On Higher Derivatives of Expectations

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Abstract

It is understood that derivatives of an expectation $E[\phi(S(T))|S(0) = x]$ with respect to x can be expressed as $E[\phi(S(T))\pi|S(0) = x]$, where $S(T)$ is a stochastic variable at time T and π is a stochastic weighting function (weight) independent of the form of ϕ. Derivatives of expectations of this form are encountered in various fields of knowledge. We establish two results for weights of higher order derivatives under the dynamics given by (1). Specifically, we derive and solve a recursive relationship for generating weights. This results in a tractable formula for weights of any order.

Key words: Price Sensitivities, Greeks, Malliavin Calculus.

1 Introductory Remarks

Consider a real valued process $\{S(t), 0 \le t \le T\}$ of which dynamics are described by the stochastic differential equation with an accompanying strong solution

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad S(t) = x \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right), \tag{1}$$

for $S(0) = x > 0$,, and associated tangent process $\{Y(t), 0 \le t \le T\}$ described by

$$\frac{dY(t)}{Y(t)} = \mu dt + \sigma dW(t), \quad Y(0) = 1. \tag{2}$$

where $\{W(t), 0 \le t \le T\}$ is a one-dimensional Brownian motion, and coefficients μ and σ are constant. Consider an expectation $u(x) = E[\phi(S(T))|S(0) =$

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[1] I am grateful to Galuh Pandita for improving the clarity of my writing.

$x]$. It is well known that derivatives with respect to the initial condition of the form $\partial_x^n u(x)$ can be expressed as $E[\phi(S(T))\pi_n|S(0)=x]$, where π_n is a stochastic weighting function (weight), see for example [4]. A surprising result is that the weights are independent of ϕ, and in fact only depend on the dynamics of the underlying variable. This has practical importance, since ϕ depends on the problem under consideration while the weights do not. This means that the weights provide a problem independent, general representation of derivatives for many applications.

In this article we establish two results on the weighting functions for derivatives of the form $\partial_x^n u(x)$ under the dynamics given by (1). Specifically, we derive and solve a recursive relationship for generating weighting functions, providing a tractable formula for functions of any order. Furthermore, we relate the derivatives considered to derivatives of the drift and diffusion coefficients, comment on efficiently evaluating these expectations, uniqueness and optimality.

2 Mathematical Preliminaries

We approach this problem using Malliavin Calculus, appealing to a number of results in the following analysis that we briefly outline in this section [2]. Let $\mathcal{S}$ be the set of stochastic functions of the form $G = f\left(\int_0^T h_1(t)dW(t), \ldots, \int_0^T h_n(t)dW(t)\right)$, where f belongs to the set of infinitely differentiable functions with all partial derivatives of at most polynomial growth, and $h_1, \ldots, h_n$ belongs to the set of infinitely differentiable functions with bounded partial derivatives. The Malliavin derivative is defined as

$$D_t G = \sum_{i=1}^{n} \nabla_i f \left(\int_0^T h_1(t)dW(t), \ldots, \int_0^T h_n(t)dW(t) \right) h_i(t), \quad t \in [0,T] \quad (3)$$

where ∇_i denotes the derivative with respect to the i-th argument [3]. The adjoint of the Malliavin derivative is the Skohorod integral. For processes that are adapted, as is the process described by (1), the Skohorod integral coincides with the Ito integral. As a consequence, the Malliavin integration by parts formula can be expressed in terms of Ito integrals as $E\left[G\int_0^T h(t)dW(t)\right] = E\left[\int_0^T D_t G h(t)dt\right]$, where we take G as one dimensional and h the same form

[2] For a comprehensive introduction to the subject consult [8] and [9].

[3] We also define the norm on $\mathcal{S}$ as $\|G\|_{1,2} = \left(E(G^2)\right)^{1/2} + (E(\int_0^T (D_t G)^2 dt))^{1/2}$. Then $\mathcal{D}^{1,2}$ denotes the Banach space that is the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2}$.

as one of h_i, $i = 1, \ldots, n$. Also note that an analog of the chain rule in deterministic calculus exists for derivatives in Malliavin calculus.

3 Analysis

In this section we derive our principal results. We are interested in the Malliavin derivative of our process $\{S(t), 0 \le t \le T\}$. Taking the derivative directly we determine that [4]

$$D_t S(\tau) = S(\tau)\sigma 1_{\{t \le \tau\}}, \quad \tau \in [0, T]. \tag{4}$$

Consider the expectation $u(x) = E_x[\phi(S(T))]$, where we introduce the convention that $E_x[\cdot] \equiv E[\cdot | S(0) = x]$. Take the normal derivative of $u(x)$ with respect to x then

$$\partial_x u(x) = E_x[\phi'(S(T))Y(T)], \tag{5}$$

and interpret the tangent process at T as $\partial_x S(T)$. From (4) notice that $Y(\tau)1_{\{t \le \tau\}} = D_t S(\tau)(x\sigma)^{-1}$ which is true for any $t \le \tau \le T$, therefore for all [5] t

$$Y(\tau) = \frac{1}{\tau} \int_0^\tau D_t S(\tau)(x\sigma)^{-1} dt. \tag{6}$$

It follows from (5) using the chain rule and performing an integrating by parts that

$$\begin{aligned} \partial_x u(x) &= E_x \left[\frac{\phi'(S(T))}{x\sigma T} \int_0^T D_t S(T) dt \right] \\ &= E_x \left[\frac{1}{x\sigma T} \int_0^T \phi'(S(T)) D_t S(T) dt \right] \end{aligned}$$

[4] Alternatively, we could arrive at (4) by noting that in general $D_t S(\tau) = Y(\tau)Y^{-1}(t)\sigma(S(t))1_{\{t \le \tau\}}$ for $\sigma(S(t)) \equiv \sigma S(t)$ and by direct inspection $S(t) = xY(t)$. See [8] for details.

[5] We could have multiplied by a function prior to integrating, chosen such that $\int_0^\tau c(s)ds = 1$. However, this generality provides no advantage here, and it suffices to take $c = 1/\tau$.

$$
\begin{aligned}
&= E_x \left[\frac{1}{x\sigma T} \int_0^T D_t \phi(S(T)) dt \right] \\
&= E_x \left[\phi(S(T)) \left(\frac{W(T)}{x\sigma T} \right) \right],
\end{aligned} \tag{7}
$$

which is also achievable by an application of the Bismut-Elworthy formula. We turn our attention to the second derivative. Differentiating (7) with respect to x and making identical manipulations shows that

$$
\partial_x^2 u(x) = E_x \left[\frac{\phi(S(T))}{x^2 \sigma T} \left(\frac{1}{\sigma T} \int_0^T W(T) dW(t) - W(T) \right) \right]. \tag{8}
$$

But notice that the weight involves the difference between two iterated Ito integrals of the form $I_k = \int_{[0,T]^{\otimes k}} (\sigma T)^{-k} dW^{\otimes k}(t)$ such that $\pi_2 = (I_2 - I_1)/x^2$. This leads to the following proposition.

Proposition 1 *The n-th weight is $\pi_n = w_n / x^n$ where w_n is recursively generated by $w_n = m(w_{n-1}) - (n-1) w_{n-1}$ for all $n = 2, 3, \ldots$, with initial condition $w_1 = I_1$, where the linear function m is defined by $m(I_k) = I_{k+1}$.*

PROOF. Proceed by induction. The trivial case $n = 0$ corresponds to no differentiation and is avoided. The recursion is verified for $n = 1, 2$. Assume that the recursion is valid for n, then show for $n + 1$

$$
\begin{aligned}
\partial_{x^{n+1}}^{n+1} u(x) &= E_x \left[\partial_x \left(\phi(F_T) \pi_n \right) \right] \\
&= E_x \left[\partial_x \left(\frac{\phi(F_T)}{x^n} \right) w_n \right] \\
&= E_x \left[\left(\frac{\phi'(F_T) Y_T}{x^n} - n \frac{\phi(F_T)}{x^{n+1}} \right) w_n \right], \\
&= E_x \left[\frac{\phi(S(T))}{x^{n+1}} \left(\frac{1}{\sigma T} \int_0^T w_n dW(t) - n w_n \right) \right] \\
&= E_x \left[\frac{\phi(S(T))}{x^{n+1}} \left(m(w_n) - n w_n \right) \right] \\
&= E_x \left[\frac{\phi(S(T))}{x^{n+1}} w_{n+1} \right],
\end{aligned}
$$

This completes the proof. $\square$

In addition, this shows that we may represent $w_n = \sum_{k=1}^{n} a_{n,k} I_k$ with constant coefficients $a_{n,k}$. So the problem of solving the recursion reduces to solving for $a_{n,k}$ and I_k for all $k \leq n$. We first turn our attention to the coefficients. From Proposition 1 it follows by inspection that $a_{n,k}$ must satisfy the recursion relation

$$a_{n,k} = \begin{cases} a_{n-1,k-1} - (n-1)a_{n-1,k}, & \text{if } 1 \leq k \leq n, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

We observe that this recursion is solved by the Stirling numbers of first kind such that $a_{n,k} = s_1(n,k)$ with closed form solution [6]

$$s_1(n,k) = \sum_{i=0}^{n-k} \sum_{j=i}^{n-k} (-1)^{i+j} \binom{j}{i} \binom{n-1+j}{n-k+j} \binom{2n-k}{n-k-j} \frac{(j-i)^{n-k+j}}{j!},$$

for $n = 1, 2, \ldots$ and $k \leq n$, as noted in [2] (Appendix 4) due to [3]. Further, we see that iterated Ito integrals of tensor power can be solved by a formula noted in [9] (Section 1) due to [7]. For our case we have

$$I_k = (\sigma\sqrt{T})^{-k} h_k\left(\frac{W(T)}{\sqrt{T}}\right), \quad h_k(y) = \sum_{i=0}^{[k/2]} \frac{y^{k-2i}}{(k-2i)!} \frac{k!}{i!(-2)^i}, \tag{10}$$

for $k = 0, 1, 2, \ldots$, where h_k are the (modified) Hermite polynomials, and $[\,\cdot\,]$ means the integer part of. Together these results imply the following lemma.

Lemma 2 *The n-th weighting function for the n-th derivative of $u(x)$ with respect to x is*

$$\pi_n = x^{-n} \sum_{k=1}^{n} s_1(n,k)(\sigma\sqrt{T})^{-k} h_k\left(\frac{W(T)}{\sqrt{T}}\right). \tag{11}$$

PROOF. The proof follows by direct substitution. □

4 Concluding Remarks

We remark that these weights are not complicated - both Stirling numbers and Hermite polynomials are effortlessly evaluated in packages like Mathematica.

[6] Alternatively, coefficients can be expressed as $a_{n,k} = (-1)^{n-k} \varphi_{n-k}(1, \ldots, n-1)$, where $\varphi_{n-k}(1, \ldots, n-1)$ denotes the $(n-k)$-th symmetric polynomial on $1, \ldots, n-1$. We exclude the proof in sake of brevity.

Also, these weights are not unique. Indeed, there exists a spectrum of weights which corresponds to different choices of c in (6). However, these weights are optimal in the sense that they are the ones that minimize the variance of the resulting expectation. To see this note that (11) proves that the weights are always a polynomial function of $W(T)$. The strong solution (1) allows us to express $W(T)$, and thus the weights, in terms of $S(T)$. This means that the weights are $S(T)$ measurable, and hence optimal, see [5], [1].

By virtue of this representation we can evaluate these expectations efficiently and exactly, without the need of computationally expensive simulations. This is important in "real world" applications. We observe that since we can write the weights in terms of $S(T)$, the expectation becomes an integral over $S(T) \in [0, \infty)$ with respect to the known density. These integrals are easily solved.

We can represent derivatives with respect to the drift and diffusion coefficients in terms of the derivatives with respect to the initial condition, see [2]. Explicitly, we have that $\partial_\mu u(x) = xT\partial_x u(x)$ and $\partial_\sigma u(x) = x^2 \sigma T \partial_x^2 u(x)$; and we can derive higher derivatives in an analogous way to the analysis presented. Also note that, from our first remark, these weights will also be optimal.

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