COMPARISONS OF CASHFLOW MAPS FOR VALUE-AT-RISK

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Abstract. This article is devoted to the study cashflow maps used in the computation of value-at-risk (VaR). Properties and characteristics of the approaches found in the literature are presented and two new approaches are introduced. The goal of this paper is to study the quality of these maps. This is done by calculating the risk induced by the difference between the mapped cashflows and the original one.

1. Introduction

Value-at-risk (VaR) is a way of predicting the maximum loss that occurs with a certain probability P in a certain interval of time. In other words, the probability that the loss will be higher than the value at risk on the time horizon is $1 - P$. The interest and the difficulty of the concept is that the risk of a portfolio is aggregated in a unique number. So we have not only to estimate the distribution of the changes of each risk factor but also the way the different factors interact, how they are correlated. A typical approach is to restrict the number of risk factors to a given set of interest rates (for example 1, 3, 6, 12 months, 2, 3, 5, 10 years swaps and 2, 3, 5, 10, 30 years government bonds) and to suppose that the joint distribution of the changes of those factors is multi-normal. Then each particular instrument is distributed between the different curves using a first order approximation. This method, called delta-normal VaR, is the RiskMetrics one popularized by JP Morgan.

The next step in the process is to map a cashflow (or the cashflow delta equivalent to the position) which has a maturity between two standard maturities to the appropriate (standard) maturities. This paper tries to answer the following questions concerning this mapping: What are the possible methods? What are their properties? Which one is the best? How are those methods coherent with other hypothesis or methodologies?

Before looking at the mapping itself, we can perhaps think to reduce its importance by adding more points to the curve to have smaller intervals between those points. However, some intermediate points are not liquid, so it is difficult to obtain good rates. If a data series is not of good quality, the statistics deduced from the series will be of poor quality also. Moreover, adding too many points can lead to some singularities in the methodology. If one uses more risk factors than the number of historical data in the computation of the covariance matrices, the matrix obtained is singular. So some positions have an estimated risk of zero. Moreover the number of

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elements of the variance-covariance matrix grows with the *square* of the dimension. So if one adds too many points, it becomes very difficult to manipulate.

With a good mapping, we can reduce the number of points and solve some of the problems describe above. A good mapping can be built if the points selected correspond to liquid, standard maturities. But often there are points which are not completely explained by the standard ones. This can be the case if we choose standard maturities of 1, 3, 6, 12 months and 2 years for example, which misses the 2 months and 18 months rates that can be quite liquid. Also for the governments bonds, the standard risk factors have a *fixed term* but the bonds in the market have a *fixed maturity*. So the 5-years rate is not directly observable in the market, and has to be calculated from a bond with maturity smaller than 5 years and one longer. This is why it is of interest to estimate the predictive quality of the mapping used.

As this paper shows, mapping can be performed with a variety of methods. We consider cashflows with a fixed *present value*. Once the present value is computed, none of the mappings we describe uses the rates. It means that the *valuation* of the cashflows and their *mapping* can be done separately with different curves. This further emphasizes that risk management methods can be quite different from pricing methods.

The plan of the paper follows. In the next section, we present six different mappings, two of which are new. For each of them, we explain how they are constructed and their main properties. The third section is devoted to a comparison between the approaches. We conclude in the fourth section.

2. Description of the cashflow maps

The notations for this section are the following. We will always consider a cashflow with present value 1. The issue is to allocate our cashflow to positions X_1 and X_2 on two standard points. This allocation doesn't preserve necessarily the present value, i.e. $X_1 + X_2$ can be different of 1. The vector of risk we want to estimate is denoted v. Its term is t. The two risk factors v_i on which the mapping is done have norms (variance) σ_i, their terms are t_i and the correlation between them is ρ. Since the risk factors are zero-coupon bonds, their term is also their duration. We will estimate v by some $\bar{v} = X_1 v_1 + X_2 v_2$.

Description of the elementary and the RiskMetrics maps can be found in Esch et al. [1996].

2.1. Elementary map.

The elementary mapping, also called duration mapping, conserves the present value ($X_1 + X_2 = 1$) and the duration (which is a way to measure the risk, see [Jorion, 1997, p. 217]).

For this elementary mapping we have

$$X_1 = \frac{t_2 - t}{t_2 - t_1} \qquad \text{and} \qquad X_2 = \frac{t - t_1}{t_2 - t_1}. \tag{1}$$

In the plane v_1-v_2, the vector $\bar{v}v$ is a convex combination (convex homotopy) of v_1 and v_2 with the duration as parameter.

The main advantages of this mapping is that it is simple (to understand, to implement and to compute) and continuous in the input. Some advantages are presented in Mina [1999] where it is called linear mapping.

Another way to derive this mapping is the following. Suppose that interest rates are compounded continuously and *forward rates* are constant between two standard terms. We obtain (after some easy calculations) a rate for the term t between t_1 and t_2 of

$$r_t = \alpha \frac{t_1}{t} r_1 + (1-\alpha) \frac{t_2}{t} r_2 \tag{2}$$

where $\alpha = (t_2 - t)/(t_2 - t_1)$. The price of a cashflow of 1 at the end of the term t has a present value

$$P_t = e^{-r_t t}.$$

We will denote by $\hat{r}_t$ and $\hat{P}_t$ the new rates and prices after the changes of the market. With those notations, we have that the gain on a position of term t and present value 1 is

$$\begin{aligned} \frac{\hat{P}_t - P_t}{P_t} &\sim \left(-\alpha \frac{t_1}{t} t(\hat{r}_1 - r_1) - (1-\alpha) \frac{t_2}{t} t(\hat{r}_2 - r_2) \right) \\ &\sim \alpha \frac{\hat{P}_1 - P_1}{P_1} + (1-\alpha) \frac{\hat{P}_2 - P_2}{P_2}. \end{aligned} \tag{3}$$

This means that the investment in a security of present value 1 and term t generates the same gain as the investment of α in a security of term t_1 and $(1-\alpha)$ in a security of term t_2.

2.2. **Rates map.** For this map, the result is obtained by interpolating interest rates. For this reason, we call it the "rates map". This map is presented in Mina [1999].

The rate r_t for the term t is interpolated linearly from r_1 and r_2, the rates for the terms t_1 and t_2

$$r_t = \alpha r_1 + (1-\alpha) r_2$$

where $\alpha = (t_2 - t)/(t_2 - t_1)$. We use continuously compounding rates. The price of a cashflow of 1 at the end of the term t has a present value

$$P_t = e^{-t r_t}.$$

We will denote by $\hat{r}_t$ and $\hat{P}_t$ the new rates and prices after a change in the market. With those notations, we have that the profit on a position of term t and present value 1 is

$$\begin{aligned} \frac{\hat{P}_t - P_t}{P_t} &\sim \left(-t\alpha e^{-r_t t}(\hat{r}_1 - r_1) - t(1-\alpha) e^{-r_t t}(\hat{r}_2 - r_2) \right) / P_t \\ &\sim -\frac{t}{t_1} t_1 \alpha (\hat{r}_1 - r_1) - \frac{t}{t_2} t_2 (1-\alpha)(\hat{r}_2 - r_2) \\ &\sim \alpha \frac{t}{t_1} \frac{\hat{P}_1 - P_1}{P_1} + (1-\alpha) \frac{t}{t_2} \frac{\hat{P}_2 - P_2}{P_2}. \end{aligned}$$

This means that the investment in a security of present value 1 and term t generate the same profit as an investment of $\alpha\frac{t}{t_1}$ in a security of term t_1 and present value 1 and $(1-\alpha)\frac{t}{t_2}$ in a security of term t_2 and present value 1.

Thus we obtain,

$$X_1 = \frac{t}{t_1}\frac{t_2 - t}{t_2 - t_1}$$

and

$$X_2 = \frac{t}{t_2}\frac{t - t_1}{t_2 - t_1}.$$

Another way to see this mapping is the following. One calculates the result of the move of the rate at a standard term on the move of the rate at the term of the cashflow (using the interpolated rates between two standard terms)

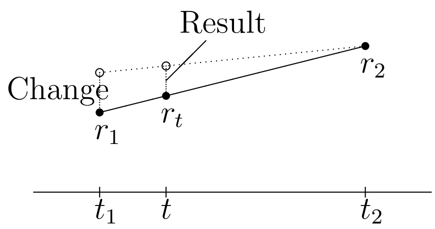

FIGURE 1. Result of the change of the rate at the standard term on an intermediate term.

Note that this mapping preserves the sensitivities with respect to the standard terms but not the present value of the cashflows $(X_1 + X_2 \neq 1)$.

Note also that the map is singular when $t_1 = 0$. A very short term rate for a period different from 0 shuold be chosen.

2.3. **RiskMetrics map.** We call RiskMetrics map the one describe in the RiskMetrics Technical Document (Mor [1996]). It is the one originally used for the computation of the value at risk. I seems that it is not used any more in the software distributed by the group. This mapping conserves the present value, the sign of the present value and the volatility obtained from a linear interpolation.

The conservation of the present value gives

$$X_1 + X_2 = 1. \tag{4}$$

We estimate the norm of v by a linear interpolation of volatilities

$$\sigma = \sigma_1 + \frac{t - t_1}{t_2 - t_1}(\sigma_2 - \sigma_1). \tag{5}$$

On the other hand, we also have

$$\sigma^2 = (X_1\sigma_1, X_2\sigma_2)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} X_1\sigma_1 \\ X_2\sigma_2 \end{pmatrix} = X_1^2\sigma_1^2 + 2\rho\sigma_1\sigma_2X_1X_2 + X_2^2\sigma_2^2.$$

Replacing X_2 by $1 - X_1$, solving the equation with respect to X_1, we have that

$$X_1 = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

where $a = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$, $b = \rho\sigma_1\sigma_2 - \sigma_2^2$ and $c = \sigma_2^2 - \sigma^2$. Between the two possible solutions we choose the one such that X_1 and X_2 are between 0 and 1.

The pictures of the geometrical representations of the elementary mapping and the RiskMetrics one are given in figure 2.

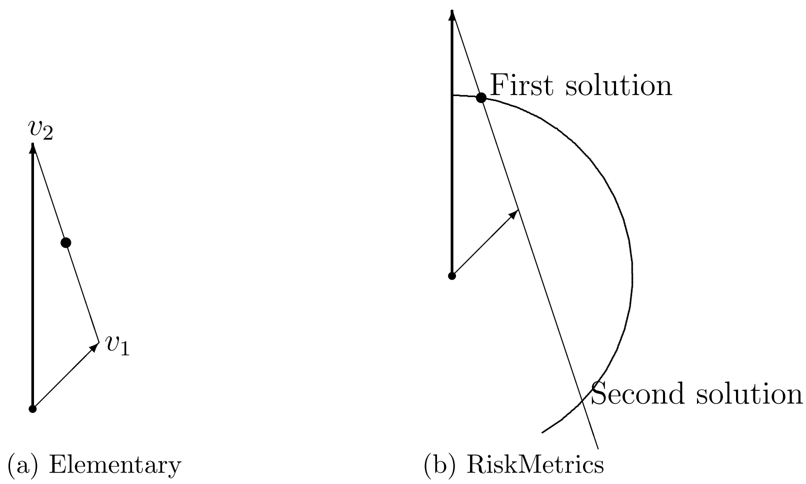

FIGURE 2. Geometrical representation of the elementary and the RiskMetrics mappings

In the plane v_1-v_2, the vector v is on the convex hull of v_1 and v_2 and the norm is given by the interpolation between the one of v_1 and the one of v_2 with the duration as parameter.

A problem with the RiskMetrics map is that the map may be non continuous in the sense that one does not necessarily have $X_1 \to 1$ and $X_2 \to 0$ when $t \to t_1$. This happens when

$$\langle v_2 - v_1 \,|v_1\rangle < 0 \quad \text{and} \quad \sigma_1 < \sigma_2 \tag{6}$$

($\langle .\,|.\rangle$ is the scalar product between two vectors), i.e. when v_2 is in the half plane with boundary perpendicular to and passing through v_1 (see figure 3). In this case for $t = t_1$, the two solutions satisfy the conditions and the mapping is ambiguous.

We prove that under the conditions (6), there is two solutions. We have $v = (1 - X_2)v_1 + X_2 v_2$. It is obvious that $X_2 = 0$ and $X_1 = 1$ satisfy (4) and (5). But we also have

$$\begin{aligned} |v|^2 &= |(1 - X_2)v_1 + X_2 v_2|^2 \\ &= |v_1|^2 + {X_2}^2|v_2 - v_1|^2 + X_2 \langle v_2 - v_1 \,|v_1\rangle \,. \end{aligned}$$

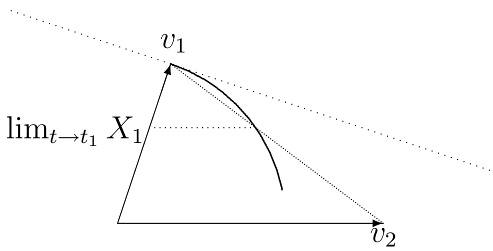

FIGURE 3. Case where the RiskMetrics map is not continuous

Risk 1	Risk 2	$\langle v_2 - v_1 \vert v_1 \rangle$	σ_1	σ_2	ρ
CAD.Z15	CAD.Z20	-0.00002278	0.0097	0.0094	0.79
EUR.Z09	EUR.Z10	-0.00000033	0.0096	0.0098	0.98
ZAR.Z02	ZAR.Z03	-0.00000106	0.0067	0.0084	0.78
ZAR.Z07	ZAR.Z09	-0.00001688	0.0160	0.0197	0.76
ZAR.Z10	ZAR.Z15	-0.00095859	0.0276	0.0635	-0.11
ZAR.Z15	ZAR.Z20	-0.00002744	0.0635	0.0657	0.96

TABLE 1. Risk factors for which the RiskMetrics mapping is discontinuous (matrices of March 15, 1999).

So there exists $\epsilon > 0$ such that for all $0 < X_2 < \epsilon$, $|v| < |v_1|$. As on the other hand $\lim_{X_2 \to 1} |v| = |v_2| > |v_1|$, by the intermediate value theorem, there exists $X_2 > 0$ such that $|v| = |v_1|$. This proves that there exists a second, distinct solution.

This can happen for example when the correlation is very small (and is always the case when it is negative) or the volatilities of the two factors of risks are similar. The mapping is made for terms between two consecutive standard terms, as the correlations are usually large for those risks and the volatilities are usually different (larger for the longer term), the phenomenon is not so frequent (see also Mina [1999] for more explanations about this phenomenon).

For example with the covariance matrix of March 15, 1999, this was the case for 5 combinations of risk factors. These are shown in table 1. The first line of this table is a case where the volatility (of the price) for the risk with the longer term is smaller that the one of the risk with shorter term. In the table 2, the risk factors at which the mapping is discontinuous for different matrices is given.

In the figure 4, we give a picture of the proportion of 1 ZAR mapped to the different risk factors as function of the term with the term between 9 and 15 years. In this extreme example, the discontinuity of the mapping appears clearly.

The elementary and the RiskMetrics mappings are characterized by the conservation of the present value. From a geometrical point of view, this means that the

Date	Risk factors					
15/3/1999	CAD.Z15	EUR.Z09	ZAR.Z02	ZAR.Z07	ZAR.Z10	ZAR.Z15
15/4/1999	CAD.Z15	DKK.Z09	ZAR.Z10			
14/5/1999	BEF.Z03	BEF.Z07	CAD.Z15	DKK.Z09	FRF.Z04	ITL.Z04
15/6/1999	BEF.Z09					
14/7/1999						
10/8/1999	FRF.Z05	FRF.Z15	GBP.Z09	GBP.Z10	ITL.Z09	JPY.Z09

TABLE 2. Risk factors for which the RiskMetrics mapping is discontinuous.

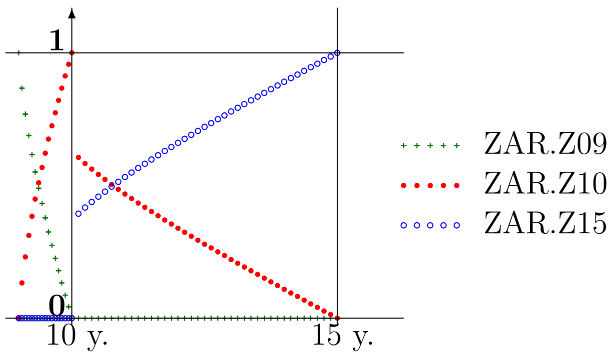

FIGURE 4. Graph of the mapping of 1 ZAR on different risk factors for terms between 9 and 15 years

estimation of v is in the convex hull of v_1 and v_2. Other mappings not having this property are possible.

2.4. **Schaller's map.** Schaller's mapping, presented in Schaller [1996], is based also on the conservation of the estimated volatility. The way the risk is distributed on the two other vectors is chosen to avoid discontinuities in the parameters that can appears in the RiskMetrics mapping.

The proportion $X_1/(X_1+X_2)$ attributed to the vector v_1 varies linearly with the duration as parameter from 1 in t_1 to 0 in t_2. So we have

$$\frac{X_1}{X_1+X_2} = \frac{t_2-t}{t_2-t_1}$$

and the conservation of the risk

$$\sigma^2 = |X_1 v_1 + X_2 v_2|^2 = X_1^2\sigma_1^2 + 2\rho\sigma_1\sigma_2 X_1 X_2 + X_2^2\sigma_2^2.$$

This gives as solution, where $\tau = (t-t_1)/(t_2-t)$,

$$X_1 = \frac{\sigma}{\sqrt{\sigma_1^2 + \sigma_2^2\tau^2 + 2\sigma_1\sigma_2\rho\tau}}$$

and

$$X_2 = \frac{\sigma}{\sqrt{\sigma_1^2/\tau^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho/\tau}}.$$

(a) Schaller (b) Polar coordinates

FIGURE 5. Geometrical representation of the Schaller and the polar coordinates mappings

2.5. **Polar coordinates mapping.** We describe now a new mapping system. The idea is as follows. The vector is constructed in the plane of the two vectors on which the mapping is done by interpolating linearly the norm (the VaR) of the vectors and the angles between them using the term as parameter.

So we estimate the norm of v by

$$\sigma = \sigma_1 + \frac{t - t_1}{t_2 - t_1}(\sigma_2 - \sigma_1).$$

We note $\alpha = \arccos \rho$. The angle between v_1 and v is then estimated by $\beta = \frac{t-t_1}{t_2-t_1}\alpha$. So

$$X_1 = \frac{\sin(\alpha - \beta)}{\sqrt{1-\rho^2}} \frac{\sigma}{\sigma_1}$$

and

$$X_2 = \frac{\sin(\beta)}{\sqrt{1-\rho^2}} \frac{\sigma}{\sigma_2}$$

The general idea of those two last mappings is the same. The estimation of v "move" from v_1 to v_2 rotating in the plane with a length given by the interpolated norm. The type of rotation is different for the two mappings.

2.6. **Three dimensional map.** This is also a new approach. The vector is not in the plane formed by the two vectors on which the mapping is done, so we have to add a third dimension. The position of this vector is obtained by estimating the covariance (correlations) with the other two vectors. This is done by using a linear interpolation of the covariance between the vectors with the term as parameter. Then this vector is projected orthogonally (to minimize the distance (the VaR)).

The picture of the construction of X_i is given in the figure 6.

The estimation of the covariance between v and v_i is

$$\rho_i = 1 + \frac{t - t_i}{t_{1-i} - t_i}(\rho - 1).$$

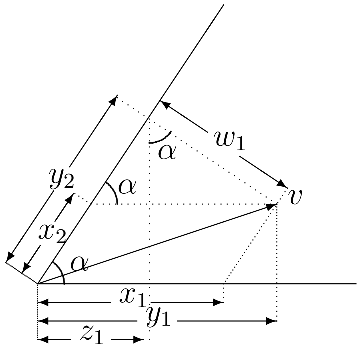

FIGURE 6. Construction used in the three dimensional mapping

Similarly the estimation of the norm is

$$\sigma = \sigma_1 + \frac{t - t_1}{t_2 - t_1}(\sigma_2 - \sigma_1).$$

Let $x_i = \sigma_i X_i$. We denote by y_i the length of the orthogonal projection of v on the line of v_i. So $y_i = \sigma\rho_i$. We denote by z_1 the length of the projection of the vector of length y_2 and of direction v_2 on the line of v_1, so $z_1 = y_2\rho$. Then $y_1 - z_1 = w_1\rho$ where $w_1 = x_1\rho$. Combining all those relations, we obtain

$$X_1 = \frac{x_1}{\sigma_1} = \frac{\sigma}{\sigma_1}\left(\frac{\rho_1 - \rho_2\rho}{1 - \rho^2}\right)$$

and

$$X_2 = \frac{\sigma}{\sigma_2}\left(\frac{\rho_2 - \rho_1\rho}{1 - \rho^2}\right).$$

Note that the last four mappings do not conserve the present value of the securities. So the sum of the mapped positions is not equal to the real present value of the positions (see also end of section 3).

Note also that the mapping is sigular when $\rho = \pm 1$. But as, in that case, the risk factors are perfectly correlated, the split between the factor is a subjective choice, not a mathematical one.

2.7. **Unused interesting property.** Before comparing the different maps, it worths mentioning an interesting property of all of them that seems unused. If we have the present value of the cashflows, the actual rates are not used any more in the different cashflow formulas. This feature is interesting if different types of products are priced on the same curve but with different spreads. Suppose you use the following algorithm to compute the VaR:

(1) Decompose the book into equivalent cashflows.
(2) Do the mapping to the risk factors using the curves of the risk factors.
(3) Compute the VaR.

Then you lose any information about the spread. An algorithm suggested by the above property is the following:

(1) Decompose the book into equivalent cashflows

(2) Discount them with the correct curve (even if it is not the one used in the risk factors).
(3) Do the mapping to the risk factors using the present value of the cashflows.
(4) Compute the VaR.

The difference between the two can be of some importance in estimating the risks. To see this we compare two situations for which the two methods give an estimated risk of 0. For the first method we take two positions with *opposite cashflows* priced on the same curve but with a fixed spread (for example a product sold to a customer and its hedging). For the second one we take two positions with *opposite present values* also priced on the same curve with a spread. The two situations have an estimated risk of 0. But what happens in the two cases if the rates change? Suppose we have a nominal of 100,000,000 for the first method and a present value of the same amount for the second method (which is a larger position) with a term of one year. If the rate is 5%, the spread of 20 bps and the change of rates of 20 bps, the change of present value is 691 in the first case and 362 in the second, almost the half.

If we look at the first order approximation in the change of rates of the change of value in the two cases, for a term of 1 year, a rate of r, a spread of s and a rate movement of ϵ we have respectively

$$\left(\frac{-1}{(1+r+s)^2}+\frac{1}{(1+r)^2}\right)\epsilon \quad \text{and} \left(\frac{-1}{1+r+s}+\frac{1}{1+r}\right)\epsilon.$$

If we take now the first order approximation in the spread, we have respectively

$$2(1+r)^{-3}s\epsilon \quad \text{and} \quad (1+r)^{-2}s\epsilon$$

As $1+r$ is close to 1, it means that the risk due to the spread hidden by the method is approximatively twice bigger with the first method.

We can see the same result through a practical case. As previously, suppose that we have a product sold to a customer with a margin and hedged. The hedge is perfect in term of sensitivities (not in term of cashflows), which means that if the cashflow of the one year product sold is C_1, the cashflow of the hedging instrument is the value C_2 such that

$$-\frac{C_2}{(1+r+s)^2}=-\frac{C_1}{(1+r)^2}.$$

Using the first methodology to evaluate the VaR, we obtain a present value of the cashflow that is used in the VaR computation of

$$\frac{C_2}{(1+r+s)^2}\left(2s+\frac{s^2}{(1+r)^2}\right).$$

By using the second methodology, we have a present value of

$$\frac{C_2}{(1+r+s)^2}s.$$

Once more, the proposed improvements reduce the error by a factor 2.

We quantify this for our example of the hedging of a liability with a one year cashflow of 100,000,000. If the sensitivity of the book is zero, the position of the book wil be seen through the VaR as beeing long of $361,778$ in the first case and $180,717$ in the second.

2.8. **Mapping of mapping.** We describe now, for the elementary and rate maps, a property that we call "the mapping of a mapping is a mapping". This property is only valid for those two maps.

Suppose that we have five times $t_1 \leq t_2 \leq t \leq t_3 \leq t_4$. If we map a cashflow at t to t_2 and t_3 and then the results to t_1 and t_4, then we obtain the same results that the mapping of the cashflow directly to t_1 and t_4.

To prove this, we use the following notations: the direct mapping on t_2 and t_3 are denoted X_2 and X_3, the mapping on t_1 and t_4 by X_1 and X_4 and the composed mappings of X_i on t_1 and t_4 by $Y_{i,1}$ and $Y_{i,4}$.

For the elementary map, we have the following equations.

$$X_2 + X_3 = 1 \quad X_2 t_2 + X_3 t_3 = t$$

and

$$Y_{i,1} + Y_{i,4} = X_i \quad Y_{i,1} t_1 + Y_{i,4} t_4 = X_i t_i.$$

Combining those equations, we have

$$(Y_{2,1} + Y_{3,1}) + (Y_{2,4} + Y_{3,4}) = X_2 + X_3 = 1$$

and

$$(Y_{2,1} + Y_{3,1})\, t_1 + (Y_{2,4} + Y_{3,4})\, t_2 = X_2 t_2 + X_3 t_3 = t.$$

This proves that $X_j = Y_{2,j} + Y_{3,j}$, as announced.

On the other hand, for the rates map, we have

$$X_2 = \frac{t}{t_2} \frac{t_3 - t}{t_3 - t_2} \quad X_3 = \frac{t}{t_3} \frac{t - t_2}{t_3 - t_2}$$

and

$$Y_{i,1} = \frac{t_i}{t_1} \frac{t_4 - t_i}{t_4 - t_1} \quad Y_{i,4} = \frac{t_i}{t_4} \frac{t_i - t_1}{t_4 - t_1}.$$

Combining those equations, we have

$$\begin{aligned} Y_{2,1} + Y_{3,1} &= \frac{1}{t_1} \frac{1}{t_4 - t_1} \left(t_2 (t_4 - t_2) X_2 + t_3 (t_4 - t_3) X_3 \right) \\ &= \frac{t}{t_1} \frac{t_4 - t}{t_4 - t_1}. \end{aligned}$$

This proves that $X_j = Y_{2,j} + Y_{3,j}$, as announced.

3. Comparisons

For the comparison between the mappings, we use the following technique. We hedge a cash-flow of present value 1 by mapping this position to the preceding and the following terms. This represent the residual VaR due to the use of the mapped cash-flow instead of the true cash-flow.

As all the data are known we have the precise value of the residual risk, i.e. the error in the computation due to the mapping.

We measure the error by the VaR of the difference. The measure of the error by the VaR of the difference is a better one than the difference of the VaR's of the two positions. We are of course interested in having a VaR as close as possible of the real one. But for a portfolio of cashflows (p), if one adds a new cashflow (v), the error of the total will be small if the distance between the true position of the new cashflow and the estimated one (v_1) is small. Adding a cashflow with the same norm that the true one (v_2) but at a very distant position will give a larger error on the estimate of the total of the VaR (see Figure 7).

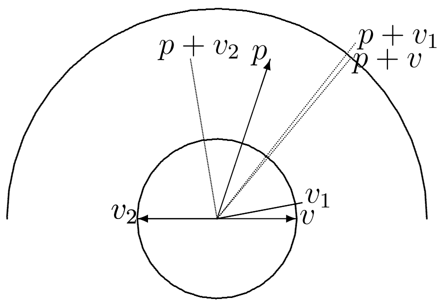

Figure 7. Different estimates of the VaR of a portfolio

We did this comparison for zero-coupon government rates. For each maturity (3, 4, 5, 7, 9 years and 10, 15, 20 years when possible), we have to compute for the different mappings the residual risk of a position which is short 1 million on that maturity and long the mapping of this same million on the two maturities surrounding it. So the figures we obtain represent the risk induced by the *mapping* on this particular position.

We did this computation with the matrix published by the RiskMetrics Group (for March 15, April 15, May 14, June 15, July 14 and August 10, 1999). We compared the results of the different mappings with the elementary one, that we use as a “benchmark” (we count the number of risk factors for which the residual risk is less than the one of the elementary map). The results are in Table 3.

The detailed results for three of the main currencies (DEM, JPY, USD) and matrix of July 14, 1999 are given in Tables 4, 5 and 6.

We see that globally the best maps seems to be the elementary, the rates, the polar coordinates and the three dimensional one. For the main currencies the results are better for the polar coordinates, the rates and the three dimensional one (16/23). Moreover the improvements can be substantial, with maximum of improvement of

Dates	Improvements *RiskMetrics*	Schaller	Rates	Polar	3 D
15/3/1999	38	45	62	64	65
15/4/1999	38	39	61	54	56
14/5/1999	41	41	64	63	65
15/6/1999	37	48	72	64	67
14/7/1999	44	46	64	62	61
10/8/1999	42	58	70	70	68

TABLE 3. Number of improvements for the different mappings with respect to the elementary mapping. Number of risk factors = 119.

	Induced risk Elem.	*RiskMetrics*	Schaller	Rates	Polar	3 dim.
Govt 3Y	667	724	720	576	750	684
Govt 4Y	811	794	782	847	741	778
Govt 5Y	1492	1547	1492	1373	1385	1385
Govt 7Y	1782	1795	1838	1994	2047	1987
Govt 9Y	1213	1211	1203	1211	1194	1206
Govt 10Y	1384	1382	1387	1444	1444	1442
Govt 15Y	1342	1487	1417	726	880	750
Govt 20Y	7022	7376	7177	5676	5678	5521

	Differences (with respect to elementary mapping)									
	RiskMetrics		Schaller		Rates		Polar		3 dim.	
	Abs.	%	Abs.	%	Abs.	%	Abs.	%	Abs.	%
Govt 3Y	-57	-9	-53	-8	91	14	-84	-13	-17	-3
Govt 4Y	17	2	29	4	-36	-4	70	9	33	4
Govt 5Y	-55	-4	1	0	120	8	108	7	107	7
Govt 7Y	-13	-1	-56	-3	-212	-12	-265	-15	-206	-12
Govt 9Y	2	0	10	1	2	0	19	2	7	1
Govt 10Y	2	0	-3	-0	-61	-4	-60	-4	-58	-4
Govt 15Y	-146	-11	-76	-6	615	46	461	34	591	44
Govt 20Y	-354	-5	-155	-2	1346	19	1344	19	1501	21
Improv.	3 / 8		3 / 8		5 / 8		5 / 8		5 / 8	

TABLE 4. The risk induced by a position of 1 million long the risk factor and short the million mapped on the adjacent risk factors, e.g. 3 year mapped on the 2 and 4 years. The value of the risk is given in the first part of the table and the improvement with respect to the elementary map in the second part. Figures for the DEM with the matrices of July 14, 1999.

Induced risk	Elem.	*RiskMetrics*	Schaller	Rates	Polar	3 dim.
Govt 3Y	362	359	357	421	295	304
Govt 4Y	346	353	349	334	335	332
Govt 5Y	588	600	586	546	526	529
Govt 7Y	1550	1522	1496	1444	1420	1482
Govt 9Y	1846	2003	1856	1716	1777	1772
Govt 10Y	2640	2748	2709	2641	2703	2652
Govt 15Y	6385	6492	6488	6242	6599	6474

Differences (with respect to elementary mapping)	*RiskMetrics*		Schaller		Rates		Polar		3 dim.	
	Abs.	%	Abs.	%	Abs.	%	Abs.	%	Abs.	%
Govt 3Y	3	1	5	1	-59	-16	67	18	57	16
Govt 4Y	-7	-2	-3	-1	12	3	11	3	14	4
Govt 5Y	-12	-2	3	0	42	7	62	11	59	10
Govt 7Y	28	2	54	3	106	7	130	8	68	4
Govt 9Y	-157	-9	-9	-1	131	7	69	4	74	4
Govt 10Y	-108	-4	-70	-3	-1	-0	-64	-2	-13	-0
Govt 15Y	-107	-2	-103	-2	143	2	-214	-3	-89	-1
Improv.	2 / 7		3 / 7		5 / 7		5 / 7		5 / 7	

TABLE 5. The risk induced by a position of 1 million long the risk factor and short the million mapped on the adjacent risk factors. The value of the risk is given in the first part of the table and the improvement with respect to the elementary map in the second part. Figures for the JPY with the matrices of July 14, 1999.

94 % for the rates map, and of 75 % for the three dimensional one for the 3-year USD, and 76 % for the rates map for the 15-year USD.

It is also worth noticing that the quality of the maps differs across the different currencies. For the same term (15-year), the figures are around 1000 for the DEM, 6000 for the JPY and 500 for the USD.

As said before, two of the mappings conserve the present value of the cashflows, the others do not. In Table 7, we give the decomposition for the USD for a cash-flow with present value 1000 across maps. The table shows X_1, X_2 and $X_1 + X_2$.

4. Conclusions

We now summarize the characteristics of the various cashflow maps.

From a numerical point of view, the elementary, the rates and the three dimensional maps are the fastest (they use only simple arithmetic operations). The RiskMetrics and the Schaller one use one (or two) square roots and the polar coordinates one use trigonometric and inverse trigonometric functions.

	Induced risk					
	Elem.	*RiskMetrics*	Schaller	Rates	Polar	3 dim.
Govt 3Y	112	122	118	6	45	28
Govt 4Y	160	173	167	91	106	95
Govt 5Y	383	375	376	419	409	415
Govt 7Y	312	318	312	250	250	250
Govt 9Y	188	196	191	159	162	158
Govt 10Y	486	483	484	489	502	504
Govt 15Y	572	558	549	136	361	399
Govt 20Y	1639	1778	1524	460	871	1043

	Differences (with respect to elementary mapping)									
	RiskMetrics		Schaller		Rates		Polar		3 dim.	
	Abs.	%	Abs.	%	Abs.	%	Abs.	%	Abs.	%
Govt 3Y	-9	-8	-5	-5	106	94	67	60	85	75
Govt 4Y	-13	-8	-7	-4	69	43	54	34	65	41
Govt 5Y	7	2	7	2	-37	-10	-27	-7	-32	-8
Govt 7Y	-6	-2	-0	-0	62	20	61	20	61	20
Govt 9Y	-7	-4	-3	-1	30	16	27	14	30	16
Govt 10Y	3	1	3	1	-3	-1	-16	-3	-18	-4
Govt 15Y	14	3	23	4	436	76	210	37	173	30
Govt 20Y	-139	-8	115	7	1179	72	768	47	596	36
Improv.	3 / 8		4 / 8		6 / 8		6 / 8		6 / 8	

Table 6. The risk induced by a position of 1 million long the risk factor and short the million mapped on the adjacent risk factors. The value of the risk is given in the first part of the table and the improvement with respect to the elementary map in the second part. Figures for the USD with the matrices of July 14, 1999.

From the point of view of the data used for the computation, no external information is needed for the elementary and the rates maps. For all the others, the volatilities of the standard terms and their correlations are used. It is worth to note that once the present value of the cashflow that we want to map is obtained, the rates are not used any more.

From a financial point of view, the elementary mapping and the RiskMetrics one conserve the present value of the securities. So the figures obtained in the analysis of the VaR are easier to present. Moreover the rates mapping is coherent with linear interpolation of continuously compounded rates and the elementary mapping is coherent with constant forward rates between two standard terms. So if VaR is used in parallel with some other methodologies for marked to market calculation and risk measures, it is better to use a mapping with the same methodology.

Term	Elementary			*RiskMetrics*			Schaller		
	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2
Govt 3Y	500	500	1000	492	508	1000	503	503	1005
Govt 4Y	500	500	1000	490	510	1000	503	503	1006
Govt 5Y	667	333	1000	661	339	1000	669	335	1004
Govt 7Y	500	500	1000	492	508	1000	502	502	1004
Govt 9Y	333	667	1000	327	673	1000	334	668	1002
Govt 10Y	833	167	1000	832	168	1000	834	167	1001
Govt 15Y	500	500	1000	495	505	1000	501	501	1003
Govt 20Y	667	333	1000	630	370	1000	679	340	1019

Term	Rates			Polar coordinates			3 dim.		
	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2
Govt 3Y	750	375	1125	778	372	1150	774	369	1143
Govt 4Y	667	400	1067	702	392	1094	698	390	1087
Govt 5Y	833	238	1071	846	236	1082	844	235	1079
Govt 7Y	700	389	1089	694	394	1088	691	392	1083
Govt 9Y	429	600	1029	429	602	1031	428	600	1028
Govt 10Y	926	111	1037	911	117	1028	911	117	1028
Govt 15Y	750	375	1125	686	395	1081	683	394	1078
Govt 20Y	889	222	1111	814	254	1069	803	249	1052

TABLE 7. Present value of the mapped cashflows for the different maps with an unmapped cashflow of present value 1000. Figures for the USD with the matrices of July 14, 1999

To summarize, the various methods could be ranked as follows, with the best at the top.

(1) Rates mapping
(2) Elementary mapping
(3) Three dimensional mapping
(4) Polar coordinates mapping
(5) Schaller mapping
(6) RiskMetrics mapping

This order is of course very subjective. The map using the rates has a lot of advantages. Its algorithm is fast and the quality of the result is very good. Moreover it is compatible with the linear interpolation of the rates between standard terms. The elementary one is similar except that the financial underlying hypothesis is probably less used. A part of the interest for the three dimensional map is probably due to its nice geometrical construction.

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