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Chow's method of optimal control: A numerical solution

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1. Introduction

We are indebted to Michael Reiter (1996) for pointing out possible shortcomings of an algorithm suggested by Chow (1993) in solving dynamic optimization problems. The main purpose of Chow (1993) was to suggest that dynamic optimization problems can be more conveniently solved by the method of Lagrange multiplier than by dynamic programming, from both the analytical and the computational points of view. The former method provides a set of first-order conditions for determining the optimal control function and the Lagrange multiplier; the latter involves solving the Bellman equation for the value function, which is bypassed when the former method is used. This major point was not challenged by Reiter (1996). Reiter (1996) examines a particular numerical implementation in solving the first-order conditions for the control function and the Lagrange function. A suggestion of Chow (1993) was to approximate the (vector) optimal control function $u(x)$ and the associated (vector) Lagrange function $\lambda(x)$ by linear functions about any value x_t for which we need to evaluate the optimal control function. The first-order conditions are then linearized at x_t and solved for the parameters of the two linear

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functions. These parameters differ according to the point x_t chosen for the linearizations.

If the linearization are performed only at the steady-state values $(\bar{x}, \bar{u}, \bar{\lambda})$ of the variables which solve the deterministic version of the optimal control problems, as done in Eqs. (6)–(9) below, the resulting linear optimal control function can serve as a (globally) linear approximation to the optimal control function. Chow (1993) suggested performing the linearizations at each value x_t for which one needs to evaluate the optimal control function and using the resulting linear control function as an approximation of the optimal control function in the neighborhood of x_t. By examples, Reiter (1996) correctly points out the possibly poor performance of this approximation. The reason for possibly poor performance is that the first-order conditions apply to the two functions $u(x)$ and $\lambda(x)$ globally. When one approximates these functions by linear functions locally at x_t, the approximations may be poor for the global optimal control functions and the Lagrange functions which are supposed to satisfy the first-order conditions. In this note, we provide a numerical method to solve the first-order conditions provided in Chow (1993) for the optimal control function and the Lagrange function.

2. Galerkin solution

The problem is to find an optimal feedback control rule $u(x)$ solving the stochastic dynamic optimization programme:

$$\max_u E_t \int_t^\infty e^{-\rho(\tau - t)} r(x(\tau), u(\tau))\, d\tau = V(x(t)) \tag{1}$$

subject to

$$dx = f(x, u)\, dt + S(x, u)\, dw, \tag{2}$$

where $x(t)$ is a $p \times 1$ vector of state variables, $u(t)$ is a $q \times 1$ vector of control variables, ρ is a discount rate, E_t is the conditional expectation operator given information at time t which includes $x(t)$, $w(t)$ is a vector Wiener process with identity covariance matrix, $r(x, u)$ a differentiable and concave utility function, both f and S in the stochastic differential equation (2) are differentiable; the covariance matrix of $S(x, u)\, dw$ equals $S(x, u)\, S(x, u)'\, dt \equiv \Sigma(x, u)\, dt$, and $V(x)$ is the value function defined by (1).

As an alternative to stochastic dynamic programming, Chow (1993) suggests a Lagrange method which converts the problem into solving a set of functional

equations for the feedback control function, $u(x) = (u_1(x), u_2(x), \ldots, u_q(x))$, and a vector Lagrange multiplier function $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_p(x))$:

$$\rho\lambda_i = \frac{\partial r}{\partial x_i} + \frac{\partial f'}{\partial x_i}\lambda + \frac{\partial \lambda'}{\partial x_i} f + \frac{1}{2}\operatorname{tr}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial x'}\right)\Sigma\right] + \frac{1}{2}\operatorname{tr}\left[\frac{\partial \lambda}{\partial x'}\frac{\partial \Sigma}{\partial x_i}\right], \quad i = 1, 2, \ldots, p. \tag{3}$$

$$\frac{\partial r}{\partial u_i} + \frac{\partial f'}{\partial u_i}\lambda + \frac{1}{2}\operatorname{tr}\left(\frac{\partial \lambda}{\partial x'} \cdot \frac{\partial \Sigma}{\partial u_i}\right) = 0, \quad i = 1, 2, \ldots, q. \tag{4}$$

In operator notation the system in (3)–(4) can be compactly written as $R(g) = 0$, where $g = (u, \lambda) \in Y$, and Y is a function space, $R: Y \to Y$ is an operator; and the zero on the right-hand side is interpreted as the zero function. A standard approach to solving functional equation is to convert the infinite-dimensional problem into a sequence of finite-dimensional subproblems from which one obtains a corresponding sequence of approximate solutions that converge to the solution of the original problem. Let Y_n be a finite-dimensional subspace of Y. For example, if Y is the space of continuous functions, Y_n may be the space of polynomials of degree n. Unfortunately, there is, in general, no discrete solution $g_n \in Y_n$ that solves the functional equation exactly, but rather the discrete solution generates an error or residual, $R(g_n) \neq 0$, even though Y_n may converge to Y. A general approach is to find g_n that makes a *projection* of the residual vanish; and different projections lead to different methods. The *Galerkin method* is characterized by an orthogonal projection while the *collocation method* is characterized by an interpolation which is also a projection onto a subspace. We refer the readers to Hackbusch (1995, pp. 75–110) and Baker (1978, pp. 719–754) for details and proofs; and also Judd (1992) for a lucid exposition.

We now briefly describe the Galerkin method. To this end it is necessary to introduce some notations. Let Y be a Hilbert space with inner product $\langle f, g\rangle$, and $P_n: Y \to Y_n$ be an orthogonal projection operator, i.e. $P_n P_n = P_n$ and $\langle P_n f, g\rangle = \langle f, P_n g\rangle$ (self-adjoint); also assume $\{\Phi_i, i = 1, 2, \ldots, n\}$ forms a basis in Y_n so that there exists representation $g_n = \alpha_1\Phi_1 + \alpha_2\Phi_2 + \cdots + \alpha_n\Phi_n$. The Galerkin method requires the projected residual to vanish, i.e. find g_n such that $P_n R(g_n) = 0$. Taking inner product with Φ_j we may write $\langle P_n R(g_n), \Phi_j\rangle = \langle 0, \Phi_j\rangle = 0$. The left-hand side can be written as

$$\langle P_n R(g_n), \Phi_j\rangle = \langle R(g_n), P_n\Phi_j\rangle$$

$$= \langle R(g_n), \Phi_j \rangle \tag{5}$$

$$= \left\langle R\left(\sum_{i=1}^{n} \alpha_i \Phi_i\right), \Phi_j \right\rangle = 0, \quad j = 1, 2, \ldots, n.$$

The last line of (5) gives an algebraic equation system to be solved for $\alpha_1, \alpha_2, \ldots, \alpha_n$, which determines the Galerkin solution.

The equation system in (5) is usually nonlinear and has to be solved by an iterative method. It is important to have a good starting value for the nonlinear equation solver. In our application, we use as the starting value a linear approximate solution obtained by linearizing the first-order conditions (3)–(4) around steady state. Since the linear solution is of independent interest and may be adequate for many purposes, we will describe the method in detail in the next section.

3. Linearizing first-order conditions

This solution is obtained by assuming certainty equivalence (i.e. setting $S(x, u) = 0$) and solving a linearized version of the deterministic first-order conditions. The linearization is to be performed around the steady state $(\bar{x}, \bar{u}, \bar{\lambda})$ which satisfies the first-order condition

$$\rho\bar{\lambda} = \frac{\partial r}{\partial x}(\bar{x}, \bar{u}) + \frac{\partial f'}{\partial x}(\bar{x}, \bar{u})\bar{\lambda} + \frac{\partial \lambda'}{\partial x}(\bar{x}, \bar{u}) f(\bar{x}, \bar{u})$$

$$\frac{\partial r}{\partial u}(\bar{x}, \bar{u}) + \frac{\partial f'}{\partial u}(\bar{x}, \bar{u})\bar{\lambda} = 0, \quad \dot{x} = f(\bar{x}, \bar{u}) = 0. \tag{6}$$

Given the steady state, one then linearizes f and the partial derivatives:

$$\frac{\partial r}{\partial x} = \frac{\partial r}{\partial x}(\bar{x}, \bar{u}) + K_{11}(x - \bar{x}) + K_{12}(u - \bar{u}),$$

$$\frac{\partial r}{\partial u} = \frac{\partial r}{\partial u}(\bar{x}, \bar{u}) + K_{21}(x - \bar{x}) + K_{22}(u - \bar{u}), \tag{7}$$

$$f = f(\bar{x}, \bar{u}) + A(x - \bar{x}) + C(u - \bar{u}).$$

Let

$$\tilde{x} \equiv x - \bar{x}, \qquad \tilde{u} \equiv u - \bar{u}, \qquad \tilde{\lambda} \equiv \lambda - \bar{\lambda} \equiv H\tilde{x}. \tag{8}$$

Eqs. (6)–(8) imply a set of linearized first-order conditions:

$$\rho H\tilde{x} = K_{11}\tilde{x} + K_{12}\tilde{u} + A' H\tilde{x} + H(A\tilde{x} + C\tilde{u}),$$

$$K_{21}\tilde{x} + K_{22}\tilde{u} + C' H\tilde{x} = 0. \tag{9}$$

Define

$$\hat{u} = \tilde{u} + K_{22}^{-1}K_{21}\tilde{x}, \qquad \hat{K}_{11} = K_{11} - K_{12}K_{22}^{-1}K_{21}, \qquad \hat{A} = A - CK_{22}^{-1}K_{21}. \tag{10}$$

The first-order conditions can be further simplified to

$$\rho H\tilde{x} = \hat{K}_{11}\tilde{x} + \hat{A}' H\tilde{x} + H(\hat{A}\tilde{x} + C\hat{u}), \quad K_{22}\hat{u} + C' H\tilde{x} = 0. \tag{11}$$

The second equation in (11) gives

$$\hat{u} = -K_{22}^{-1}C' H\tilde{x} \equiv \hat{G}x. \tag{12}$$

Substituting (12) into (11) and equating coefficients, we obtain an algebraic Riccati equation well known in linear control theory (Kwaternaak and Sivan, 1972):

$$\rho H = \hat{K}_{11} + \hat{A}' H + H\hat{A} - HCK_{22}^{-1}C'H. \tag{13}$$

In summary, we obtain approximate linear solution by the following steps:

1. Iterating (13) until convergence to obtain H. This gives

$$\tilde{\lambda} = H\tilde{x},$$

or

$$\lambda = h + Hx, \qquad h = \bar{\lambda} - H\bar{x}. \tag{14}$$

2. Compute $\hat{G} = -K_{22}^{-1}C'H$. This gives

$$\tilde{u} = \hat{u} - K_{22}^{-1}K_{21}\tilde{x} = (\hat{G} - K_{22}^{-1}K_{21})\tilde{x} \equiv G\tilde{x},$$

or

$$u = g + Gx, \qquad g = \bar{u} - G\bar{x}. \tag{15}$$

4. Optimal growth model

As an example we are going to obtain a numerical solution to the neo-classical growth model

$$\max_{c} \int_0^{\infty} \mathrm{e}^{-\rho t} \left[\frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right] \mathrm{d}t \tag{16}$$

subject to

$$\dot{k} = k(t)^{\alpha} - \delta\, k(t) - c(t), \quad k(0) \text{ given.} \tag{17}$$

The state and control are (per capita) capital stock, $k(t)$, and consumption, $c(t)$, respectively. With state and control so defined, the first-order conditions (3)–(4) become

$$\rho\lambda = (\alpha k^{\alpha-1} - \delta)\lambda + \lambda'(k^{\alpha} - \delta\, k - c), \quad c^{-\sigma} - \lambda = 0. \tag{18}$$

One can in principle apply Galerkin method directly to find $c(k)$ and $\lambda(k)$, both represented as polynomials in k. In practice, however, it is necessary to transform variables appropriately so as to enforce sign constraints and achieve better numerical stability when solving the Galerkin projection equations. To enforce positivity on c and λ as indicated by the second equation in (18), and also facilitate the use of Hermite polynomials as basis functions (to be discussed below), we transform variables as

$$x = \ln(k), \quad u(x) = \ln(c(k)) - x, \quad \phi(x) = \ln(\lambda(k)) + \sigma\, x, \tag{19}$$

and thus,

$$\phi'(x) = \frac{\lambda'}{\lambda} \exp(x) + \sigma. \tag{20}$$

The first-order conditions can now be written as

$$-\sigma u - \phi = 0 \tag{21}$$

$$\alpha \exp((\alpha-1)x) - (\delta + \rho) + (\phi' - \sigma)\{\exp((\alpha-1)x) - \delta - \exp(u)\} = 0. \tag{22}$$

Eqs. (21)–(22) is a functional equation system to be solved for $u(x)$ and $\phi(x)$.

To apply Galerkin method one has to choose a family of basis functions from which a series representation of the solution can be constructed. For example, polynomials form such a family. In practice, however, the accuracy of the solution will depend heavily on the numerical condition of the algebraic

equation system in (5). An ill-conditioned system manifests in numerical instability and non-convergence of the nonlinear equation solver. The numerical condition of the equation system deteriorates quickly when an ordinary polynomial is used, and the problem is especially serious in this example as the state space is the whole real line. It is known that an orthogonal polynomial is far more stable numerically and is thus preferred to an ordinary polynomial as basis function. In our application we use the family of Hermite polynomials $\{H_k(x), k = 0, 1, 2, \ldots\}$ whose numbers are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)g(x) \exp(-x^2)\, dx. \tag{23}$$

The explicit expression for Hermite polynomial is complicated and inconvenient to use. In practice, we use a three-term recurrence relation to generate the polynomial and its derivatives. The relevant formula can be found in Davis and Rabinowitz (1984, p. 41)

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x); \quad H_0 = 1, H_1 = 2x.$$

Further numerical efficiency can be achieved by enforcing the steady-state condition on the solution. We thus write the control function and the Lagrange multiplier as

$$u(x) = a_0 + \sum_{k=1}^{n} a_k[H_k(x) - H_k(\bar{x})], \quad \phi(x) = b_0 + \sum_{k=1}^{n} b_k[H_k(x) - H_k(\bar{x})]. \tag{24}$$

By construction the two intercept terms should be the steady state of the control and Lagrange multiplier, respectively, and thus the number of unknown parameters in each expansion can be reduced by one if such restriction is imposed. One can also let the two intercepts remain free and check if they approach the corresponding steady states as the degree of polynomial goes up. This will provide a convergence check of the numerical method.

Let $R_i(a, b; x)$, $i = 1, 2$, be the left-hand side of (21)–(22) after substituting (24); and a and b denote, respectively, the vector of unknown coefficients in the polynomial expansions. The Galerkin projections can be written as

$$\int_{-\infty}^{\infty} R_i(a, b; x)H_k(x) \exp(-x^2)\, dx = 0, \quad i = 1, 2; \; k = 0, 1, \ldots, n. \tag{25}$$

We use the nonlinear equation solver NLSYS in GAUSS to solve (25) for the $2(n + 1)$ unknowns in vectors a and b. It is important to provide good starting value for the nonlinear equation solver. A good candidate is the linear approximate solution obtained by linearizing first-order conditions as described above.

Finally the integral in (25) admits no analytical expression and has to be evaluated numerically. We find it adequate to evaluate the integral by an 8-point Gauss–Hermite quadrature rule.

Table 1 reports polynomial solutions up to degree 6 with model parameter $(\alpha, \rho, \delta, \sigma) = (0.4, 0.05, 0.025, 0.5)$. Notice that the two intercept terms indeed approach the corresponding steady states as the degree of polynomial goes up. This indicates that the discrete solution is convergent. The column labeled by 'linear' is the linear approximate solution obtained by linearizing first-order conditions. Panel C of the table reports the residual norms given a solution, which should be identically zero if the solution is exact. Comparing the residual norms, we see that the Hermite polynomial solutions are more accurate than linear approximation. Notice that the intercept term of the linear solution is precisely the steady state. This is because we have written the linear solution in the form as in (24) to be compatible with other columns. This can be checked by noting that $H_1(x) = 2x$ and using the formula in (15). Figs. 1 and 2 depict the control function and the Lagrange multiplier function, with both axes measured as deviation from the steady state, so that both functions pass through the origin. As can be seen, the gap between the linear solution and polynomial solution widens as we move away from the steady state. Figs. 3 and 4 depict the same functions from the model with the parameter σ changed to 3.

5. Endogenous growth model

Our second example is taken from Reiter (1996) which can be regarded as a stochastic version of the so-called 'AK model' in the endogenous growth literature as in Barro (1990) and Rebelo (1991) among many others. The model is

$$r(x, u) = u^{\gamma}, \quad f(x, u) = \theta x - u, \quad S(x, u) = x\sigma, \quad 0 < \gamma < 1 \text{ and } \gamma\theta < \rho. \tag{26}$$

with exact solution

$$u(x) = \left[\frac{\rho - \gamma\theta}{1 - \gamma} + \frac{1}{2}\sigma^2\gamma\right] x \tag{27}$$

and value function

$$V(x) = \left[\frac{\rho - \gamma\theta}{1 - \gamma} + \frac{1}{2}\sigma^2\gamma\right]^{\gamma - 1} x^{\gamma}. \tag{28}$$

In the endogenous growth interpretation, u and x are per capita consumption and capital stock, respectively; and θx is a production function which is linear in

Table 1
Optimal growth model

Linear	Degree 1	Degree 2	Degree 3	Degree 4	Degree 5	Degree 6
Panel A: Control function						
−1.817077	−1.806086	−1.816954	−1.816939	−1.817073	−1.817074	−1.817077
−0.297350	−0.045111	−0.069126	−0.069383	−0.067861	−0.067902	−0.068008
0.000000	0.000000	0.002324	0.002377	0.001894	0.001912	0.001968
0.000000	0.000000	0.000000	−0.000003	0.000061	0.000058	0.000042
0.000000	0.000000	0.000000	−0.000000	−0.000003	−0.000003	−0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	−0.000000	−0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
Panel B: Lagrange multiplier function						
0.908539	0.903043	0.908477	0.908470	0.908536	0.908537	0.908539
−0.168550	0.022555	0.034563	0.034691	0.033931	0.033951	0.034004
0.000000	0.000000	−0.001162	−0.001188	−0.000947	−0.000956	−0.000984
0.000000	0.000000	0.000000	0.000002	−0.000031	−0.000029	−0.000021
0.000000	0.000000	0.000000	0.000000	0.000002	0.0000001	0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	−0.000000
Panel C: Residual norm						
5.9726e-01	1.4619e-14	3.9767e-13	3.6723e-13	2.4199e-15	1.4878e-16	1.3886e-16
3.4604e-02	1.6873e-03	2.4987e-05	2.6517e-05	1.0601e-06	7.8021e-07	2.7547e-08

The rows in panel A and B are, respectively, the coefficients $(a_0, a_1, \ldots, a_6)$ and $(b_0, b_1, \ldots, b_6)$ of the polynomial expansions in (24).

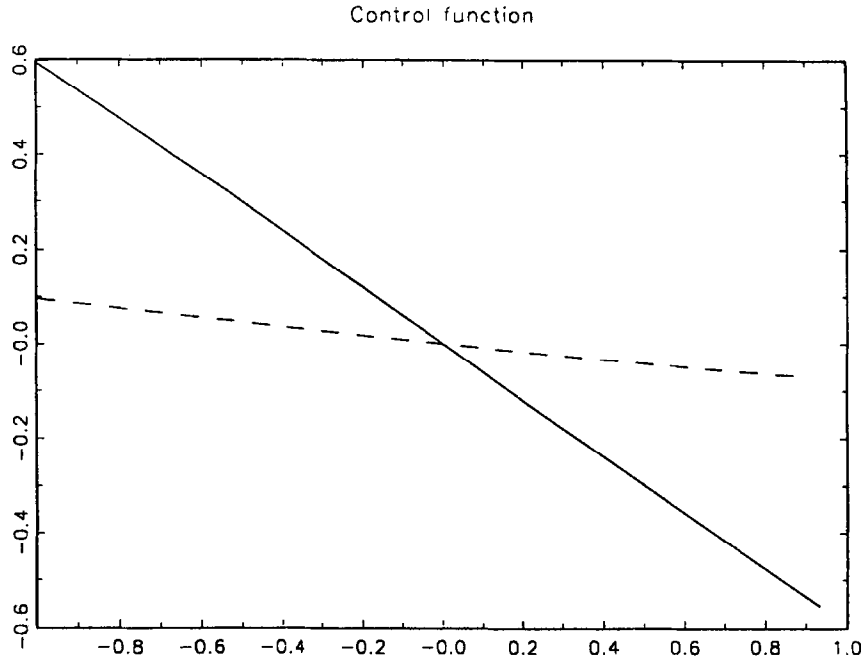

Fig. 1. $\sigma = 0.5$. Solid = linear approximation, Dashed = Hermite polynomial solution.

capital stock. Without $S(x, u)$ the state transition equation will be the capital accumulation equation with a zero depreciation rate. The first-order conditions (3)–(4) become

$$(\theta - \rho)\lambda + \lambda'\{(\theta + \sigma^2) - u\} + \tfrac{1}{2}\lambda'' x^2\sigma^2 = 0, \tag{29}$$

$$\gamma\, u^{\gamma-1} - \lambda = 0. \tag{30}$$

As we have discussed above it is important to define state and control appropriately in order to obtain accurate numerical solution. In view of (30) we redefine state and control and transform the Lagrange multiplier as follows:

$$y = \ln x, \quad v(y) = \ln(u(x)) - y, \quad \phi(y) = \ln(\lambda(x)) - (\gamma - 1)y. \tag{31}$$

Then

$$\phi' = \exp(y)\frac{\lambda'}{\lambda}, \quad \phi'' = \exp(2y)\frac{\lambda''}{\lambda} - (\phi' + \gamma - 1)(\phi' + \gamma - 2). \tag{32}$$

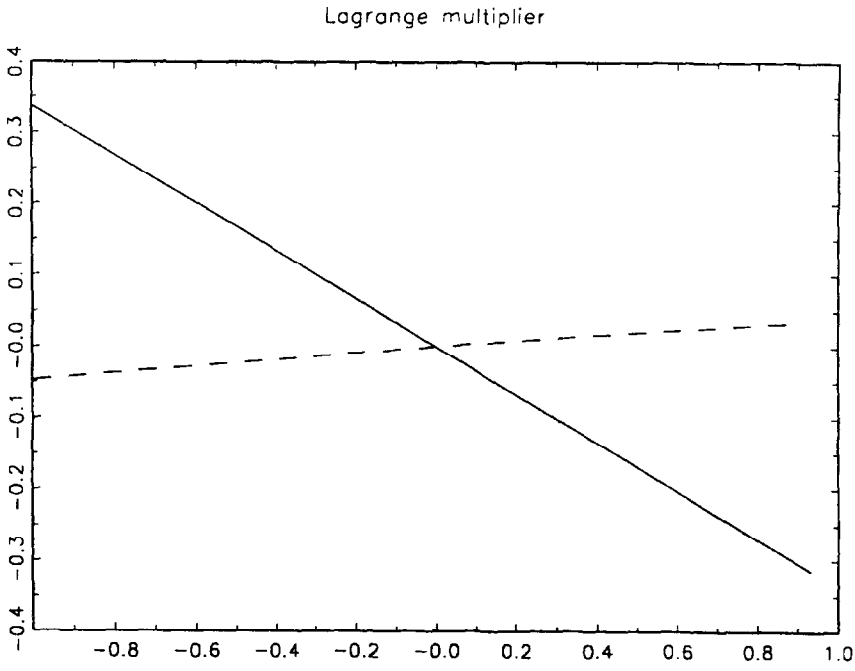

Fig. 2. $\sigma = 0.5$. Solid = linear approximation, Dashed = Hermite polynomial solution.

Thus, the first-order conditions can be written as

$$(\theta - \rho)\lambda + (\phi' + \gamma - 1)\{\rho + \sigma^2 - \exp(v)\} + \tfrac{1}{2}\sigma^2\{\phi'' + (\phi' + \gamma - 1)(\phi' + \gamma - 2)\} = 0, \tag{33}$$

$$\ln(\gamma) + (\gamma - 1)v - \phi = 0, \tag{34}$$

Eqs. (33) and (34) can be solved analytically. Consider linear solution of the form $v = g + Gx$ and $\phi = h + Hx$. The two equations can be written as

$$(\theta - \rho) + (H + \gamma - 1)(\theta + \sigma^2) + \tfrac{1}{2}\sigma^2(H + \gamma - 1)(H + \gamma - 2) - (H + \gamma - 1)\exp(g + Gy) = 0, \tag{35}$$

$$\{\ln\gamma + (\gamma - 1)g - h\} + \{(\gamma - 1)G - H\}y = 0. \tag{36}$$

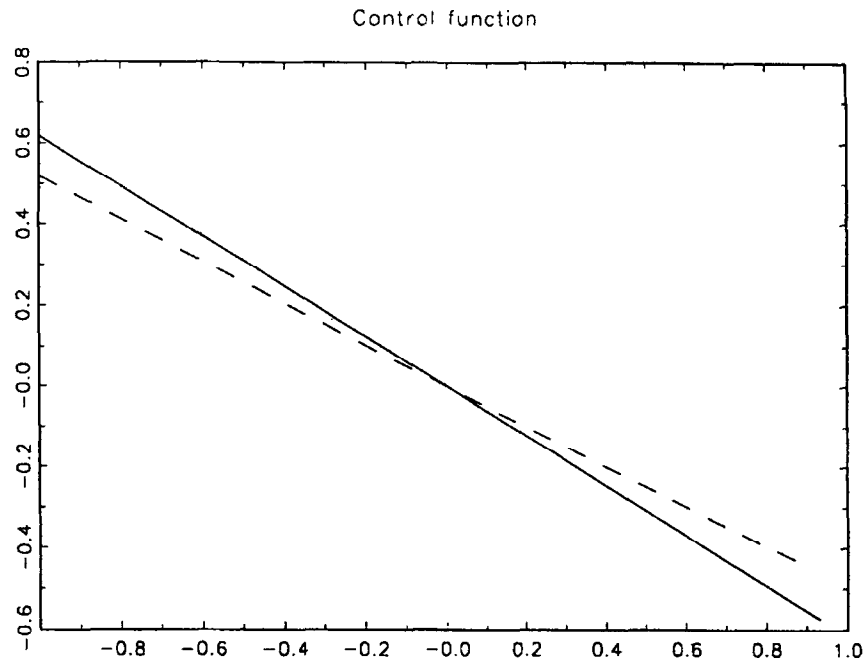

Fig. 3. $\sigma = 3$. Solid = linear approximation, Dashed = Hermite polynomial solution.

The functional equation system (35)–(36) is meant to hold for all conceivable y so that all coefficients associated with y must be zero. Thus, we have

$$h = \ln \gamma + (\gamma - 1)g, \qquad H = (\gamma - 1)G, \quad G = 0. \tag{37}$$

Setting $H = G = 0$ in (36) and solving for g, we obtain the exact solution which can be checked by taking log in (27) and using the definition in (31).

Table 2 reports Hermite polynomial solutions with model parameter $(\gamma, \rho, \theta, \sigma) = (0.5, 0.5, 0.3, 0.01)$. As can be seen, the numerical solutions are accurate up to 6 digits. The numerical computation also reveals that it is extremely important to define state and control having a steady state. For example, if the control variable is defined as $\ln(u)$ rather than $\ln(u) - y$, the numerical solution is unstable and very sensitive to initial guess, even though one can still obtain the exact solution by solving the first-order conditions analytically. In this example of endogenous growth the transformations in (31) amounts to detrending along the balanced growth path. This can be checked by differentiating the current value Hamiltonian of the deterministic version of the

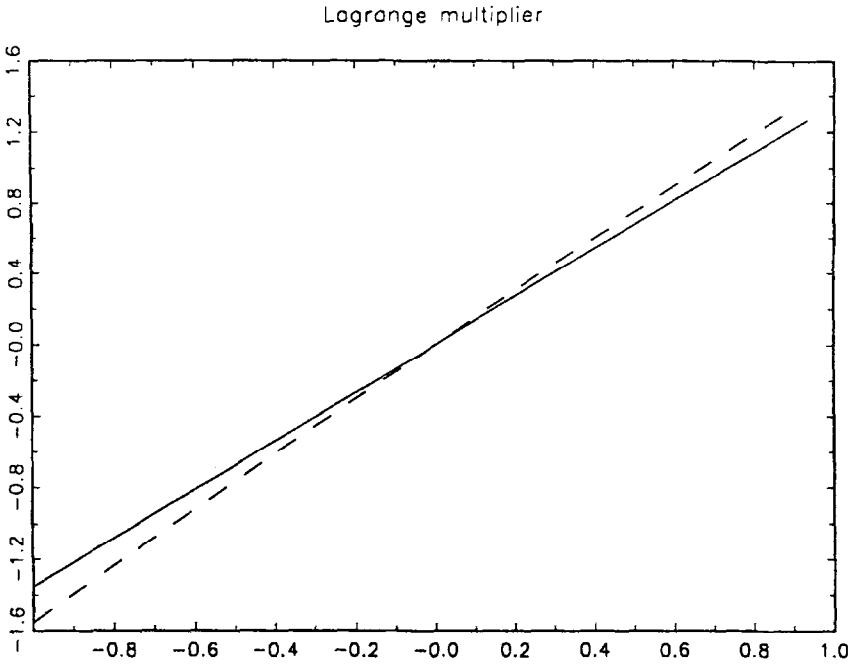

Fig. 4. $\sigma = 3$. Solid = linear approximation, Dashed = Hermite polynomial solution.

Table 2
Endogenous growth model

Exact	Degree 1	Degree 2	Degree 3	Degree 4
Panel A: Control function				
−0.356639	−0.356639	−0.356637	−0.356632	−0.356640
0.000000	0.000000	−0.000002	−0.000005	0.000000
0.000000	0.000000	0.000000	0.000001	−0.000000
0.000000	0.000000	0.000000	0.000000	0.000000
0.000000	0.000000	0.000000	0.000000	−0.000000
Panel B: Lagrange multiplier function				
−0.514828	−0.514828	−0.514829	−0.514831	−0.514827
0.000000	−0.000000	0.000001	0.000003	−0.000000
0.000000	0.000000	−0.000000	−0.000000	0.000000
0.000000	0.000000	0.000000	−0.000000	−0.000000
0.000000	0.000000	0.000000	0.000000	0.000000
Panel C: Residual norm				
0.000000	3.11542e-14	2.25596e-12	2.98192e-12	6.87953e-14
0.000000	7.78036e-08	8.12278e-07	2.63448e-06	6.81631e-08

The rows in panel A and B are, respectively, the coefficients $(a_0, a_1, \ldots, a_4)$ and $(b_0, b_1, \ldots, b_4)$ of the polynomial expansions in (24).

problem. Along the balanced growth path, it can be shown that

$$\frac{\dot{u}}{u} = \frac{\dot{x}}{x}, \frac{\dot{\lambda}}{\lambda} = (\gamma - 1)\frac{\dot{u}}{u}, \tag{38}$$

thus, suggesting the detrending scheme in (31).

References

Baker, C. T. H., 1978, The numerical treatment of integral equations (Oxford University Press, Oxford).

Barro, R., 1990, Government spending in a simple model of endogenous growth, Journal of Political Economy 98, S103–S125.

Chow, G. C., 1993, Optimal control without solving the Bellman equation, Journal of Economic Dynamics and Control 17, 621–630.

Davis, P. and P. Rabinowitz, 1984, Methods of numerical integration, 2nd edn (Academic Press, San Diego, CA).

Hackbusch, W., 1995, Integral equations: Theory and numerical treatment (Birkhauser, Basel).

Judd, K., 1992, Projection methods for solving aggregate growth models, Journal of Economic Theory 58, 410–452.

Kwakernaak, H. and R. Sivan, 1972, Linear optimal control system (Wiley-Interscience, New York).

Robelo, S., 1991 Long-run policy analysis and long-run growth, Journal of Political Economy 99, 500–521.

Reiter, M., 1996, Chow's method of optimal control, Journal of Economic Dynamics and Control, 20.