VaR and ES for Linear Portfolios with mixture of Generalized Laplace distributed Risk Factors

Jules Sadefo Kamdem *

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Abstract

In this paper, by following [3] and [4], we postpone to give an explicit estimation of Value-at-Risk and Expected Shortfall for Linear Portfolios when the risk Factors changes with mixture of generalized Laplace distributions. We therefore introduce the Delta-GLD-VaR, Delta-GLD-ES, Delta-MGLD-VaR and Delta-MGLD-ES. Note that, the GLD will give to us more flexibility to control the shape and fat tails of the risk factors in relation with the historical sample returns.

Key Words: RiskMetrics Delta-Normal VaR, Delta-GLD-VaR, Delta-MGLD-VaR, Delta-GLD-ES, Delta-MGLD ES, Hedge Fund Risk.

1 Introduction

Since Standard RiskMetrics methodology release in October 1994, its has inspired an important discussion on VaR Methodologies. One of the focal points of this discussion has been the assumption that returns follow a conditional normal distribution. Since the distribution of many observe financial risk factors series have tails that are *fatter* than those implied by conditional normality, it is necessary for risk managers to change the assumption concerning the distribution of the risk factors in relation with historical data returns. Therefore it is important to be able to modify the current RisKMetrics model to account for the possibility of such large returns.

Note that Standard RiskMetrics approach works well for the so-called *linear portfolios*, that is, those portfolios whose aggregate return is, to a good approximation, a linear function of the returns of the individual assets which make up the portfolio, and in situations where the latter can be assumed to be jointly normally distributed. This is an issue in situations demanding for real-time evaluation of financial risk. Such methods present us with a trade-off between accuracy and speed, in the sense that they are much faster than Monte Carlo, but are much less accurate unless the linear approximation is quite good and the normality hypothesis holds well. The assumption of normality simplifies the computation of VaR considerably. However it is inconsistent with the empirical evidence of assets returns, which finds that asset returns are fat tailed. This implies that extreme events are much more likely to occur in practice than would be predicted based on the assumption of normality.

Some alternative return distributions have been proposed in the world of elliptic distributions by Sadefo-Kamdem [4] and [3], that better reflect the empirical evidence. In this paper, following [4], I examine one such alternative that simultaneously allows for asset returns that are fat tailed and for tractable calculation of Value-at-Risk and Expected Shortfall, by giving attention to mixture of Generalized Laplace distributions. Note that, the particular case based on mixture of normal distributions, has been proposed by Zangari(1996)[8], Subu-Venkataraman [7] and some references therein.

*Université de Reims and Université D'evry. Address: Laboratoire de Mathématique UMR 6056, BP 1039 Moulin de la Housse 51687 Reims cedex France. sadefo@univ-reims.fr, sjules@maths.univ-evry.fr .

An obvious first generalization is to keep the linearity assumption, but replace the normal distribution by some other family of multi-variate distributions. In this paper, following Sadefo-Kamdem [4], we introduces the notion of *Delta-GLD VaR*, *Delta-Mixture GLD VaR*, *Delta-GLD ES* and *Delta-Mixture GLD ES*.

The particular subject of this paper, is to give an explicit formulas that will permit to obtain the linear VaR or linear ES, when the joint risk factors of the linear portfolios, changes with mixture of Generalized Laplace distributions. Note that, one shortcoming of the multivariate elliptic distribution is that all the marginal distributions have the same characteristic function. Since, the multivariate generalized Laplace distribution $GLD(\mu, \Sigma, \nu)$ [1] is a particular case of multivariate elliptic distributions, its marginals have the same characteristics parameter ν, the mixture of Generalized Laplace distributions will be view as a serious alternatives, to a simple Generalized Laplace distribution. Therefore, the methodology proposes by this paper seem to be interesting to controlled thicker tails.

The paper is organized, as follows: In section 2 following [4], we recall some theorems concerning the Linear combination of elliptic distributions. In section 3, following section 2, we show how to reduce the computation of the *Delta GLD VaR* to finding the zeros of a special function. In section 4, we introduces the notion of Delta mixture GLD VaR denote by *Delta MGLD VaR* , that is given via the computation of the zeros of linear combination of the incomplete Gamma special function. Our method permit to us to get the 100 percent confidence level quantile. In section 5, We introduce the *Delta GLD ES* and Delta Mixture Expected shortfall denote by *Delta MGLD ES* . Finally, in section 5 Following [3], [4], we recall some potential application areas.

2 Linear Combinations of Elliptical distributions

If and investments portfolios is a linear combination of several assets such that the joint returns of assets are assume to have elliptical distributions, then the return on a portfolio of these assets will also have an elliptical distribution.

The linearity property can be briefly summarized as follows: If $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_n)$ is the joint risk factors, with $\mathbf{X} \sim E_n(\mu, \Sigma, g_n)$, and A is the $m \times n$ matrix of rank $m \leq n$ and b some m-dimensional column vector, then

$$A\mathbf{X} + b \sim E_m(\mu, \Sigma, g_m). \tag{1}$$

Therefore, any marginal distribution $\mathbf{X}_k$ of $\mathbf{X}$ is also is also elliptical with the same characteristic generator. In other words, for $k = 1, 2, \ldots, n$, $\mathbf{X}_k \sim E_1(\mu, \Sigma, g_1)$ so that its density can be written as

$$f_k(x) = \frac{c_1}{\sigma_k} g_1 \left[\frac{1}{2} \left(\frac{x - \mu_k}{\sigma_k} \right)^2 \right]. \tag{2}$$

If the return of a linear portfolio is $\Delta\Pi = \delta_1 \cdot \mathbf{X}_1 + \cdots + \delta_n \cdot \mathbf{X}_n = \delta^T \mathbf{X}$, where $\delta = (\delta_1, \ldots, \delta_n)^T$ is a column vector with dimension n, then it immediately follows that

$$\Delta\Pi \sim E_n(\delta^T \mu, \delta^T \Sigma\, \delta, g_1). \tag{3}$$

Remark 2.1 Since the multivariate generalized distributions is a particular case of elliptic distribution, we will use (3), to estimate the VaR and ES for linear portfolio with generalized Laplace distributions Risk Factors.

[1] Here μ is the mean vector, Σ is the variance-covariance matrix, and ν is the characteristic constant given by the pdf function.

3 Linear VaR with Generalized Laplace Distributed Risk Factors

In the case where X follow a multivariate generalized Laplace distribution denote by $GLD(\mu, \Sigma, \nu)$, the pdf is in form

$$f_X(x) = \frac{C_{n,\nu}}{\sqrt{|\Sigma|}}\, g_{n,\nu}\left[(x-\mu)^T \Sigma^{-1}(x-\mu)\right], \tag{4}$$

where

$$g_{n,\nu}(u) = exp[-c_{n,\nu}\, u^{\frac{\nu}{2}}], \tag{5}$$

with

$$c_{n,\nu} = \left(\frac{\Gamma\left(\frac{n+2}{\nu}\right)}{(n)\Gamma\left(\frac{n}{\nu}\right)}\right)^{\nu/2}, \tag{6}$$

and

$$C_{n,\nu} = \frac{\nu}{2\pi^{n/2}}\left(\frac{\Gamma\left(\frac{n+2}{\nu}\right)}{(n)\Gamma\left(\frac{n}{\nu}\right)}\right)^{n/2}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{\nu}\right)}. \tag{7}$$

In the relation (3), the density function g_1 is given by

$$g_1(u) = \exp[-\frac{1}{2\,\lambda^{\nu}}u^{\nu}], \tag{8}$$

when our particular elliptic distribution is in particular the generalized Laplace distribution.

For the confidence $1-\alpha$, the VaR given as the the solution of the equation

$$\int_{\{\Delta\Pi \geq -VaR_\alpha\}} \frac{C_\nu}{\sqrt{\delta^T \Sigma\, \delta}} g_1\left[\left(\frac{r-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}\right)^2\right]\, dr = \alpha. \tag{9}$$

If we introduce the variable $u = \frac{r-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}$ into the integral (9), it becomes

$$C_\nu \int_{\frac{-VaR_\alpha-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}}^{\infty} \exp\left[-\frac{1}{2\,\lambda^\nu}\,u^\nu\right]\, dr = \alpha, \tag{10}$$

where $C_\nu = \frac{\nu}{\lambda\, 2^{1+\frac{1}{\nu}}}$ and $\lambda = 2^{\frac{1}{\nu}}\left[\frac{\Gamma(\nu^{-1})}{\Gamma(\frac{3}{\nu})}\right]^{1/2}$. If we introduce the function denote by G_ν, and changing variable $v = \frac{u^\nu}{2\,\lambda^\nu}$, we obtain the following expressions:

$$\begin{aligned} G_\nu(s) &= C_\nu \int_s^\infty \exp\left[-\frac{1}{2\,\lambda^\nu}\,u^\nu\right]\, du \\ &= \frac{1}{2\,\Gamma\left(\frac{1}{\nu}\right)} \int_{\frac{s^\nu}{2\,\lambda^\nu}}^\infty v^{\frac{1}{\nu}-1}\exp(-v)\, dv \\ &= \frac{1}{2\,\Gamma\left(\frac{1}{\nu}\right)}\,\Gamma\left(\frac{1}{\nu}, \frac{s^\nu}{2\,\lambda^\nu}\right) \end{aligned} \tag{11}$$

therefore we have the following theorem

Theorem 3.1 *Assuming that $\Delta\Pi \simeq \delta_1 X_1 + \delta_2 X_2 + ... + \delta_n X_n$ with a multivariate Generalized Laplace random vector $(X_1, X_2, .., X_n)$ with vector mean μ , and variance-covariance matrix Σ, the linear Value-at-Risk at confidence $1-\alpha$ is given by the following formula*

$$VaR_\alpha = -\delta^T\mu + q^{GLD}_{\alpha,\nu}\cdot\sqrt{\delta\Sigma\delta^t},$$

where now $s = q^{GLD}_{\alpha,\nu}$ is the unique positive solution of the transcendental equation

$$G_\nu(q^{GLD}_{\alpha,\nu}) = \alpha, \tag{12}$$

with G_ν is defined by (11).

3.1 Some values of quantiles $q^{GLD}_{\alpha,\nu}$.

In the following table, we estimate only the positive solution of $G(s) = \alpha$ for some ν. This is given, with the help of Mathematica 4 Software.

Table 1 : Some values of $q^{GLD}_{\alpha,\nu}$.

ν	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$q^{GLD}_{0.00,\nu}$	9.69240	8.72375	7.95798	7.33359	6.082028	6.39022	6.02514	5.44623
$q^{GLD}_{0.01,\nu}$	7.82240	6.18915	5.11230	4.36274	3.81777	3.40763	3.08995	2.83799
$q^{GLD}_{0.025,\nu}$	5.99146	4.81479	4.03148	3.48142	3.07860	2.77349	2.53594	2.34670
$q^{GLD}_{0.05,\nu}$	4.60515	3.75463	3.18327	2.77907	2.48126	2.25456	2.07734	1.93569

Table 2 : Some values of $q^{G.L.D}_{\alpha,\nu}$.

ν	0.80	0.90	1.80	1.90	2.00	2.50	3.00	4.00
$q^{GLD}_{0.00,\nu}$	12.6243	10.9466	5.19944	4.98283	4.88419	4.09232	3.65155	3.12822
$q^{GLD}_{0.01,\nu}$	2.90981	2.83562	2.63415	2.46641	2.32635	2.21001	2.12666	2.01599
$q^{GLD}_{0.025,\nu}$	2.13935	2.1316	2.19303	2.06617	1.95996	1.90451	1.86241	1.80408
$q^{GLD}_{0.05,\nu}$	1.58416	1.6108	1.82034	1.72490	1.64485	1.63208	1.61996	1.60092

Remark 3.2 The Delta-GLD-VaR works well for 100 percent confidence level ($\alpha = 0$).

Remark 3.3 In Practise, the Delta-GLD-VaR works well when $0 < \nu \leq 2$ (fat-tailed). Note that, in this case the density has thicker tails than the normal, whereas for $\nu > 2$ it has thinner tails (short tails distributions). When $\nu = 2$ this produces a normal density. cf. Riskmetrics (1996) [2] page 238 for more details.

4 Linear VaR under Mixture of G.L.D Risk Factors

Following [4], we now consider in detail the case where our mixture of elliptic distributions is a mixture of m multivariate Generalized Laplace distributions (GLD_{ν_j}), for j=1,...,m. We will unsurprisingly introduces the Delta-MGLD-VaR.

In the case of our mixture of multi-variate of G.L.D, the cumulative function will be given by:

$$H(VaR_\alpha) = \sum_{j=1}^{m} \beta_j G_{\nu_j}(s_j) \tag{13}$$

where

$$G_{\nu_j}(s_j) = \frac{1}{2\,\Gamma\left(\frac{1}{\nu_j}\right)}\,\Gamma\left(\frac{1}{\nu_j}, \frac{s_j^{\nu_j}}{2\,\lambda_j^{\nu_j}}\right), \tag{14}$$

with

$$s_j = \frac{-VaR_\alpha - \delta^T \mu_j}{\sqrt{\delta^T \Sigma_j\, \delta}},$$

and

$$\lambda_j = 2^{\frac{1}{\nu_j}} \left[\frac{\Gamma\left(\nu_j^{-1}\right)}{\Gamma\left(\frac{3}{\nu_j}\right)} \right]^{1/2}.$$

We then obtain the following theorem

Theorem 4.1 *Let $\Delta\Pi = \delta_1 X_1 + \ldots + \delta_n X_n$ with $(X_1, \ldots, X_n)^T$ follow a mixture of m generalized Laplace distributions (denote by GLD_{ν_j}, for $j = 1, \ldots, m$), where μ_j is the column mean vector, Σ_j is the variance-covariance matrix of the j-th component GLD_{ν_j} of the mixture and the pdf function is given by*

$$f_X(x) = \sum_{i=1}^{m} \beta_i \frac{C_{n,\nu_i}}{\sqrt{|\Sigma|}} \exp\left[-c_{n,\nu_i} \left((x-\mu_i)^T \Sigma_i^{-1} (x-\mu_i) \right)^{\frac{\nu_i}{2}} \right].$$

Then the value-at-Risk denote by Delta MGLD- VaR*, at confidence $1-\alpha$ is given as the solution of the transcendental equation*

$$\alpha = H(VaR_\alpha) \tag{15}$$

where H is defined by (13).

Corollary 4.2 *One might, in certain situations, try to model with a mixture of m Generalized Laplace distributions denote GLD_{ν_j}, for $j = 1, \ldots, m$, which all have the same variance-covariance $\Sigma = \Sigma_j$ and the same mean $\mu = \mu_j$, and obtain for example a mixture of different tail behaviors by playing with the ν_j's. In that case the VaR again simplifies to:*

$$VaR_\alpha^{MGLD} = -\delta^T \mu + q^{MGLD-VaR_\alpha} \cdot \sqrt{\delta^T \Sigma \delta},$$

with $s = q^{GLD-VaR_\alpha}$ now the unique positive solution to

$$\alpha = \sum_{j=1}^{m} \frac{\beta_j}{2\,\Gamma\left(\frac{1}{\nu_j}\right)} \Gamma\left(\frac{1}{\nu_j}, \frac{s_j^{\nu_j}}{2\,\lambda_j^{\nu_j}} \right). \tag{16}$$

where $\sum_{j=1}^{m} \beta_j = 1$.

Remark 4.3 One might, in certain situations, try to model with a mixture of m GLD_{ν_j}, for $j = 1, \ldots, m$, which all have the same $\nu_j = \nu$ and the same mean $\mu_j \approx 0$, and obtain for example a mixture of different tail behaviors by playing with the Σ_j's.

4.1 Some Numerical Result of Delta Mixture-GLD VaR coefficient

Here we give some numerical results when applying the corollary 4.2, in the situation where $m = 2$.

By introducing the function F such that

$$F(s, \beta, \nu_1, \nu_2) = \beta \cdot G_{\nu_1}(s) + (1-\beta) \cdot G_{\nu_2}(s), \tag{17}$$

where G_{ν_j} is define in (14), for $j = 1, 2$, for given as inputs β, ν_1 and ν_2, we give a table that contains some solutions $s = q_{\beta,\nu_1,\nu_2} = q_{\beta,\nu_1,\nu_2}^{MGLD-VaR_\alpha}$ of the following transcendental equation:

$$F(s, \beta, \nu_1, \nu_2) = \alpha.$$

For given Σ, μ, and δ, these solutions will permit to calculate VaR_α, when the confidence is $1-\alpha$.

- In the case where $\alpha = 0$, we obtain some solutions of (16) in the following table:

Table 3: Coefficients $q^{MGLD-VaR_\alpha}_{\beta,\nu_1,\nu_2}$ of $VaR^{MGLD}_{\alpha,\beta,\nu_1,\nu_2}$, when $\alpha = 0$.

(ν_1, ν_2)	(0.75,1.00)	(0.90,1.20)	(1.20,1.50)	(1.60,1.95)	(1.40,1.80)	(1.10, 2.00)	(1.30, 1.90)
$q^{MGLD-VaR_0}_{0.05,\nu_1,\nu_2}$	10.1487	8.3493	6.51656	5.07643	5.61692	6.63993	5.81607
$q^{MGLD-VaR_0}_{0.10,\nu_1,\nu_2}$	10.6437	8.74449	6.7128	5.26422	5.80926	7.05997	6.11923
$q^{MGLD-VaR_0}_{0.15,\nu_1,\nu_2}$	11.0239	9.03451	6.86394	5.35476	5.94519	7.30695	6.30095
$q^{MGLD-VaR_0}_{0.20,\nu_1,\nu_2}$	11.3283	9.26021	6.98501	5.42709	6.04851	7.48343	6.43054
$q^{MGLD-VaR_0}_{0.25,\nu_1,\nu_2}$	11.5810	9.44410	7.08530	5.48685	6.13131	7.62142	6.53138
$q^{MGLD-VaR_0}_{0.30,\nu_1,\nu_2}$	11.7967	9.59909	7.17061	5.53755	6.20022	7.73514	6.61402
$q^{MGLD-VaR_0}_{0.35,\nu_1,\nu_2}$	11.9850	9.73306	7.24473	5.58147	6.25917	7.83218	6.68414
$q^{MGLD-VaR_0}_{0.40,\nu_1,\nu_2}$	12.1522	9.85113	7.31020	5.62016	6.31067	7.91704	6.74512
$q^{MGLD-VaR_0}_{0.45,\nu_1,\nu_2}$	12.3027	9.9567	7.36882	5.65471	6.35639	7.99263	6.79913
$q^{MGLD-VaR_0}_{0.50,\nu_1,\nu_2}$	12.4397	10.0525	7.42189	5.68591	6.39751	8.06095	6.84767

Remark 4.4 Note that, the precedent results are available when $\alpha = 0$. This means that with our model, one would calculate the linear VaR with mixture of generalized Laplace distributions, for 100 percent confidence level.

5 Linear Expected Shortfall with G.L.D Risk Factors

Expected Shortfall is a sub-additive risk statistic that describes how large losses are on average when they exceed the VaR level. Expected Shortfall will therefore give an indication of the size of extreme losses when the VaR threshold is breached. We will evaluate the Expected Shortfall for a linear portfolio under the hypothesis of elliptically distributed risk factors. Mathematically, the Expected Shortfall associated with a given VaR is defined as:

$$\text{Expected Shortfall } = \mathbb{E}(-\Delta\Pi | -\Delta\Pi > VaR),$$

see for example [4] and the references therein. Assuming that $\Delta\Pi \sim E_n(\delta^T\mu, \delta^T\Sigma\delta, g_1)$, the Expected Shortfall at confidence level $1-\alpha$ is given by

$$\begin{aligned}
-ES_\alpha &= \mathbb{E}(\Delta\Pi \mid \Delta\Pi \leq -VaR_\alpha) \\
&= \frac{1}{\alpha}\mathbb{E}\left(\Delta\Pi \cdot 1_{\{\Delta\Pi \leq -VaR_\alpha\}}\right) \\
&= \frac{C_{1,\nu}}{\alpha\sqrt{\delta^T\Sigma\,\delta}} \int_{\mathbb{R}} u\, 1_{\{u \leq -VaR_\alpha\}}\, g_1\left[\left(\frac{u-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}\right)^2\right] du \\
&= \frac{C_{1,\nu}}{\alpha\sqrt{\delta^T\Sigma\,\delta}} \int_{-\infty}^{-VaR_\alpha} u\, g_1\left[\left(\frac{u-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}\right)^2\right] du. \qquad (18)
\end{aligned}$$

In the particular case where g_1 is given by the generalized Laplace distributions denote by GLD_ν, it suffices to replace g_1 in (18) by $g_{1,\nu}$ as define in (8), therefore we obtain

$$
\begin{aligned}
-ES_\alpha &= \frac{C_\nu}{\alpha\,\sqrt{\delta^T \Sigma\,\delta}} \int_{-\infty}^{-VaR_\alpha} u\; g_1 \left[\left(\frac{u-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}} \right)^2 \right] du \\
&= \frac{C_\nu}{\alpha\,\sqrt{\delta^T \Sigma\,\delta}} \int_{-\infty}^{-VaR_\alpha} u\;\exp\left[-\frac{1}{2\,\lambda^\nu} \left(\frac{u-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}} \right)^\nu \right] du \\
&= \frac{C_\nu}{\alpha} \int_{-\infty}^{\frac{-VaR_\alpha-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}} (\delta^T\mu\; + \;\sqrt{\delta^T\Sigma\,\delta}\; v)\;\exp\left[-\frac{1}{2\,\lambda^\nu}\; v^\nu \right] dv \\
&= \delta^T\mu\; + \;\frac{C_\nu}{\alpha}\sqrt{\delta^T\Sigma\,\delta} \int_{-\infty}^{\frac{-VaR_\alpha-\delta^T\mu}{\sqrt{\delta^T\Sigma\,\delta}}} v\;\exp\left[-\frac{1}{2\,\lambda^\nu}\; v^\nu \right] dv \qquad (19)\\
&= \delta^T\mu\; - \;\frac{C_\nu}{\alpha}\sqrt{\delta^T\Sigma\,\delta} \int_{q_{\alpha,\nu}^{GLD}}^{+\infty} w\;\exp\left[-\frac{1}{2\,\lambda^\nu}\; w^\nu \right] dw \qquad (20)\\
&= \delta^T\mu\; - \left[\frac{C_\nu}{\nu\,\alpha\,\left(\frac{1}{2\,\lambda^\nu}\right)^{2/\nu}} \int_{\left[q_{\alpha,\nu}^{GLD}\right]^\nu\,\frac{1}{2\,\lambda^\nu}}^{\infty} z^{\frac{2}{\nu}-1} e^{-z}\,dz \right] \sqrt{\delta^T\Sigma\,\delta} \qquad (21)
\end{aligned}
$$

where

$$
q_{\alpha,\nu}^{GLD} = \frac{\delta^T\mu\; + \;VaR_\alpha}{\sqrt{\delta^T\Sigma\,\delta}}
$$

using the incomplete Γ-function $\Gamma(a,x) = \int_x^\infty t^{a-1}\exp(-t)\,dt$, we arrive at the following result:

$$
\begin{aligned}
ES_\alpha &= -\delta^T\mu\; + \left[\frac{C_\nu}{\nu\,\alpha\,(\frac{1}{2\,\lambda^\nu})^{2/\nu}} \int_{\left[q_{\alpha,\nu}^{GLD}\right]^\nu\,\frac{1}{2\,\lambda^\nu}}^{\infty} z^{\frac{2}{\nu}-1} e^{-z}\,dz \right] \sqrt{\delta^T\Sigma\,\delta} \\
&= -\delta^T\mu\; + \left[\frac{C_\nu}{\nu\,\alpha\,(\frac{1}{2\,\lambda^\nu})^{2/\nu}} \Gamma\left(\frac{2}{\nu}, \left[q_{\alpha,\nu}^{GLD}\right]^\nu\,\frac{1}{2\,\lambda^\nu} \right) \right] \sqrt{\delta^T\Sigma\,\delta} \qquad (22)\\
&= -\delta^T\mu\; + \;es_{\alpha,\nu}^{GLD}\,\sqrt{\delta^T\Sigma\,\delta} \qquad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
es_{\alpha,\nu}^{GLD} &= \frac{C_\nu}{\alpha\,\nu} \frac{\Gamma\left(\frac{1}{\nu}\right)}{\Gamma\left(\frac{3}{\nu}\right)} \Gamma\left(\frac{2}{\nu}, \left(\frac{\Gamma\left(\frac{3}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)} \right)^{\nu/2} \left[q_{\alpha,\nu}^{GLD}\right]^\nu \right) \\
&= \frac{1}{2\,\alpha} \left[\frac{\Gamma\left(\frac{3}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)} \right]^{1/2} \Gamma\left(\frac{2}{\nu}, \left(\frac{\Gamma\left(\frac{3}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)} \right)^{\nu/2} \left[q_{\alpha,\nu}^{GLD}\right]^\nu \right). \qquad (24)
\end{aligned}
$$

Theorem 5.1 *Suppose that the portfolio is linear in the risk-factors $X = (X_1, \cdots, X_n)$: $\Delta\Pi = \delta \cdot X$ and that $X \sim GLD(\mu, \Sigma, \nu)$, with pdf*

$$
f(x) = \frac{C_{n,\nu}}{|\Sigma|^{-1}} \exp\left(-c_{n,\nu}\;\left[(x-\mu)^T \Sigma^{-1} (x-\mu) \right]^{\nu/2} \right).
$$

Using $q_{\alpha,\nu}^{GLD}$ which is the unique solution of (12), The Expected Shortfall at confidence level $1-\alpha$ for a Generalized Laplace Distribution is given by :

$$
ES_\alpha^{GLD_\nu} = -\delta^T\mu\; + \;es_{\alpha,\nu}^{GLD}\,\sqrt{\delta^T\Sigma\,\delta}. \qquad (25)
$$

where

$$
es_{\alpha,\nu}^{GLD} = \frac{1}{2\,\alpha} \left[\frac{\Gamma\left(\frac{3}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)} \right]^{1/2} \Gamma\left(\frac{2}{\nu}, \left(\frac{\Gamma\left(\frac{3}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)} \right)^{\nu/2} \left[q_{\alpha,\nu}^{GLD}\right]^\nu \right). \qquad (26)
$$

The Expected Shortfall for a linear G.L.D portfolio is therefore given by a completely explicit formula, once the VaR is known. Observe that, as for the VaR, the only dependence on the portfolio dimension is through the portfolio mean $\delta^T \mu$ and the portfolio variance $\delta^T \Sigma \delta$.

5.1 Some values of $es_{\alpha,\nu}^{GLD}$

With the help of Mathematica, we obtain some values of $e_{\alpha,\nu}^{GLD}$ in the following table:

ν	1.00	1.10	1.20	1.30	1.40	1.50
$es_{0.01,\nu}$	1.3382×10^{-6}	3.97021×10^{-6}	9.32964×10^{-6}	1.85266×10^{-5}	3.25035×10^{-5}	5.19176×10^{-5}
$es_{0.025,\nu}$	3.4999×10^{-5}	7.54689×10^{-5}	1.37192×10^{-4}	2.21093×10^{-4}	3.26418×10^{-4}	4.51419×10^{-4}
$es_{0.05,\nu}$	3.94307×10^{-4}	6.66236×10^{-4}	9.98301×10^{-4}	1.37652×10^{-3}	1.78852×10^{-3}	2.22448×10^{-3}

ν	1.60	1.70	1.80	1.90	2.00
$es_{0.01,\nu}$	7.71555×10^{-5}	1.08345×10^{-4}	1.45409×10^{-4}	1.88131×10^{-4}	2.36197×10^{-4}
$es_{0.025,\nu}$	5.93886×10^{-4}	7.51539×10^{-4}	9.2218×10^{-4}	1.10387×10^{-3}	1.2949×10^{-3}
$es_{0.05,\nu}$	2.67702×10^{-3}	3.14095×10^{-3}	3.61264×10^{-3}	4.08959×10^{-3}	4.57011×10^{-3}

5.2 Application: Mixture of G.L.D Expected Shortfall

It's straightforward to obtain the following theorem:

Theorem 5.2 *If the joint factors $\mathbb{X}$ of the linear portfolios follow a mixture of m multi-variate $GLD(\mu, \Sigma, \nu_i)$, for $i = 1, \dots, m$, with the pdf function*

$$f_X(x) = \sum_{i=1}^{m} \beta_i \frac{C_{n,\nu_i}}{\sqrt{|\Sigma|}} \exp\left[-c_{n,\nu_i} \left((x-\mu)^T \Sigma^{-1} (x-\mu)\right)^{\frac{\nu_i}{2}}\right],$$

then Expected Shortfall at confidence level $1-\alpha$ is given by:

$$\begin{aligned} ES_\alpha^{MGLD} &= -\delta^T \mu \sum_{i=1}^{m} \beta_i + |\delta \Sigma \delta^t|^{1/2} \sum_{i=1}^{m} \beta_i \, es_{\alpha,\nu_i}^{GLD} \\ &= -\delta \cdot \mu + |\delta \Sigma \delta^t|^{1/2} \sum_{i=1}^{m} \frac{\beta_i}{2\,\alpha} \left[\frac{\Gamma\left(\frac{3}{\nu_i}\right)}{\Gamma\left(\frac{1}{\nu_i}\right)}\right]^{1/2} \Gamma\left(\frac{2}{\nu_i}, \left(\frac{\Gamma\left(\frac{3}{\nu_i}\right)}{\Gamma\left(\frac{1}{\nu_i}\right)}\right)^{\nu_i/2} [q_{\alpha,\nu_i}^{GLD}]^{\nu_i}\right). \\ &= -\delta \cdot \mu + K_\alpha^{MGLD-ES} \cdot \sqrt{\delta \Sigma \delta^t} \end{aligned} \tag{27}$$

6 Some Areas of Applications

In this section following [3], [4] and the financial literature, we recall some areas of applications.

6.1 Delta-GLD-VaR

When the application concern portfolio with derivatives portfolio, one could use the sensitivities (Greeks) of the instruments to approximate the effect of changes in the market value. when the portfolios contains the linear instruments, one needs only to calculate the delta (first order derivative). Suppose that we are holding a portfolio of derivatives depending on n underlying assets $X^{(1)}, X^{(2)}, ..., X^{(n)}$with joint log-returns r_j (over some fixed time-window), that follow a generalized Laplace distribution. The portfolio's present value V will in general be some complicated non-linear function of the X_i's. To obtain a first approximation of its VaR, we simply approximate the present Value V of the position using a first order Taylor expansion:

$$V(X + \Delta X) \approx V(X) + \sum_{i=1}^{n} \frac{\partial V}{\partial X^{(i)}} \Delta X_i.$$

From this, we can then approximate the profit/loss function as

$$\Delta V = V(X + \Delta X) - V(X) \approx \sum_{i=1}^{n} \delta_i r^{(i)} = \delta \cdot r,$$

where we put $r = (r^{(1)}, ..., r^{(n)})$ and $\delta = (\delta_1, ..., \delta_n)$ with $\delta_i = X^{(i)} \frac{\partial V}{\partial X^{(i)}}$. The entries of the δ vector are called the "delta equivalents " for the position, and they can be interpreted as the sensitivities of the position with respect to changes in each of the risk factors. For more details see [1], where a multi-variate normal distribution for the r_i's is assumed. The discussion there generalizes straightforwardly to the GLD case or mixture of GLD denote by MGLD), where the present paper's results can be used.

6.2 Portfolios of Equities

A special case of the preceding is that of an equity portfolio, build of stock $S_1, \dots, S_n$ with joint log-returns $r = (r_1(t), \dots, r_n(t))$. In this case, the portfolio's Profit & Loss function over the time window [0,t] of interest is, to good approximation, given by

$$\begin{aligned} \Pi(t) - \Pi(0) &= \sum_{i=1}^{n} w_i S_i(0)(S_i(t)/S_i(0) - 1) \\ &\approx \sum_{i=1}^{n} w_i S_i(0) r_i(t) = \delta \cdot r^t, \end{aligned}$$

where this approximation will be good if the $r_i(t)$ are small. In this case the preceded theorems are applicable where $\delta = (w_1 S_1(0), \dots, w_n S_n(0))$ and $r_j(t) = log(X_j(t)/X_j(0))$ for j=1,...,n.

6.3 Businesses as Linear Portfolios of Business Units

An interesting way of looking upon an big enterprize, e.g. a multi-national or a big financial institution, is by considering it as a sum of its individual business units. If X_j, is the variation of price or of profitability of business unit j in one period, then the variation of price of the agglomerate in the same period will be

$$\Delta\Pi = X_1 + \cdots + X_n.$$

The entire institution is therefore modelled by a linear portfolio, with $\delta = (1, 1, \dots, 1)$, to which the results of this paper can be applied, if we model the vector of individual price variations by a multi-variate Generalized Laplace distribution. VaR, incremental VaR (see below) and Expected Shortfall will be relevant here.

6.4 Incremental VaR

Incremental VaR is defined in [1] as the statistic that provides information regarding the sensitivity of VaR to changes in the portfolio holdings. It therefore gives an estimation of the change in VaR resulting from a risk management decision. Results from [1] for incremental VaR with normally distributed risk-factors generalize straightforwardly to generalized Laplace distributed ones: if we denote by $IVaR_i$ the incremental

VaR for each position in the portfolio, with θ_i the percentage change in size of each position, then the change in VaR will be given by

$$\Delta VaR = \sum \theta_i IVaR_i$$

By using the definition of $IVaR_i$ as in [1] (2001), we have that

$$IVaR_i = \omega_i \frac{\partial VaR}{\partial \omega_i} \tag{28}$$

with ω_i is the amount of money invested in instrument i. In the case of an equity portfolio in the Generalized Laplace distributed assets, we have seen that, assuming $\mu = 0$,

$$VaR_\alpha = -q_{\alpha,\nu}^{GLD} \sqrt{\delta \Sigma \delta^t},$$

We can then calculate $IVaR_i$ for the i-th constituent of portfolio as

$$IVaR_i = \omega_i \frac{\partial VaR}{\partial \omega_i} = \omega_i \gamma_i$$

with

$$\gamma = -q_{\alpha,\nu}^{GLD} \frac{\Sigma \omega}{\sqrt{\delta \Sigma \delta^t}}.$$

The vector γ can be interpreted as a gradient of sensitivities of VaR with respect to the risk factors. This is the same as in [3], except of course that the quantile has changed from the elliptic one to the one associated to the particular case of generalized Laplace distribution.

6.5 Aggregation of risks

Suppose that we have a constituted portfolio with several under portfolios of assets from different markets. Given the Value-at-Risk of the portfolios constituting the global portfolio, under the hypothesis that the joined risks factors follow an *multivariate generalized Laplace distribution* , the question is how to get the VaR of the global portfolio.

In order to be clearer and simpler, let us consider a global constituted portfolio of 2 under portfolios from different markets with respective weights $\boldsymbol{\delta_1}$ and $\boldsymbol{\delta_2}$. $\boldsymbol{\Sigma_1}$ represents the matrix of interrelationship in the under portfolio of market 1; $\boldsymbol{\Sigma_2}$ represents the matrix of interrelationship in the under portfolio of market 2. One will be able to write the matrix of interrelationship of a global portfolio like this:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma_1} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}}^t & \boldsymbol{\Sigma_2} \end{pmatrix},$$

where $\boldsymbol{\Sigma_{12}}$ is the correlation matrix that takes into consideration the interaction between the market $\mathbf{M_1}$ and the market $\mathbf{M_2}$. If $\delta^t = (\boldsymbol{\delta_1}, \boldsymbol{\delta_2})$, we have

$$\delta^t \boldsymbol{\Sigma} \delta = \boldsymbol{\delta_1}^t \boldsymbol{\Sigma_1} \boldsymbol{\delta_1} + \boldsymbol{\delta_2}^t \boldsymbol{\Sigma_2} \boldsymbol{\delta_2} + 2 \cdot \boldsymbol{\delta_1}^t \boldsymbol{\Sigma_{12}} \boldsymbol{\delta_2}. \tag{29}$$

Therefore, since we know that when $\mu \approx 0$, we have

$$VaR_\alpha = q^{MGLD-VaR_\alpha} \cdot \sqrt{\delta \Sigma \delta^t},$$

the Value-at-Risk of the global portfolio will be given by

$$VaR_\alpha(\mathbf{M}) = \sqrt{\mathbf{VaR}_\alpha(\mathbf{M_1})^2 + \mathbf{VaR}_\alpha(\mathbf{M_2})^2 + \mathbf{2}[\mathbf{q}^{\mathbf{MGLD-VaR}_\alpha}]^2 \cdot \boldsymbol{\delta_1}^{\mathbf{t}} \boldsymbol{\Sigma_{12}} \boldsymbol{\delta_2}}. \tag{30}$$

An implicit interrelationship with the hypothesis of *Generalized Laplace distribution* is obtained in an analogous way, like in the case where one works with the hypothesis of the normal distribution. Note that, one will distinguish several situations from the behavior of $\mathbf{\Sigma_{12}}$. With some simple operations, the implicit interrelationship is

$$\phi = \frac{\delta_1{}^t \mathbf{\Sigma_{12}} \delta_2}{\sqrt{(\delta_1{}^t \mathbf{\Sigma_1} \delta_1)(\delta_2{}^t \mathbf{\Sigma_2} \delta_2)}} \tag{31}$$

with the Value-at-Risk $VaR_\alpha(\mathbf{M})$ of the global portfolio being given as follows:

$$VaR_\alpha(\mathbf{M}) = \sqrt{[\mathbf{VaR}_\alpha(\mathbf{M_1})]^2 + [\mathbf{VaR}_\alpha(\mathbf{M_2})]^2 + 2\phi \cdot \mathbf{VaR}_\alpha(\mathbf{M_1})\mathbf{VaR}_\alpha(\mathbf{M_2})}). \tag{32}$$

Also, for $\mu \approx 0$,

$$ES_\alpha = K^{GLD}_{ES,\alpha} \cdot \sqrt{\delta \Sigma \delta^t},$$

therefore by using the same technics that proves (32), we have that the Expected Shortfall of the global portfolio is given by:

$$ES_\alpha(\mathbf{M}) = \sqrt{\mathbf{ES}_\alpha(\mathbf{M_1})^2 + \mathbf{ES}_\alpha(\mathbf{M_2})^2 + 2[\mathbf{K^{GLD}_{ES,\alpha}}]^2 \cdot \delta_1{}^{\mathbf{t}} \mathbf{\Sigma_{12}} \delta_2}. \tag{33}$$

This imply that

$$ES_\alpha(\mathbf{M}) = \sqrt{[\mathbf{ES}_\alpha(\mathbf{M_1})]^2 + [\mathbf{ES}_\alpha(\mathbf{M_2})]^2 + 2\phi_{\mathbf{ES}} \cdot \mathbf{ES}_\alpha(\mathbf{M_1})\mathbf{ES}_\alpha(\mathbf{M_2})}), \tag{34}$$

where

$$\phi_{ES} = \frac{\delta_1{}^t \mathbf{\Sigma_{12}} \delta_2}{\sqrt{(\delta_1{}^t \mathbf{\Sigma_1} \delta_1)(\delta_2{}^t \mathbf{\Sigma_2} \delta_2)}} \tag{35}$$

Remark 6.1 The result about the aggregation of risks work so well in the situation where, the joint risk factors of our portfolio changes with mixture of m generalized Laplace distributions where all $\Sigma_i = \Sigma$, for $i = 1, \ldots, m$. In particular, when $\mu_i = \mu$, we have the results (34) and (32).

6.6 hedge funds risk

There is substantial empirical evidence that the distribution of hedge fund returns is typically skewed to the left and leptokurtic [2]. Therefore the unconditional return distribution shows high peaks, fat tails and more outliers on the left tail. To account for the excess Kurtosis for the data, we use the fat-tailed generalized Laplace distribution (GLD) that account for the non-normality of returns and relatively infrequent events.

6.7 Application to emerging market

Our method is applicable to a portfolio of assets in emerging markets. Therefore in relation with particular context of each emerging market, we will attribute the change the parameter ν of the $GLD(\mu, \Sigma, \nu)$ in relation with the specific historical data of the emerging market portfolio (cf. RiskMetrics [2] page 238-242 for more details.). The mixture of such distribution will bring to risk manager more flexibility to control tails distribution in relation with the sector of assets, countries etc.

7 conclusion

In this paper, we give and explicit equation in terms of incomplete Γ-function, that permit to estimate de linear Value-at-Risk and the Linear Expected Shortfall when the joint risk factors follow a mixture of multivariate generalized Laplace distribution. In analogy with Delta Normal VaR of Standard RiskMetrics, We therefore introduces the notion of Delta- GLD VAR, Delta-GLD ES, Delta-MGLD VaR and Delta-MGLD ES. We finally surveyed some potential application areas.

[2]See Fung and Hieh (1999), Agarwal and Naik(2002), Amin and Kat (2003), De Souza and Gokan(2003), and Geman and Kharoubi (2003).

References

[1] J. Mina and J.Y.Xiao (2001), Return to Riskmetrics: The Evolution of the standard (www.riskmetrics.com). 2001 RiskMetrics Group Inc.

[2] J.P. Morgan/Reuters, RiskMetrics Technical Document, 4^{th} Edition, 1996, J.P. Morgan.

[3] J. Sadefo-Kamdem, VaR and ES for Linear Portfolios of Securities with Mixture of Elliptically Distributed Risk Factors (2003), submitted.

[4] J. Sadefo-Kamdem, Value-at-Risk and Expected Shortfall for a Linear Portfolios with Elliptically Distributed Risk Factors (2003), submitted.

[5] J. Sadefo-Kamdem, Value-at-Risk and Expected Shortfall for Quadratic Portfolios of Securities with Elliptic Distributions Risk Factors (2003), see www.gloriamundi.org.

[6] M. Hamada, E. A. Valdez (2003) CAPM and Option Pricing with Elliptical Distributions.(see www.gloriamundi.org).

[7] Subu Venkataraman, (1998): Value at risk for a mixture of normal distributions: The use of quasi-bayesian estimation techniques. Economic Perspectives (FED).

[8] Zangari, Peter, " An Improved methodology for measuring VaR," RiskMetrics Monitor, 2nd quarter pp.7-25. http://www.riskmetrics.com/research/journals