What is special about the proportion?
A research report on special majority voting and the classical Condorcet jury theorem

Christian List
Research School of Social Sciences, Australian National University
and
Nuffield College, Oxford

Nuffield College Working Paper in Politics
http://www.nuff.ox.ac.uk/Politics/papers/

30 April 2003

Address for correspondence:

(until 30 August 2003)
C. List
SPT Program, RSSS
Australian National University
Canberra ACT 0200, Australia
Phone ++61 / 2 / 6125 9608
Fax ++61 / 2 / 6125 0599
clist@coombs.anu.edu.au

(from 1 September 2003)
Department of Government
London School of Economics
Houghton Street
London WC2A 2AE, U.K.
c.list@lse.ac.uk

Acknowledgments:

I am grateful to Geoffrey Brennan, Franz Dietrich, Christian Elsholtz, Robert Goodin, Frank Jackson and David Soskice for very helpful comments and suggestions, and to the participants of a Social and Political Theory seminar at the ANU in February 2003 for comments and discussion.

What is special about the proportion?
A research report on special majority voting and the classical Condorcet jury theorem

It is known that, in Condorcet's classical model of jury decisions, the proportion of jurors supporting a decision is not a significant indicator of that decision's reliability: the probability that a particular majority decision is correct given the size of the majority depends only on the absolute margin between the majority and the minority, and is invariant under changes of the proportion in the majority if the absolute margin is held fixed. Here I show that, if we relax the assumption that juror competence is independent of the jury's size, the proportion can be made significant: there are then conditions in which the probability that a given majority decision is correct depends only on the proportion of jurors supporting that decision, and is invariant under changes of the jury size. The proportion is significant in this way *if and only if* juror competence is a particular decreasing function of the jury size. However, the required condition on juror competence is not only highly special – thereby casting doubt on the significance of the proportion in realistic conditions – but it also has an adverse implication for the Condorcet jury theorem. If the proportion is significant, then the Condorcet jury theorem fails to hold; and if the Condorcet jury theorem holds, the proportion is not significant. I discuss the implications of these results for defining and justifying special majority voting from the perspective of an epistemic account of voting.

1. Introduction

Special majority voting is often used in binary decisions where a positive decision is weightier than a negative one. Under the standard definition of special majority voting, a positive decision is made *if and only if* a proportion of q or more of the votes support that decision, where q is a fixed parameter greater than 1/2, e.g. 2/3 or 5/6. For example, in jury decisions, special majorities of at least 10 out of 12 jurors are often required for a 'guilty' verdict. Constitutional amendments in the United States become valid "when ratified by the Legislatures of three fourths of the several States" (The Constitution of the United States, Article V). To change Germany's Basic Law, 2/3 majorities in both chambers of parliament, Bundestag and Bundesrat, are required. Rousseau advocated the use of special majority voting in important decisions (see Weirich 1986):

> "[T]he more the deliberations are important and serious, the more the opinion that carries should approach unanimity." (Rousseau, *The Social Contract*, bk. 4, ch. 2, cited in Weirich 1986)

> "Between the veto [i.e. unanimity] [...] and plurality [...] there are various propositions for which one can determine the preponderance of opinions according to the importance of the issue. For example, when it concerns legislation, one can demand at least three-fourths of the votes, two-thirds for matters of State, a simple plurality for elections and other affairs of the moment. This is

> only an example to explicate my idea and not a proportion that I recommend." (Rousseau, *Considerations on the Government of Poland*, ch. 9, cited in Weirich 1986)

There are (at least) two different reasons for using special majority voting.[1] Which of these applies to a given decision problem depends – among other things – on whether or not there exists an independent fact on what the correct decision is. In jury decisions, there typically exists such a fact, as it is either true or false that the defendant has committed the crime in question.[2] If false positives (e.g. convicting the innocent or making a 'bad' constitutional amendment) are considered worse than false negatives (e.g. acquitting the guilty or failing to make a 'good' constitutional amendment), then the intuition for using special majority voting is as follows. The proportion of voters supporting a particular decision appears to be a good indicator of the decision's reliability, and therefore special majority voting reduces the probability of a false positive decision, perhaps at the expense of increasing the probability of a false negative decision. Call such reasons for using special majority voting *epistemic* ones.

In other decision problems, there may not exist an independent standard of correctness. For example, different political values may lead to different judgments on the desirability of a particular constitutional amendment, and there may not exist an independent truth on which judgment is 'correct'. The justifiability of a particular decision may here depend on whether it has been reached through a procedure that has certain attractive procedural properties. A reason for using special majority voting might be that special majority voting is a procedure that protects minorities: requiring a proportion of at least q for a positive decision where $q > 1/2$ means giving veto power to any minority greater than $1-q$. Call such reasons for using special majority voting *non-epistemic* ones.

In this paper I reassess our *epistemic* reasons for using special majority voting. For this purpose, I tentatively use Condorcet's classical model of jury decisions (e.g. Grofman, Owen and Feld

[1] On epistemic and procedural accounts of voting, see, among others, Cohen (1986); Dahl (1979); Coleman and Ferejohn (1986); Estlund (1993, 1997); List and Goodin (2001).

[2] Of course, this fact is initially unknown to the jury, and the jury's task is to discover this fact as accurately as possible.

1983).[3] That model confirms the intuition that requiring more than a simple majority for a positive decision reduces the probability of false positives.[4] But does it also confirm the intuition that the *proportion* of voters supporting a particular decision is a good indicator of that decision's reliability?[5]

If the group size is fixed, a decision supported by a larger proportion is more likely to be correct than one supported by a smaller proportion (other things being equal), e.g. a decision supported by 10 out of 12 jurors (5/6) is more likely to be correct than one supported by only 8 out of 12 (2/3). But what if the group size is not fixed? Is a 5/6 majority among 24 jurors as likely to be correct as a 5/6 majority among only 12 jurors, other things being equal? And what about a 5/6 majority among 12 jurors as compared with a narrow 50.4% majority among 1000 jurors?

As noted in the literature, Condorcet's classical jury model has a remarkable implication for this question: Other things being equal, the probability that a particular majority decision is correct *given the size of that majority* is a function of the *absolute margin* between the number of votes in the majority and the number of votes in the minority (the result is stated more formally below).[6] The probability is invariant under changes in the total number of votes, so long as that absolute margin remains the same. As a 5/6 majority among 12 votes and a 50.4% majority among 1000 votes both correspond to an absolute margin of 8 (=10-2=504-496), a 5/6 majority among 12 votes would be just as likely to be correct as a 50.4% majority among 1000 votes. McLean and Hewitt (1994, p. 37) summarize this point as follows:

---

[3] Although there is a large literature on the classical Condorcet jury model, only a small part of that literature addresses special majority voting, e.g. Nitzan and Paroush (1984), Ben-Yashar and Nitzan (1997), Fey (2001). None of these papers addresses the question of when the proportion is epistemically significant.

[4] However, this intuition is not uncontested. Feddersen and Pesendorfer (1998) have constructed an alternative model of jury decisions (where jurors vote strategically) in which unanimity rule (i.e. special majority rule with $q = 1$) leads to higher probabilities of *both* false positives *and* false negatives than several less demanding majority rules, including simple majority rule.

[5] Weirich (1986) suggests that Rousseau advocated 'proportional' majority voting partly for epistemic reasons, and partly for reasons of political stability. But, while Weirich questions whether voters will always vote with the general interest in mind, he does not seem to question the availability, in cases where the Condorcet jury model applies, of an epistemic defence of 'proportional' majority voting based on the Condorcet jury theorem.

[6] To avoid a frequent misunderstanding, note that the result on the significance of the absolute margin concerns the probability of a particular state of the world (e.g. the defendant is guilty) conditional on a particular voting pattern (e.g. precisely h out of n jurors have voted for 'guilty'). The result does *not* concern the probability of a particular voting pattern conditional on a particular state of the world, where the order of conditionalization is reversed. Saying that the former probability is invariant under changes of h and n that preserve the absolute margin $m=h-(n-h)$ is *not* the same as saying that the latter probability is invariant under such changes. In fact, the latter probability depends very much on the values of n and h. See the discussion in section 2.

> "What matters is the absolute size of the majority, not the size of the electorate, nor the proportion of the majority size to electorate size. If the jury theorem is applicable, we should talk about 'a majority of 8', 'a majority of 20', etc., not 'a two-thirds majority' or 'a three-quarters majority'. Condorcet did so in most of his later work."

The significance of the absolute margin and its implications have been discussed in detail (e.g. McLean and Hewitt 1994; List 2003; the result is also implicitly used in Feddersen and Pesendorfer 1998 for computing the posterior probability that the defendant is guilty conditional on *h* out of *n* guilty signals[7]). Suppose we advocate special majority voting for epistemic reasons *and* we accept this result. Then it seems inaccurate to define special majority voting in terms of a required proportion *independently of the actual number of voters*, or to speak, as Rousseau does, of a three-fourths majority "when it concerns legislation" or a two-thirds majority "for matters of State" independently of the number of decision-makers. It seems inaccurate to require a 3/4 majority of the States to ratify a constitutional amendment independently of the actual number of States – which has increased from 13 when the US Constitution came into force to 50 at present. Likewise, it seems inaccurate to require a 2/3 majority in the German Bundestag for certain laws independently of the Bundestag's size – which has ranged from a minimum of 402 members in 1949 to a maximum of 672 in 1994, with 602 at present. Instead, it seems more accurate to spell out the special majority criterion in terms of a required margin between the majority and the minority. For example, a 3/4 majority among 13 States corresponds to a margin of 5 (=9-4), and a 2/3 majority among 402 members of the Bundestag corresponds to a margin of 134 (=402-134). Even when the number of decision-makers changes, we would have to hold the absolute margin criterion fixed, although it would then correspond to a different proportion. For example, an absolute margin of 5, corresponding to a 2/3 majority among the original 13 States, corresponds to a 55% majority among the present 50 States. Note that all this follows *only if* we advocate special majority voting for epistemic reasons *and* we accept Condorcet's classical jury model in an unmodified form. But, *if* we do, we must define special majority voting in terms of a required absolute margin.[8]

---

[7] To be precise, Feddersen and Pesendorfer use a version of theorem 4 below.

[8] If we want to resist this conclusion and defend the standard definition of special majority voting (in terms of the proportion), we have two alternatives. The first alternative is to defend that definition for non-epistemic reasons, e.g. procedural ones. The second alternative is to defend a modification of the jury model in which the proportion *is*

Although earlier work (List 2003) has identified some modifications of Condorcet's conditions under which the absolute margin ceases to be significant, these conditions do not imply the significance of the proportion. I here extend this work and ask whether we can modify Condorcet's conditions such that the proportion becomes significant, i.e. such that the probability that a particular decision is correct becomes a function of the *proportion* of the votes supporting that decision (as stated more formally below). I show that, holding all other conditions of the model fixed, the proportion becomes significant in this way *if and only if* voter competence is a particular decreasing function of the jury size. While I am unable to assess this condition empirically, I show that, *if* the condition holds, *then* the Condorcet jury theorem does *not* hold, i.e. it is no longer the case that the probability of a correct jury verdict converges to 1 as the jury size increases. Proofs of the new results in sections 3 and 4 are given in the appendix.

2. The classical Condorcet jury model and the significance of the absolute margin

In this section I introduce the classical Condorcet jury model, and state the result on the significance of the absolute margin, following List (2003).

Let 1, 2, …, n denote the n jurors ($n > 0$). We assume that there are two states of the world, represented by a binary variable X taking the value 1 for 'guilty' and 0 for 'not guilty'. The votes of the jurors are represented by the binary random variables $V_1, V_2, \ldots, V_n$, where each V_i takes the value 1 for a 'guilty' vote and 0 for a 'not guilty' vote. The vote of juror i is correct *if and only if* the value of V_i coincides with the value of X. We use capital letters to denote random variables and small letters to denote particular values. The classical Condorcet jury model assumes:

Competence (C). For all jurors $i = 1, 2, \ldots, n$, $Pr(V_i{=}1|X{=}1) = Pr(V_i{=}0|X{=}0) = p > ½$.

Independence (I). For each $x \in \{0, 1\}$, $V_1, V_2, \ldots, V_n$ are independent from each other, given the state of the world x.

---

significant for the probability of a correct majority decision. In this paper I identify such a modified model, but I remain agnostic on whether the identified model is realistic.

As these assumptions have been extensively discussed in the literature, I will not discuss them here.[9] But one point should be noted. By assumption (C), the competence parameter *p* is *constant* in two senses: the value of *p* is identical for all jurors *i*, *and* it does *not* depend on the total number of jurors *n*.

Given $V_1, V_2, \ldots, V_n$, the vote of the jury can be expressed as $V = \sum_{i=1}^{n} V_i$. Under Condorcet's assumptions, the probability distribution of *V* conditional on *X* is a binomial distribution with parameters *n* and *p*, with the following probability function:

(1) for each $h = 0, 1, 2, \ldots, n$, $Pr(V = h \mid X{=}1) = Pr(V = n{-}h \mid X{=}0) = \binom{n}{h} p^h (1{-}p)^{n-h}$.

The definition of *V* allows the following interpretation:

$V = h$:	Precisely *h* out of *n* jurors support a 'guilty' verdict.
$V > n/2$:	A simple majority supports a 'guilty' verdict.
$V \geq qn$:	A proportion of at least *q* of the jurors supports a 'guilty' verdict.
$2V{-}n \geq m$:	A majority with a margin of at least *m* between the majority and the minority supports a 'guilty' verdict. (Note that $2V{-}n = V{-}(n{-}V)$.)

Simple and special majority voting can now be defined as follows:

Simple majority voting. A positive decision (e.g. conviction) is made *if and only if* $V > n/2$.

Special majority voting / proportion definition, where $q > ½$ (hereafter *q*-voting). A positive decision (e.g. conviction) is made *if and only if* $V \geq qn$.

We also consider a non-standard definition of special majority voting:

---

[9] Cases where different jurors have different competence levels – i.e. where (C) does not hold – are discussed, for example, in Grofman, Owen and Feld (1983), Borland (1989), Kanazawa (1998). Cases where there are certain dependencies between different jurors' votes – i.e. where (I) does not hold – are discussed, for example, in Ladha (1992) and Estlund (1994). Cases where jurors vote strategically rather than sincerely – i.e. where a juror's vote does not always coincide with that juror's private signal about the state of the world – are discussed, for example, in Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1998).

Special majority voting / absolute margin definition, where $m>0$ (hereafter: m-voting). A positive decision (e.g. conviction) is made *if and only if* $2V-n \geq m$.

We assess each of these voting rules in terms of two different epistemic conditions: *truth-tracking in the limit*, and *no reasonable doubt*, as defined below. Condorcet's famous jury theorem concerns the first condition. Let me state the theorem before stating that condition.

Theorem 1. (Condorcet jury theorem; Grofman, Owen and Feld 1983) *Suppose (C) and (I) hold. Then* $Pr(V > n/2|X=1)$ ($= Pr(V < n/2 \mid X=0)$) *converges to* 1 *as n tends to infinity*.

By theorem 1, if (C) and (I) hold, simple majority voting satisfies the following condition:

Truth-tracking in the limit (T).

- The probability of a positive decision conditional on $X = 1$ converges to 1 as n tends to infinity.
- The probability of a negative decision conditional on $X = 0$ converges to 1 as n tends to infinity.

Similar results can be stated for q-voting and m-voting.

Theorem 2. (Condorcet jury theorem for q-voting; List 2003; see also Kanazawa 1998) *Suppose (C) and (I) hold, and* $q > ½$.

(i) *If* $p < q$, *then* $Pr(V \geq qn \mid X=1)$ *converges to* 0 *as n tends to infinity*.

(ii) *If* $p > q$, *then* $Pr(V \geq qn \mid X=1)$ *converges to* 1 *as n tends to infinity*.

For (i) and (ii), $Pr(V < qn \mid X = 0)$ *converges to* 1 *as n tends to infinity*.

Theorem 3. (Condorcet jury theorem for m-voting; List 2003) *Suppose (C) and (I) hold. For any* $m > 0$, $Pr(2V-n \geq m \mid X=1)$ *converges to* 1 *as n tends to infinity*; *and* $Pr(2V-n < m \mid X=0)$ *converges to* 1 *as n tends to infinity*.

By theorems 2 and 3, under (C) and (I), m-voting *always* satisfies (T), whereas q-voting satisfies (T) *only if* juror competence satisfies $p > q$, a demanding condition if q is close to 1.

The Condorcet jury theorem concerns the probability of a particular voting outcome (e.g. $V>n/2$) conditional on a particular state of the world (e.g. $X=1$). Using the theorem, we can determine, for example, how likely it is that there will be a majority (simple or special) for 'guilty' given that the defendant is truly guilty. This probability can be interpreted as the conviction rate in those cases where the defendant is guilty. Similarly, we can determine how likely it is that there will be no majority for 'guilty' given that the defendant is not guilty. This probability can be interpreted as the acquittal rate in those cases where the defendant is innocent.

But suppose we wish to assess a *particular* jury verdict. We may then wish to determine how likely it is that the defendant is truly guilty given that a majority of precisely h out of n jurors have voted for 'guilty'. Thus we may be interested *not* in the probability of a particular voting outcome (e.g. $V>n/2$) conditional on a particular state of the world (e.g. $X=1$), as addressed by the Condorcet jury theorem, *but rather* in the probability of a particular state of the world (e.g. $X=1$) conditional on a particular voting outcome (e.g. $V=h$). Note the reversed order of conditionalization. The result on the significance of the absolute margin concerns precisely this second conditional probability. Let me state the result before stating our second epistemic condition.

Let $r = Pr(X=1)$ be the prior probability that the defendant is guilty, where $0 < r < 1$. Condorcet implicitly assumed $r = ½$, but r might alternatively be defined to be the (low) probability that a randomly chosen member of the population is guilty of the crime in question.

Theorem 4. (Condorcet; List 2003) *Suppose (C) and (I) hold, and suppose $h > n/2$. Then*

$$Pr(X=1 \mid V=h) = \frac{r\,p^m}{r\,p^m + (1-r)\,(1-p)^m} = \frac{r}{r + (1-r)\,(^1/_p - 1)^m},$$

where $m = 2h-n$.

The result can be derived straightforwardly from expression (1) above and Bayes's theorem. The parameter m is the absolute margin between the majority (h) and the minority ($n-h$). By theorem 4, the probability that the defendant is guilty given that a majority of precisely h out of n jurors have voted for 'guilty' depends *only* on the value of m (in addition to p and r), but not on the values of h or n by themselves.[10] If (C) and (I) hold, m is therefore significant in the following sense:

Significance of the absolute margin (M). $Pr(X=1 \mid V = h)$ is invariant under changes of n and h that preserve $m = 2h-n$, where p and r are held fixed.[11]

In the example given in the introduction,

$$[Pr(X=1 \mid V = 10) \text{ where } n = 12] \text{ equals } [Pr(X=1 \mid V = 504) \text{ where } n = 1000].$$

Now we are in a position to state our second epistemic condition:

No reasonable doubt (D). For all h and n ($0 \leq h \leq n$), in a given situation where precisely h jurors have voted for x and n-h jurors against x, a positive decision is made *if and only if*

(2) $Pr(X=1 \mid V = h) \geq c$,

where c is a fixed "no reasonable doubt" threshold satisfying $0 < c < 1$ (e.g. $c = 0.99$).

Using theorem 4, we can determine a necessary and sufficient condition for inequality (2).

---

[10] Why should we conditionalize on $V = h$ rather than on $V \geq h$ (or on $V \geq n/2$) when we determine the degree of belief we can assign to the hypothesis that the defendant is guilty given that a majority of precisely h out of n jurors have voted for 'guilty'? Suppose we wish to assess a *particular* jury verdict, and we know that precisely h out of n jurors have voted for 'guilty'. If we were to conditionalize on $V \geq h$ (rather than on $V = h$, as we do) – i.e. on the disjunction $\phi := [V=h \textit{ or } V=h+1 \textit{ or } V=h+2 \textit{ or } \ldots \textit{ or } V=n]$ – we would fail to make use of the full information that we have. We would use *only* the information that the disjunction ϕ is true, but *not* the additional information (which we also have) on precisely which of the disjuncts in ϕ is true, namely $V = h$. Even worse, if we were to conditionalize on $V \geq h$, we would end up assigning a *greater* degree of belief to the hypothesis that the defendant is guilty than the one we can justifiably assign to that hypothesis *given our full information* (namely $V = h$). It can easily be verified that, for $p > ½$, $Pr(X=1 \mid V \geq h) \geq Pr(X=1 \mid V = h)$.

[11] This means the following. Let us hold p and r fixed. If we change n and h such that m remains the same, then $Pr(X=1 \mid V = h)$ also remains the same. Such changes of n and h can be achieved by substituting $n^* = n+2a$ for n and $h^* = h+a$ for h, where a is some integer satisfying $n+a \geq h$. Below p will be a function of n rather than a constant. Then holding p fixed means using the same function p for different values of n.

Theorem 5. (List 2003) *Suppose (C) and (I) hold. Let c be a fixed threshold such that* $0 < c < 1$. *Then*

(i) $Pr(X=1 \mid V=h) > (=)\ c$

if and only if

(ii) $2h\text{-}n > (=)\ m := \dfrac{log\left(\frac{r-cr}{c-cr}\right)}{log(^1/_p - 1)}$.

We observe that, under (C) and (I), a voting rule satisfies (D) *if and only if* it is m-voting, with the parameters as defined in theorem 5.

An absolute margin of m corresponds to a proportion of $q = {}^1/_2({}^m/_n + 1)$. Changes of n and h that preserve $m = 2h\text{-}n$ will change q. Likewise, changes of n and h that preserve $q = h/n$ will change m. As $Pr(X=1 \mid V = h)$ is an increasing function of m, it follows that, if (M) holds, the following condition does *not* hold:

Significance of the proportion (P). $Pr(X=1 \mid V = h)$ is invariant under changes of n and h that preserve $q = h/n$, where p and r are held fixed.[12]

We also observe that, under (C) and (I), there exists no parameter q such that q-voting satisfies condition (D). The only way to avoid this result would be to define q not as a single parameter, but as a function of n, i.e. $q(n)={}^1/_2({}^m/_n+1)$, where m is as defined in theorem 5. But then $q(n)$-voting is simply a notational variant of m-voting – and (P) still does not hold.

3. When is the proportion significant?

In this section I show that the proportion can be made significant if we relax the assumption that individual juror competence does *not* depend on the jury size n. We replace condition (C) in Condorcet's model with the following weaker condition:

---

[12] If p is a function of n, then holding p fixed means using the same function p for different values of n.

Non-constant competence (NC). For all jurors $i = 1, 2, \ldots, n$, $Pr(V_i=1|X=1) = Pr(V_i=0|X=0) = p(n) > ½$, for some function $p : \{1, 2, 3, \ldots\} \to (0, 1)$.

Condition (NC) is weaker than condition (C) in one respect: (NC), unlike (C), allows individual competence to depend on the total number of jurors n. But (NC) is not weaker than (C) in another respect: (NC), like (C), requires that, for each value of n, the corresponding value of $p(n)$ is identical for all jurors i. Condition (C) is simply a special case of (NC) where $p(n)$ is constant.

We first note that, if $p(n)$ is not constant in a relevant sense, the absolute margin ceases to be significant.

Theorem 6. *Suppose (NC) and (I) hold, and suppose that there exist* n_1 *and* n_2 (*either both odd or both even*) *such that* $p(n_1) \neq p(n_2)$. *Then (M) does not hold.*

But, if the absolute margin is *not* significant, this does not in general imply that the proportion *is* significant. However, we now identify a particular functional form for $p(n)$ (i.e. a class of functions $p(n)$ with one free parameter k) for which the proportion is significant.

Theorem 7. *Suppose (NC) and (I) hold, and suppose that* $p(n) = 1/(1+exp(-k/n))$ *for some constant* $k > 0$. *Suppose* $h > n/2$. *Then*

$$Pr(X=1 \mid V = h) = \frac{r}{r + (1-r)\, exp(k-2kq)},$$

where $q = h/n$.

Here the probability that the defendant is guilty given that a majority of h out of n jurors have voted for 'guilty' depends *only* on the proportion h/n (in addition to k and r), but not on the values of h or n by themselves. Therefore, if (NC) and (I) hold, where $p(n) = 1/(1+exp(-k/n))$ for some $k > 0$, then (P) also holds, i.e. the proportion is significant.

It is not only the case that the particular functional form for $p(n)$ in theorem 7 leads to the significance of the proportion. It is also the *only* functional form with this property, as the following theorem shows. The proof of the theorem also illustrates how the expression $p(n) = 1/(1+exp(-k/n))$ can be derived from the conjunction of (NC), (I) and (P).

Theorem 8. *Suppose (NC) and (I) hold. Then (P) holds if and only if* $p(n) = 1/(1+exp(-k/n))$ *for some constant* $k > 0$.

What are the properties of $p(n)$?

- For all n, $k > 0$, we have $½ < p(n) < 1$.
- For a fixed k, $p(n)$ decreases as n increases, and converges to ½ as n tends to infinity.
- For a fixed n, $p(n)$ increases as k increases, and converges to 1 as k tends to infinity. The limiting case $k = 0$ corresponds to $p(n) = ½$.

Diagram 1 shows $p(n)$ for four different values of k. The four curves correspond to $k = 1$ (bottom), $k = 5$ (2nd from bottom), $k = 10$ (2nd from top) and $k = 20$ (top). The value of n is plotted on the x-axis, the value of $p(n)$ on the y-axis.

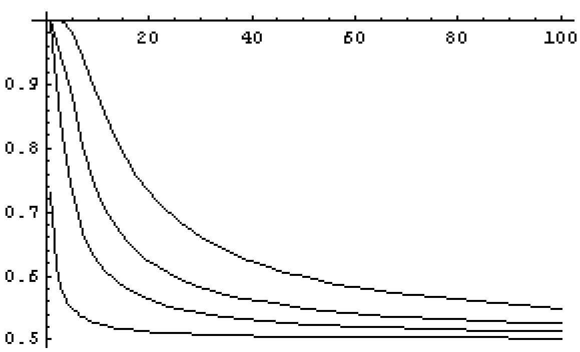

Diagram 1: *p*(*n*) for *k* = 1, *k* = 5, *k* = 10, *k* = 20 (from bottom to top curve)

Having identified conditions for the significance of the proportion, we can now revisit our epistemic condition (D), no reasonable doubt. For any threshold c (and parameters r, k), we can ask what proportion q is required for $Pr(X=1 \mid V = h) \geq c$, i.e. for inequality (2) in (D).

Theorem 9. *Let* $p(n) = 1/(1+exp(-k/n))$ *for some constant* $k > 0$. *Let* c *be a fixed threshold such that* $0 < c < 1$. *Then*

(i) $Pr(X=1 \mid V=h) > (=)\ c$

if and only if

$$\text{(ii)} \qquad h/n > (=)\ q := \tfrac{1}{2}\left(\frac{\log\left(\frac{r-cr}{c-cr}\right)}{-k} + 1\right).$$

We can observe that, under (NC) and (I) and with $p(n) = 1/(1+exp(-k/n))$, a voting rule satisfies (D) *if and only if* it is q-voting, with the parameters as defined in theorem 9.

Table 1 shows some sample calculations of q for different values of k, c and r. As explained below, the N/A entries correspond to the impossible requirement $q > 1$. To give a more intuitive interpretation of k, the table also shows some illustrative values of $p(n)$ corresponding to the shown values of k.

k	*p*(1)	*p*(12)	*p*(24)	*p*(100)	***c***	***r* = 0.001**	***r* = 0.01**	***r* = 0.3**	***r* = 0.4**	***r* = 0.5**	***r* = 0.6**
1	0.7311	0.5208	0.5104	0.5025	**0.5**	N/A	N/A	0.9236	0.7027	0.5	< 0.5
					0.75	N/A	N/A	N/A	N/A	N/A	0.8466
					0.99	N/A	N/A	N/A	N/A	N/A	N/A
					0.999	N/A	N/A	N/A	N/A	N/A	N/A
5	0.9933	0.6027	0.5519	0.5125	**0.5**	N/A	0.9595	0.5847	0.5405	0.5	< 0.5
					0.75	N/A	N/A	0.6946	0.6504	0.6099	0.5693
					0.99	N/A	N/A	N/A	N/A	0.9595	0.919
					0.999	N/A	N/A	N/A	N/A	N/A	N/A
10	0.9999	0.6971	0.6027	0.5250	**0.5**	0.8453	0.7298	0.5424	0.5203	0.5	< 0.5
					0.75	0.9003	0.7847	0.5973	0.5752	0.5549	0.5347
					0.99	N/A	0.9595	0.7721	0.75	0.7298	0.7095
					0.999	N/A	N/A	0.8877	0.8656	0.8453	0.8251
20	≈1	0.8411	0.6971	0.5498	**0.5**	0.6727	0.6149	0.5212	0.5101	0.5	< 0.5
					0.75	0.7001	0.6423	0.5486	0.5376	0.5275	0.5173
					0.99	0.7875	0.7298	0.6361	0.625	0.6149	0.6047
					0.999	0.8453	0.7875	0.6939	0.6828	0.6727	0.6625

Table 1: Values of q **corresponding to different values of** k**,** r **and** c

Theorem 9 has some natural corollaries. For these corollaries, we let $p(n) = 1/(1+exp(-k/n))$ for some $k > 0$. First, for any given proportion q and any "no reasonable doubt" threshold c (and any prior probability r), we can ask what value of the parameter k in $p(n)$ is required to ensure that $Pr(X=1 \mid V=h) \geq c$ where the proportion h/n is exactly q.

Corollary 9.1. Suppose $\frac{1}{2} < q \leq 1$, and suppose $0 < c < 1$. Then

(i) $Pr(X=1 \mid V=h) > (=)\ c$

if and only if

(ii) $$k > (=) \frac{-log\left(\frac{r-cr}{c-cr}\right)}{2q-1}$$

where $q = h/n$.

Next, given a "no reasonable doubt" threshold c and a prior probability r, we say that the threshold c is *implementable* if there exists some proportion $\frac{1}{2} < q \leq 1$ (possibly $q = 1$) such that $Pr(X=1 \mid V=h) \geq c$ where $h/n \geq q$, i.e. if it is possible to obtain a sufficiently large majority to meet the "no reasonable doubt" threshold c. If the threshold c is not implementable in a given situation (i.e. for a given value of k), then a decision rule satisfying (D) will *never* produce a 'guilty' verdict as its outcome, no matter how large the majority for 'guilty' is. In that case, no majority, however large, will support a 'guilty' verdict beyond any reasonable doubt, where the requirement of "no reasonable doubt" is represented by the condition $Pr(X=1 \mid V=h) \geq c$.

Corollary 9.2. Suppose $0 < c < 1$. Then the "no reasonable doubt" threshold c is implementable *if and only if* $k \geq -log({}^{r-cr}/_{c-cr})$.

The N/A entries in table 1 mean that, under the given values of k and r, the threshold c is *not* implementable, i.e. a proportion q greater than 1 (which is impossible) would be required for $Pr(X=1 \mid V=h) \geq c$.

4. The implications of decreasing juror competence

We have seen that the competence function $p(n)$ which leads to the significance of the proportion decreases and converges to ½ as n tends to infinity. What are the implications of this property for the Condorcet jury theorem? More precisely, if the competence function $p(n)$ has this property, does simple majority voting still satisfy our first epistemic condition (T), i.e. truth-tracking in the limit? And, more demandingly, does special majority voting satisfy condition (T)?

There are at least two plausible situations in which competence might be a decreasing function of jury size:

- There is only a limited pool of competent jurors. As we add more and more jurors, we must accept less and less competent ones.
- Each juror’s competence is a function of his/her effort, and that effort decreases as the jury size increases. This might be because (i) the juror’s marginal influence on the jury verdict decreases as the jury size increases, or because (ii) the juror’s visibility in the jury – relevant to reputation effects of the juror’s effort – decreases as the jury size increases.

The present model – where the value of $p(n)$ may depend on n but is identical for all jurors i – captures the second situation best. The second situation is consistent with an identical value of $p(n)$ for *all* jurors i, while the first situation requires different competence values for different jurors. Specifically, the first situation requires that, in any jury (regardless of its size), the first juror has competence $p(1)$, the second $p(2)$, the i^{th} $p(i)$, and so on, where p is a decreasing function of n. The result on the significance of the proportion does not hold without identical competence for all jurors.

We first note that the fact that $p(n)$ decreases and converges to ½ by itself does not undermine the Condorcet jury theorem for simple majority voting. The following theorem implies that there exist competence functions $p(n)$ with this property (decreasing and converging to ½ as n tends to infinity) for which simple majority voting still satisfies (T).

Theorem 10.[13] (Condorcet jury theorem with varying juror competence) *Suppose (NC) and (I) hold. Suppose further that p(n) satisfies the condition*

(3) $n\varepsilon(n)^2$ *tends to infinity as n tends to infinity,*

where $\varepsilon(n) := p(n)-½$. *Then* $Pr(V > n/2|X=1)$ (= $Pr(V < n/2 \mid X=0)$) *converges to* 1 *as n tends to infinity*.

Condition (3) states that the square of $p(n)$-½ (denoted $\varepsilon(n)$) tends to 0 more slowly than n tends to infinity; equivalently, $\varepsilon(n)$ tends to 0 more slowly than $1/\sqrt{n}$ tends to 0. By theorem 10, if $p(n)$ satisfies (3), simple majority voting satisfies (T). Examples of functions $p(n)$ satisfying (3) are:

$p(n) = ½ + 1/\sqrt[3]{n}$

more generally: $p(n) = ½ + 1/n^a$, where $0 < a < ½$.

By contrast, $p(n)=½+1/\sqrt{n}$ violates (3), as do $p(n)=½+1/n$ and $p(n)=½+1/n^2$. Berend and Paroush (1998) have shown, with some technical provisos, that (3) is not only sufficient for simple majority voting to satisfy (T), but also necessary. Therefore, for any $p(n)$ violating (3), simple majority voting does *not* satisfy (T).

Crucially, $p(n) = 1/(1+exp(-k/n))$ – the competence function which leads to the significance of the proportion – does *not* satisfy (3), regardless of the value of k. It converges to ½ too fast. As an illustration, diagram 2 shows $1/(1+exp(-k/n))$ for $k = 20$ in comparison with $½+1/n^{0.4999}$, which is an example of a function still satisfying (3) but converging to ½ already relatively fast.[14] While, for small values of n, $1/(1+exp(-k/n))$ lies above $½+1/n^{0.4999}$, the two curves soon intersect; and, for larger values of n, $1/(1+exp(-k/n))$ lies below $½+1/n^{0.4999}$. In essence, $1/(1+exp(-k/n))$ declines exponentially, while $½+1/n^{0.4999}$ declines only polynomially, as n increases.

[13] A proof is given in the appendix. The result is a special case of a result by Berend and Paroush (1998).

[14] As noted, an exponent of 0.5 in this functional form is the cut-off point for condition (3), i.e. $½ + 1/n^{0.5}$ converges to ½ too fast for condition (3).

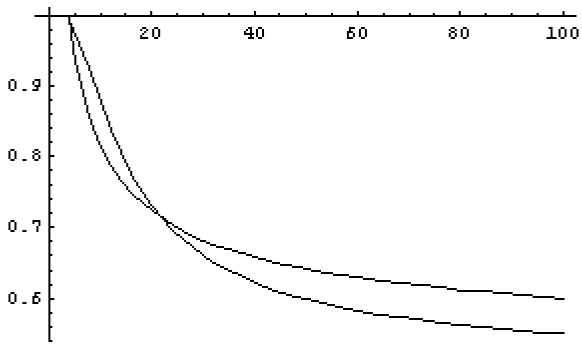

Diagram 2: 1/(1+*exp*(-20/*n*)) (the lower one of the two curves for larger values of *n*) and ½ + 1/$n^{0.4999}$ (the higher one of the two curves for larger values of *n*)

By the Berend and Paroush result, the fact that $p(n) = 1/(1+exp(-k/n))$ violates (3) implies that neither simple majority voting nor special majority voting of any form (q-voting or m-voting) satisfies (T). But it can also be shown independently that, for $p(n) = 1/(1+exp(-k/n))$, q-voting violates (T). This follows from the next result. If $p(n)$ decreases and converges to ½ as n tends to infinity (regardless of whether or not $p(n)$ satisfies (3)), then q-voting (where q>½) violates (T): specifically, the probability of a positive decision conditional on $X = 1$ converges to 0 as n tends to infinity.

Theorem 11. *Suppose (NC) and (I) hold, and suppose* $p(n)$ *converges to* ½ *as n tends to infinity. Then, for any* $q > ½$, $Pr(V \geq qn \mid X=1)$ *converges to* 0 *as n tends to infinity.*

Theorem 11 is illustrated by diagram 3, which shows the probability of obtaining a 2/3 majority for a positive decision conditional on $X = 1$, where $p(n) = 1/(1+exp(-k/n))$ with k=10. This probability can be interpreted as the conviction rate for guilty defendants under a 2/3 majority rule. The probability rapidly declines as n increases. For $r = ½$, a 2/3 majority in this scenario corresponds to $Pr(X=1 \mid V=h) = 0.965555$ (which is independent of n). So we have the following situation: *If* we get a 2/3 majority, we are justified in attaching a degree of belief of 0.965555 to its correctness (e.g. to the proposition that the defendant is truly guilty). *But* the probability of obtaining such a majority in the first place (e.g. of convicting) decreases as n increases, even when the defendant is guilty. If real-world juror competence were indeed of the form $p(n) = 1/(1+exp(-k/n))$, then we would have to prefer smaller juries to larger ones from the viewpoint of maximizing the conviction rate for guilty defendants. Put differently, if juror competence is of this form, then larger juries are subject to almost certain stalemate: irrespective of the state of the world, it is highly probable that they will not reach a positive decision.

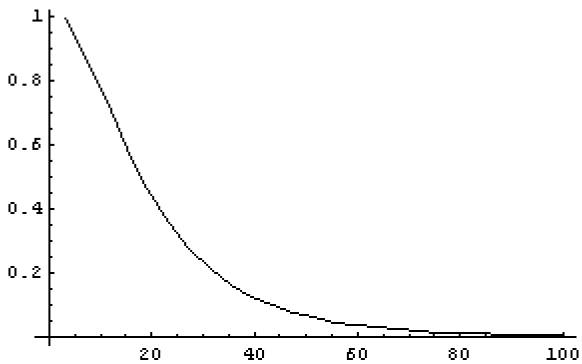

Diagram 3: Probability of obtaining a special majority of at least 2/3 conditional on X=1 where $p(n)$ = 1/(1+$exp(-k/n)$) with k=10

5. Concluding remarks

Does the classical Condorcet jury model confirm the intuition that the *proportion* of voters supporting a particular decision is a good indicator of that decision’s reliability? We need to say what we mean by a ‘good indicator’. Suppose a good indicator is one whose informational content (e.g. what it tells us about the reliability of a decision) is invariant under changes of relevant *other* parameters (e.g. the majority size h and the jury size n) that preserve the indicator itself (e.g. changes of h and n that preserve h/n). Then the answer is that the proportion is a good indicator only in special conditions.

In Condorcet’s classical model where juror competence does not depend on the jury size, the absolute margin rather than the proportion is significant, in the sense that (M) but not (P) holds. However, we have identified a modification of the model, where juror competence may depend on the jury size, in which the proportion can be made significant. In the modified model, a necessary and sufficient condition for the significance of the proportion is that juror competence is a decreasing function of n of a particular form, namely $p(n) = 1/(1+exp(-k/n))$ for some $k > 0$.

This finding leaves open the question of whether there exist real-world situations in which individual competence has this exact functional form. If the answer to this question is positive, this means that the proportion is at least sometimes significant from an epistemic perspective. If the answer is negative, then the present findings suggest that the proportion has no significance from an epistemic perspective. Let us briefly revisit the examples given in the introduction. It is implausible to think that the 'competence of each State' – in itself an aggregate agent represented by its Legislature – should depend in any systematic way on the total number of States. On the other hand, if the size of an assembly has certain incentive effects on the effort of its members, then perhaps the size of the German Bundestag does affect the competence of each member. But this is pure speculation at this point. Interestingly, Condorcet himself suggested that juror competence might decline as the jury size increases (Condorcet, cited in Waldron 1999, p. 32):

> "A very numerous assembly cannot be composed of very enlightened men. It is even probable that those comprising this assembly will on many matters combine great ignorance with many prejudices. Thus there will be a great number of questions on which the probability of the truth of each voter will be below ½. It follows that the more numerous the assembly, the more it will be exposed to the risk of making false decisions."

Condorcet's suggestion points away from the significance of the absolute margin and in the right direction for the significance of the proportion. Nonetheless, even if the suggestion were true, it would be insufficient to establish the significance of the proportion, as juror competence would have to be not only a decreasing function of jury size, but also of the exact functional form $p(n) = 1/(1+exp(-k/n))$. As an aside, we may also remark that decreasing competence by itself will not undermine the Condorcet jury theorem. As we have seen in theorem 10, even when juror competence decreases and converges to ½ as n tends to infinity, the Condorcet jury theorem may still hold, so long as the rate of decrease is sufficiently small. Of course, if competence drops below ½ as Condorcet suggests, then neither the jury theorem nor the significance of the proportion will hold.

In the classical model, m-voting, but not q-voting, satisfies the no reasonable doubt condition (D), whereas, in the modified model, q-voting, but not m-voting, satisfies that condition, provided that $p(n)$ is of the form $1/(1+exp(-k/n))$.

However, while, in the classical model, each of simple majority voting, m-voting and q-voting satisfies truth-tracking in the limit, i.e. (T) (so long as $p > q$ in the case of q-voting), this is not true in the modified model. Under the conditions in which the proportion is significant, not even simple majority voting, let alone q-voting or m-voting, satisfies that condition: the larger the jury, the higher the *acquittal* rate for *guilty* defendants. This means that, when the proportion is significant, smaller juries are better than larger ones at tracking the truth in cases where the defendant is guilty, even when we require the no reasonable doubt condition (D).

This leads to the following logical structure. In the jury models studied in this paper, we have:

- For a fixed parameter q, q-voting can satisfy *at most one* of truth-tracking the limit (T) *or* no reasonable doubt (D). Which of (T) or (D), if any, it satisfies depends on the precise functional form of $p(n)$.
  - If p does not depend on n and $p > q$, then q-voting satisfies (T) but not (D).
  - If p depends on n and $p(n) = 1/(1+exp(-k/n))$, then q-voting satisfies (D) but not (T).
  - In all other conditions, q-voting satisfies neither (T) nor (D).
- For a fixed parameter m, m-voting can satisfy *both* (T) *and* (D), so long as p does not depend on n.

Therefore, if the world is such that the proportion is epistemically significant, then it cannot also be such that the Condorcet jury theorem holds. And, if the world is such that the Condorcet jury theorem holds, then it cannot also be such that the proportion is epistemically significant. In the latter case, if we still wish to defend the use of special majority voting defined in proportional terms and independently of the jury's size, we need to defend it for reasons other than epistemic ones.

References

Austen-Smith, D. and J. S. Banks (1996) "Information Aggregation, Rationality and the Condorcet Jury Theorem", *American Political Science Review* 90: 34-45.

Ben-Yashar, R. and S. Nitzan (1997) "The optimal decision rule for fixed-size committees in dichotomous choice situations: the general result", *International Economic Review* 38: 175-186.

Berend, D. and J. Paroush (1998) "When is Condorcet's Jury Theorem valid?", *Social Choice and Welfare* 15: 481-488.

Borland, P. J. (1989) “Majority systems and the Condorcet jury theorem”, *Statistician* 38: 181-189.

Cohen, J. (1986) “An epistemic conception of democracy”, *Ethics* 97: 26-38.

Coleman, J. and J. Ferejohn (1986) “Democracy and social choice”, *Ethics* 97: 6-25.

Dahl, R. A. (1979) “Procedural democracy”, in Laslett, P. and J. Fishkin (eds) *Philosophy, Politics & Society*, 5th series, Oxford (Blackwell): 97-133.

Estlund, D. (1993) “Making truth safe for democracy”, in Copp, D., J. Hampton and J. E. Roemer (eds) *The Idea of Democracy*, New York (Cambridge University Press): 71-100.

Estlund, D. (1994) “Opinion leaders, independence and Condorcet's jury theorem”, *Theory and Decision* 36: 131-162.

Estlund, D. (1997) “Beyond fairness and deliberation: the epistemic dimension of democratic authority”, in Bohman, J. and W. Rehg (eds) *Deliberative Democracy*, Cambridge, MA (MIT Press): 173-204.

Feddersen, T. and W. Pesendorfer (1998) “Convicting the innocent: the inferiority of unanimous jury verdicts under strategic voting”, *American Political Science Review* 92: 23-36.

Fey, M. (2001) “A Note on the Condorcet Jury Theorem with Supermajority Voting Rules”, *Social Choice and Welfare* 20: 27-32.

Grofman, B., G. Owen and S. L. Feld (1983) “Thirteen theorems in search of the truth”, *Theory and Decision* 15: 261-278.

Kanazawa, Satoshi (1998) “A Brief Note on a Further Refinement of the Condorcet Jury Theorem for Heterogeneous Groups”, *Mathematical Social Sciences* 35: 69-73.

Lahda, K. (1992) “The Condorcet jury theorem, free speech and correlated votes”, *American Journal of Political Science* 36: 617-634.

List, C. (2003) “On the Significance of the Absolute Margin”, *British Journal for the Philosophy of Science*, forthcoming.

List, C. and R. E. Goodin (2001) “Epistemic Democracy: Generalizing the Condorcet Jury Theorem”, *Journal of Political Philosophy* 9: 277-306.

McLean, I. and F. Hewitt (1994) (transl’s and eds) *Condorcet : foundations of social choice and political theory*, Aldershot (Elgar).

Nitzan, S., and Paroush, J. (1984) “Are qualified majority rules special?”, *Public Choice* 42: 257-272.

Waldron, J. (1999) *The Dignity of Legislation*, Cambridge (Cambridge University Press).

Weirich, P. (1986) "Rousseau on Proportional Majority Rule", *Philosophy and Phenomenological Research* 47: 111-126.

Appendix: Proofs

Proof of theorem 6. Fix r such that $0 < r < 1$. First define a function $f_m : (0, 1) \rightarrow (0, 1)$ by

$$f_m(p) := \frac{r}{r + (1-r)\,(^1/_p - 1)^m},$$

where m is a fixed parameter such that $m \geq 1$. Then f_m is an increasing function of p. (This is easy to see. First note that, for $p \in (0, 1)$, $^1/_p > 1$, and hence $^1/_p - 1 > 0$. Next note that $^1/_p - 1$ decreases as p increases, and hence $r+(1-r)(^1/_p-1)^m$ also decreases as p increases. Therefore $f_m(p)$ increases as p increases.)

By theorem 4, for each fixed value of n, we have

$$Pr(X=1 \mid V = h) = \frac{r}{r + (1-r)\,(^1/_{p(n)} - 1)^m} \qquad \text{where } m = 2h-n$$

$$= f_m(p(n)).$$

Suppose that there exist n and n^*, either both odd or both even, such that $p(n) \neq p(n^*)$. Without loss of generality, $p(n) < p(n^*)$. Let $h = m = n$. As f_m is an increasing function of p, $f_m(p(n)) < f_m(p(n^*))$. But [$f_m(p(n)) = Pr(X=1 \mid V = h)$ where the jury size is n] and [$f_m(p(n^*)) = Pr(X=1 \mid V = h^*)$ where the jury size is n^* and $h^* = h+½(n^*-n)$]. But the change from n and h to n^* and h^* leaves the absolute margin fixed, i.e. $m = 2h-n = 2h^*-n^*$. Hence $Pr(X=1 \mid V = h)$ is *not* invariant under changes of n and h that preserve m, for the same p and r. ■

Proof of theorem 7. Let $p(n) = 1/(1+exp(-k/n))$ for some constant $k > 0$. Suppose $h > n/2$. By theorem 4, for each fixed value of n, we have:

$$Pr(X=1 \mid V = h) = \frac{r}{r + (1-r)\,(^1/_{p(n)} - 1)^m} \qquad \text{where } m = 2h-n$$

$$= \frac{r}{r + (1-r)\,(^1/_{(1/(1+exp(-k/n)))} - 1)^m}$$

$$= \frac{r}{r + (1\text{-}r)\ exp(\text{-}k/n)^m}$$

$$= \frac{r}{r + (1\text{-}r)\ exp(m(\text{-}k/n))}$$

$$= \frac{r}{r + (1\text{-}r)\ exp((2h\text{-}n)(\text{-}k/n))}$$

$$= \frac{r}{r + (1\text{-}r)\ exp(k\text{-}2k(h/n))}$$

$$= \frac{r}{r + (1\text{-}r)\ exp(k\text{-}2kq)} \qquad \text{where } q = h/n. \ \blacksquare$$

Proof of theorem 8. Suppose (NC) and (I) hold. Let $c, r \in (0,1)$ be fixed constants. By theorem 5, for each fixed value of n, we have:

$$Pr(X{=}1 \mid V{=}h) = c \qquad \text{if and only if} \quad 2h\text{-}n = m = \frac{log\left(\frac{r\text{-}cr}{c\text{-}cr}\right)}{log(^1/_{p(n)} - 1)}$$

and, using the relationship $q(n) = {}^1/_2({}^m/_n + 1)$,

$$\text{i.e.} \quad Pr(X{=}1 \mid V{=}h) = c \qquad \text{if and only if} \quad q(n) = h/n = ½\left(\frac{log\left(\frac{r\text{-}cr}{c\text{-}cr}\right)}{n\ log(^1/_{p(n)} - 1)} + 1\right).$$

Now (P) holds if and only if $q(n)$ takes the same value for *all* values of n, i.e. if and only if $q(n)$ is a constant function. Note that

$q(n)$ is constant for all n if and only if $n\ log(^1/_{p(n)} - 1)$ is constant for all n

if and only if $n\ log(^1/_{p(n)} - 1) = \text{-}k$

for some constant $k > 0$

(because $p(n) > ½$, as demanded by (NC), implies $log(^1/_{p(n)} - 1) < 0$)

if and only if $p(n) = 1/(1+exp(\text{-}k/n))$

for some constant $k > 0$. ■

Proof of theorem 9. The result follows immediately from theorem 5, the relationship $q = {}^1/_2({}^m/_n + 1)$ and the definition for $p(n)$. ■

Corollaries 9.1 and 9.2 follow immediately from theorem 9. For proving corollary 9.2, simply set $q = 1$. ■

Proof of theorem 10. By Chebyshev's inequality, if Y is a random variable with finite mean μ and finite variance σ^2, then, for all $\kappa \geq 0$, $Pr(|Y-\mu| \geq \kappa) \leq \sigma^2/\kappa^2$. I will prove that $Pr(V > n/2|X=1)$ converges to 1 as n tends to infinity. The result for $Pr(V < n/2 \mid X=0)$ is perfectly analogous. I will now conditionalize on $X = 1$. For each value of n and $p(n)$, the random variable V has mean $np(n)$ and variance $np(n)(1-p(n))$ conditional on $X = 1$. For ease of notation, we write $Pr^*(A)$ for $Pr(A|X=1)$, for any A. Since (NC) holds, we can write $p(n)=½+\varepsilon(n)$ where $\varepsilon(n) > 0$. Put $\kappa := \varepsilon(n)n$. By Chebyshev's inequality, we have:

$$(4) \qquad Pr^*(|V-n(½+\varepsilon(n))| \geq n\varepsilon(n)) \leq \frac{n(½+\varepsilon(n))(½-\varepsilon(n))}{(n\varepsilon(n))^2} = \frac{1}{4n\varepsilon(n)^2} - 1/n\ .$$

Suppose $p(n) = ½+\varepsilon(n)$ satisfies condition (3), i.e. $n\varepsilon(n)^2$ tends to infinity as n tends to infinity. Then

$$lim_{n\to\infty}(1/(4n\varepsilon(n)^2) - 1/n) = lim_{n\to\infty}(1/(4n\varepsilon(n)^2)) - lim_{n\to\infty}(1/n) = 0.$$

By the inequality (4), $Pr^*(|V-n(½+\varepsilon(n))| \geq n\varepsilon(n))$ also converges to 0 as n tends to infinity.
Note that $|V-n(½+\varepsilon(n))| \geq n\varepsilon(n)$ *if and only if* $[V \geq ½+2n\varepsilon(n)]$ or $[V \leq ½]$. Therefore

$$Pr^*(V \leq ½n) \leq Pr^*(|V-n(½+\varepsilon(n))|\geq n\varepsilon(n))$$

and hence $Pr^*(V \leq ½n)$ also converges to 0 as n tends to infinity. But $Pr^*(V \leq ½n) = 1-Pr^*(V > ½n)$, and hence

$Pr(V > n/2|X=1) = Pr^*(V > ½n)$ converges to 1 as n tends to infinity. ■

Proof of theorem 11.

Lemma (Boundedness Lemma). *Suppose $p(n) < p^*$. Then*

$$\sum_{h \geq qn} \binom{n}{h} p(n)^h(1-p(n))^{n-h} < \sum_{h \geq qn} \binom{n}{h} p^{*h}(1-p^*)^{n-h}\ .$$

Lemma (Convergence Lemma). *Suppose (C) and (I) hold. Further, suppose* $p \in S$, where $S \subseteq (0, 1)$ is an open set. Then $Pr(V/n \in S \mid X=1)$ *converges to* 1 *as n tends to infinity*.

The first lemma follows from the fact that $\sum_{h \geq qn} \binom{n}{h} p^h(1\text{-}p)^{n\text{-}h}$ is an increasing function of p. The second lemma follows from the law of large numbers.

Suppose the conditions of theorem 11 hold. Let $q > ½$. Choose p^* such that $½ < p^* < q$, e.g. $p^* = (½+q)/2$. Since $p(n)$ converges to ½ as n tends to infinity, there exists n^* such that, for all $n > n^*$, we have $p(n) < p^*$. Let $S := (0, q)$. Then S is an open set. Consider the framework where (C) and (I) hold and where the competence parameter is the constant p^*. Let V^* be the random variable representing the corresponding jury vote. Since $p^* \in S$, we can apply the convergence lemma to find that $Pr(V^*/n \in S \mid X=1)$ converges to 1 as n tends to infinity. By definition of S, $Pr(V^* \geq qn \mid X=1) \leq 1\text{-}Pr(V^*/n \in S \mid X=1)$, and hence $Pr(V^* \geq qn \mid X=1)$ converges to 0 as n tends to infinity. Further, for every $n > n^*$, we have $p(n) < p^*$. Hence, by the boundedness lemma, for every $n > n^*$,

$$\sum_{h \geq qn} \binom{n}{h} p(n)^h(1\text{-}p(n))^{n\text{-}h} < \sum_{h \geq qn} \binom{n}{h} p^{*h}(1\text{-}p^*)^{n\text{-}h}$$

The left-hand side of this inequality is $Pr(V \geq qn \mid X=1)$. The right-hand side is $Pr(V^* \geq qn \mid X=1)$, which converges to 0 as n tends to infinity. Therefore $Pr(V \geq qn \mid X=1)$ also converges to 0 as n tends to infinity. ■