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Optimal Provision of Public Goods with Altruistic Individuals

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Abstract. We study the optimal provision of public goods in the context of a special class of altruistically linked utility functions. We show that the usual Samuelson condition holds as if the utility functions were independent.

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JEL Classification System. D6, H0, H4

1. Modelling Altruism

Suppose that we have n agents with altruistically interrelated utility functions. Denote by z_i agent i's consumption bundle, and let $\mathbf{z} = (z_1, \ldots, z_n)$ represent an allocation. Each agent is assumed to have preferences over allocations, $\mathbf{z}$, which are additively separable over individual bundles, z_i. Thus, we assume that agent i's utility index, V_i, can be represented by

$$V_i = \psi_i(\mathbf{z}) = \sum_j \beta_{ij} U_j(z_j) = \underbrace{\beta_{ii} U_i(z_i)}_{ego} + \underbrace{\sum_{j \neq i} \beta_{ij} U_j(z_j)}_{alter}, \tag{1}$$

where $U_i(\cdot)$ is a twice-differentiable, strictly quasi-concave, and monotonically increasing function. We will always assume that $\beta_{ii} > 0$. Utility results from an *ego* part, $\beta_{ii} U_i$, and an *alter* part, $\sum_{j \neq i} \beta_{ij} U_j$. If $\beta_{ij} = 0$ for all $i \neq j$, so that there is no *alter*, then we have the usual *egoistic* preferences. Otherwise, we shall say that the system is *altruistic*. While some of the results below also hold for malevolent systems, when dealing with altruistic systems we will always assume that they are *benevolent* systems so that we have $\beta_{ij} \geq 0$. Systems like (1) have been used to represent altruism by Becker (1974), and Abel and Bernheim (1991), among others. Becker (1976) uses a more general formulation —*i.e.*, he uses a utility function not necessarily separable.

As discussed, *e.g.*, in Bergstrom (1990), there is an alternative way to model interrelated utility. Instead of using a system like (1), it is sometimes more natural to specify i's preferences over his own consumption bundle and everybody else's 'happiness':

$$V_i = \phi_i(z_i, V_{\sim i}) = \underbrace{\gamma_i U_i(z_i)}_{ego} + \underbrace{\sum_{j \neq i} \delta_{ij} V_j}_{alter} = U_i(z_i) + \sum_j \alpha_{ij} V_j \tag{2}$$

where utility, V_i, is provided by the *ego* part, $\gamma_i U_i(z_i)$, and the *alter* part, $\sum_{j \neq i} \delta_{ij} V_j$; and $V_{\sim i}$ represents the vector of V_j's excluding V_i. We also have $\alpha_{ii} = (\gamma_i - 1)/\gamma_i$ and $\alpha_{ij} = \delta_{ij}/\gamma_i$ for $i \neq j$. This formulation is used, *e.g.*, in Barro (1974), Bernheim and Stark (1988), Bergstrom (1989).

Stacking the U_i's and the V_i's in column vectors U and V, a system like (2) can be expressed in matrix form as

$$V = U + AV \tag{2'}$$

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where α_{ij} is the ijth element of A. If $(I-A)^{-1}$ exists, we can write (2′) as

$$V = (I-A)^{-1}U = BU \tag{1′}$$

where the ijth element of B corresponds to β_{ij} in (1). Conversely, if we start from $V = BU$, and B^{-1} exists we can use $A = I - B^{-1}$ to transform (1′) into a system like (2′).

There are two types of issues when going from one representation to the other. There is a technical issue dealing the existence of the inverse matrices $(I-A)^{-1}$ or B^{-1}. But there is an additional issue dealing with the 'consistency' of the utility representations. It is reasonable to expect that a benevolent system like (2) with all the $\delta_{ij} \geq 0$, should have all the $\beta_{ij} \geq 0$ when transformed into a system like (1). Bergstrom (1990) establishes the conditions under which a well-behaved system like (1) can be represented by a well-behaved system like (2) and viceversa.[1] We will call those well-behaved systems *felicitous*.[2]

In what follows we only need to assume that a utility representation like (1) exists. Provided that this utility representation is also felicitous, then there exists an associated representation like (2) to which the results apply as well.

2. Optimal Public Good Provision

For the ease of exposition, suppose that there is only one private good, x_i, and one pure public good, Y; so that $z_i = (x_i, Y)$.[3] We shall assume that the public good can be produced at a constant marginal cost. Choosing units suitably, we can make the (constant) marginal rate of transformation between the private good and public good equal to one.

Let w_i represent i's endowment of the private good and let $\mathbf{w} = (w_1, \ldots, w_n)$. We will assume in what follows that B always has strictly positive diagonal elements and positive off-diagonal elements, so that it can be used to represent an altruistic system. We shall use $\mathcal{E}\langle \mathbf{w}, BU\rangle$ to represent an economy with altruistic individuals (whose preferences can be represented by a system like (1), or in matrix form as $V = BU$). We shall use $\mathcal{E}\langle \mathbf{w}, U\rangle$ to represent the same economy with the egoistic individuals that would be obtained by making the β_{ij}'s equal to zero in the altruistic system. That is, for every altruistic system we obtain an egositic system by simply dropping the *alter* part in (1).

Denote by $W = \sum_i w_i$ aggregate resources, and let $X = \sum_i x_i$; then $Y = W - X$. Pareto optimal allocations of $\mathcal{E}\langle \mathbf{w}, BU\rangle$ are maxima of

$$\mathcal{L} = \sum_i \lambda_i V_i = \sum_i \lambda_i \sum_j \beta_{ij} U_j(x_j, W - X), \tag{3}$$

for any row vector $\lambda = (\lambda_1, \ldots, \lambda_n) > 0$ —see,*e.g.*, Cornwall (1984).

Proposition 1. *Pareto efficient allocations of the altruistic economy $\mathcal{E}\langle \mathbf{w}, BU\rangle$ are also Pareto efficient allocations of the egoistic economy $\mathcal{E}\langle \mathbf{w}, U\rangle$.*

Proof. We can rewrite (3) as:

$$\mathcal{L} = \sum_j U_j(x_j, W - X) \sum_i \lambda_i \beta_{ij} = \sum_j \mu_j U_j(x_j, W - X). \tag{4}$$

[1] To get a sense of the perversities that can occur, take $n = 3$ and start out from $V_i = U_i(z_i) + \sum_{j\neq i} V_j$. This system transforms into $V_i = -0.5 \sum_{j\neq i} U_j(z_j)$. Thus, in a system of apparent benevolence, when we obtain the representation of the preferences over allocations we find that agent i: (1) does **not** care about his own consumption bundle and (2) cares negatively about other peoples' ego-happiness. A planner concerned with maximizing welfare would just need to destroy the economy's resources!

[2] A system like (1′), with $A > 0$, will be called *felicitous* if there exists a non-negative row vector η such that $\eta > \eta A$; and we shall then say that A is a felicitous matrix —in linear models, a consumption matrix A which has this property is called *productive*, see, *e.g.*, Gale (1960) or Cornwall (1984). A key property of a felicitous system is that $(I-A)^{-1}$ exists and it is non-negative so that for any $U > 0$ we have $V = (I-A)^{-1}U > 0$ —see, *e.g.*, Gale (1960). We shall say that $B > 0$ is felicitous when $B^{-1} = I - A$ and A is felicitous.

[3] This private-public terminology is valid in an egoistic economy. In an altruistic system, 'private' goods generate consumption externalities so they are not properly private.

where $\mu_j = \sum_i \lambda_i \beta_{ij} > 0$, since $\lambda_i > 0$, $\beta_{ii} > 0$, and $\beta_{ij} \geq 0$. Maxima of (4) correspond to Pareto efficient allocations of an economy where $V_i = U_i(x_i, Y)$ with welfare weights $\mu = (\mu_1, \ldots, \mu_n)$. ∎

As a corollary, efficient allocations of an altruistic system like (1) must satisfy an 'unaltered' Samuelsonian condition:[4]

$$\sum_i \frac{\frac{\partial U_i(x_i,Y)}{\partial Y}}{\frac{\partial U_i(x_i,Y)}{\partial x}} = 1; \tag{5}$$

instead of $\sum_i \frac{\partial V_i/\partial Y}{\partial V_i/\partial x} = 1$, as one might have expected.

Proposition 2. *If $U_i(x_i, Y) = v(Y)x_i + u_i(Y)$, then the optimal level of the public good in $\mathcal{E}\langle \mathbf{w}, BU\rangle$ is the same in all Pareto efficient allocations and it does not depend on the values of the β_{ij}'s.*

Proof. If $U_i(x_i, Y) = v(Y)x_i + u_i(Y)$ then $\sum_i \frac{\partial U_i/\partial Y}{\partial U_i/\partial x}$ only depends on $\sum_i x_i$ which implies that the optimal provision of Y in an egoistic system is independent of the distribution of the private good among the agents —see Bergstrom and Cornes (1981). Therefore, by proposition 1, the efficient level of Y is determined independently of B in an altruistic system. ∎

We can rewrite (3) and (4) in matrix form as $\mathcal{L} = \lambda BU = \mu U$. Since, for every $\mu > 0$, you can always find a regular $B > 0$, that guarantees that $\mu B^{-1} > 0$.[5] Then, for each Pareto efficient allocation of an egoistic system U you can always find an altruistic system BU for which that allocation is also Pareto efficient.

However, what about the reverse statement to proposition 1? Are all Pareto optima of the egoistic system $\mathcal{E}\langle \mathbf{w}, U\rangle$ also Pareto optima of the altruistic system $\mathcal{E}\langle \mathbf{w}, BU\rangle$ for any altruistic B? The answer is no. To establish the reverse proposition, we would have to show that for any $\mu > 0$ we can find $\lambda > 0$ such that $\mu = \lambda B$. Postmultiply both sides by $(I - A) = B^{-1}$ and we obtain $\mu(I - A) = \lambda$. It should be clear that, given a felicitous $A > 0$, we cannot always guarantee that $\mu(I - A) > 0$ for any arbitrary $\mu > 0$.

Proposition 3. *If $\mu B^{-1} > 0$, a Pareto efficient allocation of the egoistic economy $\mathcal{E}\langle \mathbf{w}, U\rangle$ with welfare weights μ, is a Pareto efficient allocation of the felicitous altruistic economy $\mathcal{E}\langle \mathbf{w}, BU\rangle$ with welfare weights $\lambda = \mu B^{-1}$.*

However, if preferences are of the form $U_i(x_i, Y) = v(Y)x_i + u_i(Y)$, it follows from proposition 2 that Pareto optima of $\mathcal{E}\langle \mathbf{w}, U\rangle$ are Pareto optima of $\mathcal{E}\langle \mathbf{w}, BU\rangle$ for any $B > 0$.

3. Concluding Remarks

Bernheim and Stark (1988) show that altruism can alter the utility possibilities frontier in most surprising ways. Here we derive the conditions for optimal provision of a public good and we find the intriguing result that an *unaltered* Samuelson condition must hold. That is, the sum of ego-marginal rates of substitution must equal the marginal rate of transformation. As a result, Pareto optima of an altruistic economy are also Pareto optima of the egoistic economy obtained by eliminating all altruistic links.

[4] We shall only deal with interior solutions. Corner solutions only add complication to the exposition without providing additional insights.

[5] For example, make $\beta_{ij} = 0$ if $j - 1 \neq 0, 1$ and $\beta_{ii} = 1$. Write $\mu = \lambda B$, start with $\lambda_1 = \mu_1$. Then recursively choose $\beta_{i-1,i} < \mu_i/\lambda_{i-1}$.

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