in a monetary economy. Since the model is highly stylized, future work can involve the following extensions. First, both the utility and production functions may be modified to the more standard concave specification. This would be particularly important if the mechanisms described here were to be carefully quantified and compared with the data. Second, in order to avoid excess response in nominal prices, other frictions would need to be introduced besides collateral constraints. Finally, the model studied here has not been carefully calibrated to assess its ability to match the data. A careful calibration exercise is beyond the scope of this paper and is left for future work.

In summary, what we learn from the simple, stylized model analyzed here is that collateral constraints in combination with cash-in-advance constraints, constitute a potential mechanism that transforms small monetary shocks into significant persistent output fluctuations.

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A Proof of optimal solution for borrowers

We need to prove the claim that borrowers' optimal plan is to consume only the nontradable fraction of output, i.e. $x_t = cK_{t-1}$, to borrow up to the limit and to invest all remaining resources. To do that we compare the utility achieved under the different alternative plans. The first one is to follow the proposed *investment path*. Alternatively, borrowers can consume or save. For these last two alternatives, we only consider single deviations from the investment path at date $t = 0$.[19]

Consider the borrower's marginal utility of investing p_0 dollars given that all aggregate variables remain unchanged at their steady state levels. For simplicity let $\pi = 0$. In steady state, we have $R = 1/\beta'$and $q = a/(1-\beta')$. Therefore, for given prices and aggregate variables at their steady state levels, equations (5), (6) and (7) can be rewritten as:

$$qk_t + (c-q)k_{t-1} = \frac{m_{t-1}^d}{p_{t-1}} \tag{A1}$$

$$\frac{b_t}{p_t} = q\beta' k_t \tag{A2}$$

$$\frac{m_t^d}{p_t} = (a+c)k_{t-1} + \frac{b_t}{p_t} - R\frac{b_{t-1}}{p_{t-1}}. \tag{A3}$$

Replacing the borrowing constraint into the budget constraint,

$$\frac{m_t^d}{p_t} = (a+c-q)k_{t-1} + q\beta' k_t. \tag{A4}$$

Substituting (A4) into (A1) and solving for k_t:

$$k_t = \left[\beta' + 1 - \frac{c}{q}\right] k_{t-1} + \left[\frac{a+c}{q} - 1\right] k_{t-2}. \tag{A5}$$

¿From the steady state value of q we have that $\left(\beta' + 1 - \frac{c}{q}\right) = 2 - (1-\beta') - \frac{c}{q} = 2 - \frac{a+c}{q}$ $= 2 - r^h$. Let $r^h \equiv \frac{a+c}{q}$. We can rewrite (A5) as:

$$k_t = \left(2 - r^h\right) k_{t-1} + (r^h - 1)k_{t-2} \tag{A6}$$

It is easy to check that the roots of the associated characteristic polynomial are 1 and $1 - r^h$. Therefore, k_t can be expressed as:

$$k_t = A_1 + A_2(1-r^h)^t. \tag{A7}$$

where constants A_1 and A_1 need to be determined. Under the proposed guess, the optimal strategy for borrowers is to use the extra p_0 dollars to invest in capital. With this amount, the borrower

[19] Following the logic of Kiyotaki and Moore (1997), "we appeal to the principle of unimprovability", which states that to prove that our proposed strategy of investing all the extra p_0 dollars is optimal, we need to consider only single deviations from this plan at date $t = 0$.

can buy $k_0 = 1/q$ units of capital at $t = 0$. This allows him to borrow $q\beta' k_0 = \beta'$ additional units of output.[20] At $t = 1$, consumption increases by ck_0 units so that from the additional resources, $\beta' - c/q$ can be used to buy capital. Therefore, investment is given by: $k_1 - k_0 = \frac{\beta' - c/q}{q}$, so that $k_1 = k_0(\beta' - c/q + 1)$. With these two initial values (k_0,k_1), constants $A_1$and A_2 can be determined as follows:

$$A_2 = \frac{k_0 - k_1}{r^h} \text{ and } A_1 = \frac{1}{r^h}\left[k_1 - (1 - r^h)k_0\right].$$

Utility under the investment path is given by:

$$\begin{aligned} U^{inv} &= \beta c \sum_{t=0}^{\infty} \beta^t k_t = \beta c \sum_{t=0}^{\infty} \beta^t \left[A_1 + A_2(1 - r^h)^t\right] = cA_1 \frac{\beta}{1-\beta} + cA_2 \frac{\beta}{1 - \beta(1 - r^h)} \\ &= \frac{c}{q} \frac{\beta}{1-\beta} \frac{1}{1 - \beta(1 - r^h)}. \end{aligned}$$

To show that higher utility is attained in the investment path than in the *consumption path*, we need to find conditions under which:

$$\frac{c}{q} \frac{\beta}{1-\beta} \frac{1}{1 - \beta(1 - r^h)} > 1$$

We can transform the expression above to obtain:

$$\frac{c}{a} > \frac{(1-\beta)}{\beta} \frac{(1-\beta)}{(1-\beta')} + (1-\beta)\frac{a+c}{a}.$$

Since $\frac{(1-\beta)}{\beta} < \frac{(1-\beta)}{\beta^2}$ then a sufficient condition for the utility from the investment path being higher is:

$$\frac{c}{a} > \frac{(1-\beta)}{\beta^2}\left[\frac{(1-\beta)}{(1-\beta')} + 1\right]$$

which corresponds to Assumption 2 in the text.

To complete the proof we need to show that higher utility is attained in the investment path than in the *saving path.* Borrowers can save the p_0 dollars and use the return R to commence a strategy of maximum levered investment from date $t = 1$ onwards. Then, all we need to show is that the returns from saving p_0 dollars in period $t = 0$ are lower than the return from investing at $t = 0$. Since from Assumption 1, $\beta' > \beta$, using Assumption 2 is easy to show that $\beta' > \frac{a}{a+c}$. Thus,

$$1 + r^h = 1 + \frac{a+c}{q} = 1 + \frac{(a+c)(1-\beta')}{a} > 1 + \frac{1-\beta'}{\beta'} = \frac{1}{\beta'} = R.$$

Therefore, $1 + r^h > R$, which guarantees that the investment path yields more utility than the alternative savings path. This completes the proof that the proposed solution is an equilibrium. We have presented an analytical proof for $\pi = 0$. For $\pi \neq 0$ it is not possible to provide an analytical

[20]Note that p_0 dollars are equivalent to one unit of output at $t = 0$ prices. Also, by borrowing extra $b_0 = \beta'$, the agent can demand extra β' real money balances in the third subperiod of $t = 0$, in order to buy additional capital in the first subperiod of $t = 1$.

proof. However, for all the numerical simulations in the text, we have verified in the computer that the decision rules for the borrower are optimal.

B Proof of Proposition 2

Let $\pi = 0$ so that in steady state $u = a$. Equation (15) in the text reads:

$$\theta_0 \widehat{K}_t = \theta_1 \widehat{K}_{t-1} + \theta_2 \widehat{K}_{t-2} + \mu_0 \tau^{t-2} \widehat{g}_0 \tag{B1}$$

where:

$$\theta_0 = 1 + (1 - 2\beta')\rho$$

$$\theta_1 = (1 - r^h)(1 - \rho) + 1 + (1 - 2\beta')\rho - (1 - \beta')\frac{1}{\eta}$$

$$\theta_2 = -(1 - r^h)(1 - \rho)$$

$$\mu_0 = -(R - \tau)\left[\left(1 - 2\beta'\right)\tau + (1 - r^h)\right]$$

Since the particular solution for the equation above is

$$\widehat{K}_p = \frac{\mu_0 \widehat{g}_0 \tau^t}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}$$

then the general solution is given by:

$$\widehat{K}_t = A_1 \lambda_1^t + A_2 \lambda_2^t + A_\tau \tau^t \widehat{g}_0 \tag{B2}$$

where $A_\tau = \frac{\mu_0}{\theta_0 \tau^2 - \theta_1 \tau - \theta_2}$ is a constant and the eigenvalues λ_1 and λ_2 satisfy: $\lambda_1 \lambda_2 = \frac{-\theta_2}{\theta_0}$ and $\lambda_1 + \lambda_2 = \frac{\theta_1}{\theta_0}$. Finally, the solutions for constants A_1 and A_2 can be obtained from: $\widehat{K}_1 = A_1 \lambda_1 + A_2 \lambda_2 + A_\tau \tau \widehat{g}_0$ and $\widehat{K}_0 = A_1 + A_2 + A_\tau \widehat{g}_0$.

B.1 Cycles

The dynamic properties of equation (B1) depend on the eigenvalues associated to the homogeneous difference equation $\theta_0 \widehat{K}_t = \theta_1 \widehat{K}_{t-1} + \theta_2 \widehat{K}_{t-2}$ which are given by:

$$\lambda_1, \lambda_2 = \frac{\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_0 \theta_2}}{2\theta_0}.$$

The necessary and sufficient condition for cycles is $\theta_1^2 + 4\theta_0\theta_2 < 0$. Note that $\theta_1$can be rewritten as:

$$\theta_1 = \theta_0 - \theta_2 - (1 - \beta')\frac{1}{\eta}. \tag{B3}$$

Adding and subtracting proper terms, θ_2 can be rewritten as

$$\theta_2 = \xi(\beta') - \theta_0 \tag{B4}$$

where:

$$\begin{aligned}\xi(\beta') &\equiv 2\rho(1-\beta') + r^h(1-\rho)\\ &= (1-\beta')(2\rho + \frac{a+c}{a}(1-\rho)) > 0.\end{aligned}$$

¿From (B3) and (B4), θ_1 can be written as:

$$\theta_1 = 2\theta_0 - \zeta(\beta') \tag{B5}$$

where:

$$\begin{aligned}\zeta(\beta') &\equiv (1-\beta')(2\rho + \frac{a+c}{a}(1-\rho) + \frac{1}{\eta})\\ &< (1-\beta')\left[\max\left\{2, \frac{a+c}{a}\right\} + \frac{1}{\eta}\right].\end{aligned}$$

Finally, from (B3) and (B4), $\underset{\beta'\to 1}{lim}\,\theta_2 = \theta_0$ and $\underset{\beta'\to 1}{lim}\,\theta_1 = 2\theta_0$.

B.2 Proof of Proposition

To show that for β' sufficiently large the model exhibits cycles, it needs to be proven that $\theta_1^2 + 4\theta_0\theta_2 < 0$. Use (B4) and (B5) to get:

$$\begin{aligned}\theta_1^2 + 4\theta_0\theta_2 &= (2\theta_0 - \zeta(\beta'))^2 - 4\left[\theta_0 - \xi(\beta')\right]\theta_0\\ &< -4\theta_0(1-\beta')\frac{1}{\eta} + (1-\beta')^2\left[\max\left\{2, \frac{a+c}{a}\right\} + \frac{1}{\eta}\right]^2.\end{aligned}$$

Note that $\underset{\beta'\to 1}{lim}\frac{1}{\eta} = -\frac{G''(\left(\bar{K}-K^1\right)/n)K^1}{nG'(\left(\bar{K}-K^1\right)/n)} > 0$ where K^1 is the solution of (13) for β' equal to 1 and $\pi = 0$. Therefore, the second term in the last expression approaches to zero faster than the first term as $\beta' \to 1$. Note that $\theta_0\frac{1}{\eta}$ remains bounded above since θ_0 approaches $1 - \alpha^N(K^1) > 0$ and the fact that $\frac{1}{\eta}$ approaches a constant greater than zero. Thus, for β' large enough the first term dominates and the expression is negative. ■

It is also useful to state solution (B2) in its polar representation (See Allen, 1959, page 189)

$$\widehat{K}_t = Ar^t\cos(\omega t + \phi) + A_\tau \tau^t,$$

where A and ϕ are constants that can be determined from the initial conditions, and

$$r = \sqrt{-\theta_2/\theta_0},$$

$$\omega = \cos^{-1}\left(\frac{\theta_1/\theta_0}{2r}\right).$$

Stability is guaranteed if the modulo r is less than 1, a result that follows from (B5) for large β'. In addition, r is close to 1 when β' is close to 1. Thus, the difference equation displays persistent dampening cycles.

C Forward looking solution for asset prices

This appendix gives the solution for $\widehat{q}_0$ and $\pi = 0$. From equation (14) in the text:

$$\widehat{q}_t - \beta' \widehat{q}_{t+1} = \left(1-\beta'\right)\left(\frac{1}{\eta} - \rho\right)\widehat{K}_t + \left(1-\beta'\right)\rho \widehat{K}_{t+1} - (1-\beta')\widehat{g}_{t+1}$$

iterate forward and use the transversality condition $\widehat{q}_\infty = 0$ to rule out bubbles in the price of capital to obtain:

$$\begin{aligned}
\frac{\widehat{q}_0}{(1-\beta')} &= \sum_{j=0}^{\infty} \beta'^j \left[\left(\frac{1}{\eta} - \rho\right)\widehat{K}_j + \rho \widehat{K}_{j+1} - \widehat{g}_{j+1}\right] \\
&= \left(\frac{1}{\eta} - \rho\right)\widehat{K}_0 + \sum_{j=0}^{\infty} \beta'^j \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\widehat{K}_{j+1} - \sum_{j=0}^{\infty} \beta'^j \widehat{g}_{j+1}.
\end{aligned}$$

Using the solution for the non-homogeneous second order difference equation (B1) we have:

$$\begin{aligned}
\frac{\widehat{q}_0}{(1-\beta')} &= \left(\frac{1}{\eta} - \rho\right)\widehat{K}_0 + \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\sum_{j=0}^{\infty} \beta'^j \left(A_1 \lambda_1^{j+1} + A_2 \lambda_2^{j+1} + A_\tau \tau^{j+1} \widehat{g}_0\right) \\
&\quad + (R-\tau)\widehat{g}_0 \sum_{j=0}^{\infty} \beta'^j \tau^j
\end{aligned}$$

which after some algebra yields:

$$\begin{aligned}
\frac{\widehat{q}_0}{(1-\beta')} &= \left(\frac{1}{\eta} - \rho\right)\widehat{K}_0 + \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\frac{(\lambda_1 A_1 + \lambda_2 A_2) - \beta' \lambda_1 \lambda_2 (A_1 + A_2)}{1 - \beta'(\lambda_1 + \lambda_2) + \beta'^2 \lambda_2 \lambda_2} \\
&\quad + \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\frac{\tau A_\tau \widehat{g}_0}{1 - \beta'\tau} + \frac{(R-\tau)\widehat{g}_0}{1-\beta'\tau}.
\end{aligned}$$

Finally using $\widehat{K}_0 = 0$ and the properties of λ_1, λ_2 and $\widehat{K}_1$ from Appendix B we get, after some algebra:

$$\begin{aligned}
\frac{\widehat{q}_0}{(1-\beta')} &= \frac{\left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\theta_0}{\theta_0 - \beta'\theta_1 - \beta'^2\theta_2}\widehat{K}_1 + \frac{(R-\tau)\widehat{g}_0}{1-\beta'\tau} \\
&\quad + \left(\beta'\left(\frac{1}{\eta} - \rho\right) + \rho\right)\left[\frac{\tau}{1-\beta'\tau} - \frac{(\theta_0 \tau + \beta'\theta_2)}{\theta_0 - \beta'\theta_1 - \beta'^2\theta_2}\right] A_\tau \widehat{g}_0
\end{aligned}$$

which solves for $\widehat{q}_0$ as a function of $\widehat{K}_1$. Also, the following equation relates $\widehat{q}_0$, $\widehat{q}_1$ and $\widehat{K}_1$:

$$\widehat{q}_0 = \beta' \widehat{q}_1 + \left(1-\beta'\right)\rho^\pi \widehat{K}_1 + (1-\beta')(R-\tau)\widehat{g}_0$$

D Solution for $\pi > 0$

When the steady-state inflation is not zero, but $\pi > 0$, then the simple rule that following a one-time money shock at $t = 0$ we can guarantee convergence of d_t back to the steady state by imposing $\tau < 1$ does not hold anymore. Recall that since $H_t^s = \tau H_{t-1}^s$ and when $\pi = 0$ we have $H^s = 0$, then $\tau < 1$ is enough to guarantee that H_t^s eventually converges to zero. In contrast, this is not the case when $\pi > 0$ then $d > 0$. Thus, when $\pi > 0$ the "sterilization" policy needs to be changed.

In particular, assume that the economy starts off the steady state and at time $t = 0$ there is an unexpected one-time increase in growth rate of money $\varepsilon > 0$, i.e. $\widehat{g}_0 = \frac{\varepsilon}{1+\pi}$. In this case, the government chooses a period $t = T$ such that from T on, the growth rate of money supply is zero, i.e. $\widehat{g}_t = 0$ for $t \geqslant T$. What this implies is that for $t \geqslant T$, the law of motion of $\widehat{d}_t$ is given by:[21]

$$\widehat{d}_t = \frac{1}{\beta'}\widehat{R}_t + \frac{1}{\beta'}\widehat{d}_{t-1}$$

which is clearly unstable, since $\beta' < 1$. Iterating forward on the equation above and imposing the transversality condition that $\widehat{d}_\infty = 0$, we obtain that $\widehat{d}_{T-1}$ must satisfy:

$$\widehat{d}_{T-1} = -\sum_{\tau=0}^{\infty} \beta'^{\tau} \widehat{R}_{\tau+T}$$

to guarantee convergence back to the steady-state. Further, since using the law of motion of money supply we have that $\widehat{g}_{T-1}$ is given by:

$$\widehat{g}_{T-1} = -\frac{d}{1+d}\widehat{d}_{T-1} + \frac{d}{\beta'(1+d)}\widehat{R}_{T-1} + \frac{d}{\beta'(1+d)}\widehat{d}_{T-2}$$

so that $\widehat{g}_{T-1}$ depends on $\widehat{d}_{T-1}$. In summary, when the government chooses a period T such that $\widehat{g}_T = 0$, it must also choose $\widehat{g}_{T-1}$ to satisfy the transversality condition. Further for periods $1 \leq t < T-2$ we allow the government to choose any exogenous law of motion for $\widehat{g}_t \leqslant 0$, i.e. any rule in which the monetary expansion at time $t = 0$ is reverted. For instance, a natural choice would be a gradual money contraction up to period $T-2$ and a choice of $\widehat{g}_{T-1}$ that satisfies the condition above.

When $\pi > 0$, the dynamics of capital are described by:

$$\theta_0^\pi \widehat{K}_t = \theta_1^\pi \widehat{K}_{t-1} + \theta_2^\pi \widehat{K}_{t-2} + \frac{1}{1+\pi}\left[\left(1 - 2\beta'\right)\widehat{g}_t + (1 - r^h)\widehat{g}_{t-1}\right]$$

where:

$$\theta_0^\pi = 1 + (1 - 2\beta')\frac{\rho}{(1+\pi)}$$

$$\theta_1^\pi = \frac{(1-r^h)(1-\rho)}{(1+\pi)} + 1 + (1-2\beta')\frac{\rho}{(1+\pi)} - \frac{(1-\beta')}{\eta(1+\pi)}$$

[21] This equation is the linearized version of the law of motion of the money supply when $\widehat{g}_t = 0$.

$$\theta_2^\pi = -\frac{(1-r^h)(1-\rho)}{(1+\pi)}.$$

Using the dynamic equation of capital, as well as the transversality condition for government debt, the law of motion of money supply and the forward-looking solution for capital prices it is possible to construct a system of 5 equations in 5 unknowns: $\widehat{K}_{T-1}$, $\widehat{q}_{T-2}$, $\widehat{q}_{T-1}$, $\widehat{d}_{T-1}$ and $\widehat{g}_{T-1}$. Since this system is a function of past values $\widehat{K}_{T-3}$, $\widehat{K}_{T-2}$ and $\widehat{d}_{T-2}$ an iterative procedure that starts with a guess for $\widehat{K}_1$ must be implemented to find the solution. Details on the solution procedure are available from the authors upon request.

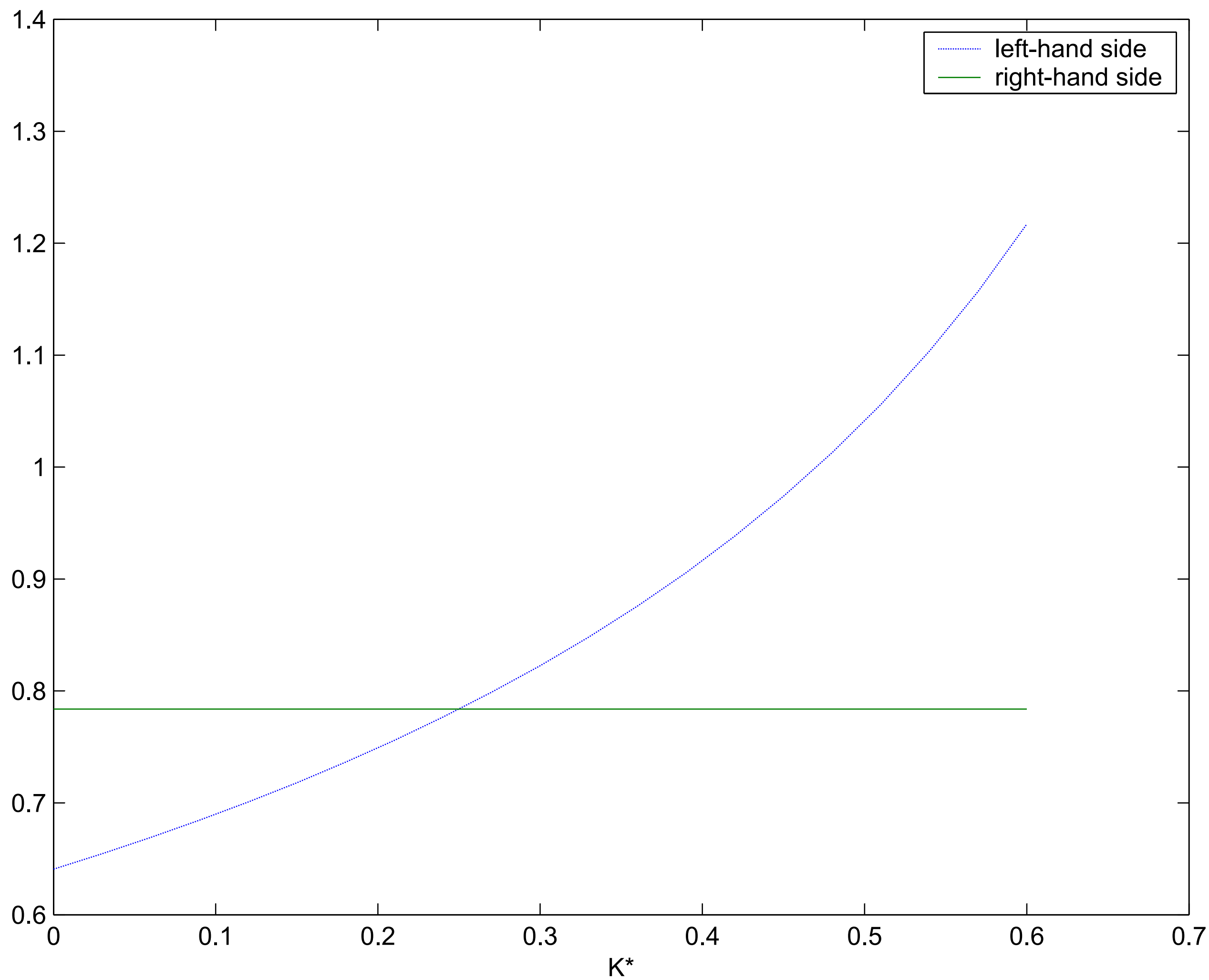

Figure 1: Steady state distribution of capital

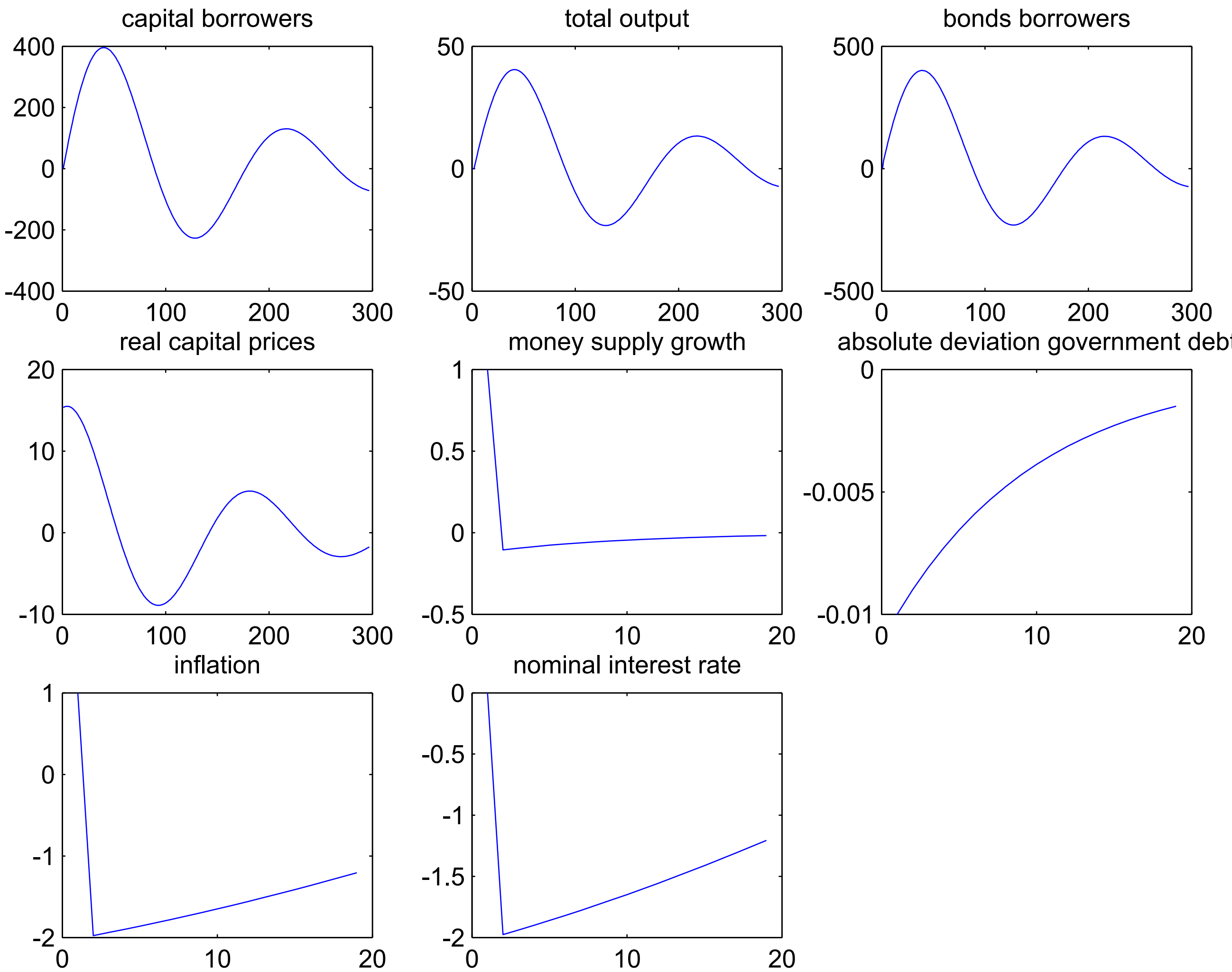

Figure 2: A 1% one-time money expansion under non-contingent debt

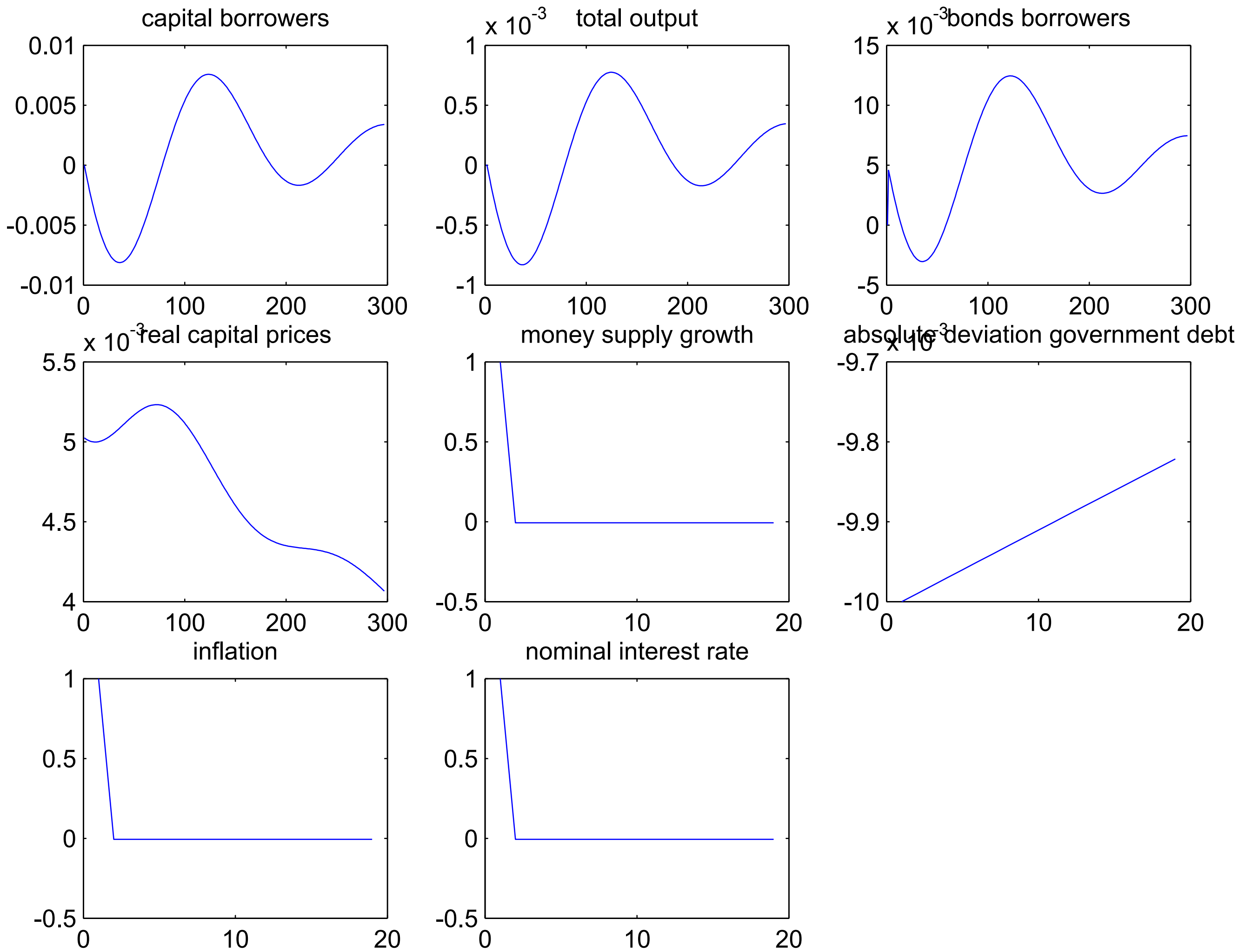

Figure 3: A 1% one-time money expansion under contingent debt

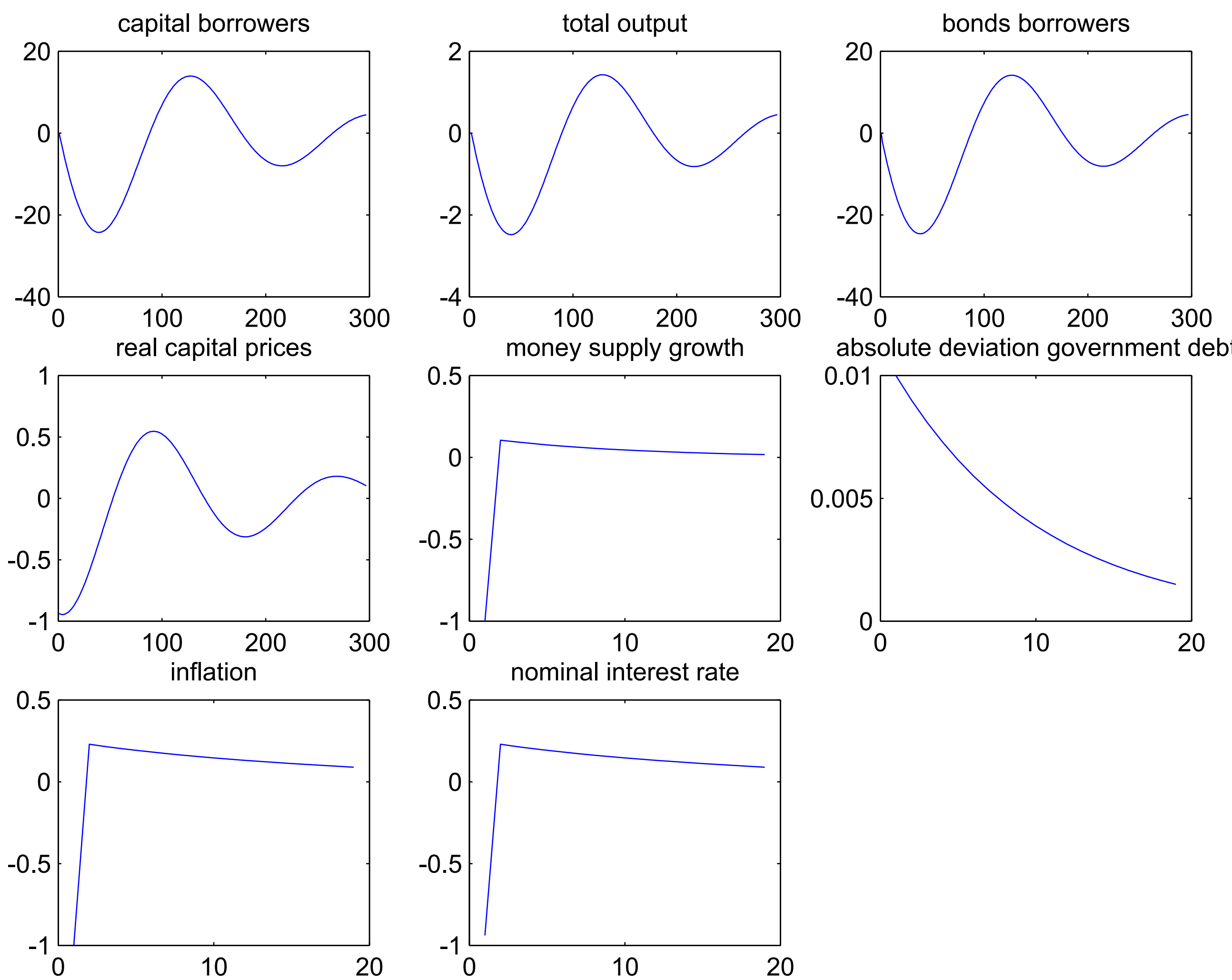

Figure 4: A 1% one-time money contraction with debt renegotiation