This version: http://econwpa.wustl.edu/eprints/pe/papers/9503/9503001.abs
Paper published in: *Investigaciones Económicas*, 20:1 (January 1996), 105–123. (Section 5.2 added December, 2002.)

On the Private Provision of Public Goods: A Diagrammatic Exposition

Eduardo Ley

IMF, Washington DC

Abstract. This paper surveys a selection of the literature on the private provision of public goods using the Kolm triangle. (The Kolm triangle is the analogue of an Edgeworth box in an economy with a public good.) We provide simple geometrical proofs of various established results using this graphical device. Our reference framework is the model of private contributions to public goods developed by Bergstrom, Blume and Varian (1986). With the Kolm triangle, we can easily study the existence and uniqueness of Nash equilibria, the effects of redistribution of the initial income, the level of provision in Stackelberg equilibria, the effects of subsidizing private contributions, and the implementation of Lindahl equilibria.

Keywords. Public Goods, Nash Equilibrium, Stackelberg Equilibrium, Lindahl Equilibrium, Kolm triangle, Redistribution, Subsidies

JEL Classification System. H41, H20

1. Introduction

Private contributions to public goods are important phenomena for many reasons. In the U.S. annual reported donations to charity amount to approximately 2% of its GDP. In Kenya, the voluntary cooperation of members of the community is essential for the provision of social infrastructure (Wilson 1992). Bergstrom, Blume and Varian (1986) developed a model to study the private provision of public goods which applies to the examples given above as well as to many other less obvious instances. Campaign funds for political parties or interest groups also fall under the scope of this model. In addition, much of the activity that takes place within the family unit can be explained as the outcome of voluntary contributions, see Becker (1981), and Konrad and Lommerud (1995). Kemp (1984), and Boadway, Pestieau and Wildasin (1989) have used this model to study multilateral foreign aid issues. The provision of national defense in alliances can also be studied within this model —see, *e.g.*, Bruce (1990). More recently, Hoel (1991) and Chichilinsky and Heal (1994) have used variants of this model to tackle global environmental issues. See Bergstrom, Blume and Varian (1986) —and references therein— for further discussion on the relevance of this model.

I wish to thank Ted Bergstrom, Sangeun Lee, Diego Moreno, Edgar Olsen, Dick Porter, Eric Rasmussen, Rafa Repullo, Marta Risueño, Hal Varian, Danyang Xie, and two anonymous referees for their useful comments and suggestions. I also thank participants in seminars at the University of Michigan, University of Hawaii, Utah State University, Resources for the Future, Universidad Carlos III de Madrid, and Universidad Complutense.

edu_ley@hotmail.com

In this paper, we provide simple geometrical proofs to various results from the public-goods literature using the Kolm triangle. The Kolm triangle is the analogue of the Edgeworth box for an economy with two agents, one private good and one pure public good. Malinvaud (1971) refers to unpublished 'research papers' by Serge-Christophe Kolm, while the triangle managed to appear a bit earlier than Malinvaud's paper in Ch. 9 (pp. 211–221) of Kolm's text on public economics (Kolm 1970). Schlesinger (1989) describes it in good detail and illustrates its use in analyzing Lindahl and Nash equilibria. Despite its potential, the Kolm triangle hardly appears in the literature.[1] Sullivan and Schlesinger (1986) analyze the relationship between various canons of 'just' taxation with the help of this graphical device. Groves and Ledyard (1987) use the triangle to illustrate incentive-compatibility problems in an economy with public goods. More recently, Chander (1993) uses the triangle to discuss dynamic procedures and incentives in public-good economies. William Thomson uses this tool in various papers dealing with allocation mechanisms (Thomson 1987), lecture notes (Thomson 1990), and concepts of equity (Thomson 1993). Leamer (1987) uses a similar device to prove factor price equalization in international trade. More surprising, perhaps, is the fact that the Kolm triangle has not found its way in public economics textbooks. An exception is Laffont (1988) who displays a few diagrams of the Kolm triangle, although he just barely refers to them in the text.

The focus of this paper is merely expositional. Our reference framework will be the model of private contributions to public goods used by Bergstrom, Blume and Varian (1986), which we describe in section 2. With the Kolm triangle (introduced in section 3), we can easily study the existence and uniqueness of Nash equilibria (section 5), the effects of redistribution of the initial income (subsection 5.1), the level of provision in Stackelberg equilibria (section 6), the effects of subsidizing private contributions (section 7) and the implementation of Lindahl equilibria (section 8).

2. The Model

We have two agents, $i = 1, 2$, each of whom consumes one private good, x_i, and one shared public good, G. Agent i has a preference ordering over the pairs (x_i, G) that can be represented by a differentiable and strictly quasi-concave utility function, $U_i(x_i, G)$. Both goods are assumed to be strictly normal goods. We shall assume that the public good can be produced at a constant marginal cost. Choosing units suitably, we can make the (constant) marginal rate of transformation between the private good and public good equal to one. Finally, let (w_1, w_2) be the agents' initial endowments of private goods.

The agents choose their private contributions, g_i, to the public good. The total amount of public good provided is determined by the sum of the individual contributions, $G =$

[1] A search made on the March 1993 *EconLit* CD (covering the *Journal of Economic Literature* since 1969) for entries containing 'Kolm' and 'triangle' returned only Schlesinger (1989).

$g_1 + g_2$. Each agent i solves

$$\max_{x_i, g_i} \quad U_i(x_i, g_1 + g_2)$$
$$\text{s.t.} \quad x_i + g_i = w_i$$
$$x_i, g_i \geq 0.$$

We can use the budget constraint to eliminate x_i and write the individual's optimization problem more compactly as

$$\begin{aligned} \max_{g_i} \quad & U_i(w_i - g_i, g_1 + g_2) \\ \text{s.t.} \quad & 0 \leq g_i \leq w_i. \end{aligned} \tag{1}$$

A more general version of this model, with any number of agents, has been extensively studied by Bergstrom, Blume and Varian (1986).

3. The Kolm Triangle

Figure 1 shows a Kolm triangle for our simple model economy. The height of the (equilateral) triangle is given by the total amount of resources available, $w_1 + w_2$. Since, in an *equilateral* triangle, the sum of the distances to the sides is constant and equals the height of the triangle,[2] then, for any point inside the triangle, we have

$$x_1 + x_2 + G = w_1 + w_2.$$

Therefore, any point inside the triangle is associated with a feasible allocation. In any allocation, z, agent i's private consumption is given by the distance from z to O_iO_0. The amount of public good, G, associated with z is simply given by the distance from z to the base of the triangle, O_1O_2.

In figure 2 we represent the agents' indifference maps. We start from a given allocation, z. To the right of the dashed line which is parallel to O_1O_0, agent one has more of the private good than at z. Above the dashed line which is parallel to O_1O_2, agent one has more of the public good than at z. It follows that any other allocation in the set B must be better than z for agent one since in B she gets more of both goods than in z. In W, on the other hand, agent one gets less from both goods so she must be worse off. The direction of the preferences is shown in figure 2. Agent i's indifference curves are convex to his origin, O_i, whenever his preferences are quasiconcave.[3]

Since along O_1O_2 we have that $G = 0$, then any point along O_1O_2 is associated with an

[2] A simple proof of this fact based on Thomson (1990) follows. Let S be the area of the triangle in figure 1, let b denote the common length of the three sides, and h denote the height of the triangle. Note that S must equal the sum of the areas of the three triangles O_1zO_2, O_2zO_0, and O_0zO_1, this implies that $S = b(x_1 + x_2 + G)/2$ or $x_1 + x_2 + G = 2S/b$, where the quantity on the right does not depend on the position of the point z. Since $S = bh/2$ we must also have $x_1 + x_2 + G = h \equiv w_1 + w_2$.

[3] As noted in Schlesinger (1989) and Thomson (1990), the transformation from the Euclidean coordinates to the barycentric coordinates in the Kolm triangle is linear. In particular, if (x_1, g) are the Euclidean coordinates associated with a consumption bundle for agent 1, then its coordinates inside the Kolm triangle (with O_1 located at the origin) *with respect to the original Euclidean axes* are given by $((2x_1 + g)/\sqrt{3}, g)$.

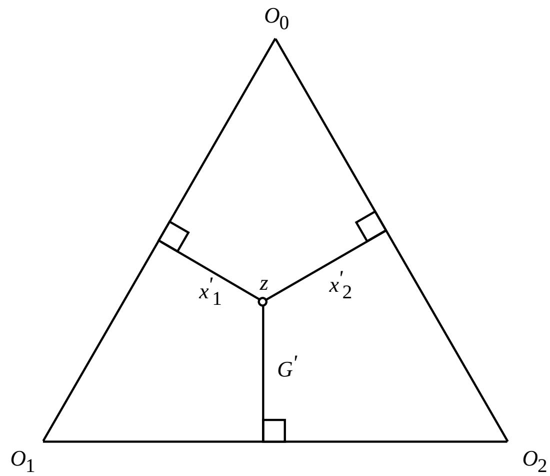

Fig. 1. A feasible allocation in a Kolm triangle.

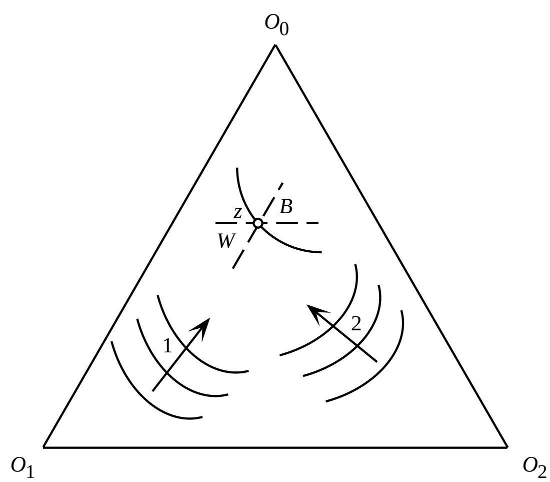

Fig. 2. The indifference maps in a Kolm triangle.

initial allocation, w. Noting that the length of O_1O_2 is $2(w_1+w_2)/\sqrt{5}$, then the distance of w from O_1 will be given by $2w_1/\sqrt{5}$.

4. Other Geometrical Representations

Dolbear (1967) used an alternative geometrical device to study this type of economies. The Dolbear triangle has been used, among others, by Shibata (1971) and Olsen (1979, 1981). Assuming a linear technology, the production possibilities set can be represented by a triangle as depicted in figure 3.

The amount of public good, G, is measured on the horizontal axis and the *total* amount of private good on the vertical axis. Once a combination (G', x') is chosen, all that remains is to divide the amount of private good, x' between the two agents. Measuring x_1 upwards from O_1 and x_2 downwards from x', any point in the segment between G' and

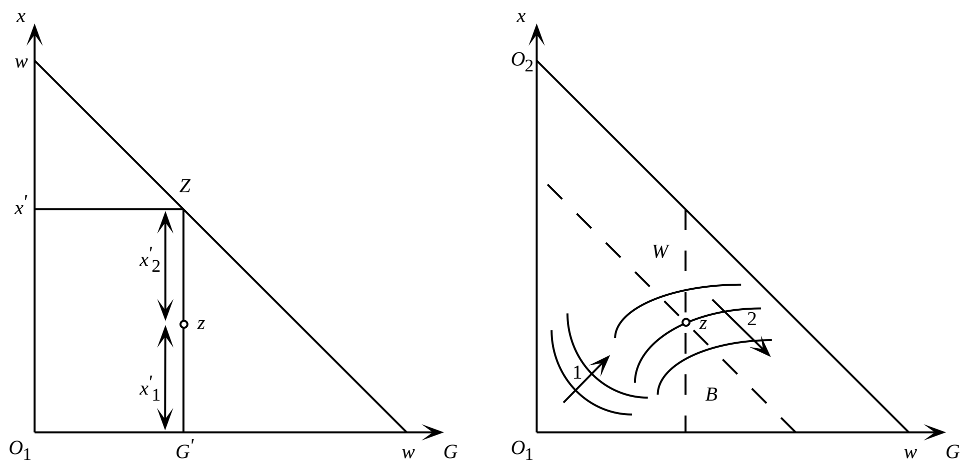

Fig. 3. The Dolbear triangle.

Z represents a feasible allocation of x' between the two agents. In figure 3, z represents one such allocation with x'_1 consumed by agent one and x'_2 by agent two; feasibility is satisfied since we have $x'_1 + x'_2 = x'$ and $x' + G' = w$. Agent one's indifference map on this triangle will have the usual representation while agent two's is harder to visualize. In figure 3, given an allocation, z, agent two will have less of both goods in region W and more of both goods in region B. The convexity assumption requires his indifference curves to be convex to O_2.

Cornes and Sandler (1985, 1986) choose yet another representation, now on the $g_1 - g_2$ plane. We show it on figure 4. At z, $G' = g'_1 + g'_2$ is provided. The dashed line through z with slope -1 represents the allocations where $g_1 + g_2 = G'$. In the region B, above the $g_1 + g_2 = G'$ line and to the left of g'_1, agent one gets more of both goods and is, therefore, better than in z. Conversely, in region W, below the $g_1 + g_2 = G'$ line and to the right of g'_1, she is worse off than in z. Representative indifference curves are shown in figure 4.

As discussed in Schlesinger (1989), an advantage that the Dolbear triangle and the Cornes-Sandler box share is that they use the familiar Euclidean coordinates unlike the Kolm triangle which uses barycentric coordinates. The Dolbear and Cornes-Sandler representations can be easily extended to more general technologies which cannot be done with the Kolm triangle. The Dolbear triangle is somewhat more difficult to read since the agents are not treated symmetrically. (The Cornes-Sandler box omits G from the graph although in the case of linear technology it can be easily recovered —*e.g.*, by drawing the line $g_1 + g_2$ through any allocation.) We shall only use the Kolm triangle in the remaining of the paper. Nevertheless, all the geometrical proofs that follow can be (although sometimes more difficulty) reproduced using Dolbear's triangle and the Cornes-Sandler box.[4]

[4] See Danziger (1976) for another representation suitable for economies with many agents.

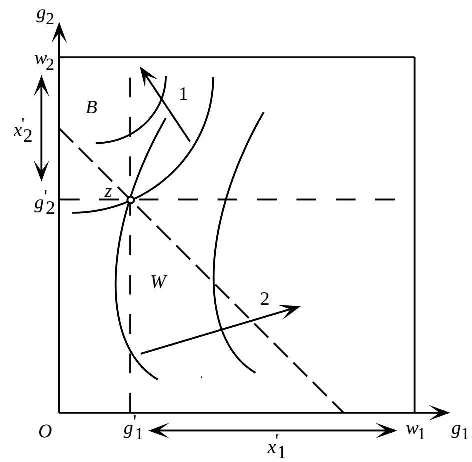

Fig. 4. The Cornes-Sandler box.

5. Nash Equilibrium

A Nash equilibrium in this model is a vector of contributions (g_1^*, g_2^*) which solves the two agents' following optimization programs:

$$\begin{array}{llcll} \max & U_1(w_1 - g_1, g_1 + g_2^*) & \text{and} & \max & U_2(w_2 - g_2, g_1^* + g_2) \\ \text{s.t.} & 0 \leq g_1 \leq w_1 & & \text{s.t.} & 0 \leq g_2 \leq w_2. \end{array}$$

Figure 5 shows a Nash equilibrium, denoted by E. Let $A = (w_1, w_2)$ represent the initial allocation. When $g_2 = 0$, agent one's opportunity locus is given by the segment AC which is parallel to O_2O_0 —*i.e.*, along AC we have that $x_2 = w_2$. When agent two is contributing $g_2^* = A'J$, agent one's opportunity locus shifts to $A'C'$. The Nash equilibrium, E, is agent one's optimal choice on her *budget line* $A'C'$. She contributes $g_1^* = A''I$ and consumes $EH = w_1 - g_1^*$ of the private good. When agent one contributes g_1^*, agent two's opportunity locus shifts from AB (where $g_1 = 0$) to $A''B''$. (Note that AB and $A''B''$ are parallel to O_1O_0.) On $A''B''$, agent two's most preferred point is E, where he contributes g_2^*. Since the agents' indifference curves cross through E, the Nash equilibrium is not Pareto optimal. (With differentiable preferences, a Pareto optimal Nash equilibrium is a possibility only at the endowment point, A.)

Let us denote by $g_1(g_2)$ and $g_2(g_1)$ agent one's and agent two's optimal solutions to (1) as functions of the other agent's gift. Thus, $g_1(g_2)$ and $g_2(g_1)$ are the agents' reaction functions. Then, if (g_1^*, g_2^*) is a Nash equilibrium, we must have $g_1^* = g_1(g_2^*)$, and $g_2^* = g_2(g_1^*)$.

We can represent agent one's reaction function in a Kolm triangle, see figure 6. Again, let $A = (w_1, w_2)$ represent the initial allocation. When $g_2 = 0$, agent one's opportunity locus is given by the segment AC. Given this constraint, agent one would choose to contribute $g_1(0)$. When $g_2 = g'$, the opportunity locus will shift to $A'C'$, and agent one will choose $g_1(g')$ for a total amount of G given by $g_1(g') + g'$. When $g_2 \geq g''$, we have that $g_1(g_2) = 0$.

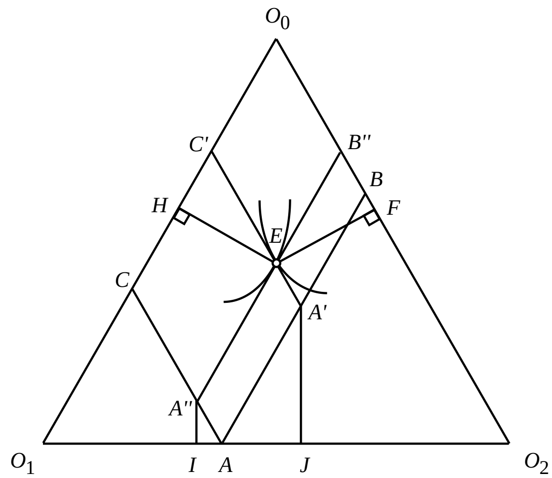

Fig. 5. A Nash equilibrium.

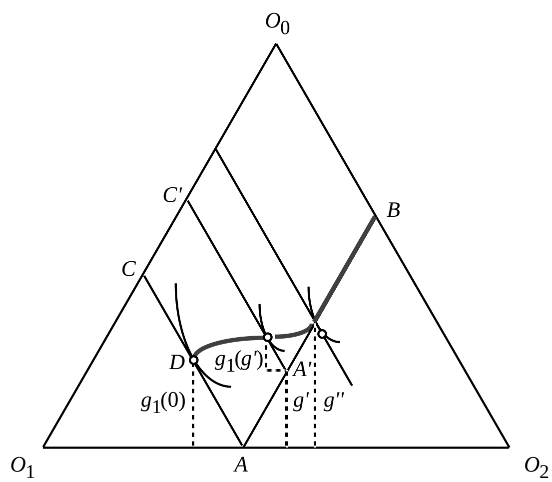

Fig. 6. Agent one's reaction function.

If both goods are normal goods, the reaction function $g_1(g_2)$ cannot be steeper than O_1O_0 (since that would imply a smaller demand of x_1 as income increases) and it cannot be flatter than O_1O_2 (since that would imply a smaller demand of G as income increases). As a result, once agent one's reaction function hits AB it has to stay on AB since AB is parallel to O_1O_0 —of course, the reaction function $g_1(g_2)$ doesn't need to ever hit AB. Said another way, once agent one contributes nothing to the public good, bigger contributions by agent two will only induce agent one to keep contributing nothing. The curve DB in figure 6 represents agent one's reaction function. (A similar derivation for agent two will tell us that $g_2(g_1)$ has to be flatter than O_2O_0 and steeper than O_1O_2.)

Given an initial distribution of income, we can plot $g_1(g_2)$ and $g_2(g_1)$. The existence of Nash equilibrium (theorem 2 in Bergstrom, Blume and Varian (1986)) will be established if we can show that the graphs of the reaction functions cross inside the triangle. Refer to figure 7. We have that $g_1(g_2)$ must start out somewhere on AC and must reach the segment BO_0. Agent two's reaction function, $g_2(g_1)$, must go from AB to CO_0. Both reaction functions must always stay inside the romboid ACO_0B, and by the assumptions made about the preferences, they both have unbroken graphs. Thus, the existence of Nash equilibrium is established.

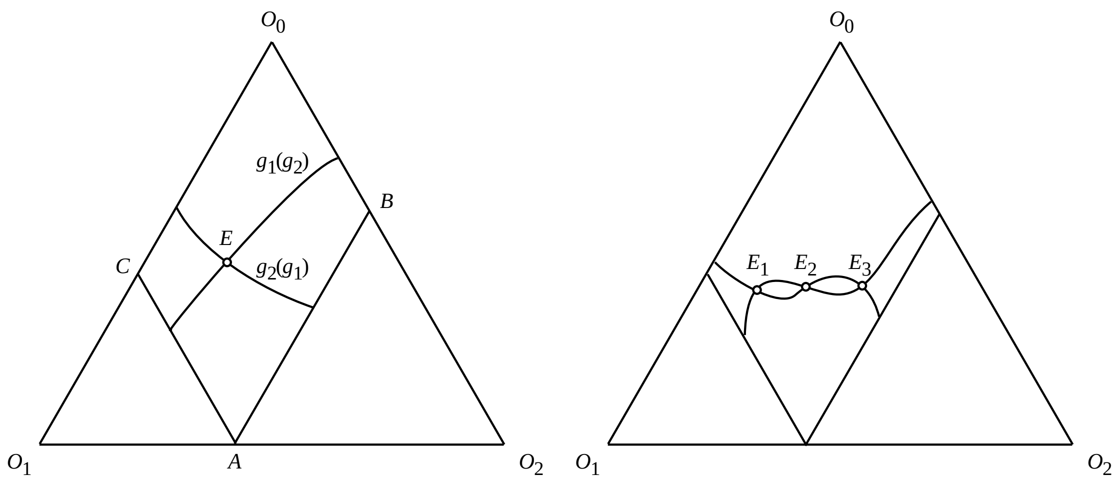

Fig. 7. Existence and uniqueness of Nash equilibrium.

Theorem 3 in Bergstrom, Blume and Varian (1986) says that: "there is a unique Nash equilibrium with a unique quantity of public good and a unique set of contributing consumers." Here the uniqueness follows from the bounds imposed by the strict normality assumption (Bergstrom, Blume and Varian (1986), page 32) on the slope of the reaction functions. The first panel in figure 7 shows a unique Nash equilibrium. The second panel gives an example of multiple equilibria when G is an inferior good.

5.1. Neutral Income Redistributions

Warr (1983) discovered an interesting neutrality theorem that was later extended by Bergstrom, Blume and Varian (1986), and Gradstein, Nitzan and Slutsky (1994). Assume that we have a Nash equilibrium, (g_1^*, g_2^*). If income is redistributed among contributing consumers in such a way that none of them loses more income than his original contribution, then there is a new Nash equilibrium, (g_1^{**}, g_2^{**}), where $g_1^{**} + g_2^{**} = g_1^* + g_2^*$, and $x_i^{**} = x_i^* = w_i - g_i^*$. That is, the same amount of public good is provided and each agent consumes the same amount of private goods that in the original equilibrium —*i.e.*, every consumer changes the amount of his gift by precisely the amount of the income transfer.

Figure 8 shows the effect of a redistribution of income from agent one to agent two that shifts the initial point from A to A'. The diagram shows the agents' reaction functions whose intersection determines the Nash equilibrium E. The portion of $g_1(g_2)$ between AC and $A'C'$ is no longer relevant after the redistribution. On the other hand, agent two's reaction function, $g_2(g_1)$, gains an additional portion between AB and $A'B'$ after the redistribution. However, the old Nash equilibrium is the Nash equilibrum of the new

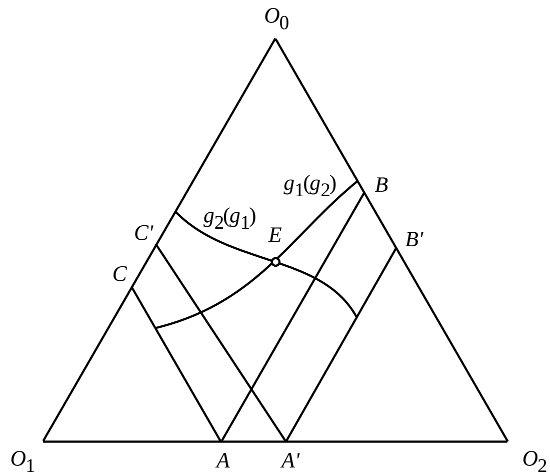

Fig. 8. Redistribution of Income.

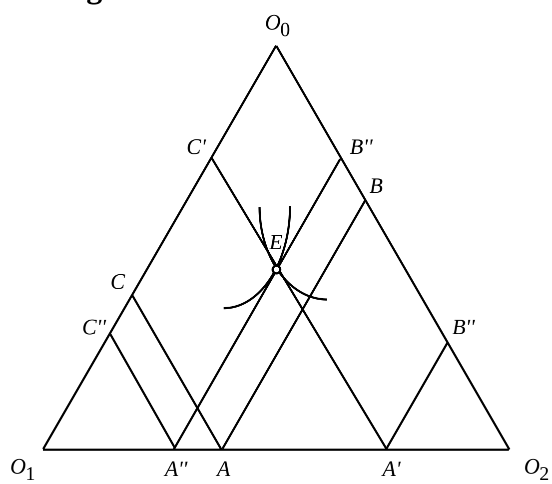

Fig. 9. Bounds on the income redistribution.

game. The agents' consumptions remain unchanged.

Figure 9 shows the bounds on income redistribution. In A' we have taken away from agent two an amount of income equal to his gift in the initial Nash equilibrium. This is the maximum amount that we can take away from him and still get the same equilibrium level of public good and private consumptions. The maximum redistribution from agent one to agent two —that will leave the equilibrium amount of G and (x_1, x_2) unchanged— will move the endowment to A''.

5.2. Inequality and Social Welfare

While the Warr neutrality result establishes that a whole range of initial distributions of income are mapped into the same final allocation, Itaya, Meza and Myles (1997, IMM henceforth) establish the remarkable result that social welfare can be raised by creating sufficient income inequality so that only the rich can afford to provide the public good. IMM show that, starting from the limit of allowable Warr-type redistributions, increasing inequality can be welfare-enhancing —as private consumption by one agent is substituted by a mix of private consumption by the other agent and public-good provision.

Take two individuals with identical preferences, $U_i = U(\cdot)$, and consider the tradeoff between agents' utilities along the different Nash equilibria obtained by redistributing income. As in IMM, let us redistribute income from agent 2 to agent 1, so that agent 2 has just enough income to afford the same private consumption, x_2^*, that he had at any of the interior Nash equilibria (*i.e.*, we make $w_2 = x_2^*$). Now, at the associated Nash equilibrium, agent 2 is contributing nothing to the public good, $g_2^* = 0$. Let us denote this initial allocation by A. Theorem 3 in IMM establishes that, starting at A, a further (small) redistribution from agent 2 (poorer) to agent 1 (richer) will always be welfare enhancing.[5]

Consider the slope of the graph relating the agents' utilities associated with the different equilibria that are obtained through redistribution of the initial incomes (Figure 10). We shall denote the absolute value of the slope of this graph by $\mathcal{N}$. Using the symmetry assumption and the fact that A leads to an interior Nash equilibrium (which implies that $U_x = U_G$), we can establish that, for $dt > 0$,

$$\mathcal{N}(t) \equiv -\frac{dU_2}{dU_1} = -\frac{\frac{\partial U_2}{\partial t}}{\frac{\partial U_1}{\partial t}} = -\frac{-U_x + U_G\frac{\partial g_1(t)}{\partial t}}{U_x\left[1 - \frac{\partial g_1(t)}{\partial t}\right] + U_G\frac{\partial g_1(t)}{\partial t}} = 1 - \frac{\partial g_1(t)}{\partial t}.$$

The assumption that both goods are normal means that $0 < \partial g_1(t)/\partial t < 1$, which, in turn, bounds the slope of the graph: $0 < \mathcal{N}(0^+) < 1$ (where 0^+ indicates a right derivative, as $t \downarrow 0$). What about $dt < 0$ —*i.e.*, transfers from 2 to 1? The Warr neutrality result implies that nothing happens until we place agent 2 in his corner solution. From there on, given the symmetry assumption, everything works out as above only with subscripts reversed, and we have that: $\mathcal{N}(0^-) > 1$.

Let $W(U_1, U_2)$ be a symmetric and differentiable social welfare function. The (absolute value of the) slope of the welfare indifference curve is given by: $\mathcal{W}(t) \equiv -\frac{dU_2}{dU_1} = \frac{\partial W}{\partial U_1} \div \frac{\partial W}{\partial U_2}$. Since at the Nash equilibrium the utility levels are equalized (Theorem 1 in IMM), and using the symmetry assumption on $W(\cdot)$, we must have that $\mathcal{W}(0^+) = \mathcal{W}(0^-) = 1$.

Starting at A, a small movement to the right (*i.e.*, $dt > 0$) will result in an equilibrium allocation that is associated with a higher social indifference curve because $\mathcal{N}(0^+) < \mathcal{W}(0^+)$. Similarly, since we have $\mathcal{N}(0^-) > \mathcal{W}(0^-)$, small movements to the left (from the corresponding initial allocation where agent 2 is just in his corner solution) will also increase welfare.

Figure 10 shows the utility possibility frontier associated with the Nash equilibria that result from different initial distributions of income, when the agents have Cobb-Douglas

[5] This result holds as long as the utility function at the individual level is not Leontieff and the social welfare function is not Rawlasian (*i.e.*, $W = \min\{U_1, U_2\}$). In either of these two cases, this type of regressive redistribution will *always* lower social welfare.

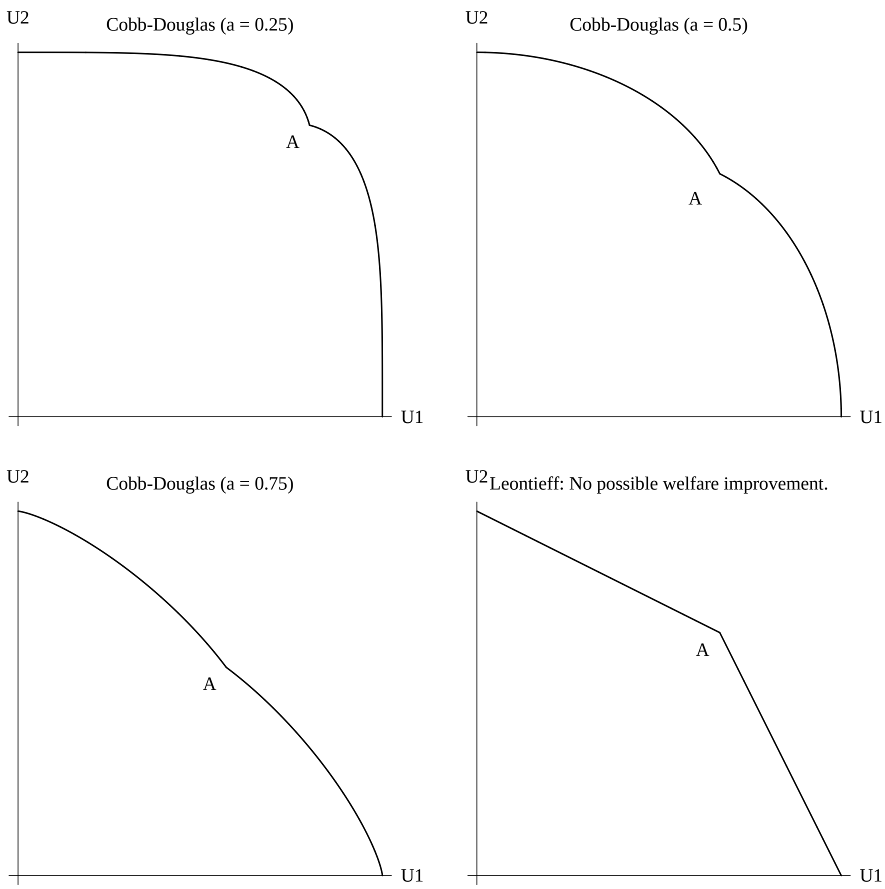

Fig. 10. Utility possibility frontier associated with the Nash equilibria resulting from different initial distributions of income.

preferences given by: $U_i = x_i^a G^{(1-a)}$.[6] The scope of regressive redistribution to enhance social welfare depends on (i) the individual preferences, and (ii) the shape of the social indifference curves. Figure 10 illustrates the effect of (i): the larger the taste for G (the smaller a), the larger the scope for redistribution to enhance welfare. When there is no substitution possible among goods (*i.e.*, preferences are of the Leontieff type at the *individual* level), then any redistribution will lower social welfare.

We do not show any social indifference curves in Figure 10, but the reader can easily imagine them to see the effects of (ii). At one extreme, a utilitarian welfare function, $W = U_1 + U_2$, would have social indifference curves that are lines with $\mathcal{W}(t) = 1$, for all t, offering the largest scope for regressive redistribution to increase welfare. At the other extreme, a Rawlsian welfare function, $W = \min\{U_1, U_2\}$, would have Leontieff-type *social* indifference curves and any redistribution would always lower social welfare.

[6] Figure 10 plots the graph given by the pairs of *indirect* utilities $(V_1(t), V_2(t))$ at the corresponding equilibria. For $dt > 0$, first place agent 1 at her corner solution A, by making $w_1 = a/(1-a) \equiv \alpha$. Then, since $g_2(t) = (1-a)(1-\alpha+t)$, we have that $V_1(t) = (\alpha-t)^a[(1-a)(1-\alpha+t)]^{1-a}$, and $V_2(t) = (1-\alpha+t)a^a(1-a)^{1-a}$.

6. Stackelberg Equilibrium

Varian (1994) studies sequential contributions to public goods. The Kolm triangle is a useful tool to gain further insights into his results. Let agent one be the leader and agent two be the follower. Then, the Stackelberg equilibrium will be determined by agent one choosing her most preferred point in agent two's reaction function. That is, agent one solves

$$\max_{g_1} \quad U_1(w_1 - g_1, g_1 + g_2(g_1))$$
$$\text{s.t.} \quad 0 \leq g_1 \leq w_1$$

where $g_2(g_1)$ is agent two's reaction function —*i.e.*, the solution to (1) for agent two.

Varian (1994)'s main result (theorem 2) states that the leader's contribution at the Stackelberg equilibrium is bounded above by her contribution at the Nash equilibrium. As a corollary, the total amount of the public good in the Stackelberg equilibrium is never bigger than the total amount provided in the Nash equilibrium. Figure 11 shows these results.

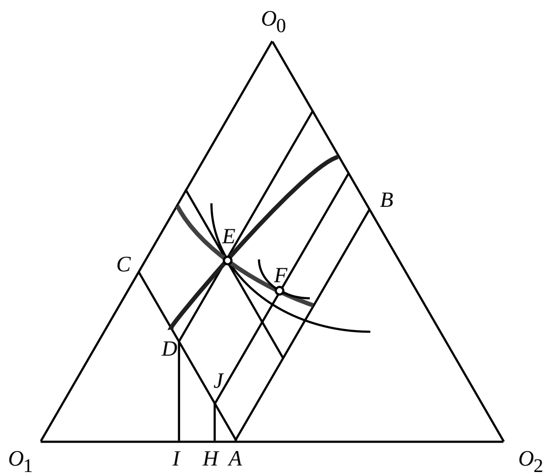

Fig. 11. Stackelberg equilibrium: Agent one is leader.

Figure 11 shows Nash and Stackelberg equilibria. The Nash equilibrium, E, is determined by the crossing of the reaction functions. The Stackelberg equilibrium, F, is given by agent one's most preferred point in agent two's reaction function. We have drawn agent one's indifference curves through those equilibria. Looking at the indifference curve through E, we see that agent one will move to points of $g_2(g_1)$ to the right of E. Since $g_2(g_1)$ has a negative slope, this movement necessarily implies less G in the Stackelberg equilibrium (Varian (1994), corollary to theorem 3). We can also easily see why agent one's contribution at the Stackelberg equilibrium can be no larger than her contribution at the Nash equilibrium. Since the Stackelberg equilibrium cannot lie to the left of the Nash equilibrium, it implies that agent one's contribution will be smaller. In figure 11, agent one contributes $g_1^* = DI$ at the Nash equilibrium and $g_1^s = JH$ at the Stackelberg equilibrium.

From figure 11 it can also be concluded that the follower's contribution at the Stackelberg equilibrium is bounded below by his contribution at the Nash equilibrium. This result is not in Varian (1994). However, once it is noticed, it can be easily proved with the analytical apparatus developed there.

7. Subsidizing Contributions

Back in a Nash model, Roberts (1987) discovered the puzzling result that rich people might be made worse off when their contributions are subsidized at a higher rate than poor people —*e.g.*, when the contributions are tax-deductible in a system of progressive income taxation. This issue has been examined by Bergstrom (1989) who shows that if we have two *identical* individuals contributing to a public good, each will prefer to face a price higher than the price faced by the other individual. In Roberts (1987) and Bergstrom (1989) the subsidy is paid by a lump-sum tax on both agents. Varian (1994) shows that each agent will prefer to subsidize the other agent even if he must pay the entire amount of the subsidy himself. In Varian's model, agents have *quasi-linear utility* functions.

Boadway, Pestieau and Wildasin (1989) show (theorem 2) that for two *non-identical* individuals with *general* quasi-concave utility functions, when both goods are normal, an agent will always want to subsidize the other agent's contributions even if he must pay the entire amount of the subsidy himself. We only analyze here the case where we have interior Nash equilibria before the subsidy.

In the subsidy game, agent one will subsidize agent two at the rate s in $(0, 1)$. Agent two solves

$$\begin{aligned} \max_{g_2} \quad & U_2(w_2 - (1-s)g_2, g_1 + g_2) \\ \text{s.t.} \quad & 0 \leq (1-s)g_2 \leq w_2, \end{aligned} \tag{2}$$

and agent one's problem is

$$\begin{aligned} \max_{g_1} \quad & U_2(w_1 - sg_2 - g_1, g_1 + g_2) \\ \text{s.t.} \quad & 0 \leq g_1 \leq w_1 - sg_2. \end{aligned} \tag{3}$$

Given the subsidy rate, s, a Nash equilibrium is a vector of contributions $(g_1^*(s), g_2^*(s))$ which solves both (2) and (3).

Theorem 2 in Boadway , Pestieau and Wildasin (1989) establishes that —provided that both agents are contributing at the initial Nash equilibrium where $s = 0$— there always exists a subsidy rate, s, such that agent one —who pays the subsidy— is better off at the resulting Nash equilibrium, *i.e.*,

$$U_1(w_1 - sg_2^*(s) - g_1^*(s), g_1^*(s) + g_2^*(s)) > U_1(w_1 - g_1^*(0), g_1^*(0) + g_2^*(0)).$$

Further, agent two —who is being subsidized— is worse off than before the subsidy, *i.e.*,

$$U_2(w_2 - (1-s)g_2^*(s), g_1^*(s) + g_2^*(s)) < U_2(w_1 - g_2^*(0), g_1^*(0) + g_2^*(0)).$$

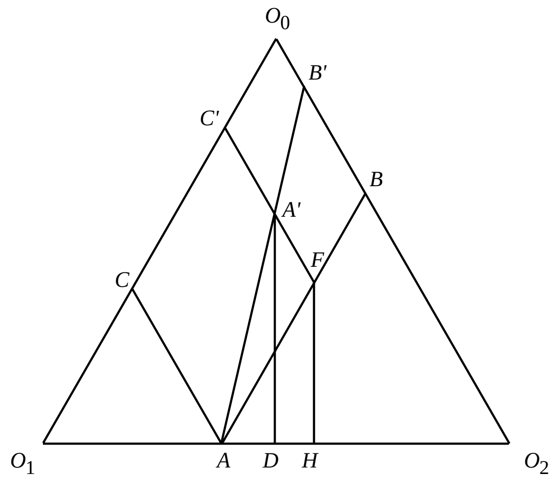

Fig. 12. The Geometry of a Subsidy.

Figure 12 shows the geometry of a subsidy. The subsidy changes the slope of agent two's opportunity loci. In particular, if α measures the angle, in radians, of AB' with respect to O_1O_2, then $s = \sin(\alpha - \pi/3)/\sin\alpha$. When $\alpha = \pi/3$ so that AB' is parallel to O_1O_0, then $s = 0$. At the other extreme, when $\alpha = 2\pi/3$ so that AB' is parallel to O_2O_0 then $s = 1$. Since agent one has to pay for the subsidy, her opportunity loci will be also affected. In figure 12, when $g_2 = A'D$, agent one has to pay $T = sg_2 = A'D - FH$. This shifts her opportunity locus to $A'C'$.

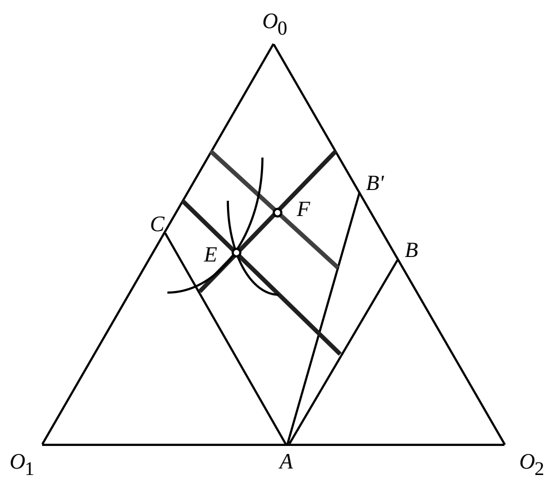

Fig. 13. Agent one subsidizes agent two.

Figure 13 shows the effects of a subsidy from agent one to agent two. We display the initial Nash equilibrium, E, where the agents' reaction functions cross. We have also represented the agents' indifference curves through E. When agent one subsidizes agent two, agent two's new reaction function has to be above his old reaction function. At the new equilibrium, F, agent one is clearly better off than at E. What about agent two?

Since the slope of agent two's indifference curve through E is parallel to O_1O_0 which is the upper bound for the slope of agent one's reaction function, $g_1(g_2)$, it follows that F must lie below agent two's indifference curve through E.

To better understand the importance of this result, let us consider a movement from the original Nash equilibrium E. At E, agent one's marginal rate of substitution between the private and public goods equals the marginal rate of transformation, 1. Suppose that by offering to match any further contributions that agent two might do, agent one gets agent two to increase his contribution by g_2. Agent one would then be effectively 'purchasing' g_2 at half price! Moreover, provided that the g_2 is not too big, both agents would improve their situations after this deal.[7] Boadway, Pestieau and Wildasin (1989)'s theorem discussed above tells us something a bit different since we are not moving from a no-subsidy Nash equilibrium but from the initial endowment point. The surprising result is that agent one is going to be willing to subsidize agent two's contribution from his very first unit, and that agent two is going to be made worse off by this scheme.

A word on corner solutions. If agent two was not contributing towards the public good at the initial Nash equilibrium, the same results hold provided that there is a subsidy that induces him to contribute a positive amount. The other corner solution, where agent one was not contributing initially, can lead to anything. It may or may not be possible to improve agent one's welfare with the subsidy; and, in either case, agent two might end up better or worse off.

8. Lindahl Equilibria

Suppose that we allow for personalized prices for the public good, agent i facing p_i, with $p_2 \equiv 1 - p_1$ and $p_1 \in [0, 1]$. Given a pair of prices, we can have the agents choosing their private demands for x_i and G. Thus, each agent i solves:

$$\begin{aligned} \max_{G} \quad & U_i(w_i - p_i G, G) \\ \text{s.t.} \quad & w_i - p_i G \geq 0 \end{aligned} \tag{4}$$

Whenever, for some p_1^*, the desired demands for G by each agent are equal then we have a Lindahl equilibrium. Since the Lindahl prices split the Kolm triangle into two separate budget sets which are disjoint except for their common boundary, it follows that Lindahl equilibria are Pareto optimal. The assumptions made on the preferences do not guarantee the existence of Lindahl equilibria. However, these assumptions guarantee that they will be unique if they exist —*i.e.*, there will be at most one Lindahl equilibrium associated with any initial endowment. See figure 14 for an illustration of a Lindahl equilibrium.

More formally, $(p_1^*, p_2^*; w_1 - p_1^* G^*, w_2 - p_2^* G^*; G^*)$ is a *Lindahl equilibrium* if G^* solves problem (4) for $i = 1, 2$, when we replace p_i by p_i^*. In a Lindahl equilibrium, the agents face personalized prices for the public good while they all consume the same

[7] If agent two increases his contribution by too much, it is possible that agent one might end up worse off. Of course, agent one's offer could always specify a limit to the matching gift.

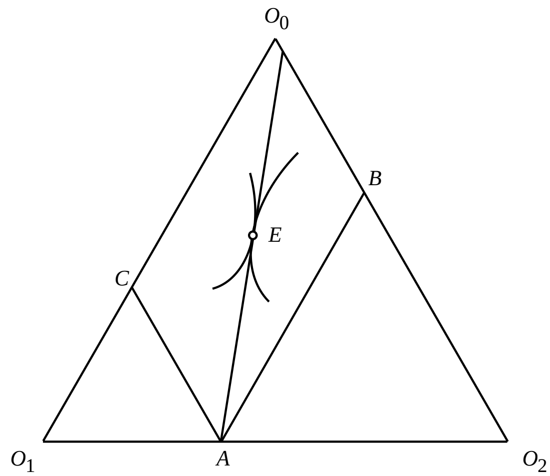

Fig. 14. Lindahl Equilibrium.

amount —in contrast with a private-goods economy Walrasian equilibrium where the agents have individual demands facing all the agents same prices. We shall define an allocation $(\bar{x}_1, \bar{x}_2, \bar{G})$ to be a *Lindahl allocation* if there exist prices, $(\bar{p}_1, \bar{p}_2)$ such that $(\bar{p}_1, \bar{p}_2; w_1 - \bar{p}_1\bar{G}, w_2 - \bar{p}_2\bar{G}; \bar{G})$ is a Lindahl equilibrium.

Personalized prices can be obtained by allowing each agent to subsidize the other agents' contributions as in section 7. What is the outcome of the two-stage game where each agent announces first a subsidy rate for the other agent and then, in a second stage, chooses his own contribution? This game is studied in Danziger and Schnytzer (1991)[8] and they find that *all* the subgame-perfect equilibria of this game are Lindahl allocations.

The two-stage game consists of:

Stage 1. Agent i chooses s_j for $j \neq 1$.

Stage 2. Each agent chooses $g_i \geq 0$. If $(1 - s_i)g_i + s_j g_j > w_i$; any rule that allocates the entire agent i's endowment to the public good will suffice.[9]

A subgame-perfect equilibrium is given by $(s_1^*, s_2^*; g_1^*(s_1, s_2), g_2^*(s_1, s_2))$ such that:

(i) $(g_1^*(\hat{s}_1, \hat{s}_2), g_2^*(\hat{s}_1, \hat{s}_2))$ is a Nash equilibrium of the second-stage game given $(\hat{s}_1, \hat{s}_2)$; *i.e.*, $g_i^*(\hat{s}_1, \hat{s}_2))$ solves

$$\begin{aligned} \phi_i(\hat{s}_i, \hat{s}_j) \equiv \max_{g_i} \quad & U_i(w_i - (1 - \hat{s}_i)g_i - \hat{s}_j g_j^*, g_i + g_j^*) \\ \text{s.t.} \quad & 0 \leq (1 - \hat{s}_i)g_i \leq w_i - \hat{s}_j g_j^* \quad \text{for } j \neq i; \end{aligned}$$

[8] Varian (1994) studied a related game where the rate at which each agent subsidizes the other agents is set by the other agents.

[9] Here, if $s_j > 0$, agent i's choice set for g_i depends on the other agent choice of g_j. We can, for example, make $g_i = (w_i - s_i g_i)/(1 - s_i)$ whenever $w_i > s_i g_i$; otherwise make $s_j = w_i/g_j$ and $g_i = 0$. See Danziger and Schnytzer (1991).

and:

(ii) $\phi_i(s_i^*, s_j^*) \geq \phi_i(s_i, s_j^*)$, for all s_i, $i, j = 1, 2; i \neq j$.

Every subgame-perfect equilibrium (SPE) of this game is a Lindahl allocation.[10]

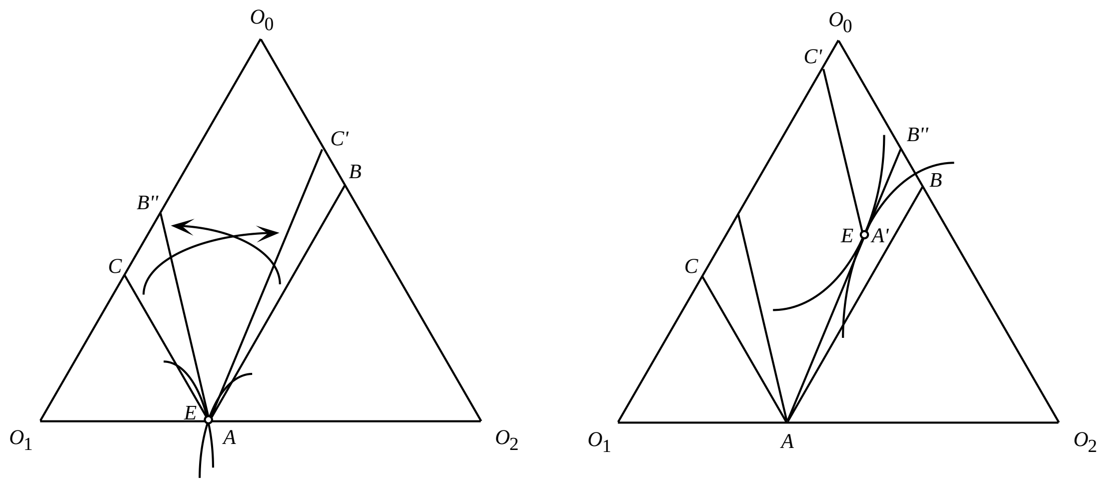

Fig. 15. Pareto Efficient Equilibria with $s_1 + s_2 \neq 1$.
Left: $s_1 + s_2 > 1$; $G = 0$. **Right:** $s_1 + s_2 < 1$; Agent 1 contributes 0.

The only possible equilibrium when $s_1 + s_2 > 1$ is at the initial allocation where both agents are contributing nothing towards the public good. This equilibrium is always a Pareto efficient allocation, provided that it exists —and there is a whole range of Lindahl prices that will support it as a Lindahl equilibrium. We illustrate one such case in the left panel of figure 15. Of course, it is possible that no equilibrium exists at all if $s_1 + s_2 > 1$.

If $s_1 + s_2 < 1$ we can possibly have a Pareto efficient SPE, which will also be a Lindahl allocation, only when $g_i = 0$ for one of the agents. Note that in the simple contributions game of section 5 we could not possibly have a Pareto efficient NE with one of the agents contributing a positive amount. We illustrate such a possibility in the right panel of figure 15. What allows for this possibility now is that AB' is steeper than AB which itself bounds the slope of agent one's indifference curves (see figure 2).

By the results in section 7 we cannot have an interior SPE whenever $s_1 + s_2 < 1$. Every equilibrium in the second-stage game which is not a Pareto efficient allocation cannot be a subgame-perfect equilibrium of the two-stage game. Figure 16 shows a general interior equilibrium of the second-stage game. Any agent can improve her welfare by incrementing the subsidy to the other agent.

Finally, let us examine a SPE, $(s_1^*, s_2^*; g_1^*(s_1, s_2), g_2^*(s_1, s_2))$, as the one depicted in figure 14, with $s_1^* + s_2^* = 1$. What would be the effect of agent two reducing agent one's subsidy to $\tilde{s}_1 < s_1^*$? We show this in figure 17. At the SPE E, with $s_1^* + s_2^* = 1$,

[10] Note that our definition of a Lindahl allocation differs from Danziger and Schnytzer (1991) since they do not allow $G = 0$ at Lindahl allocations.

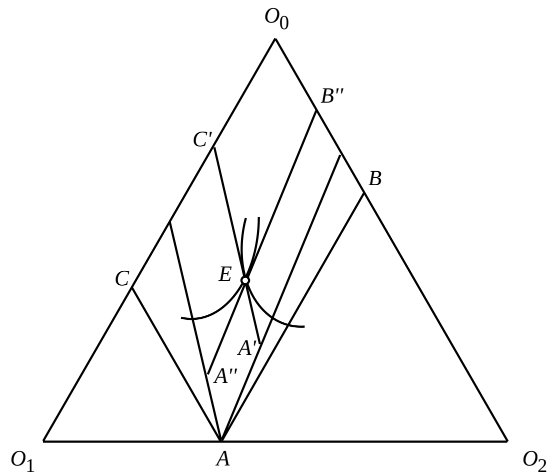

Fig. 16. Inefficient Second-Stage Game Equilibrium.

both agents are facing the common budget line AB'. Now, when s_1 falls to $\tilde{s}_1 < s_1^*$, agent one's budget line becomes AC' when agent two contributes nothing. When agent two contributes positive amounts towards the public good, agent one's budget line slides paralell to AC' from AB' to O_1O_0. The strict normality of both goods will guarantee that agent one's reaction function hits AB' below E; also, as before, once it hits AB' it stays in AB'. Therefore, the second-stage game equilibrium stays at E, now with agent two contributing $g_2^*(\tilde{s}_1, s_2^*) = g_1^*(s_1^*, s_2^*) + g_2^*(s_1^*, s_2^*)$ and agent one contributing zero, $g_1^*(\tilde{s}_1, s_2^*) = 0$. This illustrates that a given Lindahl allocation can be supported by a multitude of SPE of this game.

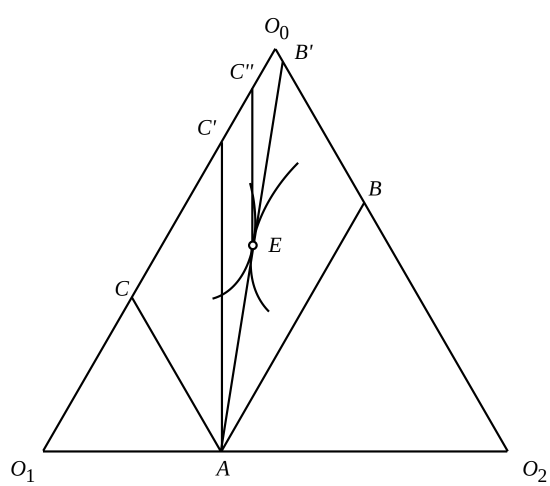

Fig. 17. Changing s_1 from the SPE E.

9. Concluding Remarks

The simple model of private contributions to public goods developed in Bergstrom, Blume and Varian (1986) has proved most fruitful to gain insights into a wide variety of problems. In this paper we provide simple geometrical proofs to a number of established results using the Kolm triangle. The Kolm triangle shows to be a powerful tool to understand the intricacies of the model and it is especially useful as a pedagogic device. The results shown in this paper were originally established using algebraic proofs, and hold in more general scenarios than the linear 2-agent 2-good world used here.

All the results discussed in this paper (except those in section 6 dealing with sequential provision) generalize to n agents. The other simplifying assumption, common linear technology, is harder to relax in some cases —specially the 'common' part. All the results hold when the public good is obtained by a single well-behaved production process, $G = f(g_1 + g_2)$, with $f'(\cdot) > 0$ and $f''(\cdot) \leq 0$. However, endowing each participant with an agent-specific technology (even linear), $g_i = f_i(w_i - x_i)$, with $G = \sum g_i$, suffices to eliminate Warr-type neutrality results (see, *e.g.*, Konrad and Lommerud (1995) for a model of the household where income transfers from the domestic partner who has comparative advantage outside the home to the other turn out to be not only efficiency-enhancing but also Pareto improving). Finally, the Warr neutrality result can also be extended to the case of many public goods (Bergstrom, Blume and Varian (1986), section 6).

10. References

Becker, G. (1981): *A Treatise on the Family*. Cambridge: Harvard University Press.

Bergstrom, T., L. Blume and H. Varian (1986): "On the Private Provision of Public Goods," *Journal of Public Economics*, **29**, 25–49.

Bergstrom, T. (1989): "Puzzles: Love and Spaghetti, The Opportunity Cost of Virtue," *Journal of Economic Perspectives*, **3**:2, 165–174.

Boadway, R., P. Pestieau and D. Wildasin (1989): "Tax-Transfer Policies and the Voluntary Provision of Public Goods," *Journal of Public Economics*, **39**, 157–176.

Bruce, N. (1990): "Defense Expenditures by Countries in Allied and Adversial Relationships," *Defence Economics*, **1**, 179–195.

Chander, P. (1993): "Dynamic Procedures and Incentives in Public Good Economies," *Econometrica*, **61**:6, 1341–1354.

Chichilnisky, G. and G. Heal (1994): "Who should abate carbon emissions? An international viewpoint," *Economics Letters*, **44**:4, 443–450.

Cornes, R. and T. Sandler (1985): "The Simple Analytics of Public Good Provision," *Economica*, **52**, 103–16.

Cornes, R. and T. Sandler (1986): *The Theory of Externalities, Public Goods and Club Goods*. London: Cambridge University Press.

Danziger, L. (1976): "A graphic representation of the Nash and Lindahl Equilibria in an Economy with a Public Good," *Journal of Public Economics*, **6**, 295–307.

Danziger, L. and A. Schnytzer (1991): "Implementing the Lindahl voluntary-exchange mechanism," *European Journal of Political Economy*, **7**, 55–64.

Dolbear, F.T. (1967): "On the Theory of Optimal Externality," *American Economic Review*, **57**, 90–103.

Gradstein, M., S. Nitzan and S. Slustky (1994): "Neutrality and the Private Provision of Public Goods with Incomplete Information," *Economics Letters*, **46**, 69–75.

Groves, T. and J.O. Ledyard (1987): "Incentive Compatibility since 1972," in *Information, Incentives and Economic Mechanisms. Essays in Honor of Leonid Hurwicz*, ed. T. Groves, R. Radner and S. Reiter. Minneapolis: University of Minnesota Press.

Hoel, M. (1991): "Global Environmental Problems: The Effects of Unilateral Actions Taken by One Country," *Journal of Environmental Economics and Management*, **20**:1, 55–70.

Itaya, Jun-ichi, David de Meza, and Gareth D. Myles (1997): "In praise of Inequality: Public Good Provision and Income Redistribution," *Economics Letters*, **57**, 289–96.

Kemp, M. (1984): "A Note on the Theory of International Transfers," *Economics Letters*, **14**:2–3, 259–62.

Kolm, S.-C. (1970): *L'État et Le Système des Prix*. Paris: Dunod–C.N.R.S..

Konrad, K.A. and K.E. Lommerud (1995): "Family Policy with non-cooperative Families," *Scandinavian Journal of Economics*, **97**:4, 581–601.

Laffont, J.-J. (1988): *Fundamentals of Public Economics*. London: The MIT Press.

Leamer, E. (1987): "Factor Scarcity, Factor Abundance and Attitudes toward Protection," *Journal of Political Economy*, **95**:5, 961–999.

Malinvaud, E. (1971): "A Planning Approach to the Public Goods Problem," *Scandinavian Journal of Economics*, **11**, 96–112.

Olsen, E.O. (1979): "The Simple Analytics of External Effects," *Southern Economic Journal*, **45**:3, 847–854.

Olsen, E.O. (1981): "The Simple Analytics of the Externality Argument for Redistribution," in *Economic Prespectives: An Annual Survey of Economics, volume 2*, ed. M. B. Ballabon. New York: Harwood Academic.

Roberts, R. (1987): "Financing Public Goods," *Journal of Political Economy*, **95**, 420–437.

Schlesinger, H. (1989): "On the Analytics of Pure Public Good Provision," *Public Finance*, **44**:1, 102–109.

Shibata, H. (1971): "A Bargaining Model of the Pure Theory of Public Expenditure," *Journal of Political Economy*, **79**:1, 1–29.

Sullivan, D. and H. Schlesinger (1986): "Canons of Just Taxation: Efficiency and Fairness in an Economy with a Public Good," *Public Finance Quaterly*, **14**:4, 448-465.

Thomson, W. (1987): "Monotonic Allocation Mechanisms in Economies with Public Goods," University of Rochester Working Paper # 117.

Thomson, W. (1990): "Economies with Public Goods: An Elementary Treatment," mimeo, University of Rochester.

Thomson, W. (1993): "Notions of Equal, or equivalent, opportunities," *Social Choice and Welfare*, **10**, 1–20.

Varian, H. (1994): "Sequential Provision of Public Goods," *Journal of Public Economics*, **53**:2, 165–186.

Warr, P. (1983): "The Private Provision of a Public Good is Independent of the Distribution of Income," *Economics Letters*, **13**, 207–211.

Wilson, L.S. (1992): "The *Harambee* movement and efficient public good provision in Kenya," *Journal of Public Economics*, **48**:1, 1–20.