Solving Models with Imperfect and Asymmetric Information

preliminary version

Paweł Kowal*

May 28, 2005

Abstract

We consider linear dynamic models with rational expectations in case of incomplete and asymmetric information as well as agents heterogeneity. This problem requires solving infinite dimensional matrix equations. We propose asymptotic expansion method to reduce this problem to the finite dimensional problem.

JEL classification: C61, C63, E17

Keywords: Computational Methods, Imperfect Information, Asymmetric Information, Linear Rational Expectations Model, Asymptotic Expansion.

1 Introduction

We consider linear dynamic models with rational expectations in case of incomplete and asymmetric information as well as agents heterogeneity.

Standard rational expectations approach in Lucas spirit, where agents perfectly observe all variables in the economy, while offering very convenient and powerful framework, lacks of realism, since in the economy most of the variables are unobservable or can be observed only with noise. On can however easily reconcile rational expectations with imperfect knowledge within bayesian inference framework, where agents gradually and optimally learn

*Department of Economics, Warsaw School of Economics, email: pkowal3@sgh.waw.pl

about the true values of variables in the economy over time based on available information. Unfortunately introducing incomplete information or, more generally, bayesian inference greatly complicates model solution techniques.

There exists few algorithm of solving models with imperfect information e.g. Svensson and Woodford (2004), Dellas (2004), Gerali and Lippi (2003). These algorithms concentrate however only on case of symmetric information, when all agents observe the same small set of economic variables, or on the special case of asymmetric information or when agents can be ranked with respect to information set they have[1]. The second important drawback of existing methods of solving models with imperfect information is microeconomic incoherence. Only observed variables may influence agents' decisions. Thus, if some variable influences the economy, then at least one agent must perfectly observe this variable. But this is inconsistent with the assumption that only fraction of variables is observed in the economy. Additionally, since all agents observe their individual states and controls, and these variables generally are informative with respect to aggregated variables, hence current algorithms are not well suited for rigorous microeconomic analysis of macroeconomic phenomena.

In this paper we present a new method of solving models with incomplete and asymmetric information. We analyze each agent individually and allow for quite a large form of heterogeneity, in particular agents may differ with respect to type, individual state, and information set. Agents observe variables in the economy with individual and aggregated noise. All agents observe their individual state.

This problem requires solving infinite dimensional matrix equations and cannot be solved exactly. We propose asymptotic expansion method around full information case to reduce this problem to finite dimensional problem. In this paper we concentrate on first order expansion. Proposed algorithm requires solving matrix equations witch are standard in linear rational expectations problems. Moreover we avoid necessity of recursive computations, which typically arise in models with asymmetric information.

In section 2 we outline the model structure, section 3 presents matrix equations, which determine solution, in sections 4-6 we formulate and solve asymptotic expansion of the optimal policy function, section 7 discusses computational issues, finally section 8 concludes.

[1]Let i and j are any two agents in the economy and I_t^i, I_t^j are information sets of agents i and j respectively. Then $I_t^i \subset I_t^j$ or $I_t^j \subset I_t^i$.

2 The problem

Consider an economy with I types of agents. For $i \in I$ let us denote set of type i agents as Ω^i. Each agent $\alpha \in \Omega^i$ faces the same optimization problem. Different agents belonging to set Ω^i may however differ in their states $x_t^{\alpha,i}$ and information set $I_t^{i,\alpha}$. An agent α of type i in state $x_t^{\alpha,i}$ makes decisions about variables $y_t^{\alpha,i}$ taking aggregate variables, w_t, as given. We assume that optimization problem of the agent leads to linearized first order conditions in the form

$$\begin{aligned} x_{t+1}^{i,\alpha} &= A_1^i x_t^{i,\alpha} + A_2^i y_t^{i,\alpha} + A_3^i \epsilon_{t+1}^{i,\alpha} + B_1^i w_t + B_3^i \epsilon_{t+1} \\ 0 &= E\{C_1^i x_t^{i,\alpha} + C_2^i y_t^{i,\alpha} + D_1^i w_t | I_t^{\alpha,i}\} + E\{G_1^i x_{t+1}^{i,\alpha} + G_2^i y_{t+1}^{i,\alpha} + H_1^i w_{t+1} | I_t^{\alpha,i}\} \end{aligned} \tag{1}$$

Information set $I_t^{\alpha,i}$ determines information available to agent α of type i in period t. This information set will be described later. The variable $\epsilon_t^{i,\alpha}$ represents individual shocks, the variable ϵ_t represents aggregated shocks.

We assume that for each $i \in I$, $\alpha \in \Omega^i$, $\{\epsilon_t^{\alpha,i}\}$ and $\{\epsilon_t\}$ are martingale difference sequences with respect to any information set $I_{t-1}^{\beta,j}$ with increments normally distributed, $\mathrm{var}(\epsilon_t) = \mathbf{I}$, $\mathrm{var}(\epsilon_t^{i,\alpha}) = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix, and $E_{\Omega^i}\{\epsilon_t^{\alpha,i}\} = 0$.

We allow for incomplete information. Agent $\alpha \in \Omega^i$ does not observe all variables in the economy but imperfectly observes only a fraction of variables. However each agent perfectly observes individual state and control variables. The agent $\alpha \in \Omega^i$ information set $I_t^{\alpha,i}$ consists of all model parameters and set of variables $\Xi_t^{i,\alpha}$ where

$$\Xi_t^{i,\alpha} = \{Z_\tau^{i,\alpha}, x_\tau^{i,\alpha}; \tau \leq t\}$$

and variable Z_τ^i is given by

$$Z_\tau^{i,\alpha} = w_\tau + L_1^i \upsilon_\tau + L_2^i \upsilon_\tau^{i,\alpha} \tag{2}$$

Information set contain all private state variables, and thus, all private variables, and a fraction of aggregated variables possibly observed with individual as well as aggregated noise. Variables $\{\upsilon_t\}$ and $\{\upsilon_t^{i,\alpha}\}$ are martingale difference sequences with respect to any information set $I_{t-1}^{\beta,j}$ with increments normally distributed, $\mathrm{var}(\upsilon_t) = \mu\mathbf{I}$, $\mathrm{var}(\upsilon_t^{i,\alpha}) = \mu\mathbf{I}$ and $E_{\Omega^i}\{\upsilon_t^{i,\alpha}\} = 0$, and μ is a small variance scaling parameter. Additionally let $\mathrm{cov}(\upsilon_t, \epsilon_t) = \sqrt{\mu}V$ and $\mathrm{cov}(\upsilon_t^{i,\alpha}, \epsilon_t^{i,\alpha}) = \sqrt{\mu}W$.

2.1 Aggregate variables

Let us denote by x_t^i mean state of agents of type i, $x_t^i = E_{\Omega^i}\{x_t^{\alpha,i}\}$. Similarly, $y_t^i = E_{\Omega^i}\{y_t^{\alpha,i}\}$. Let us denote concentration of vectors or matrices, z^i, indexed by elements of set I as $\operatorname{col} z^i$, and block-diagonal matrix composed with vectors or matrices, z^i, indexed by elements of set I as $\operatorname{diag} z^i$

$$\operatorname{col}(z^i) \equiv \begin{bmatrix} z^1 \\ z^2 \\ \cdots \\ z^I \end{bmatrix} \qquad \operatorname{diag}(z^i) \equiv \begin{bmatrix} z^1 & 0 & \cdots & 0 \\ 0 & z^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^I \end{bmatrix}$$

We have $x_t = \operatorname{col}(x_t^i)$ and $y_t = \operatorname{col}(y_t^i)$.

To close the model we must yet specify evolution of aggregate variables, w_t. We assume that

$$w_t = \begin{bmatrix} x_t \\ y_t \\ p_t \end{bmatrix} \tag{3}$$

and

$$p_{t+1} = N_1 p_t + N_2 w_t + N_3 \epsilon_{t+1} \tag{4}$$

where p_t is a vector of aggregated variables, which does not come from aggregation of individual variables.

3 Solution

In this paper we are going to find some solution to (1) but not all. Thus, we use a method "guess and verify". Besides the constructed solution there may exists many other, e.g. possibly even infinite set of sunspot dynamics.

We are going to solve the problem (1) using the method of undetermined coefficients. In particular, we are going to find solution to (1) in the form

$$y_t^{i,\alpha} = \sum_{k=0}^{\infty} P_k^i x_{t-k}^{i,\alpha} + \sum_{k=0}^{\infty} Q_k^i Z_{t-k}^{i,\alpha} \tag{5}$$

Agents are heterogenous with respect to information sets. This heterogeneity has two sources – individual states, not observable for other agents in the economy, and individual shocks in observation of aggregate variables. The first source of this heterogeneity leads to infinite dimension of the problem since it generally would required to take into account all historical information concerning individual states to make forecasts about aggregated states. Since agents have different information sets their must forecast other agents forecasts too. These forecasts usually have different updating schemes which prohibits existing low dimensional markovian representation of the solution to (1). Observe that if we assume that only current individual state and new information, $Z_{t-k}^{i,\alpha}$ matter, then we can express the second equation under (1) as

$$0 = A^i x_t^{i,\alpha} + B^i Z_t^{i,\alpha} + CE\{w_t|I_t^{i,\alpha}\} + DE\{w_{t+1}|I_t^{i,\alpha}\} \tag{6}$$

where A, B, C, D are appropriate matrices. In case of imperfect information $E\{w_t|I_t^{i,\alpha}\}$ generally depends on all past individual states and information as is shown later. Then equation (6) cannot hold for all sequence of shocks $\{\epsilon_t\}$, $\{\epsilon_t^{i,\alpha}\}$, $\{\upsilon_t\}$, $\{\upsilon_t^{i,\alpha}\}$ and the method of undetermined coefficients breaks down. Thus, such a low dimensional representation is impossible.

Only observed variables matter for policy rule (5). Any observed variable under information set $I_t^{i,\alpha}$ may be expressed as a weighted sum of all current and lagged variables in the information set, thus, may have representation in the form (5).

We can express the second condition under (1) using (2) as

$$\begin{aligned} 0 = \sum_{k=0}^{\infty} T_k^i x_{t-k}^{i,\alpha} + \sum_{k=0}^{\infty} U_k^i Z_{t-k}^{i,\alpha} + \Big(D_1^i + (G_1^i + G_2^i P_0^i) B_1^i\Big) E\{w_t|I_t^{i,\alpha}\} \\ + (H_1^i + G_2^i Q_0^i K_1^i) E\{w_{t+1}|I_t^{i,\alpha}\} \end{aligned} \tag{7}$$

where

$$\begin{aligned} T_0^i &= C_1^i + C_2^i P_0^i + (G_1^i + G_2^i P_0^i)(A_1^i + A_2^i P_0^i) + G_2^i P_1^i \\ T_k^i &= C_2^i P_k^i + (G_1^i + G_2^i P_0^i) A_2^i P_k^i + G_2^i P_{k+1}^i \\ U_m^i &= C_2^i Q_m^i + (G_1^i + G_2^i P_0^i) A_2^i Q_m^i + G_2^i Q_{m+1}^i \end{aligned} \tag{8}$$

for $k > 0$ and $m \geq 0$.

Now, we are going to expand expressions $E\{x_{t-k}|I_t^{\alpha,i}\}$, $E\{\upsilon_{t-k}|I_t^{\alpha,i}\}$ in equation (7). Using (37) we can express expectations under private information set $I_t^{\alpha,i}$ as

$$E\{w_{t+1-k}|I_t^{i,\alpha}\} = \sum_{m=0}^{\infty} \Phi_{k,m}^i x_{t-m}^{i,\alpha} + \sum_{m=0}^{\infty} \Psi_{k,m}^i Z_{t-m}^{i,\alpha} \tag{9}$$

for any $k \geq 0$ and $i \in I$ where matrices $\Phi^i_{k,m}$ and $\Psi^i_{k,m}$ satisfy

$$\begin{aligned}
\sum_{p=0}^{\infty} \Phi^i_{k,p} \operatorname{cov}(x^{i,\alpha}_{t-p}, x^{i,\alpha}_{t-m}) + \sum_{p=0}^{\infty} \Psi^i_{k,p} \operatorname{cov}(Z^{i,\alpha}_{t-p}, x^{i,\alpha}_{t-m}) = \operatorname{cov}(w_{t+1-k}, x^{i,\alpha}_{t-m}) \\
\sum_{p=0}^{\infty} \Phi^i_{k,p} \operatorname{cov}(x^{i,\alpha}_{t-p}, Z^{i,\alpha}_{t-m}) + \sum_{p=0}^{\infty} \Psi^i_{k,p} \operatorname{cov}(Z^{i,\alpha}_{t-p}, Z^{i,\alpha}_{t-m}) = \operatorname{cov}(w_{t+1-k}, Z^{i,\alpha}_{t-m})
\end{aligned} \tag{10}$$

thus

$$\begin{aligned}
0 = \sum_{m=0}^{\infty} \Big(T^i_m + (D^i_1 + (G^i_1 + G^i_2 P^i_0) B^i_1) \Phi^i_{1,m} + (H^i_1 + G^i_2 Q^i_0 K^i_1) \Phi^i_{0,m} \Big) x^{i,\alpha}_{t-m} \\
+ \sum_{m=0}^{\infty} \Big(U^i_m + (D^i_1 + (G^i_1 + G^i_2 P^i_0) B^i_1) \Psi^i_{1,m} + (H^i_1 + G^i_2 Q^i_0 K^i_1) \Psi^i_{0,m} \Big) Z^{i,\alpha}_{t-m}
\end{aligned}$$

In this way we have a system of matrix equations

$$\begin{aligned}
0 \equiv T^i_m + (D^i_1 + (G^i_1 + G^i_2 P^i_0) B^i_1) \Phi^i_{1,m} + (H^i_1 + G^i_2 Q^i_0 K^i_1) \Phi^i_{0,m} \\
0 \equiv U^i_m + (D^i_1 + (G^i_1 + G^i_2 P^i_0) B^i_1) \Psi^i_{1,m} + (H^i_1 + G^i_2 Q^i_0 K^i_1) \Psi^i_{0,m}
\end{aligned} \tag{11}$$

for $i \in I$ and $m \geq 0$ which determine matrices P^i_m and Q^i_m.

3.1 Aggregated states

Let us assume that dynamics of aggregate variables is

$$w_{t+1} = \sum_{k=0}^{\infty} \bar{P}_k w_{t-k} + \sum_{k=0}^{\infty} \bar{Q}_k \upsilon_{t-k} + \bar{R} \upsilon_{t+1} + \bar{S} \epsilon_{t+1} \tag{12}$$

We must find yet matrices $\bar{P}_k$, $\bar{Q}_k$, $\bar{R}$, $\bar{S}$. From (4) we have

$$p_{t+1} = \Big(N_1 \mathbf{J}_3 + N_2 \Big) w_t + N_3 \epsilon_{t+1} \tag{13}$$

where $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$ are selecting matrices such that $\mathbf{J}^1 w_t = x_t$, $\mathbf{J}^2 w_t = y_t$, $\mathbf{J}^3 w_t = p_t$. From (1)

$$x_{t+1} = \Big(\operatorname{diag}(A^i_1) \mathbf{J}^1 + \operatorname{diag}(A^i_2) \mathbf{J}^2 + \operatorname{col}(B^i_1) \Big) w_t + \operatorname{col}(B^i_3) \epsilon_{t+1} \tag{14}$$

From (5), (2) and (12) we have

$$y_{t+1} = \sum_{k=0}^{\infty} (\Lambda_0 \bar{P}_k + \Lambda_{k+1}) w_{t-k} + \sum_{k=0}^{\infty} \Big(\operatorname{diag}(Q^i_{k+1}) \operatorname{col}(L^i_1) + \Lambda_0 \bar{Q}_k \Big) \upsilon_{t-k} \\ + \Lambda_0 \bar{S} \epsilon_{t+1} + \Big(\Lambda_0 \bar{R} + \operatorname{diag}(Q^i_0) \operatorname{col}(L^i_1) \Big) \upsilon_{t+1} \tag{15}$$

where

$$\Lambda_k = \operatorname{diag}(P^i_k) \mathbf{J}_1 + \operatorname{diag}(Q^i_k) \tag{16}$$

thus, from (3), (13), (14), (15) and (12) we have

$$\bar{R} = \begin{bmatrix} 0 \\ \Lambda_0 \bar{R} + \operatorname{diag}(Q^i_0) \operatorname{col}(L^i_1) \\ 0 \end{bmatrix} \qquad \bar{S} = \begin{bmatrix} \operatorname{col}(B^i_3) \\ \Lambda_0 \bar{S} \\ N_3 \end{bmatrix} \tag{17}$$

$$\bar{P}_0 = \begin{bmatrix} \operatorname{diag}(A^i_1) \mathbf{J}^1 + \operatorname{diag}(A^i_2) \mathbf{J}^2 + \operatorname{col}(B^i_1) \\ \Lambda_0 \bar{P}_0 + \Lambda_1 \\ N_1 \mathbf{J}_3 + N_2 \end{bmatrix} \quad \bar{P}_k = \begin{bmatrix} 0 \\ \Lambda_0 \bar{P}_k + \Lambda_{k+1} \\ 0 \end{bmatrix} \tag{18}$$

and for $k \geq 0$

$$\bar{Q}_k = \begin{bmatrix} 0 \\ \operatorname{diag}(Q^i_{k+1}) \operatorname{col}(L^i_1) + \Lambda_0 \bar{Q}_k \\ 0 \end{bmatrix} \tag{19}$$

4 Asymptotic expansion

In this section we are going to construct asymptotic expansion of general solution to (1) around $\mu \to 0$. We are looking for solution to (1) in the form

$$y^{i,\alpha}_t = \sum_{k=0}^{\infty} P^i_k(\mu) x^{i,\alpha}_{t-k} + \sum_{k=0}^{\infty} Q^i_k(\mu) Z^{i,\alpha}_{t-k}$$

and we are going to find asymptotic series

$$y^{i,\alpha}_t \sim \sum_{k=0}^{\infty} \Big(P^{i,0}_k + P^{i,1}_k \mu + \frac{1}{2} P^{i,2}_k \mu^2 + \dots \Big) x^{i,\alpha}_{t-k} \\ + \sum_{k=0}^{\infty} \Big(Q^{i,0}_k + Q^{i,1}_k \mu + \frac{1}{2} Q^{i,2}_k \mu^2 + \dots \Big) Z^{i,\alpha}_{t-k} \tag{20}$$

for $\mu \to 0$. Let us express (11) as

$$
\begin{aligned}
&0 \equiv T^i_m(\mu) \\
&\quad + \Big(D^i_1 + (G^i_1 + G^i_2 P^i_0(\mu))B^i_1\Big)\Phi^i_{1,m}(\mu) + \Big(H^i_1 + G^i_2 Q^i_0(\mu)\Big)\Phi^i_{0,m}(\mu) \\
&0 \equiv U^i_m(\mu) \\
&\quad + \Big(D^i_1 + (G^i_1 + G^i_2 P^i_0(\mu))B^i_1\Big)\Psi^i_{1,m}(\mu) + \Big(H^i_1 + G^i_2 Q^i_0(\mu)\Big)\Psi^i_{0,m}(\mu)
\end{aligned}
\tag{21}
$$

5 Zero order terms

Assume that $\mu = 0$. Then the problem (1) reduces to the full information case. In this case $Z^{i,\alpha}_t = w_t$, thus $E\{w_{t-k}|I^{i,\alpha}_t\} = w_{t-k}$ for $k \geq 0$ and from (9) for any $k, m \geq 0$ we have

$$
\Phi^i_{k+1,m}(0) = 0 \qquad\qquad \Psi^i_{k+1,m}(0) = \mathbf{I}\delta_{k,m}
$$

Let us suppose that $P^{i,0}_k = 0$ and $Q^{i,0}_k = 0$ for $k > 0$. Then, from (16), $\Lambda_k(0) = 0$ and $\bar{P}_k(0) = 0$ for $k > 0$, $\bar{Q}_k(0) = 0$ for $k \geq 0$. Thus, for $\mu = 0$ $E\{w_{t+1}|I^{i\alpha}_t\} = \bar{P}_0(0)w_t$ since $\upsilon_{t+1-k}(0) = 0$, for any $k \geq 0$ and

$$
\Phi^i_{0,m}(0) = 0 \qquad\qquad \Psi^i_{0,m}(0) = \bar{P}_0(0)\delta_{0,m}
$$

Hence, (21) reduces to

$$
\begin{aligned}
&0 \equiv T^i_m(0) \\
&0 \equiv U^i_m(0) + \Big(D^i_1 + (G^i_1 + G^i_2 P^{i,0}_0)B^i_1\Big)\delta_{0,m} + \Big(H^i_1 + G^i_2 Q^{i,0}_0\Big)\bar{P}_0(0)\delta_{0,m}
\end{aligned}
$$

and using (8) for $m > 0$ we have

$$
\begin{aligned}
0 &\equiv C^i_2 P^{i,0}_m + (G^i_1 + G^i_2 P^{i,0}_0)A^i_2 P^{i,0}_m + G^i_2 P^{i,0}_{m+1} \\
0 &\equiv C^i_2 Q^{i,0}_m + (G^i_1 + G^i_2 P^{i,0}_0)A^i_2 Q^{i,0}_m + G^i_2 Q^{i,0}_{m+1}
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
0 &\equiv C^i_1 + C^i_2 P^{i,0}_0 + (G^i_1 + G^i_2 P^{i,0}_0)(A^i_1 + A^i_2 P^{i,0}_0) + G^i_2 P^{i,0}_1 \\
0 &\equiv C^i_2 Q^{i,0}_0 + (G^i_1 + G^i_2 P^{i,0}_0)A^i_2 Q^{i,0}_0 + G^i_2 Q^{i,0}_1 \\
&\quad + \Big(D^i_1 + (G^i_1 + G^i_2 P^{i,0}_0)B^i_1\Big) + \Big(H^i_1 + G^i_2 Q^{i,0}_0\Big)\bar{P}_0(0)
\end{aligned}
\tag{23}
$$

Matrices $P^{i,0}_k = 0$, $Q^{i,0}_k = 0$ for $k > 0$ satisfy (22). Then the fist condition under (23) determines $P^{i,0}_0$, the second one determines $Q^{i,0}_0$.

6 First order terms

Differentiating (21) with respect to μ yields

$$0 \equiv \frac{d}{d\mu}T^i_m(0) + \Big(D^i_1 + (G^i_1 + G^i_2 P^{i,0}_0)B^i_1\Big)\frac{d}{d\mu}\Phi^i_{1,m}(0) + \Big(H^i_1 + G^i_2 Q^{i,0}_0\Big)\frac{d}{d\mu}\Phi^i_{0,m}(0)$$

$$0 \equiv \frac{d}{d\mu}U^i_m(0) + \Big(D^i_1 + (G^i_1 + G^i_2 P^{i,0}_0)B^i_1\Big)\frac{d}{d\mu}\Psi^i_{1,m}(0) + \Big(H^i_1 + G^i_2 Q^{i,0}_0\Big)\frac{d}{d\mu}\Psi^i_{0,m}(0) + G^i_2 Q^{i,1}_0 \bar{P}_0(0)\delta_{0,m} \tag{24}$$

We have

$$\begin{aligned} E\{w_{t+1}|I^{i,\alpha}_t\} &= \sum_{k=0}^{\infty} \bar{P}_k E\{w_{t-k}|I^{i,\alpha}_t\} + \sum_{k=0}^{\infty} \bar{Q}_k E\{\upsilon_{t-k}|I^{i,\alpha}_t\} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \bar{P}_k \Phi^i_{k+1,m} x^{i,\alpha}_{t-m} + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \bar{P}_k \Psi^i_{k+1,m} Z^{i,\alpha}_{t-m} + \sum_{k=0}^{\infty} \bar{Q}_k E\{\upsilon_{t-k}|I^{i,\alpha}_t\} \end{aligned}$$

differentiating with respect μ, and taking into account that $\bar{Q}_k(0) = 0$ for $k \geq 0$, yields

$$\begin{aligned} \frac{d}{d\mu}E\{w_{t+1}|I^{i,\alpha}_t\}(0) = &\sum_{m=0}^{\infty} \bar{P}_0(0)\frac{d}{d\mu}\Phi^i_{1,m}(0)x^{i,\alpha}_{t-m}(0) \\ &+ \sum_{k=0}^{\infty}\frac{d}{d\mu}\bar{P}_k(0)Z^{i,\alpha}_{t-k}(0) + \sum_{m=0}^{\infty}\bar{P}_0(0)\frac{d}{d\mu}\Psi^i_{1,m}(0)Z^{i,\alpha}_{t-m}(0) \\ &+ \bar{P}_0(0)\frac{d}{d\mu}Z^{i,\alpha}_t(0) \end{aligned}$$

since value of the variables may depend on μ we explicitly denote that we consider value of given variable under $\mu = 0$. On the other hand

$$\begin{aligned} \frac{d}{d\mu}E\{w_{t+1}|I^{i,\alpha}_t\}(0) = &\sum_{m=0}^{\infty}\frac{d}{d\mu}\Phi^i_{0,m}(0)x^{i,\alpha}_{t-m}(0) + \sum_{m=0}^{\infty}\frac{d}{d\mu}\Psi^i_{0,m}(0)Z^{i,\alpha}_{t-m}(0) \\ &+ \bar{P}_0(0)\frac{d}{d\mu}Z^{i,\alpha}_t(0) \\ \frac{d}{d\mu}E\{w_t|I^{i,\alpha}_t\}(0) = &\sum_{m=0}^{\infty}\frac{d}{d\mu}\Phi^i_{1,m}(0)x^{i,\alpha}_{t-m}(0) + \sum_{m=0}^{\infty}\frac{d}{d\mu}\Psi^i_{1,m}(0)Z^{i,\alpha}_{t-m}(0) \\ &+ \bar{P}_0(0)\frac{d}{d\mu}Z^{i,\alpha}_t(0) \end{aligned}$$

hence for $m \geq 0$

$$\begin{aligned}\frac{d}{d\mu}\Phi^i_{0,m}(0) &= \bar{P}_0(0)\frac{d}{d\mu}\Phi^i_{1,m}(0)\\ \frac{d}{d\mu}\Psi^i_{0,m}(0) &= \bar{P}_0(0)\frac{d}{d\mu}\Psi^i_{1,m}(0) + \frac{d}{d\mu}\bar{P}_m(0)\end{aligned} \tag{25}$$

From (10) we have

$$\begin{aligned}0 &= \operatorname{cov}(W^{i,\alpha}_t, x^{i,\alpha}_{t-m})(0) + \frac{d}{d\mu}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, x^{i,\alpha}_{t-m})(0)\\ 0 &= \operatorname{cov}(W^{i,\alpha}_t, Z^{i,\alpha}_{t-m})(0) + \frac{d}{d\mu}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, Z^{i,\alpha}_{t-m})(0)\end{aligned} \tag{26}$$

where

$$W_t = \sum_{p=0}^{\infty}\frac{d}{d\mu}\Phi^i_{1,p}x^{i,\alpha}_{t-p} + \sum_{p=0}^{\infty}\frac{d}{d\mu}\Psi^i_{1,p}Z^{i,\alpha}_{t-p}$$

thus, for $m > 0$, $E\{W^{i,\alpha}_t | I^{i,\alpha}_{t-m}\} = 0$ and there exists matrices Ξ^i_1, Ξ^i_2, $\bar{\Xi}^i_1$, $\bar{\Xi}^i_2$ such that $W_t = \Xi^i_1\epsilon_t + \Xi^i_2\epsilon^{i,\alpha}_t + \bar{\Xi}^i_1\upsilon_t + \bar{\Xi}^i_2\upsilon^{i,\alpha}_t$, thus for $m > 0$

$$\frac{d}{d\mu}\Phi^i_{1,m}(0) = 0 \qquad\qquad \frac{d}{d\mu}\Psi^i_{1,m}(0) = 0$$

and (26) reduces to

$$\begin{aligned}0 = {} & \frac{d}{d\mu}\Phi^i_{1,0}(0)\operatorname{var}(x^{i,\alpha}_t)(0) + \frac{d}{d\mu}\Psi^i_{1,0}(0)\operatorname{cov}(Z^{i,\alpha}_t, x^{i,\alpha}_t)(0)\\ & + \frac{d}{d\mu}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, x^{i,\alpha}_t)(0)\\ 0 = {} & \frac{d}{d\mu}\Phi^i_{1,0}(0)\operatorname{cov}(x^{i,\alpha}_t, Z^{i,\alpha}_t)(0) + \frac{d}{d\mu}\Psi^i_{1,0}(0)\operatorname{var}(Z^{i,\alpha}_t)(0)\\ & + \frac{d}{d\mu}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, Z^{i,\alpha}_t)(0)\end{aligned}$$

We have from (12) and (1) that

$$\begin{aligned}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, Z^{i,\alpha}_t) &= \mu(L^i_1 + L^i_2 I^i)\bar{R}' + \mu L^i_1 L^{i\prime}_1 + \mu L^i_2 L^{i\prime}_2\\ \operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, x^{i,\alpha}_t) &= 0\end{aligned}$$

where $I^i\upsilon_t = E_{\Omega^i}\{\upsilon^{i,\alpha}_t\}$. Hence

$$\begin{aligned}\frac{d}{d\mu}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, Z^{i,\alpha}_t) &= (L^i_1 + L^i_2 I^i)\bar{R}' + L^i_1 L^{i\prime}_1 + L^i_2 L^{i\prime}_2\\ \frac{d}{d\mu}\operatorname{cov}(L^i_1\upsilon_t + L^i_2\upsilon^{i,\alpha}_t, x^{i,\alpha}_t) &= 0\end{aligned}$$

and

$$\frac{d}{d\mu}\Psi^i_{1,0}(0) = \Big((L^i_1 + L^i_2 I^i)\bar{R}(0)' + L^i_1 L^{i\prime}_1 + L^i_2 L^{i\prime}_2\Big)$$
$$\times \Big(\operatorname{cov}(Z^{i,\alpha}_t, x^{i,\alpha}_t)(0)\operatorname{var}(x^{i,\alpha}_t)(0)^{-1}\operatorname{cov}(x^{i,\alpha}_t, Z^{i,\alpha}_t)(0) - \operatorname{var}(Z^{i,\alpha}_t)(0)\Big)^{-1}$$
$$\frac{d}{d\mu}\Phi^i_{1,0}(0) = -\frac{d}{d\mu}\Psi^i_{1,0}(0)\operatorname{cov}(Z^{i,\alpha}_t, x^{i,\alpha}_t)(0)\operatorname{var}(x^{i,\alpha}_t)(0)^{-1} \tag{27}$$

Observe that matrices $d/\mu\Psi^i_{1,0}(0)$ and $d/d\mu\Phi^i_{1,0}(0)$ are already known. Conditions (24) take now the form for $m > 0$

$$\begin{aligned} 0 &\equiv \frac{d}{d\mu}T^i_m(0) \\ 0 &\equiv \frac{d}{d\mu}U^i_m(0) + \Big(H^i_1 + G^i_2 Q^{i,0}_0\Big)\frac{d}{d\mu}\bar{P}_m(0) \end{aligned} \tag{28}$$

from (8) we have

$$\begin{aligned} \frac{d}{d\mu}T^i_0(0) &= C^i_2 P^{i,1}_0 + G^i_2 P^{i,1}_0 (A^i_1 + A^i_2 P^{i,0}_0) + (G^i_1 + G^i_2 P^{i,0}_0) A^i_2 P^{i,1}_0 + G^i_2 P^{i,1}_1 \\ \frac{d}{d\mu}T^i_k(0) &= C^i_2 P^{i,1}_k + G^i_2 P^{i,1}_0 A^i_2 P^{i,0}_k + (G^i_1 + G^i_2 P^{i,0}_0) A^i_2 P^{i,1}_k + G^i_2 P^{i,1}_{k+1} \\ \frac{d}{d\mu}U^i_m(0) &= C^i_2 Q^{i,1}_m + G^i_2 P^{i,1}_0 A^i_2 Q^{i,0}_m + (G^i_1 + G^i_2 P^{i,0}_0) A^i_2 Q^{i,1}_m + G^i_2 Q^{i,1}_{m+1} \end{aligned}$$

and from (18), for $k \geq 0$

$$\mathbf{J}_2 \frac{d}{d\mu}\bar{P}_k = (\mathbf{I} - \Lambda_0)^{-1}(\operatorname{diag}(P^{i,1}_{k+1})\mathbf{J}_1 + \operatorname{diag}(Q^{i,1}_{k+1})) \tag{29}$$

the first equation under (28) takes now the form

$$0 \equiv C^i_2 P^{i,1}_k + G^i_2 P^{i,1}_0 A^i_2 P^{i,0}_k + (G^i_1 + G^i_2 P^{i,0}_0) A^i_2 P^{i,1}_k + G^i_2 P^{i,1}_{k+1}$$

this equation is fulfilled by

$$P^{i,1}_k = 0 \qquad\qquad \text{for } k > 0$$

suppose that $d/d\mu\bar{P}_k(0) = 0$ for $k \geq 0$. Then, the second equation under (28) takes the form

$$0 \equiv C^i_2 Q^{i,1}_m + (G^i_1 + G^i_2 P^{i,0}_0) A^i_2 Q^{i,1}_m + G^i_2 Q^{i,1}_{m+1}$$

this equation is fulfilled by

$$Q_k^{i,1} = 0 \qquad\qquad \text{for } k > 0$$

thus, indeed $d/d\mu \bar{P}_k(0) = 0$ for $k \geq 0$.

Let us now consider the case $k = 0$. Equations (24) take the form

$$\begin{aligned}
0 &\equiv \Big(C_2^i + (G_1^i + G_2^i P_0^{i,0})A_2^i\Big)P_0^{i,1} + G_2^i P_0^{i,1}(A_1^i + A_2^i P_0^{i,0}) \\
&\quad + \Big(D_1^i + (G_1^i + G_2^i P_0^{i,0})B_1^i + (H_1^i + G_2^i Q_0^{i,0})\bar{P}_0(0)\Big)\frac{d}{d\mu}\Phi_{1,0}^i(0) \\
0 &\equiv \Big(C_2^i + (G_1^i + G_2^i P_0^{i,0})A_2^i\Big)Q_0^{i,1} + G_2^i Q_0^{i,1}\bar{P}_0 + G_2^i P_0^{i,1} A_2^i Q_0^{i,0} \\
&\quad + \Big(D_1^i + (G_1^i + G_2^i P_0^{i,0})B_1^i + (H_1^i + G_2^i Q_0^{i,0})\bar{P}_0(0)\Big)\frac{d}{d\mu}\Psi_{1,0}^i(0)
\end{aligned} \tag{30}$$

Equations (30) determine $P_0^{i,1}$ and $Q_0^{i,1}$ and finish computation of first order correction terms.

7 Notes on computational issues

The first equation under (23) is fairly standard and appears while analyzing linear dynamic rational models. This matrix equation can be reduced to appropriate generalized eigenvalue-eigenvector problem and then solved using generalized Schur decomposition, see Kowal (2005) for further details. Generalized Schur decomposition is available e.g. in Lapack library.

After solving the first equation under (23) we can determine the matrix $\bar{P}_0(0)$ using (18). Thus, the second equation under (23) becomes generalized Sylvester matrix equation. This equation is a special case of the generalized Lyapunov matrix equation

$$\begin{aligned}
AX - YB &= C \\
DX - YE &= F
\end{aligned} \tag{31}$$

Equation (31) can be solved using e.g. Slicot library. Having determined matrices $P_0^{i,0}$ and $Q_0^{i,0}$ we can find all covariance matrices required by the equation (27) and thus also matrices $d/d\mu\Phi_{1,0}^i(0)$ and $d/d\mu\Psi_{1,0}^i(0)$. Now equations under (30) become generalized Sylvester matrix equation. Notice that we can determine full information solution as well as first order correction terms without using any recursive computations.

8 Conclusions

In this paper we have presented a method of solving linear rational models with imperfect and asymmetric information. General imperfect information problem even in linear framework requires solving infinite dimensional problems. Proposed asymptotic expansion method is able to deliver approximate to such problems. We have concentrated on expansion around full information case since it allows for direct comparisons with full information case and thus for assessment of influence of imperfect information on the economy. Additionally such an expansion allows for relatively easy solution. We have limited ourselves to first order terms only, but higher order expansions seems to be possible.

A Conditional expectations

In this section we closely follow Pollock (1999). Let x and y are random vectors whose joint distribution is normal. Then there exists a vector α and a matrix B such that

$$E(y|x) = \alpha + B'x \tag{32}$$

thus $E(y|x)$ is also normally distributed. We are going to find α and B. Multiplying $E(y|x)$ by the marginal density function of x and by integrating with respect to x yields

$$E\{E(y|x)\} = E(y)$$

On applying this condition to (32) we find that

$$E(y) = \alpha + B'E(x), \qquad \text{or} \qquad \alpha = E(y) - B'E(x) \tag{33}$$

Next, by multiplying $E(y|x)$ by x' and by the marginal density function of x, and by integrating with respect to x, we obtain the joint moment $E(xy')$. Thus, from equation (32), we get

$$E(yx') = \alpha E(x) + B'E(xx') \tag{34}$$

multiplying the first equation under (33) by $E(x')$ gives

$$E(y)E(x') = \alpha E(x') + B'E(x)E(x') \tag{35}$$

substracting (35) form (34) yields

$$\text{cov}(y, x) = B'\{E(xx') - E(x)E(x')\} = B' \text{var}(x)$$

where $\text{cov}(y, x) \equiv E(yx') - E(y)E(x')$ and $\text{var}(x) \equiv \text{cov}(x, x)$. Thus

$$B' = \text{cov}(y, x) \times \text{var}(x)^{-1} \tag{36}$$

Substituting (36) and the second equation under (33) to (32) yields

$$\begin{aligned} E(y|x) &= E(y) + \Psi \times (x - E(x)) \\ \Psi &= \text{cov}(y, x) \times \text{var}(x)^{-1} \end{aligned} \tag{37}$$

substracting y form both sides of equation (37) and multiplying by $z' - E(z|x)'$ yields

$$\text{cov}(y, z|x) = \text{cov}(y, z) - \text{cov}(y, x)\, \text{var}(x)^{-1} \text{cov}(x, z) \tag{38}$$

where $\text{cov}(y, z|x) \equiv \text{cov}((y - E(y|x))(z - E(z|x))')$.

References

[1] F. Collard and H. Dellas. The new keynesian model with imperfect information and learning. working paper, March 2004.

[2] A. Gerali and F. Lippi. Optimal control and filtering in linear forward-looking economies: A toolkit. CEPR Discussion Papers, Januay 2003.

[3] P. Kowal. An algorithm for solving arbitrary linear rational expectations model. working paper, January 2005.

[4] D.S.G. Pollock. *A Handbook of Time-Series Analysis, Signal Processing and Dynamics.* Academic Press, London, 1996.

[5] L.E.O. Svenson and M. Woodford. Indicator variables for optimal policy under asymmetric information. *Journal of Economic Dynamics and Control*, 48(4):661–690, 2004.