A Conditional Defense of Plurality Rule: Generalizing May's Theorem in a Restricted Informational Environment

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Abstract: May's theorem famously shows that, in social decisions between two options, simple majority rule uniquely satisfies four appealing conditions. Although this result is often cited as a general argument for majority rule, it has never been extended beyond pairwise decisions. Here we generalize May's theorem to decisions between many options where voters each cast one vote. We show that, surprisingly, plurality rule uniquely satisfies May's conditions. Our result suggests a conditional defense of plurality rule: If a society's balloting procedure collects only a single vote from each voter, then plurality rule is the uniquely compelling procedure for electoral decisions.

Social choice theorists in Condorcet's and Borda's tradition are idealistic electoral reformers in at least two respects. First, they propose certain ideals with respect to the *information* we should collect from voters in a *balloting procedure*. Second, they propose certain ideals with respect to how we should *aggregate* that information in an *aggregation procedure*, so as to make a decision on its basis. A fully fledged *voting procedure* consists of both a balloting procedure and an aggregation procedure.[2]

In this article, we want to be only half as idealistic. Here we take balloting procedures as they are, and consider aggregation procedures as they might be. We offer a *conditional* defense of "plurality rule." *If* a society's balloting procedure collects only a single vote from each voter, *then* plurality rule, which always chooses the option

[1]We are grateful to Dennis Mueller, Franz Dietrich and two anonymous referees for helpful comments on an earlier version of this paper.

[2]The same balloting procedure may go along with different aggregation procedures (e.g., full-preference balloting with pairwise majority voting, STV, AV, the Borda count etc.) and the same aggregation procedure with different balloting procedures (e.g., plurality rule with secret single-vote balloting, open single-vote balloting, and even full-preference balloting).

with the most votes, is the uniquely compelling aggregation procedure; we show that it is so, in the sense that it uniquely satisfies May's well-known minimal conditions on a democratic procedure generalized to decisions over any number of options.[3] Our result thus constitutes a many-option generalization of May's classical theorem on majority rule in the distinctive, but politically common, informational environment of single-vote balloting. May's conditions are widely regarded as normatively compelling in the two-option case, for which they were originally formulated, and we suggest that they remain compelling in the many-option case considered here.[4]

Our result should be of interest in at least two respects. First, to the best of our knowledge, plurality rule has never been associated with May's theorem or a many-option version of May's conditions. Second, and perhaps more importantly, plurality rule has traditionally been held in low esteem, among both formal social choice theorists and philosophical theorists of democracy. Thus any positive argument for plurality rule appears to go against the grain.

So what about plurality rule's well-known defects? For example, plurality rule violates several desiderata that social choice theorists often expect aggregation procedures for many-option decisions to meet, including Condorcet consistency ("the Condorcet winner should be selected if it exists") and consistency of the winning option under a contraction or expansion of the set of available options. But formulating these desiderata requires referring to voters' full preference orderings, and implementing them in an aggregation procedure requires a balloting procedure that

[3]Specifically, we generalize May's conditions from the case where voters each submit one vote over two options to the case where they each submit one vote over any number of options.

[4]To be precise, we focus on regular electoral decisions, where there is no normatively admissible asymmetry between voters or options, and claim that May's conditions are defensible in such decisions. In decisions with a normatively admissible asymmetry between voters (e.g. some expert panels) or options (e.g. some jury decisions or referenda with a status-quo bias), some of May's conditions may need to be relaxed.

collects (enough of) this information. Given only a single vote from each voter, we simply have insufficient information to implement them. For example, we can only guess as to how expanding or shrinking the set of available options affects voters' single votes, their revealed "first choices". Likewise, voters may have incentives to vote strategically under plurality rule. But without further information about voters' preferences, we have no choice but to take voters' revealed first choices at face value.[5] To emphasize, we do not unconditionally defend plurality rule. In particular, we do not defend single-vote balloting procedures; we only make a conditional claim: *If* single-vote balloting procedures are used – as they often are, in practice – *then* plurality rule is the way to go.

Of course, collecting only a single vote from each voter is not ideal. For example, "the number of voters who think each candidate the worst ... is no less important ... than the number of voters who think each candidate the best" (Dummett, 1997, 51-52). A balloting procedure that collects voters' revealed "first choices" alone takes no account of that. It would undeniably be ideal in many cases to collect voters' full preferences, top to bottom, over all the available options. It would at least be an improvement to collect more (even short of "full") information about these preferences, through a two-ballot runoff procedure, such as in France, for example. Such richer informational environments would allow us to use more sophisticated aggregation procedures than plurality rule.

Realistically, however, that is simply not the way ballots are conducted in many

[5] Vulnerability to strategic voting is not unique to plurality rule, but shared by all reasonable aggregation procedures over more than two options, which are typically not strategy-proof (by the Gibbard-Satterthwaite theorem). But under plurality rule, strategic voting is limited as follows: voters have an incentive to vote for their preferred option among those two options that they think are most likely to win. In particular, plurality rule has the property of being "immune to insincere manipulation" (van Hees and Dowding, 2005).

places in the world. Many real-world balloting procedures collect only information about voters' (revealed) "first choices". Instead of ranking alternatives best-to-worst, people are only asked to vote for a single option. The US, UK and Canada are the most famous examples of countries employing such balloting procedures in their legislative elections. But such procedures are also used in many other (sometimes surprising) countries and international organizations.[6]

Moreover, as a practical political fact, it is often far easier to change the rules of how votes are aggregated, in aggregation procedures, than it is to change how votes are collected, in balloting procedures. The latter involves changing the formal rules governing the behavior of millions of voters.[7] The former involves changing the formal rules governing the behavior of perhaps only a few hundred election officials.

Suppose then that, as a practical real-world constraint, the single-vote balloting procedure is given. The other half of the social choice theorists' ideal nonetheless remains in play. We still need to ask what the best way is to aggregate the information collected by this balloting procedure into a social decision. This question has not been addressed in the social-choice-theoretic literature. We seek to give an answer. After introducing May's theorem in Section 1, we prove our new theorem in Section 2, which generalizes May's result to decisions over more than two options in the given restricted informational environment. In Section 3, we discuss the informational basis

[6] Other countries include Bangladesh, Belize, Bhutan, Botswana, Cameroon, Central African Republic, Ivory Coast, Cuba, North Korea, India, Gambia, Grenada, Jordan, Kenya, Kuwait, Lebanon, Latvia, Malawi, Maldives, Malaysia, Mongolia, Morocco, Myanmar, Nepal, Nigeria, Pakistan, Papua New Guinea, Rwanda, Saint Kitts and Nevis, Saint Lucia, Saint Vincent and the Grenadines, Samoa (Western), Sao Tome and Principe, Seychelles, Solomon Islands, South Africa (for directly elected seats), Thailand (mixed system), Trinidad and Tobago, Tonga, Tuvalu, Uganda, Tanzania (the directly elected members), Zambia, Zimbabwe.

[7] Of course, a change in the aggregation procedure may also induce a change in electoral behavior; but at least the formal rules on the collection of votes remain the same.

of voting. Finally, to illustrate the conditional nature of our argument for plurality rule, we show in Section 4 that, if we use the richer informational environment of approval balloting instead of single-vote balloting, our May-style argument for plurality rule becomes a novel May-style argument for approval voting (Brams and Fishburn, 1978). Our findings highlight that the question about the most compelling aggregation procedure depends on the informational environment in which this procedure is meant to operate. In Section 5, we make some concluding remarks.

1 May's theorem

May's (1952) classical theorem states that, in social decisions between two options, simple majority rule, uniquely among all aggregation procedures, satisfies the four normatively appealing conditions of being: open to all inputs ("universal domain"); not biased in favor of any particular voter ("anonymity"); not biased in favor of any particular outcome ("neutrality"); and "positively responsive" to people's votes (if one or more voters change their votes in favor of one option and no others change theirs, then the social decision does not change in the opposite direction; and if the outcome was a tie before the change, then the tie is broken in the direction of the change). In both formal social choice theory and democratic theory more generally, this result occupies a prominent place as an argument for democratic rule, in the form of simple majority rule.

In the formal social choice literature, May's theorem "is deservedly considered a minor classic" (Barry and Hardin, 1982, 298). It is one of the first things said on the subject of "normative properties of social decision rules", in all the classic overviews, beginning with Luce and Raiffa's *Games and Decisions* (1957, 357-358) and running through Mueller's *Public Choice III* (2003, 133-136). Arrow comments on it in the notes appended to the second edition of *Social Choice and Individual*

Values (1951/1963, 100-102). Sen discusses it in *Collective Choice and Social Welfare* (1970, 68, 70-1) and extends it in a paper with Pattanaik (Sen and Pattanaik, 1969). Several other social choice theorists offer derivations and extensions (e.g., Murakami, 1966, 1968; Pattanaik, 1971, 50-52; Fishburn, 1973, 50, 57 ff.; Shepsle and Bonchek, 1997, 160-162; Cantillon and Rangel, 2002).

Because its proof is relatively straightforward, May's theorem may count only as a "minor" classic in the formal social choice literature, but it has been received as a major finding within democratic theory more generally. In a field replete with negative findings (impossibility, instability and manipulability results),[8] May's theorem stands out as a powerful positive result supporting democratic rule.

Consider these examples, to get a sense of how deeply May's theorem has penetrated non-formal democratic theory. In *Democracy and Its Critics*, the capstone of Robert Dahl's lifelong work on democratic theory, May's theorem is invoked as the second of his "four justifications for majority rule" (Dahl, 1989, 139-141). May's theorem is one of the first considerations that William Riker (1982, 59 ff.) feels the need to neutralize, in his argument against populist democracy in *Liberalism Against Populism*. May's theorem is also one of the first considerations offered in defense of liberal democracy in Ackerman's *Social Justice and the Liberal State* (1980, 277-285). It is a key element in "political equality" as conceptualized by both Beitz (1989, 59) and Christiano (1990, 154-157; 1993, 183; 1996), and continual reference is made to it across democratic theory (Coleman and Ferejohn, 1986, 18-19; Martin, 1993, 367-368 n. 5; Saward, 1998, 69; Waldron, 1999, 148, 189 n. 38; Risse, 2004, 51-5).

However, May's theorem, as proven by May and used in the subsequent literature,

[8] The most prominent negative results are Arrow's theorem (1951/1963) and results about cyclical social preferences and electoral disequilibrium (Sen, 1970, 1977; Schofield, 1976; McKelvey, 1979; cf. Mackie, 2003) and the Gibbard-Satterthwaite theorem on strategic manipulability (Gibbard, 1973; Satterthwaite, 1975). For an overview, see Austen-Smith and Banks (1999).

applies only to decisions between two options or a sequence of two-option decisions. But in the real world our choices are rarely between only two options (or if they are, that is often the result of an undemocratic agenda setting process; e.g. Riker, 1982, 59-60). Curiously, in the formal social choice literature, May's theorem has never been extended beyond pairwise decisions.[9] In more informal discussions, the problem is occasionally noted but never pursued. Thus, for example, Coleman and Ferejohn (1996, 18) remark upon the need to extend May's theorem to "admissible choice on larger sets of alternatives [than just two] by voting rules that are extensions of binary majority rule"; but they "leave aside" the question of "what would constitute an extension of simple majority rule" in the case of more-than-two options. Risse (2004, 51-52 n. 25) observes that "May's Theorem only applies when groups decide on two options", adding in a note that, "There surely could be some mathematical generalization of May's Theorem, but no such generalization is likely to preserve the elementary character of the assumptions of May's Theorem".

The proof below puts paid to that speculation. We show that May's theorem can be extended to the many-option case, where voters each cast a single vote for one option. Our conditions are straightforward generalizations of May's original ones. We prove that – perhaps surprisingly, given its bad reputation – plurality rule uniquely satisfies those conditions.

While there are other axiomatic characterizations of plurality rule in the literature

[9] Pattanaik (1971, pp 50-51) generalizes May's *conditions* to the many-option case, but applies those conditions to other problems rather than May's theorem itself. Also, much work has been done formally extending May's theorem – just not in the present way, to imply plurality rule over more than two options. For example, pairwise majority voting can be characterized by imposing May's original conditions on each pairwise ranking (under independence of irrelevant alternatives), e.g. Cantillon and Rangel (2002). Such an extension preserves the original pairwise format of May's theorem, but also involves a richer informational environment than we are often in politically, namely full-preference balloting.

(e.g., Roberts, 1991; Ching, 1996), we are not aware of any contribution – formal or informal – that associates May's theorem, or a many-option version of May's precise conditions, with plurality rule.

2 Generalizing May's theorem to more than two options

2.1 An informal statement

There are n individuals and k options. May's original theorem addresses the special case $k = 2$. Our result holds for any positive $k \geq 2$. Each individual submits a vote for one option or abstains. A combination of votes across individuals is called a *profile*. This captures the informational environment of *single-vote balloting*. An *aggregation procedure* is a function that assigns to each such profile a corresponding outcome. The outcome is either a single winning option or a tie between two or more options. May's conditions can be generalized to the k-option case as follows:

Universal domain. The aggregation procedure accepts all logically possible profiles of votes as admissible input.

Anonymity. The outcome of the aggregation procedure is invariant under a permutation of the votes across individuals.

Neutrality. If the votes are permuted across options, then the outcome is permuted accordingly.

Positive responsiveness. If one or more voters change their votes in favor of an option that is winning or tied and no other voters change theirs, then that option is uniquely winning after the change.

We define *plurality rule* as follows. For any profile, the option, if unique, that receives the largest number of votes, is chosen as the winner; if there is no unique such option, then all the options that receive an equal largest number of votes are tied.

Theorem. An aggregation procedure satisfies universal domain, anonymity, neutrality and positive responsiveness if and only if it is plurality rule.

2.2 A formal statement

There are n individuals, labeled $1, \ldots, n$, and k options, labeled $1, \ldots, k$. Each individual votes for precisely one option or abstains. Individual i's vote is denoted v_i and represented by one of the following column vectors:

a vote for option 1	a vote for option 2	...	a vote for option k	an abstention
$\begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ \ldots \\ 0 \end{pmatrix}$	...	$\begin{pmatrix} 0 \\ 0 \\ \ldots \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$

A *profile (of votes)* is a matrix $v = (v_1, \ldots, v_n)$, i.e. a row vector of column vectors. We write v_{ij} to denote the entry in column i and row j in v. So $v_{ij} = 1$ means that individual i votes for option j. (In particular, if $v_{ij} = 1$ then $v_{ih} = 0$ for all $h \neq j$.)

An *aggregation procedure* is a function f that maps each profile v to an outcome of the form $f(v) = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_k \end{pmatrix}$, where each x_j is either 0 or 1 and at least one x_j is 1. For each j, we write $f(v)_j = x_j$. Informally, $f(v)_j = 1$ means that option j is winning or tied, and $f(v)_j = 0$ means that option j is non-winning. By definition, for any profile, there is at least one option j with $f(v)_j = 1$. If there is exactly one such option, this option is the unique winner; if there is more than one such option, all the options j with $f(v)_j = 1$ are tied.

Now May's conditions generalized to k-option choices can be formally stated as follows:

Universal Domain. The domain of f is the set of all logically possible profiles.

Anonymity. Let σ be any permutation of the n individuals, represented by a permutation of columns. For any profile v, $f(v) = f(\sigma(v))$.

Neutrality. Let τ be any permutation of the k options, represented by a permutation of rows. For any profile v, $f(\tau(v)) = \tau(f(v))$.

For any two profiles v and w and any option j, we write $v \succ_j w$ if and only if there exists some option h such that

for some individuals i, $v_{ij} > w_{ij}$ and $v_{ih} < w_{ih}$,[10]

and for all other individuals i, $v_i = w_i$.

Informally, if a profile changes from w to v, then $v \succ_j w$ means that at least one individual's vote changes towards option j from some other option h, while all other votes remain the same.

Positive responsiveness. For any two profiles v and w and any j, if $f(w)_j = 1$ and $v \succ_j w$, then $f(w)_j = 1$ and, for all $h \neq j$, $f(w)_h = 0$.

If $k = 2$, these conditions reduce to the standard conditions of May's theorem.

[10] And $v_{il} = w_{il}$ for all options $l \neq j, h$. Under single-vote balloting, this clause is already implied by $v_{ij} > w_{ij}$ and $v_{ih} < w_{ih}$. However, in the alternative environment of approval balloting discussed below, we need to add this clause explicitly.

Define *plurality rule* as the aggregation procedure f, where, for any profile v and any option j,

$$f(v)_j = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} v_{ij} \geq \sum_{i=1}^{n} v_{ih} \text{ for all } h \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

We can now state our result.

Theorem. An aggregation procedure satisfies universal domain, anonymity, neutrality and positive responsiveness if and only if it is plurality rule.

2.3 The proof

It is easy to see that plurality rule satisfies all the conditions. Universal domain is satisfied because plurality rule is defined for all logically possible profiles. Anonymity is satisfied because the plurality winner (or tied set of options) depends only on the number of votes for each option, not on the voters' identities. Neutrality is satisfied because the question of whether an option is winning, tied or losing depends only on the number of votes for this option and its contenders, not on these options' labels. Finally, positive responsiveness is satisfied because, under plurality rule, any additional votes for a winning option do not hurt that option, and any additional votes for a tied option break the tie in favor of that option.

Suppose, conversely, that an aggregation procedure f satisfies all the conditions. For any profile v, we call the column vector $a = \sum_{i=1}^{n} v_i$ a *votes vector.* Every such votes vector is of the form $a = \begin{pmatrix} a_1 \\ a_2 \\ \ldots \\ a_k \end{pmatrix}$ where each $a_j \geq 0$ and $a_1 + \ldots + a_k \leq n$. Here a_1 is the number of votes for option 1, a_2 the number of votes for option 2, and so on.

Claim 1. The aggregation procedure f can be represented by a function g whose domain is the set of all possible votes vectors and whose co-domain is the same as that of f. For each v, we have $f(v) = g(\sum_{i=1}^{n} v_i)$.

Proof of claim 1. Claim 1 follows from the anonymity of f, which implies that, for any two profiles v and w, if $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} w_i$ then $f(v) = f(w)$.

Claim 2. For any votes vector a and any permutation of rows τ, $g(\tau(a)) = \tau(g(a))$.

Proof of claim 2. Given claim 1, claim 2 follows from the neutrality of f.

For any two votes vectors a and b and any option j, we write $a \succ_j b$ if and only if there exists some option h such that

$$a_j = b_j + e \text{ and } a_h = b_h - e \text{ where } e > 0,$$
$$\text{and for all options } l \neq j, h,\ a_l = b_l.$$

Informally, if a votes vector changes from b to a, then $a \succ_j b$ means that option j gains e votes at the expense of some other option h, while all other options receive an equal number of votes.

Claim 3. For any two votes vectors a and b and any j, if $g(b)_j = 1$ and $a \succ_j b$, then $g(a)_j = 1$ and, for all $h \neq j$, $g(a)_h = 0$.

Proof of claim 3. Given claim 1, claim 3 follows from the positive responsiveness of f.

Claim 4. For each votes vector a, $g(a)_j = 1$ if and only if $a_j \geq a_h$ for all $h \neq j$.

Proof of claim 4. First, consider a votes vector a such that $g(a)_j = 1$. Assume, for a contradiction, that $a_j < a_h$ for some $h \neq j$. Write $a_h = a_j + e$ (with $e > 0$). Let

τ be the row permutation which swaps rows j and h and leaves all other rows fixed. By claim 2, $g(\tau(a)) = \tau(g(a))$. Hence $g(\tau(a))_h = \tau(g(a))_h = g(a)_j = 1$. Note that $\tau(a)_j = a_h = a_j + e$ and $\tau(a)_h = a_j = a_h - e$; also, for all $l \neq j, h$, we have $\tau(a)_l = a_l$. Therefore $\tau(a) \succ_j a$. By claim 3, $g(\tau(a))_j = 1$ and, for all $l \neq j$, $g(\tau(a))_l = 0$. In particular, this implies $g(\tau(a))_h = 0$, a contradiction.

Next, consider a votes vector a such that $a_j \geq a_h$ for all $h \neq j$. Assume, for a contradiction, that $g(a)_j = 0$. As g has the same co-domain as f, there must exist some h such that $g(a)_h = 1$ (and $h \neq j$ as $g(a)_j = 0$). Again let τ be the row permutation which swaps rows j and h and leaves all other rows fixed. By claim 2, $g(\tau(a)) = \tau(g(a))$ and hence $g(\tau(a))_h = \tau(g(a))_h = g(a)_j = 0$. Now either $a_j = a_h$ or $a_j > a_h$. If $a_j = a_h$, then $\tau(a) = a$, in which case $g(a)_h = g(\tau(a))_h = 0$, a contradiction. If $a_h < a_j$, write $a_j = a_h + e$ (with $e > 0$) and note that $\tau(a)_h = a_j = a_h + e$ and $\tau(a)_j = a_h = a_j - e$; also, for all $l \neq j, h$, we have $\tau(a)_l = a_l$. Therefore $\tau(a) \succ_h a$. By claim 3, $g(\tau(a))_h = 1$, a contradiction.

Claims 1 and 4 now imply that f is plurality rule.

3 The informational environment of voting

We have generalized May's classical theorem to many-option decisions in the informational environment in which voters each submit a single vote for one of the options. As noted, this informational environment is very different from one in which voters reveal their full preference orderings over all options, top to bottom.

Following Sen's pioneering work (1970, 1982), questions about the appropriate informational basis of social choice have received considerable attention. But much of this debate has focused on welfare-economic applications of social choice theory rather than voting-theoretic ones. For example, a much discussed question is how to measure the effects of alternative policies or social states on individual welfare, in order to arrive at a social preference ordering over these policies or states. In particular, it is debated whether only ordinal and interpersonally non-comparable measures of individual welfare are feasible, as Arrow originally suggested, or whether we can construct cardinal and/or interpersonally comparable measures of welfare.

By contrast, questions about the appropriate informational basis of voting have received less attention in the social-choice-theoretic literature. Many formal models in mainstream voting theory, following Arrow's own model (1951/1963), are based on an informational environment in which voters' full preference orderings are available. There has been little work on the question of what aggregation procedures to use for electoral decisions in restricted informational environments. By showing that plurality rule uniquely satisfies some compelling normative conditions in the restricted but empirically important informational environment of single-vote balloting, we have thus contributed towards filling this gap in the literature.

A side effect of this restricted informational environment is that the notorious problem of cyclical majority preferences remains hidden here. Given a set of revealed "first-choice" votes, plurality rule always produces a determinate winning option (or a tied set of options); no cycle can be observed. Nonetheless, in terms of voters' underlying full preferences, there may well be majority cycles. Single-vote balloting does not solve – it merely hides – the problems raised by standard social choice paradoxes.[11]

[11] But, arguably, many prominent responses to these paradoxes, such as Shepsle and Weingast's (1981) structure-induced equilibrium, also only hide these problems. Although we define plurality rule in a restricted informational environment, where a vector of single votes is aggregated into a winning option or tied set, it can also be defined in a richer environment, where a vector of preference orderings is aggregated into one of the following outputs: (i) a social preference ordering (Arrow's framework), (ii) a choice function that assigns to each set of available options a winning option or tied set (the "collective choice rule" framework), (iii) a single winning option (the Gibbard-Satterthwaite

Notice that many standard critiques of plurality rule do not actually undermine our conditional claim – that plurality rule should be used as an aggregation procedure *if* a single-vote balloting procedure is used – but they are rather directed against the antecedent of this conditional, i.e. they criticize the use of single-vote balloting procedures. (And, as we have noted, we do not defend single-vote balloting here.) Recall Dummett's above quoted point that "the number of voters who think each candidate the worst ... is no less important ... than the number of voters who think each candidate the best" (see further Dummett, 1984, ch. 6; 1997, 51-57). Borda (1784/1995, 83) begins his critique of plurality rule as follows:

> There is a widespread feeling, which I have never heard disputed, that in a ballot vote, the plurality of votes always shows the will of the voters. That is, that the candidate who obtains this plurality is necessarily preferred by the voters to his opponents. But I shall demonstrate that this feeling, while correct when the election is between just two candidates, can lead to error in all other cases.

In elaboration, Borda focuses on the balloting procedure underlying plurality rule: "If a form of election is to be just, the voters must be able to rank each candidate according to his merits, compared successively to the merits of each of the others..." Plurality rule "is highly unsatisfactory" in those terms, precisely "because in this type

framework). Then Arrow's theorem applies in cases (i) and (ii) (suitably reformulated in (ii)), and the Gibbard-Satterthwaite theorem applies in case (iii). Hence plurality rule, suitably defined, violates some of Arrow's conditions (i.e. IIA and Pareto) and some of Gibbard's and Satterthwaite's (i.e. strategy-proofness). But Arrow's and Gibbard's and Satterthwaite's theorems cannot be formulated in the present restricted informational environment, as conditions such as contraction or expansion consistency, IIA, strategy-proofness etc. are not expressible here. Thus the problems raised by these theorems remain "hidden" in that environment, although they occur in a richer environment into which the restricted one can be embedded.

of election, the voters cannot give a sufficiently complete account of their opinions of the candidates...".

Fishburn illustrates how single-vote balloting can fail to record important information. Consider a 100-person electorate with preferences (best to worst) over options x, y and z as follows: 34 voters have $x \succ y \succ z$; 33 have $y \succ z \succ x$; 33 have $z \succ y \succ x$. As Fishburn (1973, 162) observes, "Plurality selects x. ... [But] x has 34 first-place votes and 66 third-place votes, whereas y has 33 first-place votes, 57 second-place votes and no third-place votes. Also, options y and z are each preferred to option x by a majority of 66 out of 100 voters. Plurality rule decides outcomes on the basis of first-preferences alone. If second- and third-preferences are to count for anything much at all, then surely there should be a strong case for option y being socially chosen rather than x."

In short, standard critiques of plurality rule are in fact critiques of single-vote balloting; they do not undermine the conditional claim we defend here.

4 From single-vote balloting to approval balloting

To further emphasize the conditional nature of our argument for plurality rule, we finally show that, if we enrich the informational environment and use, for example, approval balloting instead of single-vote balloting, our May-style argument for plurality rule becomes a May-style argument for approval voting as defined by Brams and Fishburn (1978). Although we state this result primarily to show that our main theorem depends on the given informational environment, our May-style characterization of approval voting can also be seen as a novel result in its own right.

In the formal framework introduced above, we now assume that each individual votes not only for a single option or abstains, but votes for all those options he or she approves of (which may be any number of options between 0 and k). Individual

i's *approval vote* is denoted v_i and represented by a column vector $\begin{pmatrix} v_{i1} \\ v_{i2} \\ \ldots \\ v_{ik} \end{pmatrix}$ where $v_{ij} = 1$ if and only if individual i votes (i.e. indicates approval) for option j. All other definitions, including that of an aggregation procedure, remain as stated above, except that the concept of a *profile* now refers to a *profile of approval votes*, defined as a matrix $v = (v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are the approval votes of individuals $1, ..., n$. This captures the informational environment of *approval balloting.*

Under this modification, universal domain becomes the condition that the aggregation procedure accepts all logically possible profiles of approval votes as admissible input; the other three conditions retain their original interpretation.

The formal analogue of plurality rule under approval balloting is *approval voting*: for any profile of approval votes, the option, if unique, that receives the largest number of votes (i.e. individual approvals), is chosen as the winner; if there is no unique such option, then all the options that receive an equal largest number of votes (individual approvals) are tied. The functional form which defines approval voting is the same as that which defines plurality rule above – except that it is now applied to the informational environment of approval balloting. Formally, *approval voting* is the aggregation procedure f, where, for any profile of approval votes v and any option j,

$$f(v)_j = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} v_{ij} \geq \sum_{i=1}^{n} v_{ih} \text{ for all } h \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

It is immediately obvious in this framework that approval voting satisfies all of universal domain, anonymity, neutrality and positive responsiveness as formally stated above. Does it satisfy these conditions uniquely? Not quite. The conditions are also satisfied by a variant of approval voting in which the votes (i.e. indications of approval) cast by each individual are discounted by the total number of options for which that individual votes. For example, under such a variant, if an individual votes for only one option, then this vote might be given a weight of 1, but if he or she votes for two options, then these votes might each be given a weight of only 1/2, and so on (other methods of weighting or discounting might also be compatible with May's conditions).

However, if anonymity is strengthened subtly, then we obtain a May-style theorem on approval voting.

Optionwise anonymity. Let σ be any permutation of the votes cast for some option j (holding fixed the votes cast for other options), represented by a permutation of the entries in row j in a profile (holding fixed all other rows). For any profile v, $f(v) = f(\sigma(v))$.

Optionwise anonymity is the separate application of May's original anonymity condition to each option; it prevents differential treatment of an individual's votes on some option depending on his or her votes on other options. Informally, we can describe the difference between anonymity and optionwise anonymity as follows. Anonymity is compatible with a procedure whereby each individual submits a single anonymous ballot paper on which he or she indicates which options he or she approves of and which not. These ballot papers are then put into a ballot box and shuffled. Yet, although all information about the voters' identity is eliminated, it is still possible to associate anonymous voters with combinations of approved options; it is possible to see, for example, that one voter has voted for options 1, 3 and 5, a second for options 2 and 3, a third for option 4 alone, and so on. Optionwise anonymity, by contrast, requires a procedure whereby each individual submits a separate ballot paper for each option, indicating approval or disapproval of that option. These ballot papers are then put into separate ballot boxes, one ballot box for each option, and shuffled inside these separate boxes. This eliminates not only all information about the voters' identity, but also all information about combinations of approved options. Under single-vote balloting, where each voter can only vote for one option, anonymity and optionwise anonymity are equivalent, but, under approval balloting, the two conditions come apart, and optionwise anonymity is stronger than anonymity *simpliciter*.

Now a straightforward adjustment of our proof above leads to the following result.

Theorem. Under approval balloting, an aggregation procedure satisfies universal domain, optionwise anonymity, neutrality and positive responsiveness if and only if it is approval voting.

This result not only shows that, in the richer informational environment of approval balloting, a May-style argument can be given in support of approval voting – a result not yet known in the literature – but it also illustrates the conditional nature of our argument for plurality rule above. It should now be clear that the question of what aggregation procedure to use depends crucially on the informational environment in which this procedure is meant to operate.

5 Concluding remarks

Democratic theorists defend the use of "majority rule," often without saying precisely which of a large range of broadly majoritarian voting procedures they mean (Spitz, 1984). Moreover, when giving May's theorem pride of place in their arguments for majority rule, they often gloss over the theorem's restriction to decisions between two options.

In the real world, "our standard voting system ... is ... the plurality vote, where a voter votes for his favorite candidate and the candidate with the most votes wins" (Saari, 2006). Yet, of all the broadly majoritarian voting procedures that have been proposed in theory or are used in practice, plurality rule is perhaps the one that is held in lowest esteem by theorists of democracy. As we have noted, plurality rule is criticized in particular for focusing solely on voters' revealed "first choices" and not taking into account their full preferences.

However, just as the most commonly used aggregation procedure in the real world is plurality rule, so the most commonly used balloting procedure is single-vote balloting. Surprisingly, it is has never been noticed before that, under single-vote balloting, plurality rule is what May's theorem, in a simple generalization to decisions over many options, supports.

Our result fills an important gap in the literature; it not only constitutes the first generalization of May's theorem beyond pairwise decisions, but also provides a conditional defence of plurality rule in the restricted, but empirically prominent informational environment of single-vote balloting. If single-vote balloting is used, as it often is in practice, then plurality rule is indeed the way to go.

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