SEARCH, LAYOFFS AND RESERVATION WAGES WHEN JOB OFFERS FOLLOW A STOCHASTIC PROCESS

SVETLANA BOYARCHENKO

Department of Economics, The University of Texas at Austin,
1 University Station C3100, Austin, TX 78712, U.S.A.
E-mail: sboyarch@eco.utexas.edu
Phone: (512)-475-8521; fax: (512)-471-3510

Abstract. Despite the fact that the empirical data indicate the presence of non-stationarity in wage offer distributions, the majority of job-search models are stationary. We model logs of wage offers as a Markov process with i.i.d. increments and solve two typical job-search models for reservation wages, value functions and expected individual duration of unemployment. All solutions are in the closed form and admit interpretation in terms of expected present values of certain streams of payoffs.

Key words: reservation wage, supremum process, Wiener-Hopf factorization.

1. Introduction

Over the past two decades, many studies in labor economics have attempted to analyze empirically the determinants of individual employment spell duration of workers looking for a job (see, for example, Van den Berg (1999) and the bibliography therein). Many of these studies have used the job search model as a workhorse: for surveys, see, for example, Devine and Kiefer (1991) and Wolpin (1995). Indeed, the former review of the empirical literature cites over 500 studies.

The basic job-search model contains three exogenous variables: the Poisson rate of arrival of job offers, the wage distribution and the unemployment compensation. If none of these variables change over time and the time horizon is infinite, then the model is stationary. The majority of existing job search models are stationary. At the same time, various empirical studies find significant duration dependence of

I am thankful to D. Corbae, S. Freeman, B. Smith, and other participants of a seminar at UT Austin for discussion and comments.

the probability of being (re)-employed. This dependence indicates the presence of non-stationarity (see, e.g., Van den Berg (1990) and the bibliography therein). Moreover, stationary wage distribution is inconsistent with the productivity shocks modeled as a process in the neoclassical growth model.

We construct the first job-search model where wage offers follow a stochastic Markov process. To be more specific, we assume that log-wages follow a random walk, that is a process with independently and identically distributed increments. The characteristics of the random walk are fairly general. Two job-search models are considered in the paper: the benchmark model, where a worker remains employed forever if she accepts an offer at some point in time; and the model where the worker faces a positive probability of being laid off every period after the first period on the job. Both models are solved for the reservation wages, value functions, and expected duration of unemployment.

The method of the paper is straightforward and can be summarized as follows: we assume that an unemployed worker accepts a wage offer if and only if the wage is not less than a certain barrier called the reservation wage. We fix an arbitrary candidate for the reservation wage and write the corresponding dynamic programming problem. It turns out to be the case that the Bellman equation is an integral equation known as the Wiener-Hopf equation. The latter can be solved for the value function by the Wiener-Hopf factorization method. The central point of the method in the form suggested in the paper is that practically every step of the solution of the optimization problem can be interpreted as the calculation of the expected present value of a certain stream of payoffs. In particular, in the benchmark model, the expectations are taken under assumption that wages follow the supremum process $\bar{w}_t = \max_{0\le t\le s} w_s$ (here w_t is the wage offer at date t). In the model with layoffs, we also use the process for wages offered every other period and the supremum of this process. The processes with two time periods as the basic time unit become involved because the worker can be laid off and if this happens, she has to wait one period before a new offer arrives.

After an explicit formula for the value function has been obtained by the Wiener-Hopf factorization method, we guess the reservation wage and verify that the latter maximizes the value function. Also, we prove the uniqueness of the solution. The reservation wages and value functions in both models admit closed form solutions in terms of stochastic integrals. Moreover, explicit analytical solutions can be obtained as well (see Boyarchenko and Levendorskiǐ (2002)). In the paper, we provide such solutions for the reservation wages and the expected waiting time before an offer is accepted for a particular case of the probability distribution function.

So far, only a few papers have been published that allow for non-stationarity in job-search models (see, for example, Burdett (1979), Gronau (1971), Heckman and Singer (1982), Lippman and McCall (1976) and Mortensen (1986)). These papers consider only very specific departures from stationarity and they lack a rigorous derivation of the reservation wage dynamics. Van den Berg (1990) examines the dynamics of an individual's reservation wage in a more general non-stationary job-search model. In his model, all exogenous variables may vary over time in a rather general way. He derives a differential equation that describes the evolution of the reservation wage over time. However, there are some unrealistic assumptions in Van den Berg's model. First of all, it is assumed that once a job offer is accepted, it will be held forever. The only reason the author does not allow workers to quit or to be laid off is the intractability of the corresponding model. Second, and more important, even though Van den Berg allows the wage distribution to vary over time, the wage which a worker may be offered in the future is independent of the current state variables in that model. When wage offers evolve as a stochastic process, the current state affects possible future wages, therefore the way of modeling of the wage evolution suggested in the paper is much more realistic than traditional random draws from independent stationary or even non-stationary distributions.

The rest of the paper is organized as follows. In Section 2, the reader is reminded about the specification of the benchmark job-search model, the process for wages is specified and main results for the reservation wage and value function are given. In Section 3, the rigorous derivation of results for the benchmark model is presented. Section 4 contains the solution for the model with layoffs. In Section 5, we derive explicit analytical formulas for the reservation wages for the case when the probability distribution function for increments of log-wages is given as (or can be approximated by) a simple exponential polynomial. In this case, the problem reduces to computation of the roots of a quadratic polynomial. At the same time, the model is rather flexible: it admits jumps in wages in both directions and the relative sizes of large and small jumps can be controlled by the parameters of the PDF. The reservation wage formulas allow one to analyze how the workers respond to changes in the parameters of the underlying stochastic process. In Section 6, the formula for the expected waiting time is derived, and for the case of exponential polynomials, the explicit solution is given. For the latter case, we also provide the necessary and sufficient condition (in terms of the parameters of the PDF) of finiteness of the expected waiting time. The result is particularly simple in the case of continuous PDF: the expected waiting time is finite if and only if the drift of the log-wage is positive, and if it is positive, then the expected waiting time is inverse proportional to the drift as in the deterministic continuous time model. Section 7 contains final remarks. The most technical issues are delegated to the Appendix.

2. Benchmark model: main results

2.1. **Problem specification.** Consider a problem of intertemporal job search. Time is discrete and the horizon is infinite: $t = 0, 1, \dots$. An unemployed worker devises a strategy to maximize the present value of her expected life-time utility:

$$\max E \sum_{t=0}^{\infty} \beta^t u(z_t), \tag{2.1}$$

where E is the expectation conditioned on information available at $t = 0$, $0 < \beta < 1$ is the discount factor, u is the instantaneous utility function such that $u' > 0$ and $u'' \le 0$, and z_t is the instantaneous income. In this paper, we consider the linear utility: $u(z) = z$. Generalization to the case of Cobb-Douglas utility is straightforward and more general utility functions can be considered as well.

If the worker is unemployed at date t, she receives the unemployment income: $z_t = b \ge 0$, where b is the unemployment insurance benefit less of search costs. At the same time, the worker gets a wage offer w_t, one at each date t.[1] The worker has an option of rejecting an offer and waiting until next period for a new wage offer to arrive. Alternatively, the worker can accept a wage offer w, in which case she will be paid the wage w every period starting from the date of acceptance.[2] Thus the value function defined by (2.1) is a function of the current wage offer w, which is the state variable in the model. Notice that the value of being unemployed forever is equal to

$$\sum_{t=0}^{\infty} \beta^t b = \frac{b}{1-\beta}.$$

[1]For simplicity, we assume that job offers arrive every period when the worker is unemployed, otherwise, we could have introduced a probability of random offer arrivals.

[2]We disregard layoffs, quits and recalls in this model; the model with layoffs will be considered in Section 4.

We assume that wage offers follow a stochastic process; to be more specific, we assume that the log-wage $X_t = \ln w_t$ admits a decomposition $X_t = X_0 + Y_1 + \ldots + Y_t$, where $Y_1, \ldots, Y_t$ are independently and identically distributed random variables on the probability space Ω (one says that X is a random walk on $\mathbf{R}$). We impose the following condition on the wage process:

$$\beta E[w_t/w_{t-1}] = \beta E[w_1/w_0] < 1. \tag{2.2}$$

Equation (2.2) ensures that the value function (2.1) is finite, which can be argued as follows. The value function does not exceed the sum of the value of being unemployed forever (which is finite) and $\sum_{t=0}^{\infty} \beta^t E[w_t]$, the sum of the expected present values of accepting an offer at date $t = 0, 1, \ldots$. By the law of iterated expectations, we have for i.i.d. Y_j's:

$$\begin{aligned} E[w_t] &= E\left[e^{X_0+Y_1+\ldots+Y_t}\right] \\ &= e^{X_0} E\left[e^{Y_1}\right] \ldots E\left[e^{Y_t}\right] \\ &= w_0 E[w_1/w_0] \ldots E[w_t/w_{t-1}]. \end{aligned}$$

Therefore,

$$\sum_{t=0}^{\infty} \beta^t E[w_t] = w_0 \sum_{t=0}^{\infty} \beta^t E[w_1/w_0]^t = \frac{w_0}{1 - \beta E[w_1/w_0]} < \infty$$

if and only if (2.2) holds. Hence if (2.2) is satisfied, the value function (2.1) is finite. It can be shown that if (2.2) fails, then the value function is infinite.

If the wage offer is sufficiently low, then it is advantageous to remain unemployed. Define w^* to be the smallest wage offer such that the unemployed worker is better off accepting than rejecting the offer; w^* is known as the reservation wage. That is, the worker accepts the offer w if and only if $w \geq w^*$. Let w be the current wage offer, and $V(w; \hat{w})$ be the value of the offer when $\hat{w}$ is chosen as a candidate for the reservation wage. If the offer is accepted, the worker gets w from now on, therefore

$$V(w; \hat{w}) = \frac{w}{1-\beta}, \quad \text{if } w \geq \hat{w}; \tag{2.3}$$

otherwise, the worker gets b this period and a new offer next period, hence

$$V(w; \hat{w}) = b + \beta E[V(w_1; \hat{w}) \mid w_0 = w], \quad \text{if } w < \hat{w}. \tag{2.4}$$

Notice that in the traditional labor search model, the continuation payoff in (2.4) is $E[V(w; \hat{w})]$. This implies that current prices/wages do not convey any information about expected future payoffs, which is unrealistic. In that model, the value of rejecting an offer is independent of w, and the value of accepting is increasing in w, therefore the reservation wage is the one that equates these two values.

2.2. **Reservation wage and value of an offer.** We want to find the reservation wage, w^*, that maximizes the value function:

$$V(w; w^*) \geq V(w; \hat{w}), \quad \forall\ w \text{ and } \hat{w}. \tag{2.5}$$

To this end, we fix an arbitrary $\hat{w}$ and solve equations (2.3) and (2.4) for V. By comparing solutions for various $\hat{w}$, we find a unique w^* satisfying (2.5). In this subsection, we present and discuss the formulas for w^* and $V(w; w^*)$. The rigorous derivation of the formulas is presented in Section 3.

Introduce

$$\underline{w}_t = \min_{0 \leq s \leq t} w_s \quad \text{and} \quad \bar{w}_t = \max_{0 \leq s \leq t} w_s,$$

the processes $\underline{w} = \{\underline{w}_t\}$ and $\bar{w} = \{\bar{w}_t\}$ will be called the infimum and supremum wage processes respectively. Let T be an exponentially distributed random variable, independent of $\{Y_t\}$, with the mean $\beta/(1-\beta)$ (in other words, T takes values $t = 0, 1, \ldots$ with probability $(1-\beta)\beta^t$). The reservation wage is given by

$$\frac{w^*}{b} = E[\bar{w}_T |\ w_0 = 1] = E[\bar{w}_T |\ w_0 = w]/w, \tag{2.6}$$

or equivalently,

$$\frac{w^*}{b} = (1-\beta)\sum_{t=0}^{\infty} \beta^t E[\bar{w}_t |\ w_0 = 1] = \frac{\sum_{t=0}^{\infty} \beta^t E[\bar{w}_t |\ w_0 = w]}{\sum_{t=0}^{\infty} \beta^t w}. \tag{2.7}$$

If there is no uncertainty, then trivially, $w^* = b$. If wages follow a stochastic process, then there is a risk: if the worker accepts the offer $b \leq w$, she misses the opportunity of receiving higher offers in the future. Hence there is the hurdle w^*/b the wage offer must clear in order to be accepted. By (2.6), the hurdle equals the ratio of the expected wage accepted at random time T, given the wages follow the supremum process, and the current wage w. According to (2.7), the hurdle is the ratio of two expected present values of wage income streams: the one in the numerator is for supremum wages while the value in the denominator is calculated for the constant stream $w_t = w$. If the probability of negative jumps in wages increases, $E[\bar{w}_t]$ decreases, hence the hurdle decreases as well. This agrees with the fact that people are willing to take lower paid jobs when they expect wages to drop. Similarly, if the probability of positive jumps increases, the hurdle increases as well, hence increasing upward uncertainty in wages may cause reduction in the supply of labor. Recall that in the standard job search model (see, e.g., Sargent (1987)),

$$w^* = b + \frac{\beta}{1-\beta}\int_{w^*}^{\infty}(w' - w^*)dw'.$$

This is a non-linear equation which does not admit a closed form solution except for special cases. Our approach provides an explicit formula for the reservation wage together with the natural interpretation of the formula.

Our next result is the formula for the value function in the region $w < w^*$. The value function admits the following decomposition:

$$V(w; w^*) = \frac{w}{1-\beta} + V_s(w; w^*), \quad w < w^*, \tag{2.8}$$

where the first term is the value of accepting the current offer, and $V_s(w; w^*)$ is the option value of searching. The option value $V_s(w; w^*)$ is given by (see Section 3)

$$V_s(w; w^*) = \frac{w}{E[\bar{w}_T|\ w_0 = w]} \cdot \frac{E[(w^* - \bar{w}_T)_+|\ w_0 = w]}{1-\beta}, \tag{2.9}$$

where $(w^* - \bar{w}_T)_+ \equiv \max\{w^* - \bar{w}_T, 0\}$. We see that the first factor in (2.9) is the reciprocal of the hurdle introduced by (2.6), and the second factor is proportional to the expected value of the European put option on the supremum of wage with the strike price w^* and the random expiration date T. Equivalently, we can write (2.9) as

$$V_s(w; w^*) = \frac{w}{E[\bar{w}_T|\ w_0 = w]} \sum_{t=0}^{\infty} \beta^t E[(w^* - \bar{w}_t)_+|\ w_0 = w]. \tag{2.10}$$

The last equation makes it clear that the option value of searching is positive as long as the supremum of wage is below the reservation wage, and at the reservation wage, the option value of searching vanishes.

Alternatively, the value of the offer can be decomposed as

$$V(w; w^*) = \frac{b}{1-\beta} + V_e(w; w^*), \quad w < w^*; \tag{2.11}$$

here the first summand is the value of staying unemployed forever, and $V_e(w; w^*)$ is the option value of future employment opportunities. The option value is

$$V_e(w; w^*) = \frac{w}{E[\bar{w}_T|\ w_0 = w]} \cdot \frac{E[(\bar{w}_T - w^*)_+|\ w_0 = w]}{1-\beta}. \tag{2.12}$$

The second factor in (2.12) is determined by the expected value of the European call option on the supremum of wage with the strike price w^* and the random date of expiry T.

Finally, we can write

$$V_e(w;w^*) - V_s(w;w^*) = \frac{w-b}{1-\beta}.$$

The last equation says that the difference between the option value of future employment opportunities and the option value of searching equals the present value of the gain of accepting the current offer.

3. Benchmark model: proofs

3.1. **Reduction to the Wiener-Hopf equation.** Here we present the rigorous derivation of the results described in Section 2; the most technical parts of the proof are delegated to the Appendix. In Section 2, we used the current wage as the state variable, which is non-negative, therefore the state space there is $\mathbf{R}_+$. In order to prove the results, we use the state space for log-wages, which is $\mathbf{R}$, because the choice of the half-line as the state space requires much more difficult technique. So, instead of characterizing the state by w, we use $x = \ln w$ as a generic state variable; and $w(x) = e^x$ is the current wage offer. Fix a candidate for the log-reservation wage, $h \in \mathbf{R}$, and set $W(x;h) = V(w(x); w(h))$, $W_e(x;h) = V_e(w(x); w(h))$, $W_s(x;h) = V_s(w(x); w(h))$.

In order to find the value function $W(\cdot;h)$, we want to reduce the original optimization problem to the Wiener-Hopf equation and solve the latter by the Wiener-Hopf factorization method. This method can be applied and explained in the easiest way when one deals with bounded functions. However in our model, the value function is unbounded because with the optimal choice of the reservation wage, the value function grows in the same way as the wage, that is exponentially. To deal with the problem, we introduce the wage ceiling $w^n(x) \equiv \min\{w(x), w(n)\}$, where $n > \ln b$. Let W^n be the value function defined by (2.1) for the case when the worker faces the wage offer w^n, and $w(h)$ is chosen as a candidate for the reservation wage. Since the instantaneous utility of the worker cannot be greater than $w(n)$, function W^n is bounded by $w(n)/(1-\beta)$. Clearly, for each x, $W^n(x;h)$ increases with n, and by the monotone convergence theorem,

$$\lim_{n\to\infty} W^n(x;h) = W(x;h).$$

After the analytic expression for $W^n(x;h)$ is obtained, we will pass to the limit and find $W(x;h)$.

For a Markov process X, denote a family of operators, $\{P_t\}$, acting in $L^\infty(\mathbf{R})$ as follows:

$$P_t f(x) = E[f(X_t) \mid X_0 = x].$$

Set $P \equiv P_1$. Clearly, $\|P_t\| = 1$, $\forall\ t$, and by the law of iterated expectations, $P_t = P^t$. Following the same reasoning as in Section 2, we conclude that W^n satisfies the following system:

$$W^n(x;h) = \frac{w^n(x)}{1-\beta}, \qquad \text{if } x \geq h; \tag{3.1}$$

$$W^n(x;h) = b + \beta P W^n(x;h), \quad \text{if } x < h. \tag{3.2}$$

As before, define the option value of searching by

$$W_s^n(x;h) = W^n(x;h) - \frac{w^n(x)}{1-\beta}. \tag{3.3}$$

Notice that W_s^n is bounded and

$$\lim_{n\to\infty} W_s^n(x;h) = W(x;h) - \frac{w(x)}{1-\beta} = W_s(x;h).$$

Equations (3.1)–(3.3) imply that

$$(I-\beta P)W_s^n(x;h) = -(I-\beta P)g_0^n(x), \quad x < h; \tag{3.4}$$

$$W_s^n(x;h) = 0, \qquad x \geq h, \tag{3.5}$$

where $g_0^n(x) \equiv (w^n(x) - b)/(1-\beta)$. Introduce

$$g^n(x) \equiv g_0^n(x+h), \tag{3.6}$$

$$\tilde{W}^n(x) \equiv W_s^n(x+h;h), \tag{3.7}$$

and rewrite the problem (3.4)–(3.5) as

$$(I-\beta P)\tilde{W}^n(x) = -(I-\beta P)g^n(x), \quad x < 0; \tag{3.8}$$

$$\tilde{W}^n(x) = 0, \qquad x \geq 0. \tag{3.9}$$

Equation (3.8) subject to (3.9) is called the Wiener-Hopf equation; it can be solved by the Wiener-Hopf factorization method. The method can be applied in different (essentially equivalent) analytical and stochastic forms - see the discussion in Boyarchenko and Levendorskiǐ (2002). Unlike in the above monograph, here we use the stochastic form till the end when it is necessary to obtain explicit formulas and the analytical tools become indispensable. We believe that this form is more suitable for applications in Economics.

We know that $\tilde{W}^n$ and g^n are bounded, therefore we may look for a solution to the system (3.8)–(3.9) in $L^\infty(\mathbf{R})$. For $\tilde{W}^n$, a solution to (3.8)–(3.9), define function g_1 by

$$(I-\beta P)\tilde{W}^n = -(I-\beta P)g^n + g_1. \tag{3.10}$$

By construction, $g_1 \in L^\infty(\mathbf{R})$ and vanishes on $\mathbf{R}_-$; hence $g_1 \in L^\infty(\mathbf{R}_+)$. [3] It is obvious that the problem (3.8)–(3.9) is equivalent to the following problem: find $W \in L^\infty(\mathbf{R}_-)$ and $g_1 \in L^\infty(\mathbf{R}_+)$ which satisfy (3.10).

3.2. **Expected present value and resolvent.** If g_1 in (3.10) had been known, $\tilde{W}^n$ could have been found easily by using the inverse to $I - \beta P$:

$$(I - \beta P)^{-1} = I + \beta P + \beta^2 P^2 + \cdots = \sum_{t=0}^{\infty} \beta^t P^t. \tag{3.11}$$

The series converges because $\|\beta P\| = \beta \|P\| = \beta \in (0, 1)$. By applying (3.11) to (3.10) we would have obtained

$$\tilde{W}^n = -g^n + (I - \beta P)^{-1} g_1 = -g^n + \sum_{t=0}^{\infty} \beta^t P^t g_1. \tag{3.12}$$

Recall that $P^t g_1 = E[g_1(X_t)|\ X_0 = x]$, hence we can rewrite the last equation as

$$\tilde{W}^n = -g^n + U_X^\beta g_1,$$

where U_X^β is the resolvent operator of the process X, defined by

$$(U_X^\beta f)(x) \equiv E\left[\sum_{t=0}^{\infty} \beta^t f(X_t)|\ X_0 = x\right] = \sum_{t=0}^{\infty} \beta^t E[f(X_t)|\ X_0 = x].$$

In other words, the resolvent operator applied to a function f gives the expected present value of the stochastic stream $f(X_t)$. The argument above shows that for the random walk X,

$$U_X^\beta (I - \beta P) = (I - \beta P) U_X^\beta = I \quad \text{or}$$
$$U_X^\beta = (I - \beta P)^{-1} \quad \text{and} \quad (U_X^\beta)^{-1} = I - \beta P. \tag{3.13}$$

3.3. **Infimum and supremum processes and Wiener-Hopf factorization.** If g_1 had been known, to find $\tilde{W}^n$, it would have sufficed to compute the expected present value of the stream $g_1(X_t)$. Unfortunately, g_1 is unknown, so (3.12) does not help. Nevertheless, $\tilde{W}^n$ can be written in terms of resolvents (expected present values) not of the random walk X, but of the processes $N_t = \min_{0 \le s \le t} X_s$ and $M_t = \max_{0 \le s \le t} X_s$ which are called the infimum and supremum processes respectively, as their analogs in continuous time. The Wiener-Hopf factorization theorem (see the Appendix) allows one to factorize

[3] Recall that one writes $f \in L^\infty(\mathbf{R}_\mp)$, if $f \in L^\infty(\mathbf{R})$ vanishes on $\mathbf{R}_\pm$. Clearly, $L^\infty(\mathbf{R}_\mp) \subset L^\infty(\mathbf{R})$ is a subspace.

U_X^β into the product of resolvents of the processes $N = \{N_t\}$ and $M = \{M_t\}$:

$$(1-\beta)^{-1}U_X^\beta = U_M^\beta U_N^\beta = U_N^\beta U_M^\beta. \tag{3.14}$$

Using (3.13) and (3.14) we can rewrite (3.10) as

$$(U_N^\beta)^{-1}(U_M^\beta)^{-1}\tilde{W}^n = -(U_N^\beta)^{-1}(U_M^\beta)^{-1}g^n + (1-\beta)g_1. \tag{3.15}$$

Notice that to justify (3.15), we need to know that U_N^β and U_M^β are invertible in $L^\infty(\mathbf{R})$. We prove the boundedness of the inverses to U_M^β and U_N^β as follows: first, $I - \beta P$ is bounded. Second, U_M^β and U_N^β are bounded (it suffices to notice that $|E[f(x + M_t)]| \le ||f||$, hence the norm of U_M^β is bounded by $1 + \beta + \beta^2 + \cdots = (1-\beta)^{-1}$, and the same holds with N instead of M), and finally, on the strength of (3.14), the inverses

$$(U_M^\beta)^{-1} = (1-\beta)(I - \beta P)U_N^\beta \tag{3.16}$$

and

$$(U_N^\beta)^{-1} = (1-\beta)(I - \beta P)U_M^\beta$$

are bounded as well.

For the next step, we need the following lemma.

Lemma 1. *Let $z \in (0,1)$. Then*

a) For any $f \in L^\infty(\mathbf{R}_-)$, we have $U_M^z f \in L^\infty(\mathbf{R}_-)$, and moreover, $U_M^z : L^\infty(\mathbf{R}_-) \to L^\infty(\mathbf{R}_-)$ is invertible;

b) For any $f \in L^\infty(\mathbf{R}_+)$, we have $U_N^z f \in L^\infty(\mathbf{R}_+)$, and moreover, $U_N^z : L^\infty(\mathbf{R}_+) \to L^\infty(\mathbf{R}_+)$ is invertible.

Proof. a) Let $x > 0$. Then for each t, and each realization $M_t(\omega)$, $\omega \in \Omega$, of M_t, we have $f(x + M_t(\omega)) = 0$, and hence $E[f(x + M_t)] = 0$. Thus, $U_M^z f(x) = 0$. To prove that $(U_M^z)^{-1} f(x) = 0$ as well, a more detailed study of the structure of U_M^z is needed (see the Appendix).

b) is proved similarly. □

Now we can explicitly solve (3.15). We have $g_1 \in L^\infty(\mathbf{R}_+)$, hence by applying U_N^β to (3.15), we get

$$(U_M^\beta)^{-1}\tilde{W}^n = -(U_M^\beta)^{-1}g^n + g_2, \tag{3.17}$$

where $g_2 \in L^\infty(\mathbf{R}_+)$. By construction, $\tilde{W}^n \in L^\infty(\mathbf{R}_-)$, and on the strength of Lemma 1, the LHS in (3.17) belongs to $L^\infty(\mathbf{R}_-)$. Hence, by multiplying (3.17) with $\mathbf{1}_{(-\infty,0)}$, the indicator function of $(-\infty, 0)$, we obtain

$$(U_M^\beta)^{-1}\tilde{W}^n = -\mathbf{1}_{(-\infty,0)}(U_M^\beta)^{-1}g^n. \tag{3.18}$$

Next, we apply U_M^β to (3.18):

$$\tilde{W}^n = -U_M^\beta \mathbf{1}_{(-\infty,0)}(U_M^\beta)^{-1}g^n,$$

and using (3.6) and (3.7), we obtain

$$W_s^n(x;h) = -\left(U_M^\beta \mathbf{1}_{(-\infty,h)}(U_M^\beta)^{-1} g_0^n\right)(x). \tag{3.19}$$

We have found a unique bounded solution to the Wiener-Hopf equation; hence, it is the option value of searching W_s^n. In the Appendix, we show that

$$\lim_{n\to\infty} W_s^n(x;h) = -\left(U_M^\beta \mathbf{1}_{(-\infty,h)}(U_M^\beta)^{-1} g_0\right)(x). \tag{3.20}$$

Hence, the option value of searching for the original problem is given by

$$W_s(x;h) = -\left(U_M^\beta \mathbf{1}_{(-\infty,h)}((1-\beta)U_M^\beta)^{-1}(e^{\cdot} - b)\right)(x). \tag{3.21}$$

To derive explicit formulas for the reservation wage and value function, we introduce

$$\phi^+(\beta,\xi) = E\left[e^{i\xi M_T} \mid X_0 = 0\right] = (1-\beta)U_M^\beta(e^{i\xi x})|_{x=0}$$

Notice that

$$\begin{aligned}(1-\beta)U_M^\beta e^{\gamma x} &= (1-\beta)\sum_{t=0}^{\infty}\beta^t E\left[e^{\gamma X_t} \mid X_0 = x\right]\\ &= e^{\gamma x}(1-\beta)\sum_{t=0}^{\infty}\beta^t E\left[e^{\gamma X_t} \mid X_0 = 0\right]\\ &= e^{\gamma x}(1-\beta)U_M^\beta e^{\gamma x}|_{x=0},\end{aligned}$$

whence

$$(1-\beta)U_M^\beta e^{\gamma x} = \phi^+(\beta,-i\gamma)e^{\gamma x}, \tag{3.22}$$

and one easily derives that $((1-\beta)U_M^\beta)^{-1}e^{\gamma x} = \phi^+(\beta,-i\gamma)^{-1}e^{\gamma x}$. Now we can rewrite (3.21) as

$$W_s(x;h) = \left(U_M^\beta \mathbf{1}_{(-\infty,h)}(b - \phi^+(\beta,-i)^{-1}e^{\cdot})\right)(x). \tag{3.23}$$

Theorem 2. *Let h^* be a (unique) solution to*

$$\phi^+(\beta,-i)^{-1}e^h - b = 0. \tag{3.24}$$

Then $w^* = e^{h^*}$ is the reservation wage.

Proof. First notice that the optimality condition (2.5) can be written in terms of log-wages and the option value of searching as

$$W(x;h^*) \ge W(x;h) \quad \forall\ x \text{ and } h \tag{3.25}$$

or equivalently,

$$\frac{w}{1-\beta} + W_s(x;h^*) \ge \frac{w}{1-\beta} + W_s(x;h) \quad \forall\ x \text{ and } h.$$

Therefore to prove the optimality of the choice of $h = h^*$, we need to check condition (3.25). If in addition, we show that (3.25) holds as a strict inequality for some x and $h \neq h^*$, then the choice of the reservation wage is unique. Let $y(x) = b - \phi^+(\beta, -i)^{-1}e^x$. Consider the difference

$$\begin{aligned} W(x;h^*) - W(x;h) &= W_s(x;h^*) - W_s(x;h) \\ &= \left(U_M^\beta(\mathbf{1}_{(-\infty,h^*)} - \mathbf{1}_{(-\infty,h)})y\right)(x). \end{aligned}$$

Notice that $y(x)$ is positive for $x < h^*$ and negative for $x > h^*$. Hence the function $\left((\mathbf{1}_{(-\infty,h^*)} - \mathbf{1}_{(-\infty,h)})y\right)(x)$ is non-negative and positive on (h, h^*) if $h < h^*$ (or on (h^*, h) if $h > h^*$)). Since the resolvent operator maps non-trivial non-negative functions into non-trivial non-negative ones, we conclude that h^* satisfies the optimality condition (3.25) which holds as a strict inequality for some x and $h \neq h^*$.

□

It remains to substitute

$$\phi^+(\beta, -i) = E[e^{M_T} |\; X_0 = 0] = E[\bar{w}_T |\; w_0 = 1]$$

into (3.24) to get (2.6). Thus the first main result obtains.

To derive the formula for the option value of searching, we use (3.24) to rewrite (3.23) as

$$W_s(x;h^*) = \phi^+(\beta, -i)^{-1}\left(U_M^\beta \mathbf{1}_{(-\infty,h)}(e^{h^*} - e^{\cdot})\right)(x).$$

Using the definition of $\phi^+(\beta, -i)$ and that of the resolvent, one obtains the following formula:

$$\begin{aligned} W_s(x;h^*) &= E\left[e^{M_T} |\; X_0 = 0\right]^{-1} \sum_{t=0}^{\infty} \beta^t E\left[\left(e^{h^*} - e^{M_t}\right)_+ |\; X_0 = x\right] \\ &= \frac{e^x}{E\left[e^{M_T} |\; X_0 = x\right]} \cdot \frac{E\left[\left(e^{h^*} - e^{M_T}\right)_+ |\; X_0 = x\right]}{1-\beta} \end{aligned}$$

Substituting wages for the log-wages in the last equation, one gets (2.9).

Finally, recall that

$$W_e(x;h) = W(x;h) - \frac{b}{1-\beta} = W_s(x;h) + \frac{e^x - b}{1-\beta},$$

and use (3.21) to derive

$$\begin{aligned} W_e(x;h) &= \frac{e^x - b}{1-\beta} - \left(U_M^\beta \mathbf{1}_{(-\infty,h)}((1-\beta)U_M^\beta)^{-1}(e^{\cdot} - b)\right)(x) \\ &= \left(U_M^\beta \mathbf{1}_{[h,\infty)}((1-\beta)U_M^\beta)^{-1}(e^{\cdot} - b)\right)(x). \end{aligned}$$

Following the same steps as before, we arrive at

$$W_e(x;h^*) = \frac{e^x}{E\left[e^{M_T} \mid X_0 = x\right]} \cdot \frac{E\left[\left(e^{M_T} - e^{h^*}\right)_+ \mid X_0 = x\right]}{1-\beta}.$$

Thus, the formulas for the option value of searching and option value of the future employment opportunities obtain.

4. Model with layoffs

Suppose that each period after the first period on the job, the worker faces a probability $0 < \lambda < 1$ of being laid off. The probability λ of being laid off next period is assumed to be independent of tenure. If the worker is laid off, she gets the unemployment income b immediately and sits out a period before a new offer may arrive. As in Section 2, we look for the reservation wage w^*. Fix an arbitrary $\hat{w}$, a candidate for the reservation wage. Let $A(w;\hat{w})$ be the value of accepting the current offer w and $U(w;\hat{w})$ be the value of rejecting the offer. Then the value function of the worker is given by

$$V(w;\hat{w}) = \max\{A(w;\hat{w}), U(w;\hat{w})\}.$$

Now we specify $U(w;\hat{w})$ and $A(w;\hat{w})$. if the worker rejects the offer, she receives b this period and a new offer next period, therefore the value of rejecting the offer is

$$U(w;\hat{w}) = b + \beta E[V(w_1;\hat{w}) \mid w_0 = w]. \tag{4.1}$$

If the worker accepts the offer, she receives w immediately; with probability λ she is laid off the next period and becomes unemployed, and with probability $1-\lambda$ she remains on the job. Therefore the value of accepting is

$$A(w;\hat{w}) = w + \lambda\beta E[U(w_1;\hat{w}) \mid w_0 = w] + (1-\lambda)\beta A(w;\hat{w}).$$

Substituting (4.1) for U, we can solve the last equation for $A(w;\hat{w})$:

$$A(w;\hat{w}) = \frac{w + \lambda\beta b}{1-\beta(1-\lambda)} + \frac{\lambda\beta^2}{1-\beta(1-\lambda)} E[V(w_2;\hat{w}) \mid w_0 = w]. \tag{4.2}$$

Here we used the law of iterated expectations:

$$E\left[E[V(w_2;\hat{w}) \mid w_1] \mid w_0 = w\right] = E[V(w_2;\hat{w}) \mid w_0 = w].$$

Write (4.1) and (4.2) as

$$\begin{aligned} V(w;\hat{w}) &= b + \beta E[V(w_1,\hat{w}) \mid w_0 = w], && \text{if } w < \hat{w}; \\ V(w;\hat{w}) &= \frac{w + \lambda\beta b}{1-\beta(1-\lambda)} + \tilde{\beta} E[V(w_2,\hat{w}) \mid w_0 = w], && \text{if } w \geq \hat{w}, \end{aligned}$$

where $\tilde{\beta} \equiv \frac{\lambda\beta^2}{1-\beta(1-\lambda)}$. Introduce, as before, the option value of searching $V_s(w;\hat{w}) = V(w;\hat{w}) - A(w;\hat{w})$.

Further we proceed as in Section 3. We change the state space from $\mathbf{R}_+$ to $\mathbf{R}$, the state space of log-wages, and keep the notation for the value functions as in Section 3. Next, we consider the value function $W^n(x;h)$ for the optimization problem with the wage ceiling $w^n(x)$ (see Section 3). This value function satisfies the following equations:

$$\begin{aligned} W^n(x;h) &= b + \beta P W^n(x;h), \quad \text{if } x < h; && (4.3)\\ W^n(x;h) &= \frac{w^n(x) + \lambda\beta b}{1-\beta(1-\lambda)} \\ &+ \tilde{\beta} E[W^n(x+Y_1+Y_2;h)], \quad \text{if } x \geq h. && (4.4) \end{aligned}$$

On the RHS of (4.4), we see the sum of two copies of i.i.d. Y_j; hence it is natural to consider not only the process $X = \{X_t\}_{t\geq 0}$, but the process $\tilde{X} \equiv \{X_{2t}\}_{t\geq 0}$ as well. Let $\{\tilde{P}_t\}$ be the corresponding family of operators: $\tilde{P}_t f(x) = E[f(\tilde{X}_t)|\ X_0 = x]$. Set $\tilde{P} = \tilde{P}_1$ and notice that by the law of iterated expectations, $\tilde{P} = P^2$.

For the problem with the wage ceiling, the option value of searching is defined by

$$W^n_s(x;h) = W^n(x;h) - A^n(x;h),$$

which (on the strength of (4.4)) is equivalent to

$$W^n_s(x;h) = (I - \tilde{\beta}\tilde{P})W^n(x;h) - g^n_1(x), \tag{4.5}$$

where $g^n_1(x) = (w^n(x) - b)/(1-\beta(1-\lambda))$. As in Section 3, we will solve the problem (4.3)–(4.4) for the option value of searching. First, by (4.5),

$$W^n(x;h) = (I - \tilde{\beta}\tilde{P})^{-1}W^n_s + (I - \tilde{\beta}\tilde{P})^{-1}g^n_1(x), \tag{4.6}$$

next, by substituting (4.6) into (4.3)–(4.4) we arrive at the system

$$(I-\beta P)(I-\tilde{\beta}\tilde{P})^{-1}W^n_s(x;h) = -(I-\beta P)(I-\tilde{\beta}\tilde{P})^{-1}g^n_1(x),\ x < h;$$
$$W^n_s(x;h) = 0, \qquad x \geq h.$$

Set

$$\begin{aligned} g^n(x) &= g^n_1(x+h); && (4.7)\\ \tilde{W}^n(x) &= W^n_s(x+h;h), && (4.8) \end{aligned}$$

then $\tilde{W}^n_s(x)$ is a solution to

$$(I-\beta P)(I-\tilde{\beta}\tilde{P})^{-1}\tilde{W}^n(x) = -(I-\beta P)(I-\tilde{\beta}\tilde{P})^{-1}g^n(x),\ x < 0; \tag{4.9}$$
$$\tilde{W}^n(x) = 0, \qquad x \geq 0. \tag{4.10}$$

Problem (4.9)–(4.10) is equivalent to; find $\tilde{W}^n \in L^\infty(\mathbf{R}_-)$ and $g_2 \in L^\infty(\mathbf{R}_+)$ satisfying

$$(I-\beta P)(I-\tilde{\beta}\tilde{P})^{-1}\tilde{W}^n(x) = -(I-\beta P)(I-\tilde{\beta}\tilde{P})^{-1}g^n(x)+g_2(x). \quad (4.11)$$

Let $U^{\tilde{\beta}}_{\tilde{X}}$ be the resolvent and $\tilde{M}$, $\tilde{N}$ be the supremum and infimum processes for $\tilde{X}$. Then

$$(I-\tilde{\beta}\tilde{P})^{-1} = U^{\tilde{\beta}}_{\tilde{X}}, \quad (4.12)$$

and by the Wiener-Hopf factorization,

$$(1-\tilde{\beta})^{-1}U^{\tilde{\beta}}_{\tilde{X}} = U^{\tilde{\beta}}_{\tilde{M}}U^{\tilde{\beta}}_{\tilde{N}} = U^{\tilde{\beta}}_{\tilde{N}}U^{\tilde{\beta}}_{\tilde{M}}.$$

Also, we have

$$(U^{\tilde{\beta}}_{\tilde{M}})^{-1} = (1-\tilde{\beta})(I-\tilde{\beta}\tilde{P})U^{\tilde{\beta}}_{\tilde{N}}. \quad (4.13)$$

Therefore, we may rewrite (4.11) as

$$\begin{aligned}(U^{\beta}_{N})^{-1}(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{N}}U^{\tilde{\beta}}_{\tilde{M}}\tilde{W}^n(x) &= -(U^{\beta}_{N})^{-1}(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{N}}U^{\tilde{\beta}}_{\tilde{M}}g^n(x)\\ &+ (1-\beta)(1-\tilde{\beta})^{-1}g_2(x). \quad (4.14)\end{aligned}$$

Using Lemma 1, we can explicitly solve (4.14). First, notice that the resolvents of the supremum and infimum processes of X and $\tilde{X}$ commute (see the Appendix), hence the factors in (4.14) do. Second, recall that $g_2 \in L^\infty(\mathbf{R}_+)$, hence by applying U^{β}_{N} and $(U^{\tilde{\beta}}_{\tilde{N}})^{-1}$ to (4.14), we obtain

$$(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{M}}\tilde{W}^n(x) = -(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{M}}g^n(x) + G_2(x), \quad (4.15)$$

where $G_2 \in L^\infty(\mathbf{R}_+)$. By construction, $\tilde{W}^n \in L^\infty(\mathbf{R}_-)$, and therefore by Lemma 1, the LHS in (4.15) belongs to $L^\infty(\mathbf{R}_-)$. Hence, multiplying (4.15) by $\mathbf{1}_{(-\infty,0)}$, we arrive at

$$(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{M}}\tilde{W}^n(x) = -\mathbf{1}_{(-\infty,0)}(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{M}}g^n(x),$$

whence we derive

$$\tilde{W}^n(x) = -(U^{\tilde{\beta}}_{\tilde{M}})^{-1}U^{\beta}_{M}\mathbf{1}_{(-\infty,0)}(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{M}}g^n(x). \quad (4.16)$$

Finally, we return to the original variables and on the strength of (4.7) and (4.8) obtain from (4.16)

$$W^n_s(x;h) = -(U^{\tilde{\beta}}_{\tilde{M}})^{-1}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}(U^{\beta}_{M})^{-1}U^{\tilde{\beta}}_{\tilde{M}}g^n_1(x). \quad (4.17)$$

Thus, we have got a unique bounded solution to the Wiener-Hopf equation (4.9)–(4.10), therefore W^n_s is the option value of searching in the problem with the wage ceiling. Similarly as it was done in Section 3, it is possible to show that one can use (3.16), (4.13), and the monotone convergence theorem and pass to the limit in (4.17) as $n \to \infty$ to compute the option value of searching in the original problem:

$$\begin{aligned} W_s(x;h) &= -(U^{\tilde\beta}_{\tilde M})^{-1}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}(U^{\beta}_{M})^{-1}U^{\tilde\beta}_{\tilde M}g_1(x) \\ &= -\left((U^{\tilde\beta}_{\tilde M})^{-1}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}(U^{\beta}_{M})^{-1}U^{\tilde\beta}_{\tilde M}\frac{e^{\cdot}-b}{1-\beta(1-\lambda)}\right)(x). \end{aligned}$$

Notice that $(1-\beta(1-\lambda))^{-1} = (1-\tilde\beta)(1-\beta)^{-1}(1+\lambda\beta)^{-1}$, hence we can rewrite the last equation as

$$\begin{aligned} &(1+\lambda\beta)W_s(x;h) = \\ &= -\left((U^{\tilde\beta}_{\tilde M})^{-1}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}((1-\beta)U^{\beta}_{M})^{-1}(1-\tilde\beta)U^{\tilde\beta}_{\tilde M}(e^{\cdot}-b)\right)(x). \end{aligned}$$

Introduce

$$\tilde\phi^+(\tilde\beta,\xi) = (1-\tilde\beta)U^{\tilde\beta}_{\tilde M}e^{i\xi x}|_{x=0},$$

and use (3.22) to derive the following formula:

$$W_s(x;h) = (1+\lambda\beta)^{-1}(U^{\tilde\beta}_{\tilde M})^{-1}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}(b-\phi^+(\beta,-i)^{-1}\tilde\phi(\tilde\beta,-i)e^{\cdot})(x).$$

By (4.13), the last formula is equivalent to

$$\begin{aligned} W_s(x;h) &= (1+\lambda\beta)^{-1}(1-\tilde\beta)(I-\tilde\beta\tilde P)U^{\tilde\beta}_{\tilde N}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}(b- \\ &- \phi^+(\beta,-i)^{-1}\tilde\phi(\tilde\beta,-i)e^{\cdot})(x). \end{aligned} \tag{4.18}$$

Also, we may pass to the limit as $n \to \infty$ in (4.6) and using (4.12), derive

$$W(x;h) = (I-\tilde\beta\tilde P)^{-1}W_s + \left(U^{\tilde\beta}_{\tilde X}\left(\frac{w(\cdot)+\lambda\beta b}{1-\beta(1-\lambda)}\right)\right)(x), \tag{4.19}$$

whence we obtain the value of the offer using (4.18) and the definition of the resolvent:

$$\begin{aligned} W(x;h) &= \frac{\lambda\beta b}{(1-\beta)(1+\lambda\beta)} + \frac{U^{\tilde\beta}_{\tilde X}w(x)}{1-\beta(1-\lambda)} \\ &+\frac{1-\beta}{1-\beta(1-\lambda)}U^{\tilde\beta}_{\tilde N}U^{\beta}_{M}\mathbf{1}_{(-\infty,h)}(b-\phi^+(\beta,-i)^{-1}\tilde\phi(\tilde\beta,-i)e^{\cdot})(x). \end{aligned} \tag{4.20}$$

Theorem 3. *Let h^* be a (unique) solution to*

$$\phi^+(\beta,-i)^{-1}\tilde\phi(\tilde\beta,-i)e^h - b = 0. \tag{4.21}$$

Then $w^* = e^{h^*}$ is the reservation wage.

Proof. Is similar to the proof of Theorem 2, and the same uniqueness (of the reservation wage) result obtains. ∎

Substituting the values of factors $\phi(\beta,-i)$ and $\tilde{\phi}(\tilde{\beta},-i)$ into (4.21), one gets

$$\frac{w^*}{b}=\frac{(1-\beta)\sum_{t=0}^{\infty}\beta^t E\left[e^{M_t}\middle|\ X_0=0\right]}{(1-\tilde{\beta})\sum_{t=0}^{\infty}\tilde{\beta}^t E\left[e^{\tilde{M}_t}\middle|\ X_0=0\right]}. \tag{4.22}$$

Let $\varpi_t = e^{\tilde{M}_t}$ be the supremum of the wage process with two time units as a basic time unit. Then in terms of wages,

$$\frac{w^*}{b}=\frac{(1-\beta)\sum_{t=0}^{\infty}\beta^t E\left[\bar{w}_t\middle|\ w_0=1\right]}{(1-\tilde{\beta})\sum_{t=0}^{\infty}\tilde{\beta}^t E\left[\varpi_t\middle|\ w_0=1\right]}. \tag{4.23}$$

The last equation says that in the model with layoffs, the hurdle is proportional to the ratio of two expected present values: the one in the numerator is computed for the stream of payoffs of the supremum of wages, and the value in the denominator is calculated for the stream of payoffs of the supremum of wages offered every other period and discounted by $\tilde{\beta}$.

Equivalently, one can write the reservation wage equation (4.23) as follows:

$$\frac{w^*}{b}=\frac{\sum_{t=0}^{\infty}\beta^t E\left[\bar{w}_t\middle|\ w_0=w\right]}{\sum_{t=0}^{\infty}\beta^t w}\cdot\frac{\sum_{t=0}^{\infty}\tilde{\beta}^t w}{\sum_{t=0}^{\infty}\tilde{\beta}^t E\left[\varpi_t\middle|\ w_0=w\right]}.$$

Hence the hurdle can be written as a product of two factors: the first one accounts for the risk of future positive jumps in wage offers (it is the hurdle in the benchmark model). The second factor is the reciprocal of the hurdle in the benchmark model with the discount factor $\tilde{\beta}$ and wage offers arriving every other period. This factor compensates the risk of future positive jumps in wages because the worker can be laid off with positive probability, and if this happens, the worker gets a new offer in a period from the moment of layoff.

To obtain the formula for the value function, we introduce

$$\mathcal{V}(w)=\sum_{t=0}^{\infty}\beta^t E[(w^*-\bar{w}_t)_+|\ w_0=w],$$

which is one of the factors in the option value of searching in the benchmark model (see (2.10)). Let $\{\tilde{w}_t\}=\{w_{2t}\}$ and $\omega_t=e^{\tilde{N}_t}$ be respectively the wages and the infimum of wages which are offered every other period. Set

$$\Upsilon(w)=\sum_{t=0}^{\infty}\tilde{\beta}^t E[\mathcal{V}(\omega_t)|\ w_0=w].$$

From (4.20) and (4.22), one derives

$$\begin{aligned} V(w;w^*) &= \frac{\lambda\beta b}{(1-\beta)(1+\lambda\beta)} + \frac{\sum_{t=0}^{\infty}\tilde{\beta}^t E[\tilde{w}_t|\ w_0 = w]}{1-\beta(1-\lambda)} \\ &+ \frac{b(1-\beta)}{w^*(1-\beta(1-\lambda))}\Upsilon(w). \end{aligned} \tag{4.24}$$

If $\lambda = 0$, then (4.22) and (4.24) reduce to (2.7) and (2.8) and (2.10) for the reservation wage and value function in the benchmark model.

5. Numerical example

Consider the random walk with the probability distribution $P(dx) = p(x)dx$, where

$$p(x) = c_-(-\lambda_-)e^{\lambda_- x}\mathbf{1}_{[0,+\infty)}(x) + c_+\lambda_+ e^{\lambda_+ x}\mathbf{1}_{(-\infty,0)}(x), \tag{5.1}$$

and $\lambda_- < -1 < 0 < \lambda_+$, $c_\pm > 0$, $c_+ + c_- = 1$. Here c_+ and c_- characterize the intensity of negative and positive jumps in wages respectively, and the relative intensity of large positive (respectively negative) jumps increases in $-\lambda_-^{-1}$ (respectively, λ_+^{-1}). Denote by $\hat{p}$ the Fourier transform of p, and calculate

$$\begin{aligned} \hat{p}(-\xi) &= \int_{-\infty}^{+\infty} e^{ix\xi}p(x)dx \\ &= c_-(-\lambda_-)\int_0^{\infty} e^{ix\xi+\lambda_- x}dx + c_+\lambda_+\int_{-\infty}^0 e^{ix\xi+\lambda_+ x}dx \\ &= \frac{c_-(-\lambda_-)}{-\lambda_- - i\xi} + \frac{c_+\lambda_+}{\lambda_+ + i\xi} \\ &= \frac{-\lambda_-\lambda_+ - i\xi(c_-\lambda_- + c_+\lambda_+)}{(-\lambda_- - i\xi)(\lambda_+ + i\xi)}. \end{aligned}$$

Let $z \in (0,1)$. To calculate $\phi^+(z,-i)$, which enters the reservation wage equation (3.24) for $z = \beta$ in the benchmark model, we need to factorize

$$1 - z\hat{p}(-\xi) = \frac{(1-z)(-\lambda_-\lambda_+) - i\xi(\lambda_+ + \lambda_- - z(c_-\lambda_- + c_+\lambda_+)) + \xi^2}{(-\lambda_- - i\xi)(\lambda_+ + i\xi)}$$

Set $\xi = -i\alpha$, then the equation

$$1 - z\hat{p}(-\xi) = 0 \tag{5.2}$$

turns into

$$\alpha^2 + \alpha(\lambda_+ + \lambda_- - z(c_-\lambda_- + c_+\lambda_+)) - (-\lambda_-\lambda_+)(1-z) = 0. \tag{5.3}$$

The roots of (5.3) are

$$\alpha_\pm(z) = \frac{1}{2}[-(\lambda_- + \lambda_+) + z(c_-\lambda_- + c_+\lambda_+) \pm \sqrt{\mathcal{D}}], \tag{5.4}$$

where

$$\mathcal{D} = [(\lambda_- + \lambda_+) - z(c_-\lambda_- + c_+\lambda_+)]^2 + 4(1-z)(-\lambda_-\lambda_+).$$

Since $z < 1$ and $-\lambda_-\lambda_+ > 0$, we conclude that $\alpha_+(z) > 0$ and $\alpha_-(z) < 0$, hence (5.2) has one root $-i\alpha_+(z)$ in the lower half-plane, and one root $-i\alpha_-(z)$ in the upper half-plane. Recall the Wiener-Hopf factorization formula (A.7)

$$\frac{1-z}{1-z\hat{p}(-\xi)} = \phi^+(z,\xi)\phi^-(z,\xi).$$

Since the LHS in the last formula is a rational function, it is possible to show that $\phi^+(z,\xi)$ (respectively, $\phi^-(z,\xi)$) is a rational function which has neither zeros nor poles in the half-plane $\Im\xi > 0$ (respectively, $\Im\xi < 0$), and $\phi^\pm(z,0) = 1$ (see, e.g., Boyarchenko and Levendorskiǐ (2002), Chapter 13). Therefore,

$$\phi^+(z,\xi) = \frac{(-\lambda_- - i\xi)\alpha_+(z)}{-\lambda_-(\alpha_+(z) - i\xi)}, \tag{5.5}$$

and

$$\phi^-(z,\xi) = \frac{(\lambda_+ + i\xi)(-\alpha_-(z))}{\lambda_+(-\alpha_-(z) + i\xi)}. \tag{5.6}$$

Now, for the benchmark model, $z = \beta$, hence by (3.24),

$$\frac{w*}{b} = \phi^+(\beta, -i) = \frac{(-\lambda_- - 1)\alpha_+(\beta)}{-\lambda_-(\alpha_+(\beta) - 1)}. \tag{5.7}$$

Recall, that in the model with layoffs $\tilde{P} = P^2$, therefore in this model, we have to factorize

$$\begin{aligned} \frac{1-\tilde{\beta}}{1-\tilde{\beta}\hat{p}(-\xi)^2} &= \frac{\left(1-\tilde{\beta}^{1/2}\right)\left(1+\tilde{\beta}^{1/2}\right)}{\left(1-\tilde{\beta}^{1/2}\hat{p}(-\xi)\right)\left(1+\tilde{\beta}^{1/2}\hat{p}(-\xi)\right)} \\ &= \tilde{\phi}^+(\tilde{\beta},\xi)\tilde{\phi}^-(\tilde{\beta},\xi). \end{aligned}$$

By the same reasoning as above, we conclude that

$$\tilde{\phi}^+(\tilde{\beta},\xi) = \phi^+\left(\tilde{\beta}^{1/2},\xi\right)\phi^+\left(-\tilde{\beta}^{1/2},\xi\right), \tag{5.8}$$

and

$$\tilde{\phi}^-(\tilde{\beta},\xi) = \phi^-\left(\tilde{\beta}^{1/2},\xi\right)\phi^-\left(-\tilde{\beta}^{1/2},\xi\right). \tag{5.9}$$

Using (4.21), we derive

$$\begin{aligned}
\frac{w^*}{b} &= \frac{\phi^+(\beta,-i)}{\tilde{\phi}^+(\tilde{\beta},-i)} \\
&= \frac{\phi^+(\beta,-i)}{\phi^+\left(\tilde{\beta}^{1/2},-i\right)\phi^+\left(-\tilde{\beta}^{1/2},-i\right)} \\
&= \frac{(-\lambda_- - 1)\alpha_+(\beta)(\alpha_+(\tilde{\beta}^{1/2})-1)(\alpha_+(-\tilde{\beta}^{1/2})-1)}{-\lambda_-\alpha_+(\tilde{\beta}^{1/2})\alpha_+(-\tilde{\beta}^{1/2})(\alpha_+(\beta)-1)}. \qquad (5.10)
\end{aligned}$$

Notice that (5.1) can be viewed as the simplest way of approximation of the empirical probability density of log-wages by exponential polynomials. More sophisticated approximations with exponential polynomials can be used as well which still keep the model analytically tractable. Even the simple four parameter family of processes considered here contains jumps in both directions and allows to control sizes of large and small jumps. At the same time, the factors in the Wiener-Hopf factorization formula can be calculated explicitly in terms of roots of a quadratic equation. The analytical formulas (5.7) and (5.10) for the reservation wages in the benchmark model and the model with layoffs can be used for comparative statics analysis. By the latter, one can infer how unemployed workers respond to the changes in parameters of the underlying stochastic process. Moreover, in the next section, for the same PDF, we derive an explicit formula for the expected waiting time till a job offer is accepted, which also depends on the parameters of the process for log-wages.

6. Expected waiting time

Assume that the current wage offer w is less than w^*, set $y = \ln w^* - \ln w = h^* - x$, and consider the waiting time R_y till the job offer is accepted. This is the random variable defined by

$$R_y = \min\{t > 0 \mid X_t \geq h^*\}.$$

The expected waiting time can be calculated as follows:

$$\begin{aligned} E[R_y] &= E\left[\sum_{t=0}^{\infty} \mathbf{1}_{(-\infty,h^*)}(M_t)|\ X_0 = x\right] \\ &= E\left[\sum_{t=0}^{\infty} \mathbf{1}_{(-\infty,y)}(M_t)|\ X_0 = 0\right] \\ &= \sum_{t=0}^{\infty} E\left[\mathbf{1}_{(-\infty,y)}(M_t)|\ X_0 = 0\right] \\ &= \lim_{\beta\to 1-0} \sum_{t=0}^{\infty} \beta^t E\left[\mathbf{1}_{(-\infty,y)}(M_t)|\ X_0 = 0\right], \end{aligned}$$

and finally,

$$E[R_y] = \lim_{\beta\to 1-0} (U_M^{\beta}\mathbf{1}_{(-\infty,y)})(0). \tag{6.1}$$

An analytical form for the last expression can be derived for wide classes of random walks, but in general, the resulting formula is rather cumbersome, and uses the explicit formulas for the factors in the Wiener-Hopf factorization formulas. Here we restrict ourselves to a special case when the probability density is given by (5.1).

Theorem 4. *a) The expected waiting time till a job offer is accepted is finite if and only if*

$$C_0 \equiv \lambda_+(1-c_+) + \lambda_-(1-c_-) > 0. \tag{6.2}$$

b) If $C_0 > 0$ then

$$E[R_y] = \frac{-\lambda_-\lambda_+}{C_0}(y - 1/\lambda_-). \tag{6.3}$$

Proof. See the Appendix. □

In the case of a continuous probability density, when the tails of probability density p match at the origin, we have $c_+ = c_- = 1/2$, and (6.2) and (6.3) become simpler:

$$\lambda_+ + \lambda_- > 0,$$

equivalently,

$$m \equiv E[X_1] = 1/(-2\lambda_-) - 1/(2\lambda_+) > 0, \tag{6.4}$$

and

$$E[R_y] = \frac{1}{m}(y - 1/\lambda_-). \tag{6.5}$$

Condition (6.4) has a simple interpretation: the expected waiting time is finite if and only if the drift of the log-wage, m, is positive, and if it is positive, then (6.5) says that the expected waiting time is inverse proportional to the drift, as in the deterministic continuous time model. Notice however that neither the term $-1/\lambda_-$ nor the answer in the generic case $c_+ \neq c_-$ admit such simple interpretations.

7. Conclusion

The paper is the first attempt to model wage offers evolving as a geometric random walk as opposed to random draws from independent distribution in the conventional job-search models. Such non-stationarity may be due to business cycle effects or idiosyncratic effects (if a worker remains unemployed for sufficiently long time, the job market may start viewing her as a loser, which may result in lower wages) or both. From our point of view, the suggested modeling is more realistic, because the state of the model today affects its state tomorrow. Hence the agents when deciding whether to work or not have to take into consideration, what wages they may be offered tomorrow, given the offers they are facing today. For the job-search models with and without layoffs, we obtained closed form solutions for reservation wages, value functions and the expected waiting time before a job offer is accepted. The results admit meaningful economic interpretation in terms of expected present values of relevant payoff streams. The results are mainly driven by the supremum process for wages, which indicates that for a worker's decision whether to accept an offer or not, record setting wages rather than all wage movements matter.

For a special case of the PDF given by an exponential polynomial, simple analytical formulas for reservation wages and the expected waiting time are derived, which are suitable for comparative statics analysis. Nobody would argue that it is crucial to study how labor force participants respond to variations in exogenous factors. The determination of the factors affecting the length of time spent out of work by unemployed individuals is an important matter with significant applications for the design and impact of policies such as unemployment compensation, for instance.

We restricted the analysis for the case of linear utility function for expositional simplicity. The model can be easily generalized for the case of Cobb-Douglas utility, which is one of the ways to depart from a representative agent. By varying parameters of the utility function, one can introduce heterogeneous workers in the job-search models and consider aggregate labor force fluctuations, which is a prerequisite for understanding how fluctuations in the nation's output of goods and services propagate over time.

To be more consistent with Macro Economic Theory, it would be necessary to model log-wages not as a random walk, but as an AR(1) process. Such extension of our model is also feasible; the solution is accessible only by quantitative methods, including the finite time horizon case.

Appendix A

A.1. **The Wiener-Hopf factorization.** Let $z \in (0,1)$, let $Y_1, Y_2, \cdots$ be i.i.d. random variables with the probability distribution $\mu(dx)$. Let $X_t = X_0 + Y_1 + \cdots + Y_t$ be the random walk started at 0: $X_0 = 0$, and denote by $\mu_t(dx)$ the probability distribution of X_t. Let T be a random variable independent of X and taking values in $\{0, 1, \dots\}$, with $P(T = t) = (1-z)z^t$. Consider the random variable X_T.

Theorem 5. *(Spitzer (1964))*

$$E\left[e^{i\xi X_T}\right] = E\left[e^{i\xi M_T}\right] E\left[e^{i\xi N_T}\right]. \tag{A.1}$$

Moreover, we have the Spitzer identities

$$E\left[e^{i\xi M_T}\right] = \exp\left[\sum_{t=1}^{\infty} \frac{z^t}{t} \int_0^{\infty} (e^{ix\xi} - 1)\mu_t(dx)\right], \tag{A.2}$$

and

$$E\left[e^{i\xi N_T}\right] = \exp\left[\sum_{t=1}^{\infty} \frac{z^t}{t} \int_{-\infty}^{0} (e^{ix\xi} - 1)\mu_t(dx)\right]. \tag{A.3}$$

Set

$$\phi^+(z,\xi) \equiv E\left[e^{i\xi M_T}\right] \equiv (1-z)U_M^z(e^{ix\xi})|_{x=0}, \tag{A.4}$$

$$\phi^-(z,\xi) \equiv E\left[e^{i\xi N_T}\right] \equiv (1-z)U_N^z(e^{ix\xi})|_{x=0}. \tag{A.5}$$

Denote by $\hat{\mu}(\xi)$ the Fourier transform of $\mu(dx)$:

$$\hat{\mu}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}\mu(dx).$$

Since Y_j are i.i.d., we have

$$E\left[e^{i\xi X_t}\right] = \hat{\mu}(-\xi)^t,$$

and therefore

$$\begin{aligned} E\left[e^{i\xi X_T}\right] &= (1-z)\sum_{t=0}^{\infty} z^t E\left[e^{i\xi X_t}\right] \\ &= (1-z)\sum_{t=0}^{\infty} z^t \hat{\mu}(-\xi)^t \\ &= (1-z)/[1-z\hat{\mu}(-\xi)]. \end{aligned} \tag{A.6}$$

By using (A.6), (A.4) and (A.5), we can rewrite (A.1) as

$$(1-z)/[1-z\hat{\mu}(-\xi)] = \phi^+(z,\xi)\phi^-(z,\xi). \tag{A.7}$$

The factors $\phi^\pm$ enjoy the following important property.

Lemma 6. *For any $z \in (0,1)$, $\phi^\pm(z,\xi)$ and $1/\phi^\pm(z,\xi)$ are holomorphic and bounded in the half-plane $\pm\Im\xi > 0$ and continuous up to the boundary of the half-plane.*

Proof. It suffices to notice that the expression under the exponent sign in (A.2) (resp., (A.3)) is holomorphic in the half-plane $\Im\xi > 0$ (resp., $\Im\xi < 0$) and bounded up to the boundary of the half-plane. □

A.2. **Resolvents as PDO.** Let u be a sufficiently regular function, say, $u \in \mathcal{S}(\mathbf{R})$ (that is, $u(x)$ and each of its derivatives decay at infinity faster than any power of x). Let $\hat{u}$ be the Fourier transform of u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(\xi) d\xi.$$

By the Fourier inversion formula,

$$u(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\hat{u}(\xi)d\xi, \tag{A.8}$$

therefore

$$\begin{aligned} (U_X^z u)(x) &= E^x\left[\sum_{t=0}^{\infty} z^t (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{iX_t\xi}\hat{u}(\xi)d\xi\right] \\ &= (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\sum_{t=0}^{\infty} z^t E[e^{iX_t\xi}]\hat{u}(\xi)d\xi \\ &= (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\sum_{t=0}^{\infty} z^t \hat{p}(-\xi)^t\hat{u}(\xi)d\xi \\ &= (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}(1-z\hat{p}(-\xi))^{-1}\hat{u}(\xi)d\xi. \end{aligned}$$

Let an operator A be defined by

$$Au(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi)\hat{u}(\xi)d\xi.$$

Then one says that A is a pseudo-differential operator (PDO) with the symbol a and writes $A = a(D)$ (in some cases, the integration along a different line $\Im\xi = \sigma$ in the complex plane must be used - see e.g. Boyarchenko and Levendorskiĭ (2002)). Thus, the resolvent U_X^z is a PDO with the symbol $(1 - z\hat{p}(-\xi))^{-1}$:

$$U_X^z = (1 - z\hat{p}(-D))^{-1}.$$

By using (A.4) and (A.5), we similarly conclude that

$$(1 - z)U_M^q = \phi^+(z, D), \quad (1 - z)U_N^q = \phi^-(z, D). \tag{A.9}$$

A.3. **Proof of (3.14).** Now we can rewrite (A.7) as

$$(1 - z)(1 - z\hat{p}(-D))^{-1} = \phi^+(z, D)\phi^-(z, D),$$

or equivalently,

$$(1 - z)U_X^z = (1 - z)U_M^z(1 - z)U_N^z.$$

Which gives (3.14).

A.4. **Proof of Lemma 1.** To finish the proof for U_M^z, we have to show that for any $f \in L^\infty(\mathbf{R}_-)$ and any $g \in C_0^\infty((0, +\infty))$,

$$((U_M^z)^{-1}f, g) \equiv \int_{-\infty}^{+\infty} (U_M^z)^{-1}f(x)g(x)dx = 0.$$

Let $-M$ be the infimum process for the dual process $-X$; then

$$\int_{-\infty}^{+\infty} ((U_M^z)^{-1}f)(x)g(x)dx = \int_{-\infty}^{+\infty} f(x)((U_{-M}^z)^{-1}g)(x),$$

therefore it suffices to show that for any $x < 0$,

$$(1 - z)^{-1}((U_{-M}^z)^{-1}g)(x) \equiv (\tilde{\phi}^-(z, D)^{-1}g)(x) = 0,$$

where $\tilde{\phi}^-$ is the minus-factor in the Wiener-Hopf factorization formula for the resolvent of the process $-X$ (the reader should not confuse the notation $\tilde{\phi}^-$ here with the same notation introduced for the process $\tilde{X}$ in the main body of the paper). By using the definition of PDO, we have

$$\tilde{\phi}^-(z, D)^{-1}g(x) = (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{ix\xi}\tilde{\phi}^-(z, \xi)\hat{g}(\xi)d\xi, \tag{A.10}$$

where $\sigma = 0$. Since $g \in C_0^\infty((0,+\infty))$, its Fourier transform admits the analytic continuation into the half-space $\Im\xi < 0$, and in the closed half-plane, it satisfies an estimate

$$|\hat{g}(\xi)| \le C_N(1+|\xi|)^{-N}, \tag{A.11}$$

for any N, where C_N depends on N but not on ξ. By Lemma 6, $\tilde{\phi}^-(z,\xi)^{-1}$ is bounded in the same closed half-plane, therefore the integrand in (A.10) admits the estimate (A.11). By the Cauchy theorem, we may push the line of integration in (A.10) down: $\sigma \to -\infty$; in the limit, the integral (A.10) vanishes, and we are done.

A.5. **Proof of (3.20).** First, we use (3.16) to rewrite (3.19) as

$$\begin{aligned} W_s^n(x;h) &= \beta(1-\beta)U_M^\beta \mathbf{1}_{(-\infty,h)} P U_N^\beta g_0^n(x) \\ &\quad -(1-\beta)U_M^\beta \mathbf{1}_{(-\infty,h)} U_N^\beta g_0^n(x). \end{aligned}$$

By construction, $g_0^n(x) \le w(x) = e^x$, therefore, on the strength of (A.5), $U_N^\beta g_0^n(x) \le (1-\beta)^{-1}\phi^-(\beta,-i)e^x$. Using (A.4), one gets

$$(1-\beta)U_M^\beta \mathbf{1}_{(-\infty,h)} U_N^\beta g_0^n(x) \le (1-\beta)^{-1}\phi^+(\beta,-i)\phi^-(\beta,-i)e^x. \tag{A.12}$$

Notice that $\phi^-(\beta,-i) < \infty$ by Lemma 6, and using (A.2) and (A.4), it is straightforward to show that $\phi^+(\beta,-i) < \infty$ due to (2.2). Since both U_M^β and U_N^β are operators with non-negative kernels, and (A.12) holds, we can use the monotone convergence theorem to get

$$\lim_{n\to\infty} U_M^\beta \mathbf{1}_{(-\infty,h)} U_N^\beta g_0^n(x) = U_M^\beta \mathbf{1}_{(-\infty,h)} U_N^\beta g_0(x) < \infty.$$

Similarly,

$$\lim_{n\to\infty} U_M^\beta \mathbf{1}_{(-\infty,h)} P U_N^\beta g_0^n(x) = U_M^\beta \mathbf{1}_{(-\infty,h)} P U_N^\beta g_0(x) < \infty,$$

hence (3.20) obtains.

A.6. **Commutativity of resolvents of supremum and infimum processes of M, N, $\tilde{M}$ and $\tilde{N}$.** Each of these resolvents is a PDO on $\mathbf{R}$ with the symbol independent of the state variable. By applying the Fourier transform, we see that the product of such two PDO's is a PDO whose symbol is the product of the symbols. Hence, these PDO's commute.

A.7. **Proof of Theorem 4.** Fix $y > 0$, and define the function $u(z) = \mathbf{1}_{(-\infty,y)}(z)$. Let $\hat{u}(\xi)$ be the Fourier transform of u. It is defined in the half-plane $\Im\xi > 0$, therefore the Fourier inversion formula reads

$$u(z) = (2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{iz\xi}\hat{u}(\xi)d\xi, \tag{A.13}$$

where $\sigma > 0$ is arbitrary. By applying U^β_M to (A.13), and using the formula $U^\beta_M = (1-\beta)^{-1}\phi^+(\beta, D)$ and the definition of PDO, we obtain

$$(U^\beta_M \mathbf{1}_{(-\infty,y)})(0) = \frac{1}{(1-\beta)2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{e^{-iy\xi}\phi^+(\beta,\xi)}{-i\xi}d\xi, \tag{A.14}$$

Then use the explicit formula (5.5) for the factor $\phi^+(\beta,\xi)$ and substitute (A.14) into (6.1):

$$\begin{aligned} E[R_y] &= \lim_{\beta\to 1-0}\frac{1}{(1-\beta)2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{e^{-iy\xi}\phi^+(\beta,\xi)}{-i\xi}d\xi \\ &= \lim_{\beta\to 1-0}\frac{1}{(1-\beta)2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{e^{-iy\xi}(-\lambda_- - i\xi)\alpha_+(\beta)}{(-\lambda_-)(\alpha_+(\beta)-i\xi)(-i\xi)}d\xi. \end{aligned}$$

We can push the line of integration down. It crosses two poles of the integrand at $\xi = 0$ and $\xi = -i\alpha_+(\beta)$, and by using the residue theorem, we obtain for any $\sigma_1 < -\alpha_+(\beta)$:

$$\begin{aligned} E[R_y] &= \lim_{\beta\to 1-0}\frac{1}{1-\beta}\left(1 - e^{-y\alpha_+(\beta)}\frac{-\lambda_- - \alpha_+(\beta)}{-\lambda_-}\right) \\ &\quad + \lim_{\beta\to 1-0}\frac{1}{(1-\beta)2\pi}\int_{-\infty+i\sigma_1}^{+\infty+i\sigma_1} \frac{e^{-iy\xi}(-\lambda_- - i\xi)\alpha_+(\beta)}{(-\lambda_-)(\alpha_+(\beta)-i\xi)(-i\xi)}d\xi. \end{aligned} \tag{A.15}$$

By integrating by part in the last integral, we obtain an absolutely converging integral; moreover, the new integrand admits the bound via

$$y^{-1}\left|e^{-iy\xi}\frac{\partial}{\partial\xi}\left(\frac{(-\lambda_- - i\xi)\alpha_+(\beta)}{(-\lambda_-)(\alpha_+(\beta)-i\xi)(-i\xi)}\right)\right| \le \frac{Ce^{y\sigma_1}}{1+|\xi|^2}.$$

Hence, the integral in (A.15) vanishes in the limit $\sigma_1 \to -\infty$, and we conclude that the last term in (A.15) is zero.

If $C_0 < 0$, then from (5.4), we conclude that $\alpha_+(\beta) > 0$ remains bounded away from 0 as $\beta \to 1-0$: $\alpha_+(\beta) \ge d$, where $d > 0$ is independent of β. Hence,

$$E[R_y] \ge \lim_{\beta\to 1-0}\frac{1-e^{-yd}}{1-\beta} = +\infty.$$

If $C_0 = 0$, then from (5.4), $\alpha_+(\beta) \sim d\sqrt{1-\beta}$, where $d > 0$, and hence,

$$E[R_y] = \lim_{\beta \to 1-0} \frac{1 - e^{-yd\sqrt{1-\beta}}}{1-\beta} = +\infty.$$

Finally, if $C_0 > 0$, then $\alpha_+(\beta) \to 0$ as $\beta \to 1-0$, and moreover,

$$\frac{\alpha_+(\beta)}{1-\beta} \to \frac{4(-\lambda_-\lambda_+)}{2(C_0 + \sqrt{C_0^2 + 0})} = \frac{-\lambda_-\lambda_+}{C_0}.$$

Hence,

$$\begin{aligned}\frac{1}{1-\beta}\left(1 - e^{-y\alpha_+(\beta)}\frac{-\lambda_- - \alpha_+(\beta)}{-\lambda_-}\right) \quad &\sim \quad \frac{1 - e^{-y\alpha_+(\beta)}}{1-\beta} + \frac{\alpha_+(\beta)}{(1-\beta)(-\lambda_-)} \\ &\sim \quad (y + 1/(-\lambda_-))\frac{\alpha_+(\beta)}{1-\beta} \\ &\sim \quad (y + 1/(-\lambda_-))\frac{-\lambda_-\lambda_+}{C_0}.\end{aligned}$$

Theorem 4 has been proved.

References

[1] Boyarchenko, S.I. and S.Z. Levendorskiĭ (2002b): *Non-Gaussian Merton-Black-Scholes theory*. Singapore: World Scientific.

[2] Burdett, K. (1979), “Search, leisure and individual labor supply”, in Lippman, S.A. and McCall, J.J. (eds), *Studies in Economics of Search*, Amsterdam: North Holland.

[3] Devine, T.J. and Kiefer, N.M. (1991), *Empirical Labor Economics*, Oxford: Oxford University Press.

[4] Gronau, R. (1971), “Information and frictional unemployment”, *American Economic Review*, **61**, pp. 290-301.

[5] Heckman, J. and Singer, B. (1982), “The identification problem in econometric models for duration data”, in Hildenbrand W. (ed), *Advances in Econometrics: Proceedings of World Meetings of the Econometric Society, 1980*, Cambridge: Cambridge University Press.

[6] Lippman, S.A. and McCall, J.J (1976), “Job search in a dynamic economy”, *Journal of Economic Theory*, **12**, pp. 365-90.

[7] Mortensen, D.T. (1986), “Job search and labor market analysis”, in Ashenfelter, O. and Layard, R. (eds), *Handbook of Labor Economics*, Amsterdam: North-Holland.

[8] Sargent, T.J. (1987), *Dynamic Macroeonomic Theory*, 2nd ed. New York: Academic Press.

[9] Van den Berg, G.J. (1990), “Nonstationarity in job search theory”, *Review of Economic Studies*, **57**, pp. 255-77.

[10] Van den Berg, G.J. (1999), “Empirical inference with equilibrium search models of the labor market”, *The Economic Journal*, **109**, F283-F306.

[11] Wolpin, K.I. (1995), *Empirical Methods for the Study of Labor Force Dynamics*, Luxembourg: Harwood Academic Publishers.