PARAMETER RISK IN THE BLACK AND SCHOLES MODEL

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Abstract. We study parameter or estimation risk in the hedging of options. We suppose that the world is such that the price of an asset follows a stochastic differential equation. The only unknown is the (future) volatility of the asset. Options are priced and hedged according to the Black and Scholes formula. We describe the distribution of the profit and loss of the hedging activity when the volatility of the underlying is misestimated. A financial interpretation of the results is provided. Analytical bounds and numerical results for call, put, and portfolios complete our description.

1. Introduction

We suppose that we are in a world with Black and Scholes belief. By this we mean that there exists an asset that follows the stochastic equation

$$dS_t = g(t, \omega)S_t dt + v(t, \omega)S_t dW_t$$

where $g, v : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ and W_t is a Brownian motion. We suppose that v is positive and g and v are regular enough for the existence of a solution of the equation. We do not explicitly give such conditions; it can be some standard Lipschitz condition (to satisfy Ito theorem [7, Chap. 5]) or a more specific one. The volatility can depend on the time, the level of the asset but also of an independant random component.

But people act as if the asset was following a pure geometric Brownian motion

$$dS_t = \mu(t)S_t\, dt + \sigma(t)S_t\, dW_t$$

with $\mu, \sigma : \mathbb{R}^+ \to \mathbb{R}$. We suppose also that continuous rebalancing is possible. People think they know the volatility, sell options at the price given by the Black and Scholes formula and hedge it using the Δ-neutral self-financing strategy and hope that the profit at maturity (or at any time between the initial transaction and maturity) will be 0. It means that not only one sells the option at a wrong price but also that the hedging used will be Δ-neutral with respect to an incorrect

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Δ. So the final profit will be non zero, not only due to the initial error but also due to accumulating errors during the life of the option.

In this paper we study the theoretical profit. We compute the profit using continuous rebalancing and asset movement given by the stochastic differential equation. Study of the profit using discrete rebalancing, best estimate of volatility and real data can be found in Figlewski and Green [3]. Our approach is very different to this in the sense that we study the error coming from the misestimation of the parameter of the model not from the inadequacy of the model with reality.

We give a description of the distribution of this profit. Then we obtain lower and upper bounds in the case of call and put. Numerical results describing the influence of different parameters are then provided. We develop the result for the case where the asset is paying continuously a dividend.

This note doesn't present the more complex model. But it includes the Hull and White model for zero coupon bonds. Its goal is to present one type of risk that appears when trading options. In particular the parameters that influence the risk are underlined.

All the parameters of the *model* influence the distribution of the profit. This list includes a parameter that is not present in the *pricing* formula: the growth rate μ.

We insist also on the fact that the profit does not only comes from the misprizing of the option trades but also from all the hedging process. Even if all the option trades are done internally within a bank and each component of the bank believes that its risk is non-existent, the total profit of the bank can be non zero (see 6.4).

We also present some results on portfolios of options. A small change of the pay-off profile can lead to large differences of the profit distribution. This comes from the difficulty to hedge some positions with instable hedging strategies (large gamma).

2. Notation an preliminary remarks

As we have two parameters for the volatility (v and σ), we will use a double notation for all the discussion. Any variable with a bar ($\bar{H}$, $\bar{V}$, $\bar{\pi}$, ...) will refer to the value using the estimated volatility σ.

The following variables are used:

T	Maturity of the option.
X	Strike of the option.
r_0	The (short term) interest rate (continuously compounded).
r_1	Rate of the dividend (continously compounded).
V_t	Value of the replication portfolio in t.
H_t	The hedging ratio (delta) in t.
H_t^0	The quantity of cash in the hedging strategy.
S_t	The value of the underlying.
S_t^D	Discounted value of S_t: $e^{-\int_0^t r_0(s)ds} S_t$.
V_t^D	Discounted value of V_t: $e^{-\int_0^t r_0(s)ds} V_t$.
$\pi(t, S^D)$	Discounted price of the option in t when the discounted price of the underlying is S^D using the Black and Scholes formula. The usual formula for the price of an option is in term of its price (not discounted price). If we denote by $P(t, S)$ the usual price, we have $\pi(t, S^D) = e^{-\int_0^t r_0(s)ds} P(t, e^{\int_0^t r_0(s)ds} S^D)$.

For the explanations, we consider that the dividend is continuously paid in the asset. This means that if we have one asset at some stage, after a period of t we have $e^{r_1 t}$ assets. If the dividend is paid in local currency instead, from a theoretical point of view nothing is changed. As we can continuously rebalance, we can buy the quantity of assets we need. This last type of payment is usually the case for gold options in USD, where the interest on the gold deposits are paid in USD.

The model we present includes the Garman-Kohlhagen model for currency options if one considers the interest on the foreign currency as a dividend. The meaning of local and foreign currencies is not related to the citizenship of the trader or its base currency. The local currency is the one in which the premium is payed, the foreign currency is the other one. For example for a USD based trader which sell a EUR/CHF call, receiving the premium in CHF, the local currency is the CHF. So if he delta hedges this position, he will buy Δ EUR and sell the equivalent in CHF. Its position in CHF will be $-\Delta S_t + P$.

3. The profit

Consider a derivative with pay-off $P(T, S_T)$ at time T. If the value of this derivative exists between 0 and T and depend only on t and S_t, we denote it $P(t, S_t)$. If the price doesn't exists between 0 an T, the formulae containing $P(t, S_t)$ have to be considered only for $t = T$.

The profit[1] at any moment between the initial transaction and the maturity of the option is

$$\mathrm{PnL}_t = \bar{V}_t - P(t, S_t)$$

where $\bar{V}_t$ is the value of the hedging portfolio using the estimated parameter σ. To simplify the writing, we will use the discounted values in the computations. The value of the hedging portfolio is

$$\bar{V}_t^D = \bar{H}_t^0 + \bar{H}_t S_t^D$$

where $\bar{H}_t = D_2\bar{\pi}(t, S_t^D)$. Note that for call and put, $D_2\bar{\pi}$ is bounded and C^∞ in (t, S), so regular enough for the integration. We suppose that the strategy we use is self-financing:

$$\bar{V}_t^D = \bar{V}_0^D + \int_0^t r_1(\theta)\bar{H}_\theta S_\theta^D d\theta + \int_0^t \bar{H}_\theta dS_\theta^D$$

In the case where v is different from σ the profit will come in $\bar{H}_t^0$ that will no longer be the one predicted.

The estimated price $\bar{\pi}(t, S_t^D)$ satisfies the partial differential equation

$$D_1\bar{\pi}(t, x) - r_1(t)xD_2\bar{\pi}(t, x) + \frac{1}{2}x^2\sigma^2(t)D_{22}\bar{\pi}(t, x) = 0.$$

Applying the Itô formula to $\Pi_t = \bar{\pi}(t, S_t^D)$ and this last result, we have

$$\begin{aligned} d\Pi_t &= D_1\bar{\pi}(t, S_t^D)dt + D_2\bar{\pi}(t, S_t^D)dS_t^D + \frac{1}{2}D_{22}\bar{\pi}(t, S_t^D)v^2(t, \omega)S_t^D dt \\ &= D_2\bar{\pi}(t, S_t^D)(r_1(t)S_t^D dt + dS_t^D) \\ &\quad + \frac{1}{2}S_t^{D^2}(v^2(t, \omega) - \sigma^2(t))D_{22}\bar{\pi}(t, S_t^D)dt \end{aligned}$$

and then

$$(1) \quad \begin{aligned} \int_0^t \bar{H}_\theta dS_\theta^D &= \bar{\pi}(t, S_t^D) - \bar{\pi}(0, S_0^D) - \int_0^t D_2\bar{\pi}(\theta, S_\theta^D)r_1(t)S_\theta^D\, d\theta \\ &\quad - \frac{1}{2}\int_0^t D_{22}\bar{\pi}(\theta, S_\theta^D)S_\theta^{D^2}(v^2(\theta, \omega) - \sigma^2(\theta))d\theta. \end{aligned}$$

Combining (1) with the self-financing strategy and the fact that the portfolio is Δ-hedged ($\bar{H}_\theta = D_2\bar{\pi}$), we have

$$\begin{aligned} \mathrm{PnL}_t^D &= \bar{V}_0^D - \bar{\pi}(0, S_0^D) \\ &\quad + \bar{\pi}(t, S_t^D) - \pi(t, S_t^D) \\ &\quad + \frac{1}{2}\int_0^t S_\theta^{D^2}(\sigma^2(\theta) - v^2(\theta, \omega))D_{22}\bar{\pi}(\theta, S_\theta^D)d\theta. \end{aligned}$$

[1]We call profit the value of the strategy. This value can be positive or negative. So a sentence like "profit accumulate slowly" should be interpreted as "the value change slowly". It can also be a loss accumulating. So profit can be a very bad thing!

4. Interpretation

The profit is

$$\begin{aligned}\mathrm{PnL}_t &= e^{\int_0^t r_0(s)ds}\left(\bar{V}_0 - \bar{P}(0,S_0)\right) \\ &\quad +\bar{P}(t,S_t) - P(t,S_t) \\ &\quad +\frac{1}{2}\int_0^t {S_\theta}^2(\sigma^2(\theta) - v^2(\theta,\omega))D_{22}\bar{P}(\theta,S_\theta)d\theta\end{aligned}$$

and is made of three parts.

The first one is the (actualized) difference between the premium paid for the option and its value using the estimated volatility σ. It can be interpreted as the initial margin made on the trade or the fees. This is a fixed amount decided by the trader. There is no (model) risk associated to this amount. So we usually suppose that it is zero.

The second one is the difference, at the moment of the computation of the profit, between the value computed using the estimated volatility of the option and the true one. It is the profit that appears when the real volatility is discovered. At the maturity of the option, as the value of the option is its intrinsic value whatever the volatility is, this part is zero. If the option is never hedged by another option and never marked-to-market with its true value, this part is never known.

The third one is due to the incorrect Δ used for the Δ-hedging. It is the integral of the gamma of the option (second derivative of the price with respect to the underlying) times the square of the price multiplied by the difference between the square of the expected volatility and the true one. This is the most insidious part of the risk. If the position is held to maturity this profit accumulate slowly and this source of profit remains unknown.

This last part can be interpreted as the cumulative cost of the non-self-financing part of the theoretical strategy. It is the amount that has to be injected in the portfolio to maintain the theoretical quantity of asset and cash. In [4, Section 7.1.1], it is obtained for constant σ using this idea, even it is not obvious that this number does not only include the necessary injections but also the financing cost of such injections.

4.1. **Convexity.** When the price is convex in S_t, Γ is always positive and thus an overestimation of the volatility gives always a positive profit even if the wrong volatility is used for the hedging. In the case were the payoff is a convex function of the price, the price of the option is itself convex ([7, Section 4.5, p. 83]). The call and the put are particular examples of this.

4.2. **Gamma reduction.** Suppose that one is hedging a call. Looking at the profit, it can be tempting to try to reduce Γ to reduce the importance of the third part. This can be done making σ close to 0. Then Γ converges to 0 almost everywhere. From there one can be tempted to deduce that when σ tend to 0 the risk disappears. From a financial point of view this would mean that you can hedge the option without risk by choosing a volatility of (very close to) 0. This correspond to a stop-loss strategy (see [6, Section 13.3, p. 309]). From a technical point of view this is not correct. It can not be deduced from the fact that Γ tend to 0 almost everywhere that its integral tends to 0. Lebesgue dominated convergence theorem can not be used here because the functions are not dominated when $S = Xe^{-r(T-t)}$ (at the money option). This strategy is cost-free except when the option is at the money (which will be the case with a non-zero probability). In this case there is an infinite number of rebalancing of all the portfolio (Δ jump from 0 to 1 and inversely, Γ is infinite) and the cost is high.

Note nevertheless that the gamma used is the one with respect to the estimated Black and Scholes model, so it is known. If the option is chosen to reduce the gamma, the risk can be reduced (see Section 6.6).

4.3. **Structural robustness.** At maturity, the value of the option is model independent. The only part left is the third one. In particular we don't even need the *existence* of a true *price* of the option between the initial transaction and the maturity to have this result. The only thing we need is the existence of the price of the *asset*. The volatility can be very *wild* (stochastic, time dependant, depending of S) and the result still hold. So even if the true equation of the underlying is not as nice as described by the Black and Scholes model, some bound can be obtain for the profit. It is enough to know an upper bound of the volatility to obtain surely a positive profit. This *structural robustness* is very important from a practical point of view. Without such a property a trader would have to be convinced that the Black and Scholes model he uses to trade and hedge options is perfect and he knows the exact volatility. With the property it is enough that he believes that he can estimate well enough the volatility to obtain an upper bound or at least not to be too below the reality. It is strange (as noted in [2]) that such a robustness property which is very important for the practicality of the technic is rarely mentioned in standard books on the subject.

4.4. **Growth rate.** Even if it doesn't appear explicitely in the formula, the growth rate μ plays a role through S_t. It is not use for the pricing and the rebalancing but influences the final profit and thus the *risk*. This can be viewed more explicitely by writing the stochastic derivative of PnL^D (see also [5]):

$$\begin{aligned} d\,\mathrm{PnL}^D &= r_1(t)\bar{\Delta}_t S_t^D dt + \bar{\Delta}_t dS_t^D - (r_1(t)\Delta_t S_t^D dt + \Delta_t dS_t^D) \\ &= \left(\bar{\Delta}_t - \Delta_t\right) S_t^D \left((\mu - r_0(t) + r_1(t))\,dt + \sigma\,dW_t\right). \end{aligned}$$

5. Bound on the profit

In this section we describe a bound on the profit for calls (or puts) under certain restrictions on the volatility and the rates. The estimated equation for the underlying is a pure geometric Brownian motion (σ constant). The true volatility is bounded by two constants, i.e. there exists $\sigma_- > 0$ and σ_+ such that

$$\sigma_- \leq v(t, \omega) \leq \sigma_+.$$

The interest rates (r_0 and r_1) are constant[2] We denote by X the strike price of the option.

In this case

$$\Gamma = \frac{N'(d_1)e^{-r_1(T-t)}}{S\sigma\sqrt{T-t}}$$

where

$$d_1 = \frac{\ln(S/X) + (r_0 - r_1 + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The first bound we can obtain has already been described in the previous section. As the payoff of the call is convex, when $\sigma \geq \sigma_+$, the profit is positive and when $\sigma \leq \sigma_-$, the profit is negative.

To present the second bound we need a longer argument. To obtain the bound, we maximize $I_t(S) = \Gamma(t, S)S^2$ for a fixed t. For any t, it can be proved that $I_t \geq 0$ (due to convexity), $\lim_{S\to 0} I_t(S) = 0$ and $\lim_{S\to +\infty} I_t(S) = 0$ (details are given in Appendix A). Then there exists a maximum of I_t in $[0, +\infty)$. We denote it S_{max}. As I_t is differentiable in the interval, this maximum is such that $I'_t(S_{\mathsf{max}}) = 0$. A strait-forward computation gives

$$S_{\mathsf{max}} = Xe^{-(r_0 - r_1 - \sigma^2/2)(T-t)}.$$

In S_{max}, we have $d_1 = \sigma\sqrt{T-t}$ and thus

$$I_t(S_{\mathsf{max}}(t)) = \frac{N'(\sigma\sqrt{T-t})}{\sigma\sqrt{T-t}} S_{\mathsf{max}} = \frac{X}{\sqrt{2\pi}\sigma} \frac{e^{-(r_0-r_1)(T-t)}}{\sqrt{T-t}}.$$

[2] The same construction is possible if $\sigma, \sigma_-, \sigma_+, r_0$ and r_1 are time dependent, but then we don't have the explicit formula anymore.

From this we obtain the bound on the profit. When $r_0 \neq r_1$, we have

$$\begin{aligned}
\mathrm{PnL}_t - (\bar{P} - P) \\
&\leq \frac{1}{2}(\sigma^2 - \sigma_-^2)\int_0^t I_\theta(S_{\mathsf{max}}(\theta))d\theta \\
&= \frac{1}{2}(\sigma^2 - \sigma_-^2)\frac{1}{\sqrt{2\pi}}\frac{X}{\sigma}\int_0^t \frac{e^{-(r_0-r_1)(T-\theta)}}{\sqrt{T-\theta}}d\theta \\
&= \frac{1}{2}(\sigma^2 - \sigma_-^2)\frac{1}{\sqrt{2\pi}}\frac{X}{\sigma}\frac{\sqrt{2}}{\sqrt{r_0-r_1}}\int_{\sqrt{2(r_0-r_1)}\sqrt{T}}^{\sqrt{2(r_0-r_1)}\sqrt{T-t}} e^{-\frac{1}{2}\xi^2}\,d\xi \\
&= \frac{1}{\sqrt{2}}(\sigma^2 - \sigma_-^2)\frac{X}{\sigma\sqrt{r_0-r_1}}\left(N(\sqrt{2(r_0-r_1)}\sqrt{T}) - N(\sqrt{2(r_0-r_1)}\sqrt{T-t})\right)
\end{aligned}$$

So if we denote by $f(X,T,r_0,r_1)$ the two factors on the right, we have

$$\frac{1}{\sqrt{2}}(\sigma^2 - \sigma_+^2)f(X,T,r_0,r_1) \leq \mathrm{PnL}_t - (\bar{P} - P) \leq \frac{1}{\sqrt{2}}(\sigma^2 - \sigma_-^2)f(X,T,r_0,r_1).$$

In particular if the volatility is overestimated ($\sigma \geq \sigma_+$), the profit is always positive.

When $r_0 = r_1$, we have

$$\begin{aligned}
\mathrm{PnL}_t - (\bar{P} - P) &\leq \frac{1}{2}(\sigma^2 - \sigma_-^2)\int_0^t I_\theta(S_{\mathsf{max}}(\theta))d\theta \\
&= \frac{1}{2}(\sigma^2 - \sigma_-^2)\frac{1}{\sqrt{2\pi}}\frac{X}{\sigma}\int_0^t (T-\theta)^{-\frac{1}{2}}d\theta \\
&= 2(\sigma^2 - \sigma_-^2)\left(\sqrt{T} - \sqrt{T-t}\right)
\end{aligned}$$

This bound can not be improved as $S_t = S_{\mathsf{max}}(t)$ is a possible path. It is a (too) smooth growth at rate $\sigma^2/2 - r_0 + r_1$ to the strike price.

6. Numerical results

As the profit depends in a non-trivial way of all the path of the price (and not only its final value), an analytical description of its distribution seems out of reach.

We give some numerical results[3] in the case where the true equation is a pure geometric Brownian motion (v constant). The methodology we use to obtain the results is the following. For each result, we have simulated a large number of path of the underlying (typically 10,000). Each path is made of 100 or more time-steps. We used those simulated paths to numerically compute the integral.

In the different figures, we have represented the minimum, the 25%, 50%, the 75% percentile, the average, and the maximum of the different profit realizations. The minimum and maximum realizations are

[3] Matlab code available

represented by the dashed line. The 25, 50 and 75% percentile are in dotted line. The average is in continuous thin line.

We have also computed the profit that would be realized if the transaction is done at the estimated volatility and the hedging is done from the start with the correct volatility. This correspond the initial difference of value with the estimated volatility and the true one actualized (at the constant rate r_0). This profit is called the initial profit and in the figures it is represented by the thick line. The theoretical maximum and minimum are also represented, in continuous thin lines.

We describe the influence of several parameters on the profit.

Volatility error: For a fixed v, how the profit changes with σ.

True volatility: For a given volatility error, how the profit changes with v.

Time: How the profit evolves between the initial transaction and the maturity of the option.

Growth rate: The growth rate μ does not influence the price and the hedging of the option, but influences the final profit. This show that risk management is more difficult than pricing.

Portfolio: The maximum profit on a portfolio of options is not the sums of the maximum profits on the components. We give a example of reduction and increase of profit.

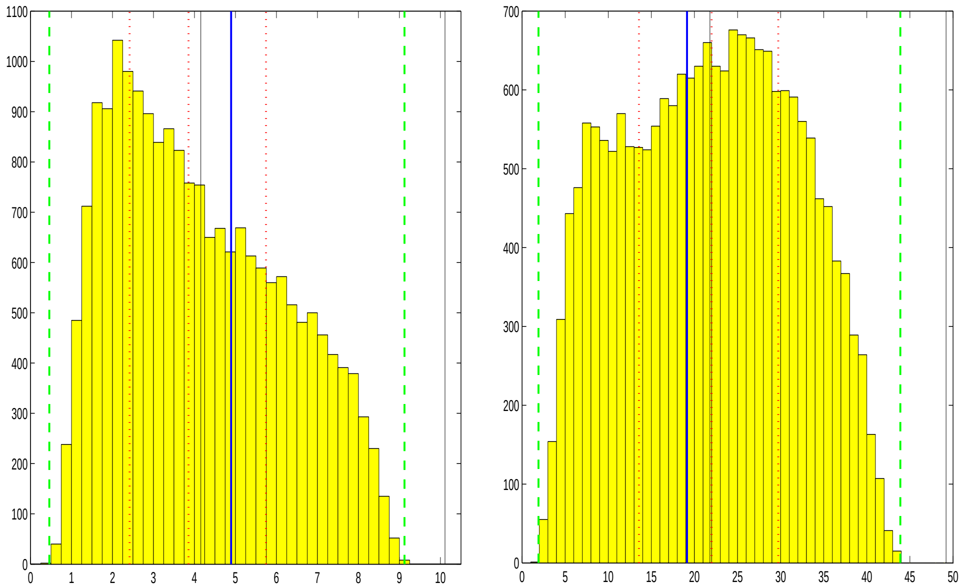
The maturity of the option can be viewed as a scaling factor. All the options we study have a maturity of 1. An option with a maturity of 0.5 and a volatility of σ is just an option with maturity 1 and volatility $\sigma/4$ in a world with a different unit of time.

6.1. **Distribution of the error.** Figure 1 represents the distribution of the profit due to pricing and hedging of an option with a wrong volatility. The initial asset price is 100. The strike price are respectively 90 and 115. The true volatility is 20% and the estimated one is 22%. The growth rate is 10%, the risk free rate and the dividend rate are 5% and the time to maturity is 1 year. The sample consist of 20,000 simulations of 200 time-steps.

The distribution is not symmetric around the average. The shape of the distribution depends of the parameters and is discussed in the rest of this section.

6.2. **Volatility error.** Figure 2 represents the profit due to pricing and hedging of an option with a wrong volatility for different errors. The strike price is 90 and the time to maturity is 1 year. The true volatility is 20%, the growth rate is 5% and the dividend rate 3%. The initial price is 100. The sample consists of 10,000 simulations of 200 steps.

As expected the profit depends strongly on the volatility error. The profit has the same sign as $\sigma - v$. The average profit is almost symmetric around the true volatility. This is also the case for the initial profit.



(a) Strike price of 90 (b) Strike price of 115

FIGURE 1. Numerical simulation of the distribution of the profit (as percentage of the true price).

Nevertheless the extreme profits are larger for underestimations of the volatility than for overestimation. This non-symmetry can be observe in the bound on the profit in the factor $1/\sigma$.

6.3. **True volatility.** If we suppose that the error is fixed (absolute or relative), the profit changes with the true volatility.

Figure 3 shows the profit for an error of 1% of the annual volatility. The other parameters are: $\mu = 0.10$, $r_0 = 0.05$, $r_1 = 0.03$, $X = 100$, $T = 1$, and $S_0 = 80$. The simulation uses 10,000 iterations of 200 steps. When the true volatility is 10%, the average profit is 0.1362; when the volatility is 30%, the average profit is 0.3163. It means the for an equal absolute error, the profit changes from the simple to more than the double. When the volatility is high, delta-hedging is riskier; but also the same volatility margin is generating a larger profit.

Figure 4 shows the profit for an error of one tenth of the volatility. The other parameters are: $\mu = 0.10$, $r_0 = 0.05$, $r_1 = 0.03$, $X = 90$, $T = 1$, and $S_0 = 1000$. Then the result is even worse; the average profit range from 0.0880 when v is 10% to 0.9628 when v is 30%, which is more than 10 times higher.

6.4. **Growth rate.** Figure 5 represent the profit for different growth rates. The option we used is a call with a strike of 100 and a maturity of one year. The true volatility of the geometric Brownian motion is

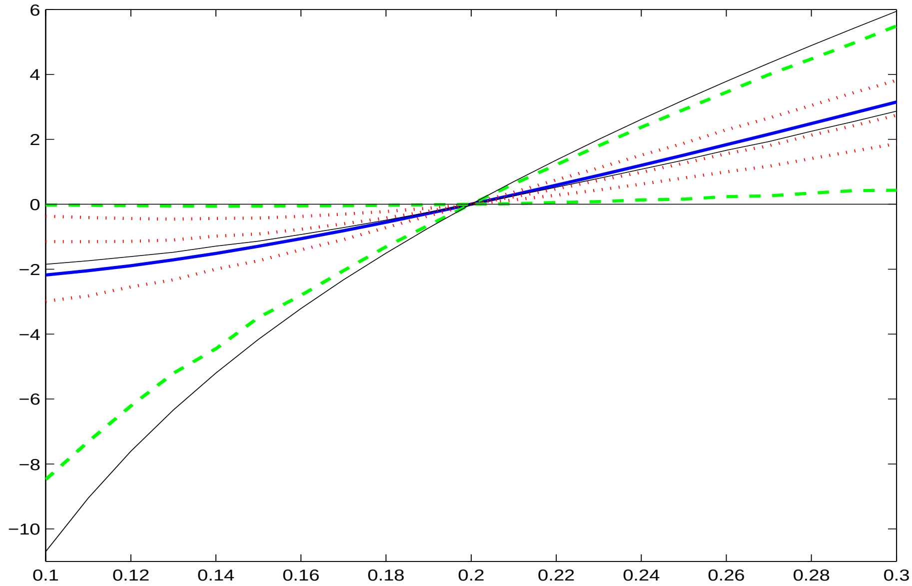


FIGURE 2. Numerical simulation of the profit (as percentage of the true price) for different values of the estimated volatility σ.

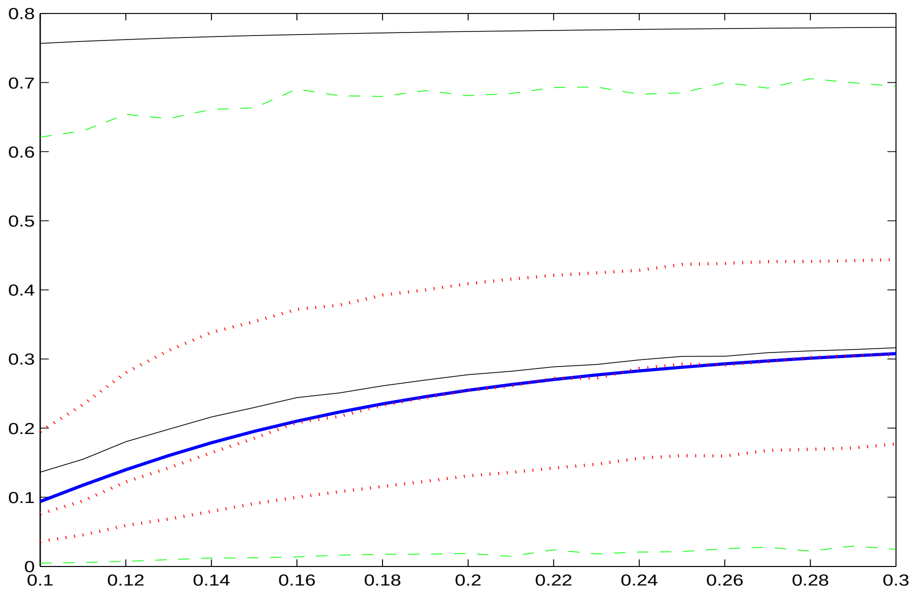


FIGURE 3. Numerical simulation of the profit for different values of the true volatility and an error of 1% of annual volatility.

20%. The true initial price of this option is 8.4333. All the values we use in this subsection are percentages of this value.

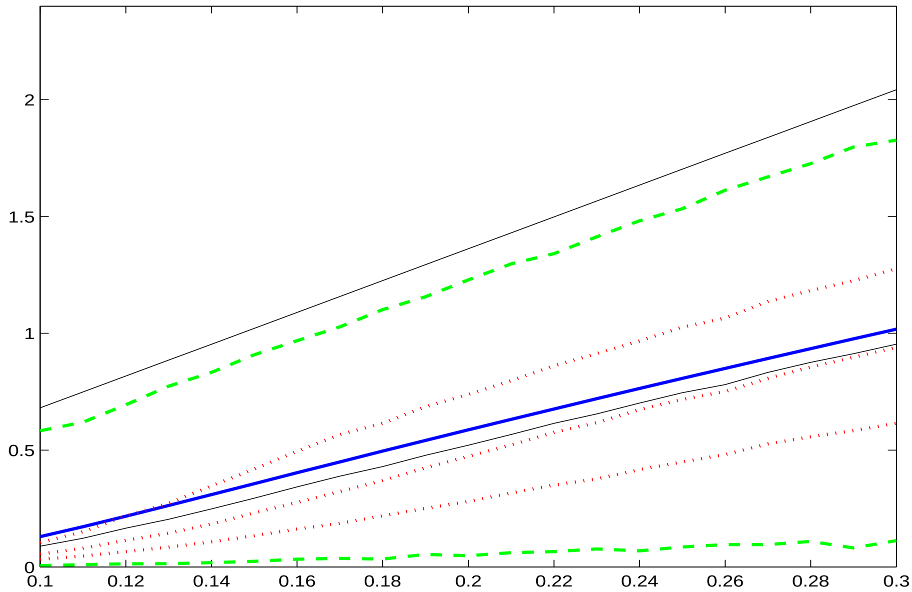


FIGURE 4. Numerical simulation of the profit for different values of the true volatility and an error of one tenth of the true volatility.

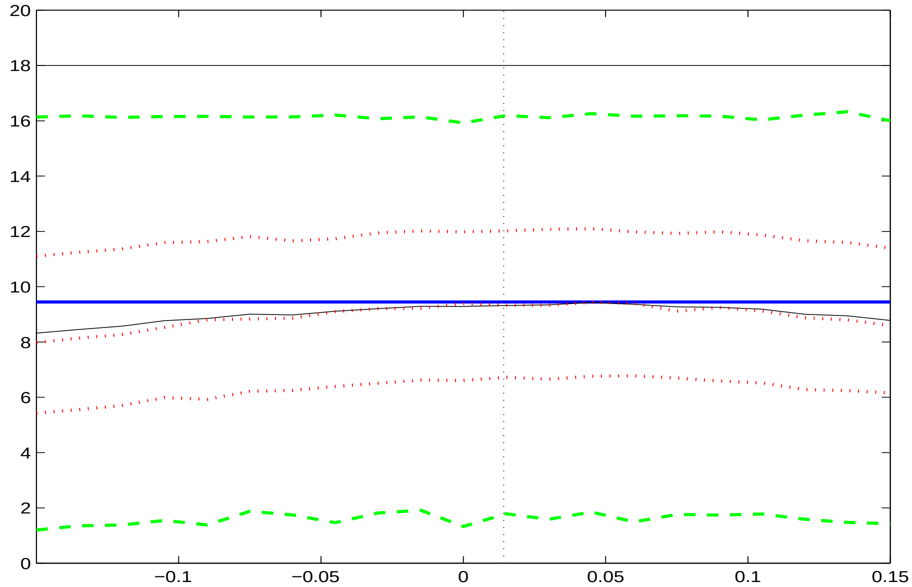


FIGURE 5. Numerical simulation of the profit (as percentage of the true price) for different values of the growth rate μ.

The estimated volatility used for pricing and hedging the option is 22%. If the option had been sold with a price corresponding to this volatility of 22% but hedge (perfectly) with the true volatility, the profit would have been 9.4485% of the true initial price. As the estimated volatility is higher than the true one, the minimum profit is zero (see Section 5). The maximum profit is given by the horizontal thin line and is 18.0021%.

We have estimated the values of the profit of different realizations of the stochastic equation and the hedging. This has been done for 21 different growth rates (μ) between -15% and 15%.

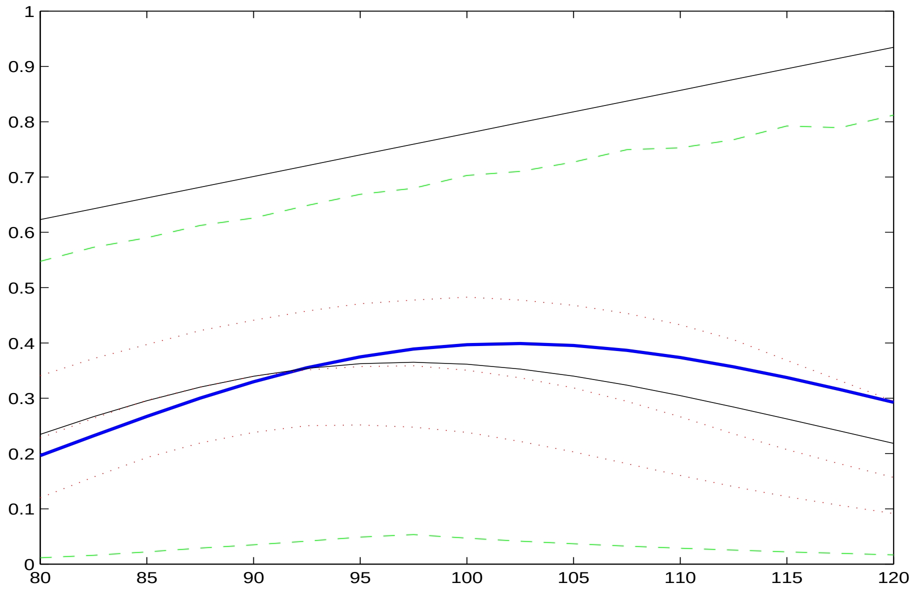
The average and the percentile are influenced by the growth rate. The growth of the smooth path for which the profit is maximum (see Section 5) is given by the vertical dotted line. When the actual growth rate is close to this rate, on average the paths will be closer to this profit-maximizer path. That is why the average profit is higher when the growth rate is close to profit-maximizer rate. In the numerical results we obtain, the difference between the highest average (9.4276%) and the minimum (8.3179%) reaches 1.1097%.

This parameter, which is usually not estimated when dealing with options using the Black and Scholes model, can change the average profit by about 1% of the value of the product sold. This is about 11.5% of the initial margin that could have been obtained by hedging with the true volatility.

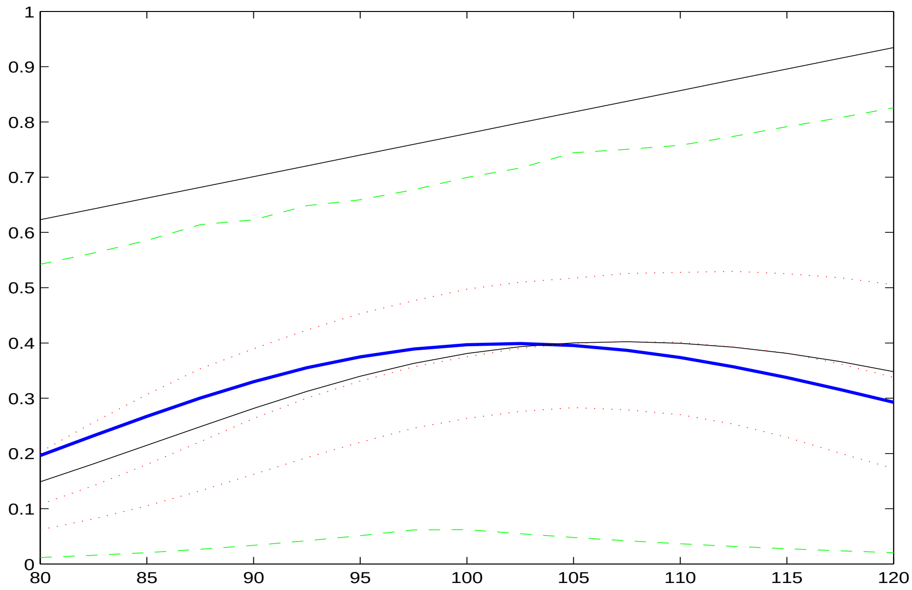
The difference between the average profit and the initial one depends also of the strike price. Figure 6 shows the absolute profit when $\mu = -0.1$ and $\mu = 0.1$ for strikes between 80 and 120 and $S_0 = 100$. The figures were done using 17 different strikes; for each of them 10,000 simulations with 200 steps were used. As can be seen from the graphs, the average is above or below the initial profit depending of the strike and this relation depends itself from the growth rate.

Suppose that in a bank two traders (A and B) have a superior predictive ability with respect to the market to estimate the volatilities. Suppose also there is a set of independent underlying (or the same underlying on several periods). The volatility for each of them is 20 or 21% and for each of them the market price is 20 or 21%. By saying that the hedgers as a superior predictive ability with respect to the market, we mean that when the market estimation is correct, the hedgers are correct also and when the market is wrong, the hedgers can be wrong like the market or correct. The market price can be the price available outside the bank or any price that is used for internal transactions.

The traders are trading options among themselves at the market price and delta hedge them with the outside market. All the option transactions are done internally. So if the price is not perfect, it is only an internal transfer of profit. The market price is only used to compute the profit of the individual traders. For the total of the bank, the market price is irrelevant. The only thing that change the total



(a) $\mu = -0.1$



(b) $\mu = +0.1$

FIGURE 6. Numerical simulation of the profit for different values of the strike price.

profit is the fact that one trader is correctly estimating the volatility and the other not[4].

[4]If both traders are wrong, but one less than the other, the result of this section stays similar.

If both traders have the same estimates, they will always have opposite position, not only in term of options but also in term of hedging. So the total profit for the bank will always be zero. If they don't have the same estimates, one is correct and one is wrong. As the traders are superior to the market, the market is wrong. The possibilities are summarized in the following table.

	Trader A / Trader B	
Market – True	20/21	21/20
20 – 21	short/long -average/+initial	short/long +initial/-average
21 – 20	long/short +initial/-average	long/short -average/+initial

Let's comment on the first line; the second is just the symmetric situation. If the market price is 20 then B, the trader estimating the volatility at 21, perceives the price as underestimated and so will be long. As he is correct in his estimate and uses it for hedging, he will make the initial profit. The other trader will hedge the opposite position with the wrong hedging and will make the average profit (if we suppose that there are enough independent positions to use the average).

We apply this to the case describe in Figure 6 for calls largely out-of-the-money with a trike price of 115. If $\mu = 0.1$, the profit will be -0.044: 0.3373 of initial profit minus 0.3813 of average profit. If $\mu = -0.1$, then the profit will be $+0.0748$: 0.3373 of initial profit minus 0.2625 of average profit. Note also that in the case where $\mu = -0.1$, the initial profit is at the level of the 75% percentile, i.e. 75% of the time the profit will be below the profit that would have been obtained by hedging the option correctly.

This means that in the situation described here, even if the traders have a superior predictive ability, a risk manager should not allow the traders to trade internally. That type of position will generate losses for the bank when the growth rate is positive. This further emphasizes the fact that risk management methods are not only pricing methods. The internal trades are not only transfers of *profit*, but also transfer of *risk* from traders with superior predictive ability to less capable traders.

6.5. **Time.** To study the effect of time on profit, we change slightly our approach. We suppose that a call is sold at the true volatility (21%) but hedge with a lower volatility (20%). This represents the situation where someone sell an option above its estimated volatility, expecting to make a profit, but the price was the correct one and the hedging is wrong. For that scenario we compute two profits: the true one and the one estimated by the trader. For the last, the option position is valued using the estimated volatility (not the true one).

The parameters used are the following: growth rate of 10%; risk free rate of 5%, dividend rate of 3%; initial value of 100; strike price of 100; time to maturity of 1. The simulation consists of 500 path of 100 steps. As we have one computation of the profit for each time step and not only the final one, the computer time is larger for the results of this section.

The convention for the results given in Figure 7 is the same that in the previous sections. Here we have two sets of results. The one starting at 0 is the true profit, the one starting at about 0.4 is the estimated one.

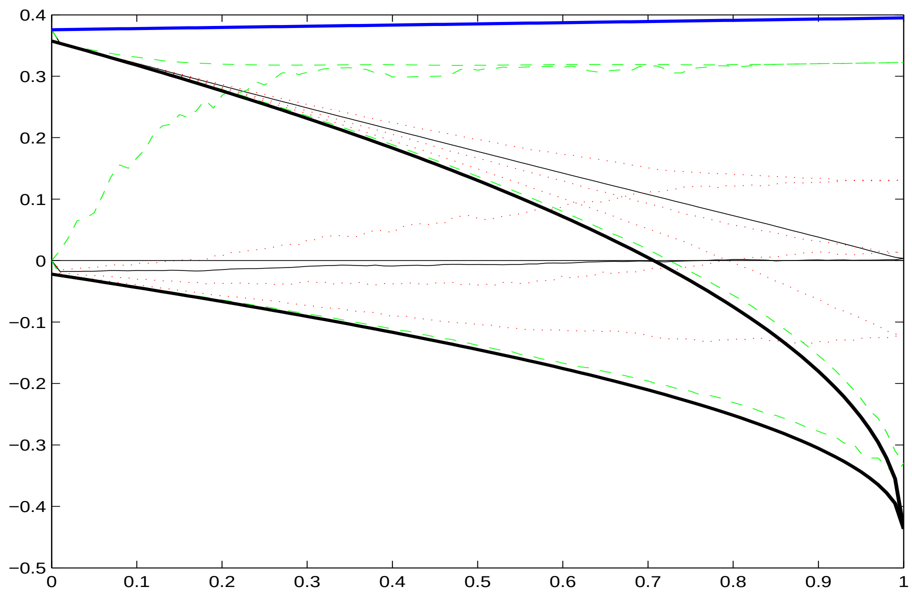


FIGURE 7. Numerical simulation of the profit between the initial transaction and the expiry of the option.

As the final value of the option is not parameter dependent, both converge to the same values. The thick almost horizontal line represents the expected profit at the start of the deal. This is the value of the position in 0 using the estimated volatility for the valuation accruing at the risk free rate. This value represents the *maximum* possible profit (both for the estimated and the true profit). This means that the profit expected by the dealer will never be achieved. The minimum value for the estimated profit is given by the initial profit plus the bound obtained in Section 5. The minimum value for the true profit required an extra-computation. The term $\bar{P}(t, S_t) - P(t, S_t)$ has also to be estimated. The minimum for that term is

$$\bar{P}(t, S_t) - P(t, S_t) = Xe^{-r_0\theta} \left(N(d_2) - N(-d_2)\right)$$

with $d_2 = \frac{1}{2}(\sigma - v)\sqrt{\theta}$.

6.6. **Portfolio.** We take two portfolios of calls that we compare with a call of strike 100. The pay-off functions of the portfolios are equal for values of the underlying between 90 and $100+5\sqrt{2}$. Between those values the pay-off of the portfolio are modified: one to reduce the parameter risk and the second one to increase it.

The first portfolio is composed of two options: long a call with strike $100-5\sqrt{2}$ on 1/2 asset and long another one with strike $100+5\sqrt{2}$ on 1/2 asset. The second portfolio is composed of three options: long a call with strike 90 on one asset, short a call with strike 95 on 2 assets and long a call with strike 100 on 2 assets. Those two portfolios are chosen such that the average of the difference between the pay-off functions is 0. The pay-off functions are represented in Figure 8.

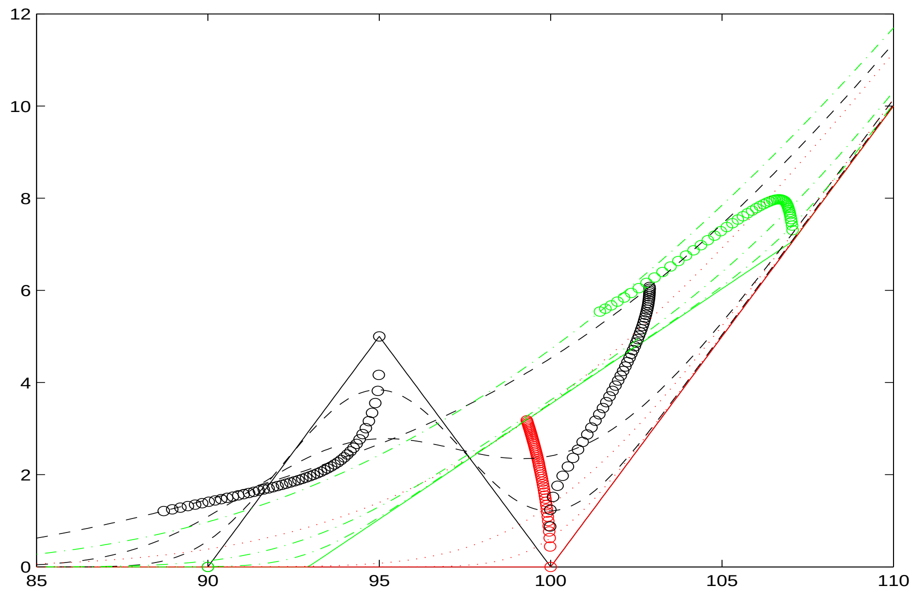


FIGURE 8. Pay-off for the portfolios. The curves represent the value of the portfolio 1 week, 1 month and 6 months before maturity. The dotted lines represent the call, the dashed-dotted lines represent the first portfolio and the dashed lines represent the second one. The circles represent the points where the Gamma is maximum or minimum.

For the rest of this section, we use a initial value of the asset of 100. The true volatility is 10%, the estimated one is 11%. The time to maturity is 0.5. Note that the initial vega (derivative of the option price with respect to the volatility) is of the same order for the different portfolios: 27.4026 for the call, 18.2937 for the reducing-risk portfolio, and 24.0470 for the increasing-risk portfolio. Note in particular that the increasing-risk portfolio has a lower vega than the call. This number is often used as a way to measure the risk with respect to change or error in volatility estimates. But it measures the instantaneous risk, not the risk during the life of the position. A deal can have little risk today but creates significantly more in the future (see comments of Donna Howe in the framework of Raroc [1]).

Figure 9 shows the distribution of profit for the 3 portfolios. The distribution is constructed with 10,000 samples of 100 time-steps. The vertical dotted lines represent the average profit; the continuous lines represents the theoretical extremum for the profit. The average profit in the sample are 0.2594 for the call, 0.1793 for the first portfolio and 0.2580 for the second. The maxima are 0.5285 for the call, 0.2941 for the first portfolio and 0.7824 for the second one. The call and the first portfolios are convex, so the minima are 0. The minimum for the second portfolio is -0.4646.

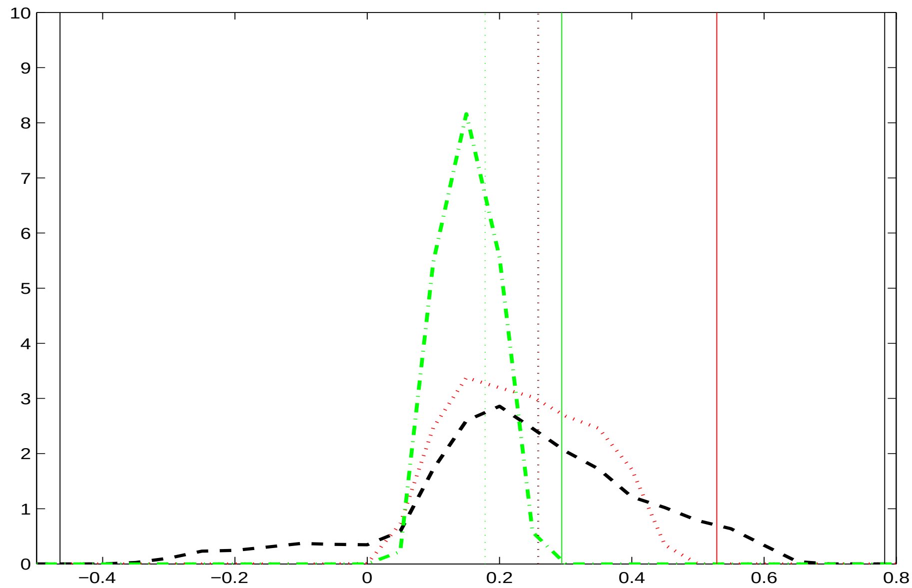


FIGURE 9. Simulated distribution for the call and the two portfolios. The dotted lines represent the call, the dashed-dotted lines represent the first portfolio and the dashed lines represent the second one. The vertical lines are the theoretical extremum.

As it is clear from Figure 9, the distributions are quite different. Even if the initial price, the initial vega and the pay-off are somehow similar. The difference lies not only in the average but mainly in the extreme profits. The reducing risk portfolio reduce the distribution of risk by about a factor of 2. The increasing risk portfolio doesn't change much the average risk but change drastically the distribution of it. Even if at the start we are globally long of a call and we overestimate the volatility, the profit can be negative!

Appendix A. Details of the computations of the upper bound on the profit

We prove that $\lim_{S\to 0} I_t(S) = 0$ and $\lim_{S\to +\infty} I_t(S) = 0$.

The first one is simple as when $S \to 0$, $d_1 \to -\infty$, so $d_1^2 \to +\infty$ and $N'(d_1) \to 0$. This shows that $I_t(S) = S^2\Gamma(t,S) \to 0$.

For the second, we need some extra computation. We have for $S > X$,

$$d_1^2 = \left(\frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)^2 \geq \frac{1}{\sigma^2(T-t)}\left(\ln(S/X)\right)^2.$$

So by posing $S/X = y$, it is enough to prove that $\lim_{y\to +\infty} y \exp(-\alpha(\ln y)^2) = 0$ for $\alpha > 0$. As this last limit can be written as $\exp(-\alpha(\ln y)^2 + \ln y)$ and that the argument of the exponential tend to $-\infty$, the result is proved.

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