How Does Systematic Risk Impact US Credit Spreads ? A Copula Study

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June 2003
Very first draft

Abstract

It is well known that some relationship between systematic risk and credit risk prevails in financial markets. In our study, the return of S&P 500 stock index is our market risk proxy whereas credit spreads represent our credit risk proxy as a function of maturity, rating and economic sector. We address the problem of studying the joint distributions and evolutions of S&P 500 return and credit spreads. Graphical and non parametric statistical analysis (i.e., Kendall's tau and Spearman's rho) show that such bivariate distributions are asymmetric and exhibit some negative relationship between S&P 500 return and credit spreads. Indeed, credit spreads widen when S&P 500 return decreases or drops under some given level. We investigate then this stylized fact using copula functions to characterize the observed dependence structures between S&P 500 return and credit spreads. We focus at least on one parameter copulas and at most on one parameter Archimedean copulas, namely Gumbel, FGM, Frank and Clayton copula functions. Starting from the empirical Kendall's tau statistics observed for each bivariate dependence structure, we estimate the related parameter values for each copula type function belonging to our copulas' set. Finally, we exhibit optimal characterizations for such dependence structures and use the optimal selected copulas to achieve a scenario analysis among which stress testing.

JEL codes : C16, C32, D81.

Keywords : systematic risk, credit risk, copulas, Archimedean copulas, stress testing.

1 Introduction

Since recent years credit risk has been one main focus of the international community and, specifically, the sound assessment of such a risk. A good assessment of credit risk requires a reliable identification of factors affecting or characterizing such a risk. Credit risk consists essentially of default risk, credit rating migration risk or downgrading, and uncertainty about the recovery rate of credit risky assets.

Some studies such as Ericsson & Renault (2000) and Baraton & Cuillere (2001) for example, show that a safe valuation of credit risk requires to take into account macroeconomic and financial factors as an explanation of some trend, credit quality factors, and liquidity factors. On the other hand, some studies consider a more specific approach of credit risk while studying its market component. Indeed, it is well known that interactions between credit risk and market risk take place in financial markets.

According to Shane (1994), Das & Tufano (1996) and Duffee (1998), credit spreads depend on the stock market since the correlation of risky bonds' returns with the stock index return is higher for high yield bonds than for low yield bonds. Dichev (1998) attempts to study if bankruptcy risk is a systematic risk. This author concludes that this risk is not purely a systematic risk and requires then some other factors to be fully explained. Along with this study, Wilson (1998), Jarrow, Lando & Yu (2001) and Gatfaoui (2002) among others focus on the systematic and idiosyncratic components of credit risk. In the works of Jarrow & Turnbull (1995a,b), who initiated the reduced form approach, credit risk and market risk are closely related. These authors also argue recently (see Jarrow & Turnbull [2000]) that credit risk and market risk are not separable. Any unexpected variation of firm assets market value due to market risk generates a credit risk since this kind of variation affects the financial profile and volatility of the firm. The reverse is also true[1]. Moreover, Elton, Gruber, Agrawal & Mann (2001) show that a large portion of corporate bonds' risk is of systematic nature rather than diversifiable or idiosyncratic nature. They find that credit spreads and risky bonds' returns change both with the same common factors which generally affect the returns of stocks.

In this paper, we address the problem of taking into account the dependence between market risk and credit risk. We focus on some simultaneous interaction between these two risks rather than to consider some causal dependence through a regression type relation. We attempt to characterize the market risk's influence on credit risk in the spirit of the work of Lucas, Klaassen, Spreij & Straetmans (2001) for example. Our goal is to study statistically the simultaneous evolution of credit risk and market risk through a random bivariate vector, and to infer the dynamic characterizing jointly the evolution process of these two risks. Therefore, we focus on a statistical modeling technique of dependence structures known as the copulas theory.

Our paper is divided into four parts. Section 2 presents the data and consid-

[1] This relationship becomes evident when considering Merton's structural model.

ers the empirical link between credit spreads and a given stock index in order to observe the empirical relationship between credit risk and market risk. Section 3 explains the usefulness of copulas and their application in our study. Section 4 goes further while investigating results of section 3 and underlines the usefulness of copulas theory for stress testing. Finally, section 5 draws some concluding remarks and possibilities to use the link prevailing between credit risk and market risk for soundly assessing credit risk.

2 Empirical behavior

In this section, we introduce the considered data and some empirical facts characterizing such data.

2.1 Data

Our monthly data are issued from Bloomberg database and cover the time period going from April 1991 to November 2000 (i.e., we have 116 observations per series). First, we consider middle aggregate yields of risky bonds on the US financial market, which are sorted by rating, economic sector and maturity. Ratings are established by Moody's rating agency and range from AAA to BAA (i.e., we consider investment grade corporate bonds). Economic sectors concern four activities such as bank & finance, industry, power and telecommunications. Maturities range from one year to ten years in order to highlight some short term and long term behaviors. Second, we also take into account corresponding US Treasury yields. This allows us to compute our credit risk proxy, namely credit spreads[2] as the difference between corporate yields and corresponding Treasury yields. Third, we consider the Standard & Poor's composite index (S&P 500) as a stock index representative of market risk or systematic risk. The S&P 500 index[3] could be assimilated to some market portfolio given that this one encompasses five hundreds stocks.

Since credit spreads have some yield nature and in order to realize an homogeneous study, we choose to consider the monthly return of the S&P 500 index rather than its level. Therefore, we will jointly analyze the S&P 500 index return and credit spreads. Before, we shall mention that credit spreads are first integrated series (i.e., their first differences are stationary whereas their levels are not) while S&P 500 index return is stationary[4].

[2] We then have 116 series of credit spreads but we decide to exclude from our sample 3 and 4 years credit spreads. Indeed, 3 years credit spreads behave like 2 years credit spreads, and 4 years credit spreads behave like 5 years credit spreads. Finally, we consider a sample of 98 distinct credit spreads.

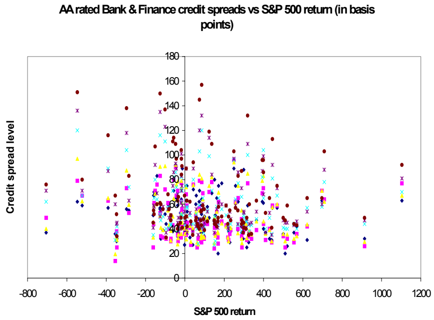
[3] Our market risk or systematic risk proxy here.

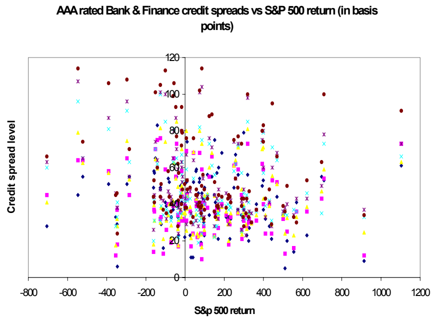
[4] We performed a Phillips & Perron (1988) statistical test which showed to be significant at a one percent level. Although statistical results are not provided in this paper, they remain available upon request.

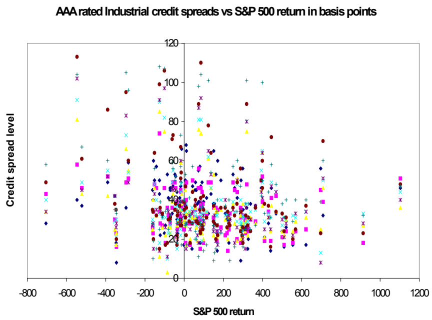
2.2 Graphical analysis

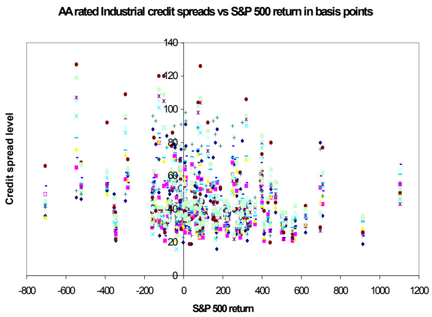
We would like to have an idea about the kind of dependence existing between credit spreads and S&P 500 index return. We then proceed in two steps to get some graphical and statistical insights. As a first step, we plot credit spreads series againts S&P 500 index return. As a second step, we compute some interesting statistics named Kendall's tau and Spearman's rho.

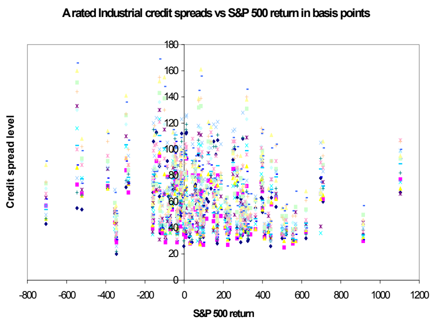
A graphical study will help us to observe the dispersion or, more precisely, the statistical distribution followed by the bidimensional random vectors composed of credit spreads and S&P 500 return. We plot underneath those graphs by tracing S&P 500 return against credit spreads of all available maturities for a given rating class and a given economic sector.

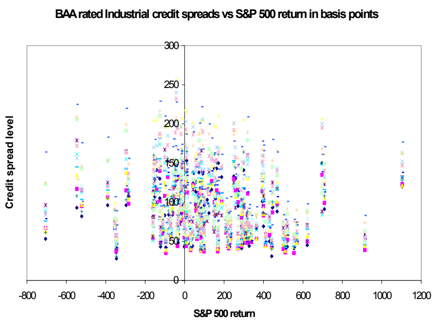


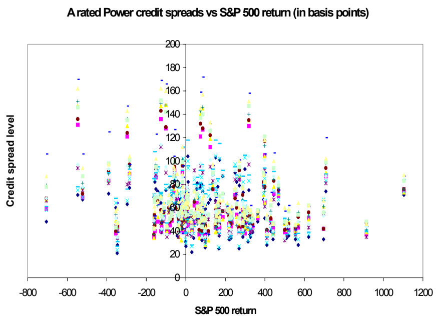


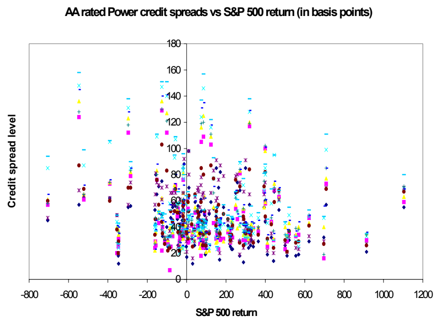


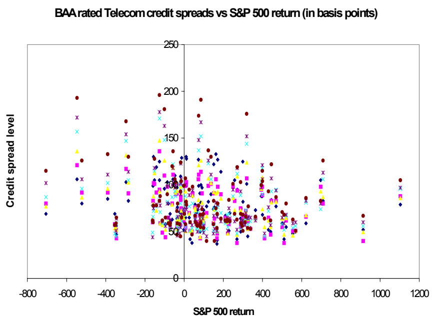


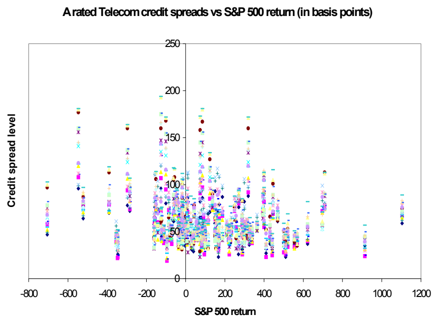


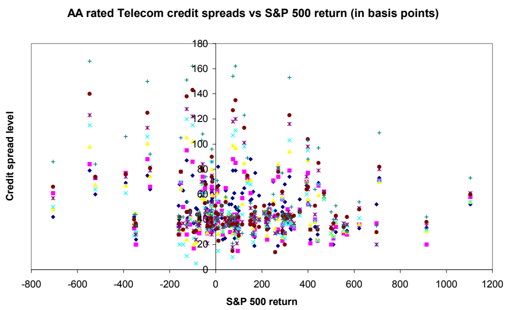












According to plots of S&P 500 index return against credit spreads, we notice two facts. First, among a same rating class, the dependence structure exhibits the same behavior whatever the rating. We only remark a slight difference in the distribution of extreme values. The lower the rating, the more important the magnitude of extreme values. Second, the dependence structure's behavior changes with the rating class. This fact is probably due to extreme values' distribution in each rating class. The better the rating of the class, the lower extreme values are and the more homogeneous credit spreads' distribution is.

Previous graphical representations of dependence help us to establish a visual link between credit risk and market risk. We notice that the joint distributions of S&P 500 return and credit spreads are clearly asymmetric. This relationship has to be confirmed through some empirical or non parametric statistics.

2.3 Preliminary analysis

Since credit spreads follow an asymmetric distribution (see Phoa [1999] for example), a dependence measure different from Pearson's linear correlation should be required. Indeed, such a measure is convenient only for elliptical distributions and becomes then biased in the asymmetric case for example[5].

To bypass such a problem, we use two concordance measures (i.e., two non parametric statistics) called Kendall's tau and Spearman's rho. Recall that a concordance measure is designed to characterize a positive as well as a negative dependence between variables whose joint distribution is non elliptical. This is currently the case of our bidimensional vectors composed of credit spreads and S&P 500 index return. For example, if we call c and d respectively the numbers of pairs of variables which are concordant and discordant, then Kendall's tau writes :

$$\tau = \frac{c-d}{c+d} = p_c - p_d \tag{1}$$

where p_c and p_d are respectively the probabilities of concordance and discordance[6].

We display underneath tables of Kendall's tau τ and Spearman's rho ϱ statistics values for each vector composed of a given credit spread and S&P 500 index return. We write $SXXnnYRRR$ for the credit spread characterized by XX sector, nn years maturity (nnY) and RRR rating. Economic sectors XX are indexed as follows : BF for bank & finance, IN for industry, PW for power and TL for telecommunications.

Spread	τ	ϱ	Spread	τ	ϱ
SBF01YAA2	-0,1313	-0,2030	SIN01YBAA1	-0,1419	-0,2212
SBF02YAA2	-0,0957	-0,1517	SIN02YBAA1	-0,1603	-0,2412
SBF05YAA2	-0,0960	-0,1509	SIN05YBAA1	-0,1221	-0,1786
SBF07YAA2	-0,1232	-0,1944	SIN07YBAA1	-0,1305	-0,1880
SBF10YAA2	-0,1093	-0,1736	SIN10YBAA1	-0,1229	-0,1882
SBF01YAAA	-0,1246	-0,1910	SIN01YBAA2	-0,1484	-0,2316
SBF02YAAA	-0,0868	-0,1357	SIN02YBAA2	-0,1393	-0,2199
SBF05YAAA	-0,0269	-0,0503	SIN05YBAA2	-0,0961	-0,1505
SBF07YAAA	-0,1500	-0,2337	SIN10YBAA2	-0,1201	-0,1768
SBF10YAAA	-0,1442	-0,2147			

[5] There is a growing debate about this topic in terms of coherency of risk measures. The interested reader could refer to Artzner et al. (1999, 2000) and Szego (2001) among others.

[6] Let $V_t = \begin{bmatrix} X_t \\ Y_t \end{bmatrix}$ be a vector of two random variables at time t. Then, two distinct observations V_t and V_s are said to be concordant if $(X_t - X_s)(Y_t - Y_s) > 0$. Conversely, if we have $(X_t - X_s)(Y_t - Y_s) < 0$, V_t and V_s are said to be discordant (i.e., negatively dependent).

Spread	τ	ϱ
SIN01YBAA3	-0,1523	-0,2400
SIN02YBAA3	-0,1336	-0,2132
SIN05YBAA3	-0,1257	-0,1884
SIN07YBAA3	-0,1391	-0,2009
SIN10YBAA3	-0,1614	-0,2364
SIN01YA1	-0,1499	-0,2372
SIN02YA1	-0,1644	-0,2557
SIN05YA1	-0,1210	-0,1836
SIN07YA1	-0,1228	-0,1972
SIN10YA1	-0,1651	-0,2473

Spread	τ	ϱ
SIN01YA2	-0,1474	-0,2295
SIN02YA2	-0,1368	-0,2058
SIN05YA2	-0,1074	-0,1665
SIN07YA2	-0,1312	-0,1965
SIN10YA2	-0,1394	-0,2065
SIN01YA3	-0,1430	-0,2218
SIN02YA3	-0,1374	-0,2106
SIN05YA3	-0,0913	-0,1402
SIN07YA3	-0,1161	-0,1739

Spread	τ	ϱ
SIN01YAA2	-0,1694	-0,2589
SIN02YAA2	-0,1385	-0,2227
SIN05YAA2	0,0063	-0,0206
SIN07YAA2	-0,1043	-0,1753
SIN10YAA2	-0,1598	-0,2433
SIN01YAA3	-0,1506	-0,2425
SIN02YAA3	-0,1278	-0,2083
SIN05YAA3	-0,0281	-0,0657
SIN07YAA3	-0,0896	-0,1546

Spread	τ	ϱ
SIN01YAAA	-0,1633	-0,2511
SIN02YAAA	-0,1320	-0,2126
SIN05YAAA	0,0105	-0,0144
SIN07YAAA	-0,1086	-0,1721
SIN10YAAA	-0,1043	-0,1580
STL01YBAA1	-0,1712	-0,2693
STL02YBAA1	-0,1380	-0,2148
STL05YBAA1	-0,0598	-0,0972
STL07YBAA1	-0,0760	-0,1287
STL10YBAA1	-0,0751	-0,1206

Spread	τ	ϱ
STL01YA1	-0,1545	-0,2399
STL02YA1	-0,1476	-0,2228
STL05YA1	-0,0639	-0,1138
STL07YA1	-0,1007	-0,1659
STL10YA1	-0,0784	-0,1374
STL01YA2	-0,1611	-0,2541
STL02YA2	-0,1500	-0,2303
STL05YA2	-0,0604	-0,1026
STL07YA2	-0,0735	-0,1215
STL10YA2	-0,0873	-0,1425

Spread	τ	ϱ
STL01YA3	-0,1566	-0,2474
STL02YA3	-0,1407	-0,2146
STL05YA3	-0,0565	-0,1037
STL07YA3	-0,0686	-0,1157
STL10YA3	-0,0829	-0,1352
STL01YAA3	-0,1241	-0,1968
STL02YAA3	-0,0529	-0,0987
STL05YAA3	0,0243	0,0034
STL07YAA3	-0,0449	-0,0865
STL10YAA3	-0,0395	-0,0845

Spread	τ	ϱ	Spread	τ	ϱ
SPW01YA1	-0,1231	-0,1962	SPW01YAA2	-0,1495	-0,2291
SPW05YA1	-0,0426	-0,0861	SPW05YAA2	0,0106	-0,0085
SPW07YA1	-0,1035	-0,1676	SPW07YAA2	-0,0657	-0,1142
SPW01YA2	-0,1137	-0,1868	SPW10YAA2	-0,0972	-0,1530
SPW02YA2	-0,1032	-0,1716	SPW01YAA3	-0,1265	-0,1938
SPW05YA2	-0,0503	-0,0892	SPW02YAA3	-0,1177	-0,1714
SPW07YA2	-0,0959	-0,1568	SPW05YAA3	-0,0270	-0,0545
SPW10YA2	-0,0809	-0,1304	SPW07YAA3	-0,0681	-0,1212
SPW01YA3	-0,1247	-0,1968	SPW10YAA3	-0,0765	-0,1241
SPW02YA3	-0,1210	-0,1874			
SPW05YA3	-0,0704	-0,1152			
SPW07YA3	-0,1262	-0,1873			

Generally speaking, we notice that correlations between credit spreads and S&P 500 return are significant. Results suggest that there is a relationship between credit risk and market risk, this link depending on the economic sector, rating and maturity. Moreover, most part of correlations are negative, which underlines the fact that a decrease in the S&P 500 return generates a widening of credit spreads, or conversely, an increase in the S&P 500 return generates a tightening of credit spreads[7].

We notice finally, at a graphical and empirical point of view, a relationship between credit spreads and S&P 500 return. This first insight exhibits the influence of market risk on credit risk according to Kendall's tau and Spearman's rho statistics. Recall that these statistics are particular cases of copulas' application[8]. Indeed, the copula notion extends the concordance notion to the continuous framework. Moreover, even if the two concordance measures above-mentionned allow to measure the association degree between two random variables, these statistics provide less information than a copula[9].

3 Copula formulation

The previous section used a particular application setting of the copula notion which is a more general dependence structure's measure. In this section, we will apply this concept to study the prevailing link between market risk and credit risk. Recall that, along with risk measures' coherency, copula is a

[7] As a comparison, we also performed a Granger causality test of S&P 500 return on credit spreads' first differences (to set a stationary universe). We get poor results in so far as S&P 500 return causes, at a five percent level, only 13 credit spreads : SBF02YAA2, SIN01YBAA1, SIN01YBAA3, SIN02YBAA1, SIN02YBAA2, SIN02YBAA3, SIN05YBAA3, SIN10YBAA1, SIN02YA3, SPW02YA2, SPW02YA3, SPW05YA1, and SPW02YAA3. Statistics are available upon request.

[8] A specific type of copula called Archimedean copula allows to compute easily non parametric statistics such as Kendall's tau and Spearman's rho. For more details, the reader is invited to refer to Nelsen (1999) and Roncalli's (2002) teaching manuscript among others.

[9] In fact, copulas also take into account phenomena due to dependent extremal events.

usefull tool since any type of bivariate (or multivariate) distribution could be characterized. Such a property is important in so far as the two first moments are insufficient to describe a multivariate distribution when this one is not elliptical.

3.1 Theoretical framework

According to Sklar (1959, 1973), each bivariate distribution could be given a copula which may be unique (i.e., continuous case). A copula gives an exhaustive description of the dependence which may take place between the distribution's marginals. Indeed, a copula could be seen as a function of a bivariate (i.e., multivariate) distribution's marginals. This function $C(u_1, u_2)$ is assumed to be continuous, non decreasing and bounded between zero and one among others (for all $(u_1, u_2) \in [0,1]^2$).

Copula's theory is the result of Fréchet classes' study in statistics. Such a study allows to define a partial order called concordance order and corresponding to the first order stochastic dominance for cumulative distribution functions (see Fréchet [1951] and Joe [1997] for example). Moreover, a copula is invariant through strictly increasing transformations of random variables, which could be usefull to study some transformations of random vectors. According to Nelsen (1999), there are three types of copula, namely absolute continuous copula, singular copula which has no density function, and finally, mixed copula which is composed of both an absolute continuous component and a singular component.

Such properties allow to define and build dependence statistics such as concordance measures, among which Kendall's tau τ and Spearman's rho ϱ. Indeed, let $C(u_1, u_2)$ be the copula[10] associated to our bivariate random vector $\begin{bmatrix} X_t \\ Y_t \end{bmatrix}$. Then the previous statistics write :

$$\begin{aligned} \tau &= 4 \iint_{[0,1]^2} C(u_1, u_2)\, dC(u_1, u_2) - 1 \qquad (2) \\ &= P\left((X_t - X_s)(Y_t - Y_s) > 0\right) - P\left((X_t - X_s)(Y_t - Y_s) < 0\right) \\ &= p_c - p_d \end{aligned}$$

and

$$\begin{aligned} \varrho &= 12 \iint_{[0,1]^2} u_1 u_2\, dC(u_1, u_2) - 3 \qquad (3) \\ &= Correlation\left(F_X(X), F_Y(Y)\right) \end{aligned}$$

where F_X and F_Y are the marginals of the distributions associated to (X) and (Y) random variables respectively. Recall that we have $\tau,\ \varrho \in [-1, 1]$ (see Schweizer & Wolff [1981] for details).

[10] We could equivalently write $C(F_X(x), F_Y(y)) = F(x, y)$ or $C(u_1, u_2) = F\left(F_X^{-1}(u_1), F_Y^{-1}(u_2)\right)$ for (x, y) belonging to the random vector's definition set and $(u_1, u_2) \in [0,1]^2$ respectively. $F(x, y)$ is assumed to be the joint cumulative distribution function of our random vector.

To go further, copulas theory allows interestingly to define tails dependence. Indeed, let $\lambda(\alpha)$ be a quantile-quantile dependence measure[11] as follows, for all $\alpha \in [0,1[$:

$$\lambda(\alpha) = P\left(Y > F_Y^{-1}(\alpha) \mid X > F_X^{-1}(\alpha)\right) = \frac{1 - 2\alpha + C(\alpha,\alpha)}{1-\alpha} \tag{4}$$

Then,the upper tail dependence measure is $\lambda_U = \lim_{\alpha \to 1^-} \lambda(\alpha)$ and the lower tail dependence measure writes $\lambda_L = \lim_{\alpha \to 0} \lambda(\alpha)$ given that $\lambda(\alpha) \in [0,1]$. When $\lambda_{U,L} = 0$ or $\lambda_{U,L} = 1$, there is respectively no tail dependence or a perfect tail dependence.

For parametric copulas such as Archimedean copulas (see Genest & MacKay [1986] among others), theory shows that tail dependence measures and concordance measures often reduce to simple parametric expressions[12]. The number of parameters could be more than one to better calibrate this kind of copula along with the empirical dependence structure under consideration.

3.2 Copula choice and estimation method

According to Deheuvels (1981), each multivariate distribution has at least one associated copula function. This copula function is unique when marginals are continuous. Here, we assume that our bivariate vectors of S&P 500 return and credit spreads exhibit continuous marginals so that each copula is defined in a unique way. Rather than specifying given marginals *ex ante* and inducing then the related copula function, we prefer to specify a given copula function consistent with the studied empirical dependence structure. Indeed, we analyze a set of bivariate random vectors composed both of a stationary distribution variable (i.e., S&P500 return) and an asymmetric distribution variable (i.e., credit spreads). Now, it seems hard to characterize exactly each distribution law of such vectors' variables. Moreover, Durrleman, Nikeghbali & Roncalli (2000) show that a misspecification of marginals leads to a biased estimation of the copula function. Such a problem leads us to directly consider the copula function without specifying any marginal law.

To be able to fit conveniently estimated copulas, we choose to use mostly one parameter Archimedean copulas. Specifically, our choice restricts to the following copulas[13] : Gumbel, FGM (i.e., Farlie-Gumbel-Morgenstern[14]), Frank

[11] This is a non signed measure.

[12] For example, it is showed that a normal copula does not allow correlation between extreme values. Moreover, Gumbel copula is the only one to be both an extreme values copula and an Archimedean copula. To go further, there also exists a copulas family which encompasses both extreme values copulas and Archimedean ones. This family, which was introduced by Capéraà, Fougères & Genest (2000), is called Archimax copulas.

[13] We did not select the Normal copula since this one does not allow extreme values' correlation. Moreover, we exluded the logistic Gumbel copula since this one does not depend on a parameter and is such that $\tau = \frac{1}{3}$, which does not correspond to our sample's observed empirical values for τ. On another hand, we also excluded Fréchet and Weibull copulas since these copulas only take into account positive extremes and negative extremes respectively.

[14] This function is a one parameter copula which does not belong to the Archimedean family.

and Clayton. These copula functions exhibit the nice following property : each copula has an analytical expression such that its related Kendall's tau is a function of its parameter θ. This simplified framework leads us to an estimation method of copulas based on dependence measures: our dependence measure is Kendall's tau τ statistic here. Indeed, since we have the expression of τ as a function of θ for each copula, and we know the empirical value of τ, we are able to invert these expressions to induce θ parameter's value. Namely, we have :

$$\tau = f(\theta) \tag{5}$$

where τ is observed here and $f(.)$ is the related function. Sometimes we are able to compute directly the parameter's value given a value of Kendall's tau such as $\theta = f^{-1}(\tau)$. When it is not the case, we have to solve numerically the equation above to get the parameter's value.

We recall underneath the expressions of Gumbel, FGM, Frank and Clayton copulas, and we also give the expressions of τ as a function of θ (with $\theta \in \mathbb{R}$ or a subset of $\mathbb{R}$).

Copula	$C(u_1, u_2; \theta)$	τ
Gumbel	$\exp\left\{-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{\frac{1}{\theta}}\right\}$	$\frac{\theta-1}{\theta}$
FGM	$u_1 u_2 [1 + \theta(1-u_1)(1-u_2)]$	$\frac{2\theta}{9}$
Frank	$-\frac{1}{\theta}\ln\left\{1 + \frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right\}$	$1 - \frac{4}{\theta}[1 - D_1(\theta)]$
Clayton	$\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}$	$\frac{\theta}{\theta+2}$

Notice that $D_1(\theta)$ is the Debye function of Abramowitz & Stegun (1970) such that $D_n(x) = \frac{n}{x^n}\int_0^x \frac{t^n}{e^t-1}dt$ for n positive integer. We then find the following expressions for each copula's parameter[15].

Copula	θ
Gumbel	$\frac{1}{1-\tau}$
FGM	$\frac{9\tau}{2}$
Frank	*Solution of* $1 - \tau - \frac{4}{\theta}[1 - D_1(\theta)] = 0$
Clayton	$\frac{2\tau}{1-\tau}$

We apply this estimation method in the following section.

3.3 Application

In this part, we give the results related to copulas parameters' estimation based on dependence measures. For Gumbel, FGM and Clayton copulas, computation is simple and immediate whereas we have to solve equation (5) for

[15] Given that $\tau \in [-1, 1]$ and given the copulas under consideration, it is immediate to deduce the range of values for θ. For Frank copula, we have $\theta \in \mathbb{R}^*$ whereas $\theta > 0$ for Clayton copula. For Gumbel and FGM copulas, we have $\theta \geq \frac{1}{2}$ and $\theta \in [-1, 1]$ respectively.

Frank copula. In Frank copula's case, numerical resolution of equation (5) relative to parameter θ is achieved using a SQR algorithm (see Wang & Tewarson [1993] for details) which is a quasi-Gauss-Newton algorithm[16].

We display in tables underneath the different values of θ for each copula. Fisrt, we give estimations related to Gumbel and FGM copulas. Then, we give results associated to Frank and Clayton copulas.

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SBF01YAA2	0,8840	-0,5907	SIN01YBAA1	0,8758	-0,6384
SBF02YAA2	0,9127	-0,4304	SIN02YBAA1	0,8618	-0,7215
SBF05YAA2	0,9124	-0,4322	SIN05YBAA1	0,8912	-0,5495
SBF07YAA2	0,8903	-0,5543	SIN07YBAA1	0,8846	-0,5873
SBF10YAA2	0,9015	-0,4919	SIN10YBAA1	0,8906	-0,5530
SBF01YAAA	0,8892	-0,5605	SIN01YBAA2	0,8708	-0,6676
SBF02YAAA	0,9201	-0,3906	SIN02YBAA2	0,8777	-0,6268
SBF05YAAA	0,9738	-0,1212	SIN05YBAA2	0,9123	-0,4325
SBF07YAAA	0,8696	-0,6748	SIN10YBAA2	0,8927	-0,5406
SBF10YAAA	0,8739	-0,6491			

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SIN01YBAA3	0,8678	-0,6855	SIN01YA2	0,8716	-0,6632
SIN02YBAA3	0,8822	-0,6010	SIN02YA2	0,8797	-0,6154
SIN05YBAA3	0,8883	-0,5657	SIN05YA2	0,9030	-0,4833
SIN07YBAA3	0,8779	-0,6257	SIN07YA2	0,8840	-0,5904
SIN10YBAA3	0,8610	-0,7263	SIN10YA2	0,8777	-0,6271
SIN01YA1	0,8697	-0,6745	SIN01YA3	0,8749	-0,6436
SIN02YA1	0,8588	-0,7397	SIN02YA3	0,8792	-0,6182
SIN05YA1	0,8921	-0,5444	SIN05YA3	0,9163	-0,4109
SIN07YA1	0,8906	-0,5526	SIN07YA3	0,8960	-0,5224
SIN10YA1	0,8583	-0,7431			

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SIN01YAA2	0,8551	-0,7624	SIN01YAAA	0,8596	-0,7349
SIN02YAA2	0,8783	-0,6233	SIN02YAAA	0,8834	-0,5938
SIN05YAA2	1,0063	0,0281	SIN05YAAA	1,0106	0,0470
SIN07YAA2	0,9055	-0,4696	SIN07YAAA	0,9020	-0,4888
SIN10YAA2	0,8622	-0,7191	SIN10YAAA	0,9055	-0,4696
SIN01YAA3	0,8691	-0,6776	STL01YBAA1	0,8538	-0,7706
SIN02YAA3	0,8866	-0,5753	STL02YBAA1	0,8787	-0,6209
SIN05YAA3	0,9727	-0,1263	STL05YBAA1	0,9436	-0,2691
SIN07YAA3	0,9177	-0,4033	STL07YBAA1	0,9293	-0,3422
			STL10YBAA1	0,9301	-0,3381

[16] The convergence criterion applied here is of 10^{-5} so that observed errors are at least of -8.6748×10^{-8} and at most of 2.2430×10^{-6}.

Spread	Gumbel	FGM	Spread	Gumbel	FGM
STL01YA1	0,8661	-0,6954	STL01YA3	0,8646	-0,7047
STL02YA1	0,8714	-0,6642	STL02YA3	0,8766	-0,6333
STL05YA1	0,9399	-0,2876	STL05YA3	0,9465	-0,2543
STL07YA1	0,9085	-0,4531	STL07YA3	0,9358	-0,3086
STL10YA1	0,9273	-0,3529	STL10YA3	0,9234	-0,3731
STL01YA2	0,8613	-0,7249	STL01YAA3	0,8896	-0,5585
STL02YA2	0,8696	-0,6748	STL02YAA3	0,9498	-0,2379
STL05YA2	0,9430	-0,2719	STL05YAA3	1,0249	0,1092
STL07YA2	0,9315	-0,3309	STL07YAA3	0,9571	-0,2018
STL10YA2	0,9197	-0,3927	STL10YAA3	0,9620	-0,1778

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SPW01YA1	0,8904	-0,5540	SPW01YAA2	0,8699	-0,6728
SPW05YA1	0,9592	-0,1915	SPW05YAA2	1,0107	0,0477
SPW07YA1	0,9062	-0,4658	SPW07YAA2	0,9384	-0,2955
SPW01YA2	0,8979	-0,5114	SPW10YAA2	0,9114	-0,4373
SPW02YA2	0,9065	-0,4644	SPW01YAA3	0,8877	-0,5695
SPW05YA2	0,9521	-0,2262	SPW02YAA3	0,8947	-0,5296
SPW07YA2	0,9125	-0,4315	SPW05YAA3	0,9737	-0,1215
SPW10YA2	0,9251	-0,3642	SPW07YAA3	0,9362	-0,3065
SPW01YA3	0,8891	-0,5612	SPW10YAA3	0,9289	-0,3443
SPW02YA3	0,8921	-0,5444			
SPW05YA3	0,9342	-0,3168			
SPW07YA3	0,8880	-0,5677			

Spread	Frank	Clayton	Spread	Frank	Clayton
SBF01YAA2	-1,1983	-0,2321	SIN01YBAA1	-1,2982	-0,2485
SBF02YAA2	-0,8673	-0,1746	SIN02YBAA1	-1,4739	-0,2764
SBF05YAA2	-0,8708	-0,1752	SIN05YBAA1	-1,1126	-0,2177
SBF07YAA2	-1,1225	-0,2194	SIN07YBAA1	-1,1911	-0,2309
SBF10YAA2	-0,9934	-0,1971	SIN10YBAA1	-1,1197	-0,2189
SBF01YAAA	-1,1354	-0,2215	SIN01YBAA2	-1,3596	-0,2584
SBF02YAAA	-0,7860	-0,1597	SIN02YBAA2	-1,2737	-0,2445
SBF05YAAA	-0,2425	-0,0524	SIN05YBAA2	-0,8715	-0,1754
SBF07YAAA	-1,3748	-0,2608	SIN10YBAA2	-1,0941	-0,2145
SBF10YAAA	-1,3205	-0,2521			

Spread	Frank	Clayton
SIN01YBAA3	-1,3973	-0,2644
SIN02YBAA3	-1,2198	-0,2357
SIN05YBAA3	-1,1461	-0,2233
SIN07YBAA3	-1,2715	-0,2442
SIN10YBAA3	-1,4841	-0,2779
SIN01YA1	-1,3741	-0,2607
SIN02YA1	-1,5127	-0,2823
SIN05YA1	-1,1019	-0,2158
SIN07YA1	-1,1190	-0,2188
SIN10YA1	-1,5201	-0,2835

Spread	Frank	Clayton
SIN01YA2	-1,3502	-0,2569
SIN02YA2	-1,2499	-0,2406
SIN05YA2	-0,9757	-0,1940
SIN07YA2	-1,1975	-0,2320
SIN10YA2	-1,2744	-0,2446
SIN01YA3	-1,3090	-0,2503
SIN02YA3	-1,2557	-0,2416
SIN05YA3	-0,8273	-0,1673
SIN07YA3	-1,0564	-0,2080

Spread	Frank	Clayton
SIN01YAA2	-1,5612	-0,2897
SIN02YAA2	-1,2665	-0,2433
SIN05YAA2	0,0563	0,0126
SIN07YAA2	-0,9475	-0,1890
SIN10YAA2	-1,4688	-0,2756
SIN01YAA3	-1,3806	-0,2617
SIN02YAA3	-1,1661	-0,2267
SIN05YAA3	-0,2528	-0,0546
SIN07YAA3	-0,8119	-0,1645

Spread	Frank	Clayton
SIN01YAAA	-1,5025	-0,2808
SIN02YAAA	-1,2047	-0,2332
SIN05YAAA	0,0941	0,0211
SIN07YAAA	-0,9870	-0,1960
SIN10YAAA	-0,9475	-0,1890
STL01YBAA1	-1,5789	-0,2924
STL02YBAA1	-1,2614	-0,2425
STL05YBAA1	-0,5398	-0,1129
STL07YBAA1	-0,6877	-0,1413
STL10YBAA1	-0,6793	-0,1398

Spread	Frank	Clayton
STL01YA1	-1,4184	-0,2677
STL02YA1	-1,3524	-0,2572
STL05YA1	-0,5772	-0,1202
STL07YA1	-0,9137	-0,1830
STL10YA1	-0,7093	-0,1454
STL01YA2	-1,4812	-0,2775
STL02YA2	-1,3748	-0,2608
STL05YA2	-0,5453	-0,1139
STL07YA2	-0,6647	-0,1370
STL10YA2	-0,7902	-0,1605

Spread	Frank	Clayton
STL01YA3	-1,4381	-0,2708
STL02YA3	-1,2873	-0,2467
STL05YA3	-0,5100	-0,1070
STL07YA3	-0,6195	-0,1283
STL10YA3	-0,7504	-0,1531
STL01YAA3	-1,1311	-0,2208
STL02YAA3	-0,4768	-0,1004
STL05YAA3	0,2184	0,0497
STL07YAA3	-0,4043	-0,0859
STL10YAA3	-0,3561	-0,0760

Spread	Frank	Clayton	Spread	Frank	Clayton
SPW01YA1	-1,1218	-0,2192	SPW01YAA2	-1,3705	-0,2601
SPW05YA1	-0,3836	-0,0817	SPW05YAA2	0,0954	0,0214
SPW07YA1	-0,9398	-0,1876	SPW07YAA2	-0,5932	-0,1233
SPW01YA2	-1,0337	-0,2041	SPW10YAA2	-0,8814	-0,1771
SPW02YA2	-0,9369	-0,1871	SPW01YAA3	-1,1539	-0,2247
SPW05YA2	-0,4533	-0,0957	SPW02YAA3	-1,0713	-0,2106
SPW07YA2	-0,8694	-0,1750	SPW05YAA3	-0,2432	-0,0526
SPW10YA2	-0,7323	-0,1497	SPW07YAA3	-0,6154	-0,1275
SPW01YA3	-1,1368	-0,2218	SPW10YAA3	-0,6918	-0,1421
SPW02YA3	-1,1019	-0,2158			
SPW05YA3	-0,6362	-0,1315			
SPW07YA3	-1,1504	-0,2241			

First, we notice that θ estimations for Clayton copula are mostly inconsistent with assumptions about this copula's parameter. Therefore, we will consider this copula function only for credit spreads for which estimations are compatible. Namely, Clayton copula will be taken into account for SIN05YAA2, SIN05YAAA, STL05YAA3 and SPW05YAA2 credit spreads. Once we have estimated θ parameters associated to each copula function, it seems natural to choose the most convenient copula type for each random vector. This selection problem is highlighted in the following section.

4 Selection process and practical use

In this section, we address the problem of choosing the optimal copula function related to each random studied bivariate vector. Furthermore, we underline some usefull application of the copula methodology such as scenario analysis.

4.1 Optimal selection

We have a set of copulas at our disposal, this set being aimed at describing bivariate dependence structures of S&P 500 return and different credit spreads. Now, our goal is to discriminate between these copula functions in order to choose the most convenient one given the dependence structures we study. We face a selection problem which could be solved through the introduction of some measure. Intuitively, we will choose the copula type function which lies the closest to the empirical dependence structure. Namely, the optimal copula is the function which minimizes the observed errors relatively to the empirical copula function.

The principle for selecting the optimal copula is simple. For this purpose, we introduce the distance of the discrete L^2 norm. Namely, let $C(u_1, u_2)$ be a theoretical copula function belonging to our copulas' set $\mathcal{C} = \{$Gumbel, FGM, Frank, Clayton$\}$ and let $\hat{C}(u_1, u_2)$ be the empirical copula function (i.e., estimated on the observed data using the methodology of Deheuvels [1981]). Then, the distance $\hat{d}_2\left(C,\hat{C}\right)$ of the discrete L^2 norm is defined as follows :

$$\hat{d}_2\left(C,\hat{C}\right)=\left\{\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\left[C\left(\frac{t_1}{T},\frac{t_2}{T}\right)-\hat{C}\left(\frac{t_1}{T},\frac{t_2}{T}\right)\right]^2\right\}^{\frac{1}{2}} \tag{6}$$

where $C\in\mathcal{C}$ and T corresponds to the number of observations (i.e., 115 in our sample). Therefore, the optimal copula function C^* describing the dependence structures we study, given our copulas' set $\mathcal{C}$, has to satisfy the next condition :

$$C^*=\min_{C\in\mathcal{C}}\left\{\frac{\hat{d}_2\left(C,\hat{C}\right)}{T}\right\} \tag{7}$$

The optimal copula function minimizes then the average observed error given the studied empirical dependence structure. We apply the discrete L^2 norm distance for each credit spread and display our results $\frac{\hat{d}_2(C,\hat{C})}{T}$ (i.e., the average distance) in the tables underneath for each copula type function.

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SBF01YAA2	1,6046E-02	1,4157E-02	SIN01YBAA1	1,5364E-02	1,4094E-02
SBF02YAA2	1,3132E-02	1,2188E-02	SIN02YBAA1	1,3832E-02	1,0804E-02
SBF05YAA2	1,5746E-02	1,4790E-02	SIN05YBAA1	1,2736E-02	1,0497E-02
SBF07YAA2	1,9632E-02	1,7538E-02	SIN07YBAA1	1,5845E-02	1,3077E-02
SBF10YAA2	2,0464E-02	1,8828E-02	SIN10YBAA1	1,4917E-02	1,2696E-02
SBF01YAAA	1,5840E-02	1,3701E-02	SIN01YBAA2	1,6005E-02	1,4082E-02
SBF02YAAA	1,5961E-02	1,5169E-02	SIN02YBAA2	1,4242E-02	1,1356E-02
SBF05YAAA	2,1468E-02	2,1468E-02	SIN05YBAA2	1,3176E-02	1,1817E-02
SBF07YAAA	1,9590E-02	1,6033E-02	SIN10YBAA2	1,4299E-02	1,2115E-02
SBF10YAAA	1,8670E-02	1,6047E-02			

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SIN01YBAA3	1,4529E-02	1,2671E-02	SIN01YA2	1,4354E-02	1,2631E-02
SIN02YBAA3	1,3885E-02	1,1980E-02	SIN02YA2	1,2726E-02	1,0875E-02
SIN05YBAA3	1,3357E-02	1,0718E-02	SIN05YA2	1,3344E-02	1,2128E-02
SIN07YBAA3	1,6018E-02	1,3120E-02	SIN07YA2	1,4754E-02	1,2489E-02
SIN10YBAA3	1,4755E-02	1,1578E-02	SIN10YA2	1,5603E-02	1,3946E-02
SIN01YA1	1,3254E-02	1,2233E-02	SIN01YA3	1,6228E-02	1,4396E-02
SIN02YA1	1,2565E-02	1,0422E-02	SIN02YA3	1,2670E-02	1,0378E-02
SIN05YA1	1,4377E-02	1,4196E-02	SIN05YA3	1,4700E-02	1,3881E-02
SIN07YA1	1,5079E-02	1,4089E-02	SIN07YA3	1,4933E-02	1,3303E-02
SIN10YA1	1,4664E-02	1,2532E-02			

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SIN01YAA2	1,4457E-02	1,2486E-02	SIN01YAAA	1,4936E-02	1,2718E-02
SIN02YAA2	1,4626E-02	1,3624E-02	SIN02YAAA	1,8713E-02	1,6643E-02
SIN05YAA2	2,3574E-02	2,3542E-02	SIN05YAAA	2,3878E-02	2,3865E-02
SIN07YAA2	1,5954E-02	1,5118E-02	SIN07YAAA	1,9452E-02	1,7833E-02
SIN10YAA2	1,7463E-02	1,5525E-02	SIN10YAAA	1,8152E-02	1,7402E-02
SIN01YAA3	1,5274E-02	1,3874E-02	STL01YBAA1	1,2659E-02	1,0884E-02
SIN02YAA3	1,4784E-02	1,3738E-02	STL02YBAA1	1,3003E-02	1,3264E-02
SIN05YAA3	1,9138E-02	1,9259E-02	STL05YBAA1	1,8418E-02	1,8015E-02
SIN07YAA3	1,8956E-02	1,8168E-02	STL07YBAA1	1,9066E-02	1,8337E-02
			STL10YBAA1	2,0934E-02	2,0801E-02

Spread	Gumbel	FGM	Spread	Gumbel	FGM
STL01YA1	1,6301E-02	1,3721E-02	STL01YA3	1,3893E-02	1,1955E-02
STL02YA1	1,5503E-02	1,4360E-02	STL02YA3	1,3036E-02	1,1957E-02
STL05YA1	2,0081E-02	1,9853E-02	STL05YA3	2,0410E-02	2,0264E-02
STL07YA1	1,8664E-02	1,7751E-02	STL07YA3	2,0033E-02	1,9530E-02
STL10YA1	2,0372E-02	2,0013E-02	STL10YA3	1,8347E-02	1,7714E-02
STL01YA2	1,4154E-02	1,2186E-02	STL01YAA3	1,5403E-02	1,4165E-02
STL02YA2	1,3446E-02	1,2368E-02	STL02YAA3	1,5124E-02	1,5556E-02
STL05YA2	1,9571E-02	1,9294E-02	STL05YAA3	2,0040E-02	2,0004E-02
STL07YA2	1,9364E-02	1,8784E-02	STL07YAA3	2,0484E-02	2,0426E-02
STL10YA2	1,9907E-02	1,9458E-02	STL10YAA3	2,0901E-02	2,0822E-02

Spread	Gumbel	FGM	Spread	Gumbel	FGM
SPW01YA1	1,8053E-02	1,6643E-02	SPW01YAA2	1,7583E-02	1,4813E-02
SPW05YA1	1,7161E-02	1,7173E-02	SPW05YAA2	1,9162E-02	1,9126E-02
SPW07YA1	1,7831E-02	1,6573E-02	SPW07YAA2	2,1264E-02	2,0743E-02
SPW01YA2	1,8917E-02	1,7357E-02	SPW10YAA2	1,9010E-02	1,8039E-02
SPW02YA2	1,3897E-02	1,2683E-02	SPW01YAA3	2,0063E-02	1,8176E-02
SPW05YA2	1,7754E-02	1,7601E-02	SPW02YAA3	1,5662E-02	1,4114E-02
SPW07YA2	1,9489E-02	1,8721E-02	SPW05YAA3	1,6440E-02	1,6418E-02
SPW10YA2	1,8467E-02	1,7893E-02	SPW07YAA3	1,9778E-02	1,9177E-02
SPW01YA3	1,7655E-02	1,5742E-02	SPW10YAA3	1,7768E-02	1,7243E-02
SPW02YA3	1,4468E-02	1,2407E-02			
SPW05YA3	1,7470E-02	1,6994E-02			
SPW07YA3	1,7550E-02	1,5302E-02			

Spread	Frank	Spread	Frank	Spread	Frank
SBF01YAA2	1,4159E-02	SIN01YBAA1	1,4204E-02	SIN01YBAA3	1,2846E-02
SBF02YAA2	1,2131E-02	SIN02YBAA1	1,0685E-02	SIN02YBAA3	1,2039E-02
SBF05YAA2	1,4737E-02	SIN05YBAA1	1,0259E-02	SIN05YBAA3	1,0581E-02
SBF07YAA2	1,7589E-02	SIN07YBAA1	1,2899E-02	SIN07YBAA3	1,2949E-02
SBF10YAA2	1,8856E-02	SIN10YBAA1	1,2535E-02	SIN10YBAA3	1,1327E-02
SBF01YAAA	1,3606E-02	SIN01YBAA2	1,4197E-02	SIN01YA1	1,2280E-02
SBF02YAAA	1,5106E-02	SIN02YBAA2	1,1341E-02	SIN02YA1	1,0227E-02
SBF05YAAA	2,1466E-02	SIN05YBAA2	1,1723E-02	SIN05YA1	1,3987E-02
SBF07YAAA	1,6024E-02	SIN10YBAA2	1,1940E-02	SIN07YA1	1,3977E-02
SBF10YAAA	1,5916E-02			SIN10YA1	1,2264E-02

Spread	Frank	Spread	Frank	Spread	Frank
SIN01YA2	1,2659E-02	SIN01YAA2	1,2483E-02	SIN01YAAA	1,2597E-02
SIN02YA2	1,0751E-02	SIN02YAA2	1,3464E-02	SIN02YAAA	1,6457E-02
SIN05YA2	1,2018E-02	SIN05YAA2	2,3542E-02	SIN05YAAA	2,3865E-02
SIN07YA2	1,2335E-02	SIN07YAA2	1,5078E-02	SIN07YAAA	1,7776E-02
SIN10YA2	1,3737E-02	SIN10YAA2	1,5372E-02	SIN10YAAA	1,7327E-02
SIN01YA3	1,4422E-02	SIN01YAA3	1,3966E-02	STL01YBAA1	1,0850E-02
SIN02YA3	1,0229E-02	SIN02YAA3	1,3626E-02	STL02YBAA1	1,3159E-02
SIN05YA3	1,3769E-02	SIN05YAA3	1,9256E-02	STL05YBAA1	1,8006E-02
SIN07YA3	1,3175E-02	SIN07YAA3	1,8159E-02	STL07YBAA1	1,8339E-02
				STL10YBAA1	2,0805E-02

Spread	Frank	Spread	Frank	Spread	Frank
STL01YA1	1,3560E-02	STL01YAA3	1,4052E-02	SPW05YA3	1,6951E-02
STL02YA1	1,3995E-02	STL02YAA3	1,5512E-02	SPW07YA3	1,5169E-02
STL05YA1	1,9820E-02	STL05YAA3	2,0000E-02	SPW01YAA2	1,4697E-02
STL07YA1	1,7734E-02	STL07YAA3	2,0421E-02	SPW05YAA2	1,9125E-02
STL10YA1	2,0009E-02	STL10YAA3	2,0822E-02	SPW07YAA2	2,0748E-02
STL01YA2	1,2124E-02	SPW01YA1	1,6756E-02	SPW10YAA2	1,7984E-02
STL02YA2	1,2095E-02	SPW05YA1	1,7160E-02	SPW01YAA3	1,8174E-02
STL05YA2	1,9266E-02	SPW07YA1	1,6533E-02	SPW02YAA3	1,3932E-02
STL07YA2	1,8774E-02	SPW01YA2	1,7443E-02	SPW05YAA3	1,6411E-02
STL10YA2	1,9436E-02	SPW02YA2	1,2575E-02	SPW07YAA3	1,9175E-02
STL01YA3	1,1885E-02	SPW05YA2	1,7582E-02	SPW10YAA3	1,7204E-02
STL02YA3	1,1720E-02	SPW07YA2	1,8671E-02		
STL05YA3	2,0247E-02	SPW10YA2	1,7862E-02		
STL07YA3	1,9526E-02	SPW01YA3	1,5758E-02		
STL10YA3	1,7684E-02	SPW02YA3	1,2228E-02		

Moreover, when Clayton copula is consistent, we have the following values :

Spread	$\frac{\hat{d}_2(C,\hat{C})}{T}$	Spread	$\frac{\hat{d}_2(C,\hat{C})}{T}$
SIN05YAA2	2,3524E-02	STL05YAA3	2,0241E-02
SIN05YAAA	2,3855E-02	SPW05YAA2	1,9029E-02

Results show that for most part of the studied dependence structures the optimal copula function is the Frank one. Exceptions are listed below (i.e., 23 dependence structures C^* follow another copula type function) :

- Gumbel : STL02YBAA1, STL02YAA3, SIN05YAA3 ;
- FGM : STL07YBAA1, STL10YBAA1, SIN01YBAA1, SIN01YBAA2, SIN01YBAA3, SIN02YBAA3, SIN01YA1, SIN01YA2, SIN01YA3, SIN01YAA3, SPW01YA1, SPW01YA2, SPW01YA3, SPW07YAA2, SBF01YAA2, SBF07YAA2, SBF10YAA2;
- Clayton : SIN05YAA2, SIN05YAAA, SPW05YAA2.

Generally speaking[17], we could conclude that the general dependence structure prevailing between S&P 500 return and credit spreads corresponds to a Frank copula function whose parameter value depends on the economic sector, rating and maturity.

4.2 Stress testing

In this subsection, we give some illustrations for using copulas as risk estimation tools. Knowing now the optimal copulas characterizing our studied dependence structures, we could use such a characterization to realize a scenario analysis.

We are interested in scenarii which could be unfavourable to credit risky assets or credit risk. To address this problem, we have to consider spreads' widening settings. Moreover, we would like to quantify the impact of systematic risk on credit risk. Following this point of view, we have to take into account spreads' widening situations due to an increase in systematic risk. We choose to represent such a risk increase by a decrease in S&P 500 return. Finally, since our aim is to quantify the influence of systematic risk on credit risk, for each bivariate vector of S&P 500 return and credit spread, we consider the probability that the considered credit spread increases given that the return of the stock market index decreases[18]. In practice, we consider the following probability :

$$\begin{aligned} P\left(U_2 > u_2 \mid U_1 \leq u_1\right) &= P\left(X_2 > x_2 \mid X_1 \leq x_1\right) \\ &= \frac{P\left(U_2 > u_2, U_1 \leq u_1\right)}{P\left(U_1 \leq u_1\right)} = \frac{u_1 - C^*\left(u_1, u_2; \theta\right)}{u_1} \end{aligned} \tag{8}$$

[17] In 76,53% of cases.

[18] For the reverse causal relationship (i.e., the influence of credit risk on systematic risk or market risk), we would have considered the probability that S&P 500 return tightens given that our credit spread widens.

where $(u_1, u_2) \in [0,1]^2$, (X_1, X_2) is our bivariate vector of S&P 500 return and credit spread respectively, and $C^*(u_1, u_2; \theta)$ is the optimal dependence structure established in previous subsection. Moreover, (U_1, U_2) corresponds to the uniform transformation of (X_1, X_2) on the subset $[0,1]^2$ given the observed marginals (i.e., empirically estimated on data with Deheuvels [1981] method). In particular, we focus on the following quantile-quantile dependence measure :

$$P(U_2 > q_\alpha \mid U_1 \leq q_\alpha) = \frac{q_\alpha - C^*(q_\alpha, q_\alpha; \theta)}{q_\alpha} = \alpha \tag{9}$$

where α represents the required probability level or critical level and q_α is the related quantile.

Therefore, a scenario analysis could lead us to consider, for example, a disturbing probability level of 10% or, differently, a stress scenario or a crisis situation with a probability level of 1%. We display in tables underneath quantiles related to each probability level for each optimal copula peculiar to our studied dependence structures. Results[19] are computed according to relation (9) whose resolution is achieved using a SQR algorithm (i.e., analogously to subsection 3.3).

Spread	$q_{1\%}$	$q_{10\%}$	Spread	$q_{1\%}$	$q_{10\%}$
SBF01YAA2	0,99006	0,90484	SIN01YBAA1	0,99006	0,90519
SBF02YAA2	0,99004	0,90329	SIN02YBAA1	0,99006	0,90496
SBF05YAA2	0,99004	0,90331	SIN05YBAA1	0,99004	0,90402
SBF07YAA2	0,99005	0,90457	SIN07YBAA1	0,99005	0,90424
SBF10YAA2	0,99005	0,90409	SIN10YBAA1	0,99005	0,90404
SBF01YAAA	0,99005	0,90409	SIN01YBAA2	0,99007	0,90541
SBF02YAAA	0,99003	0,90303	SIN02YBAA2	0,99005	0,90446
SBF05YAAA	0,99002	0,90104	SIN05YBAA2	0,99004	0,90331
SBF07YAAA	0,99005	0,90472	SIN10YBAA2	0,99004	0,90397
SBF10YAAA	0,99005	0,90458			

Spread	$q_{1\%}$	$q_{10\%}$	Spread	$q_{1\%}$	$q_{10\%}$
SIN01YBAA3	0,99007	0,90554	SIN01YA2	0,99006	0,90537
SIN02YBAA3	0,99006	0,90492	SIN02YA2	0,99005	0,90440
SIN05YBAA3	0,99005	0,90412	SIN05YA2	0,99004	0,90363
SIN07YBAA3	0,99005	0,90445	SIN07YA2	0,99005	0,90426
SIN10YBAA3	0,99006	0,90498	SIN10YA2	0,99005	0,90446
SIN01YA1	0,99007	0,90546	SIN01YA3	0,99006	0,90523
SIN02YA1	0,99006	0,90505	SIN02YA3	0,99005	0,90441
SIN05YA1	0,99004	0,90399	SIN05YA3	0,99004	0,90317
SIN07YA1	0,99005	0,90404	SIN07YA3	0,99004	0,90386
SIN10YA1	0,99006	0,90507			

[19] These results are dispayed with one more decimal point to allow to better discriminate between their peculiar quantile values.

Spread	$q_{1\%}$	$q_{10\%}$	Spread	$q_{1\%}$	$q_{10\%}$
SIN01YAA2	0,99006	0,90516	SIN01YAAA	0,99006	0,90502
SIN02YAA2	0,99005	0,90444	SIN02YAAA	0,99005	0,90428
SIN05YAA2	0,99000	0,89987	SIN05YAAA	0,99000	0,89979
SIN07YAA2	0,99004	0,90354	SIN07YAAA	0,99004	0,90366
SIN10YAA2	0,99006	0,90495	SIN10YAAA	0,99004	0,90354
SIN01YAA3	0,99007	0,90548	STL01YBAA1	0,99006	0,90520
SIN02YAA3	0,99005	0,90417	STL02YBAA1	0,99005	0,90443
SIN05YAA3	0,99038	0,90359	STL05YBAA1	0,99002	0,90219
SIN07YAA3	0,99003	0,90312	STL07YBAA1	0,99003	0,90291
			STL10YBAA1	0,99003	0,90288

Spread	$q_{1\%}$	$q_{10\%}$	Spread	$q_{1\%}$	$q_{10\%}$
STL01YA1	0,99005	0,90482	STL01YA3	0,99005	0,90487
STL02YA1	0,99005	0,90466	STL02YA3	0,99005	0,90450
STL05YA1	0,99003	0,90232	STL05YA3	0,99002	0,90208
STL07YA1	0,99004	0,90344	STL07YA3	0,99003	0,90247
STL10YA1	0,99003	0,90278	STL10YA3	0,99003	0,90292
STL01YA2	0,99006	0,90497	STL01YAA3	0,99005	0,90408
STL02YA2	0,99005	0,90472	STL02YAA3	0,99002	0,90196
STL05YA2	0,99002	0,90221	STL05YAA3	0,99000	0,89898
STL07YA2	0,99003	0,90263	STL07YAA3	0,99002	0,90168
STL10YA2	0,99003	0,90305	STL10YAA3	0,99002	0,90150

Spread	$q_{1\%}$	$q_{10\%}$	Spread	$q_{1\%}$	$q_{10\%}$
SPW01YA1	0,99005	0,90456	SPW01YAA2	0,99005	0,90471
SPW05YA1	0,99002	0,90160	SPW05YAA2	0,99000	0,89979
SPW07YA1	0,99004	0,90352	SPW07YAA2	0,99003	0,90253
SPW01YA2	0,99005	0,90424	SPW10YAA2	0,99004	0,90334
SPW02YA2	0,99004	0,90351	SPW01YAA3	0,99005	0,90414
SPW05YA2	0,99002	0,90187	SPW02YAA3	0,99004	0,90391
SPW07YA2	0,99004	0,90330	SPW05YAA3	0,99002	0,90104
SPW10YA2	0,99003	0,90286	SPW07YAA3	0,99003	0,90246
SPW01YA3	0,99005	0,90462	SPW10YAA3	0,99003	0,90272
SPW02YA3	0,99004	0,90399			
SPW05YA3	0,99003	0,90253			
SPW07YA3	0,99005	0,90413			

We could sum up our results when considering these ones on an average basis. Indeed, we notice respectively the following average values for the 1% and 10% quantiles (i.e., all maturities and ratings included for a given economic sector) : 0,99004 and 0,90352 for banking & finance, 0,99005 and 0,90422 for industrials, 0,99004 and 0,90311 for power, and finally 0,99005 and 0,90336 for telecommunications. Furthermore, after computing empirical univariate cumulative distribution functions for each studied random vector, we have estimated the empirical values of S&P 500 return and credit spreads corresponding to the quantiles above-mentioned. Our results[20] lead to the following average values for a given economic sector and for all maturities and ratings included :

Average values for the one and ten percent probability levels (in basis points) :

Average spreads	BF	IN	PW	TL
$\alpha = 1\%$	102,3750	132,5980	127,0000	134,2097
$\alpha = 10\%$	80,1667	104,3922	89,4286	94,8548

Notice that the values of S&P 500 return associated to the one and ten percent level quantiles are respectively 913,6200 and 445,3700 basis points (i.e., 9,1362% and 4,4537% respectively). The one percent scenario is clearly riskier than the ten percent scenario in terms of spreads' widening and S&P 500 return's decrease. Indeed, switching from ten percent level to one percent level generates a decrease of 51.2522% for S&P 500 return and an increase of 27.7026%, 27.0191%, 42.0127%, 41.4896% for banking & finance, industrials, power and telecommunications credit spreads respectively. Given such estimations, we are able to quantify and identify some disturbed and stressed economic and financial contexts. For example, managers focusing on extreme scenarii would consider the one percent level estimations.

5 Conclusion

In this paper, we focused on the quantification of market risk's influence on credit risk. We chose to represent, on one hand, systematic risk by S&P 500 return's evolution and, on an other hand, credit risk through different credit spreads. Credit spreads are computed through the difference between corporate bond yields and corresponding Treasury yields. They cover different maturities, ratings and economic sectors. Consequently, the quantification problem we address leads us to consider series of bivariate random vectors composed of S&P 500 return and credit spreads.

At a first stage, we observe graphically and statistically asymmetric and non elliptical distributions for such random vectors. As expected, non parametric statistics like Kendall's tau and Spearman's rho show negative relationships between S&P 500 return and credit spreads. Indeed, credit spreads widen when S&P 500 return decreases.

At a second stage, we investigate such relationships through the use of copulas and given that analytical expressions of Kendall's tau as functions of one parameter could be derived for specific copula families. Our choice focuses on Gumbel, FGM, Frank and Clayton copulas which are one parameter copulas.

[20] To spare space, we only give average values but individual values remain available upon request for any interested reader.

Knowing empirical values for Kendall's tau of all random vectors, we estimate parameter values for each type of copula and each considered bivariate random vector. Results give us a first insight and characterization of the observed dependence structures.

Given such characterizations, the third stage solves the selection problem we face with the choice of opitmal copula functions for our studied random vectors. For this purpose, we introduce the discrete L^2 norm distance and consider the average distance or, equivalently, the average estimation error. For a given dependence structure (i.e., random vector), the optimal copula is the copula function which minimizes the average error. According to our results, 76.53% of the studied dependence structures are optimally characterized by a Frank copula type function given the set of copulas we consider.

Finally, all our estimations are aimed at quantifying risks along with copulas. We give some illustrations while achieving a scenario analysis. Given our optimal characterizations, we are able to estimate vectors' values related to a given risk level. Specifically, we consider probabilities that credit spreads widen given that S&P 500 return tightens. For given levels of ten percent and one percent for such probabilities, we induce the corresponding values of S&P 500 return and credit spreads. We therefore quantify jointly systematic risk and credit risk for the two scenarii type above-mentioned.

Consequently, we underline the usefulness of copulas for quantifying risks. To go further, we could explore a greater range of copula families to test on our framework. Moreover, we could consider higher frequency data such as weekly or daily ones to better fit to continuous case. We could therefore improve our characterizations and estimations of the dependence structure prevailing between market risk and credit risk. On an other hand, our setting could be generalized to multivariate cases in order to take into account other factors such as liquidity risk or business cycle.

To conclude, copulas are a powerful tool commonly used in finance for stress testing and value-at-risk analysis. For example, Jouanin et *al.* (2003) apply copulas to credit derivatives whereas Granger et *al.* (2002) use copulas to study common factors.

6 Appendix

In this section, we give some details about copulas and dependence measures.

6.1 Sklar's theorem

In 1959, Sklar defined a copula function to characterize the link between a joint distribution function and its univariate margins.

Theorem 1 *Let F be a joint distribution function with margins F_X and F_Y.*

Then there exists a copula C such that for all x,y in $\bar{\mathbb{R}}$,

$$F(x,y) = C(F_X(x), F_Y(y)) \tag{10}$$

If F_X and F_Y are continuous, then C is unique; otherwise, C is uniquely determined on $RanF_X \times RanF_Y$. Conversely, if C is a copula and F_X and F_Y are distribution functions, then the function F defined by (10) is a joint distribution function with margins F_X and F_Y.

6.2 Kendall and Spearman statistics

We describe here Kendall's tau τ and Spearman's rho ρ statistics used to assess dependence between random variables.

Kendall's tau is based on concordance and corresponds to the probability of concordance minus the probability of discordance. Let $(X_1, Y_1)'$ and $(X_2, Y_2)'$ be two independent and identically distributed continuous random vectors, with joint distribution F. Then, Kendall's tau is defined by:

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0) \tag{11}$$

Spearman's rho is based on concordance and discordance. Let $(X_1, Y_1)'$, $(X_2, Y_2)'$ and $(X_3, Y_3)'$ be three independent continuous random vectors with a common distribution function F, whose margins are F_X and F_Y respectively. Let C be a copula function. Spearman's rho is proportional to the probability of concordance minus the probability of discordance for two vectors $(X_1, Y_1)'$ and $(X_2, Y_3)'$. In this pair of vectors whose margins are identical, $(X_1, Y_1)'$ has a distribution function $F(x,y)$ whereas $(X_2, Y_3)'$ has a distribution function $F_X(x) F_Y(y)$ since its components are independent. Therefore, Spearman's rho writes:

$$\rho = 3\left[P((X_1 - X_2)(Y_1 - Y_3) > 0) - P((X_1 - X_2)(Y_1 - Y_3) < 0)\right] \tag{12}$$

We then have two definitions of dependance between variables. However, other association measures exist and are presented in the book of Nelsen (1999).

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