Arrow's Theorem, Weglorz' Models and the Axiom of Choice

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Abstract. Applying Weglorz' models of set theory without the axiom of choice, we investigate Arrow-type social welfare functions for infinite societies with restricted coalition algebras. We show that there is a reasonable, nondictatorial social welfare function satisfying "finite discrimination", if and only if in Weglorz' model there is a free ultrafilter on a set representing the individuals.

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1. Introduction.

Arrow's theorem [2], as formulated by [18], is the assertion that the decisive coalitions of a reasonable[1] social welfare function F form an ultrafilter[2] $\mathcal{U}$ on the set I of *individuals*. Here the (two or more) voters in I decide about the "social preference" (the output of the social welfare function) on three or more alternatives. We assume that a "reasonable social welfare function" (section 2.2) satisfies certain ethical and informational requirements.

If I is finite, then $\mathcal{U}$ is principal ($\cap\mathcal{U} = \{i\}$) and therefore F is dictatorial (i is the dictator, whose strict preferences are obeyed).

If I is infinite, then (assuming SPI introduced below) there exist free (nonprincipal) ultrafilters $\mathcal{U}$ on I. Infinite societies I may represent future generations, finitely many people who extend into the indefinite future or (c.f. [23]) finitely many people who face (infinitely many states of) uncertainty. In view of [18] a free ultrafilter $\mathcal{U}$ defines a nondictatorial social welfare function.

The axiom SPI, "each infinite set I carries a free ultrafilter", depends on the *axiom of choice*[3] AC (c.f. [14]). This assertion is an example of an independence theorem[4] whose proof applies Weglorz' models.

Roughly speaking (section 2.1), a *model* of (a language of) set theory gives a "meaning" to the formulas (in that language). While "conventional" models satisfy each axiom of ZFC (Zermelo-Fraenkel set theory with AC) and hence SPI, there are other models which satisfy ZF (or its variant ZFA of section 2.1) and not SPI. It is therefore of interest to study the existence of social welfare functions from the viewpoint of set theory, applying different models, as is exemplified by the work of H. J. Skala ([28], [29] and [30]; c.f. section 2.3).

For Weglorz' models $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ (of ZFA set theory) the validity of SPI depends on parameters which can be investigated in "conventional" ZFC. The

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1 In this paper "reasonable" has a technical meaning which needs not relate to "rational".

2 [11] uses the equivalent notion of a two-valued finitely additive probability measure.

3 Each family of nonempty sets admits a choice function which picks an element of each set of the family.

4 "Independence theorems" (section 2.3) determine the relative strength of set theoretical axioms. (Economists are advised to consult [16].) In order to avoid confusion (and because we use a more general framework) we shall rename welfare axioms which mention "independence" (IIA corresponds to monotonicity of section 2.2).

parameters are a group Γ of permutations of the individuals and a Boolean subalgebra $\mathbb{B} \subseteq \mathcal{P}(I)$ of the powerset (algebra) of I. (Thus $\mathbb{B}$ with the constants $\emptyset$ and I is a Boolean algebra w.r. to the operations $\cup$, $\cap$ and complementation $-$.) In the present paper we apply this observation to relate social welfare theory in Weglorz' models to (Armstrong's generalization [1] of) social welfare theory within the "conventional" models of ZFC: Weglorz' models will provide a framework for the investigation of different notions of anonymity and discrimination.

In social welfare theory (section 2.2), $\mathbb{B}$ describes the "observable" coalitions and Γ defines "Γ-equal treatment of" and "finite Γ-discrimination among" the individuals (Γ-anonymity resp. topological Γ-anonymity of section 3.1). The latter condition requires that the individuals be partitioned into finitely many (observable) classes ("equally treated components"), each consisting of individuals that are treated equally in the following sense: The social preference does not change by any permutation in Γ, as long as it permutes individuals within each class (component), not across different classes. (In section 4.1 we discuss the problem, if there are interesting topologically Γ-anonymous welfare functions with more than two components.) We regard topological Γ-anonymity as an informational (but not an ethical) requirement about the simplicity of the welfare function: It is satisfied both by dictatorial functions (but not by all; c.f. lemma 23) and by functions which satisfy very strong forms of anonymity (c.f. section 3.1).

Our main result (section 3.2) is a translation between (i) assertions about (the existence of) social welfare functions within the "mathematical universe" (by this we mean a particular model V of ZFC which we use as a carrier of mathematics) and (ii) assertions about (the existence of) ultrafilters in Weglorz' models. The latter are instances of independence theorems (section 2.3). Note that assertion (ii) may be true or not, depending on $\mathbb{B}$ and Γ; c.f. problem 1 in sectwelfareion 4.1. (AC fails in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$, but some axioms close to AC do not.)

Exemplary result. *The following assertions are equivalent:*
(i) There is a reasonable finitely (Γ-)discrimatory nondictatorial social welfare function F which observes $\mathbb{B}$.
(ii) In $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ there is a free ultrafilter on a set which represents I. □

We conclude that Weglorz' models provide meaningful information about finite discrimination because they explain why Arrow's theorem is true relative to certain combinations of $\mathbb{B}$ and Γ. A similar analysis applies to ecological reasoning (section 3.3). So the problem is now: Is finite discrimination such a big deal?

We answer this question in the affirmative by deriving finite discrimination from a set theoretic condition of "symmetry" which in turn we view as a minimal requirement for "empirical meaningfulness" (c.f. [8]) or "describability". (So an informational condition for social welfare functions is related to notions of philosophy.) The paper thereby complements the studies (c.f. [23] and [22]) in computability analysis of social choice, since computability requires the existence of an algorithm to "describe" the social welfare function.

2. Preliminaries.

2.1. Some set theory. The first application of independence theorems about AC has been an analysis of different definitions of "finite": I is *finite*, if it is *equipollent* to (of the same cardinality as) an element of $\omega = \{0, 1, \ldots\}$. I is *Dedekind finite*, if each countable subset is finite. In the presence of AC this is another definition of "finite". A more restrictive definition is *amorphous*: each subset is finite or its complement is finite. (An infinite set is amorphous, iff the *Frechet filter* of the cofinite subsets is an ultrafilter.)

In the absence of AC the latter two assertions are no longer definitions of "finite". Instead they form a hierarchy of notions of different strength: finite, amorphous, Dedekind finite. The following observation ([26], Proposition 1.1) about "Dedekind finite powerset" (c.f. [16]: if I is amorphous, then $\mathcal{P}(I)$ is Dedekind finite) is useful and it applies in particular to $\mathbb{R} = \mathcal{P}(\omega)$:

LEMMA 1. *Without AC, if $\mathcal{P}(I)$ is Dedekind finite and α is an ordinal number, then the range of each function $f : I \to \mathcal{P}(\alpha)$ is finite.* □

Also Arrow's theorem (and its analysis by [11] and [18]) gives rise to a notion of finiteness: A set I of individuals (with at least two elements) is *Arrow finite*, if and only if each reasonable social welfare function on I (the definition in section 2.2 includes the requirement that there are three or more alternatives) is dictatorial (equivalently: if and only if each ultrafilter on I is principal). An application of the Frechet ultrafilter shows that an infinite amorphous set is Arrow infinite.

In ZF set theory without the axiom of choice AC, the following assertions BA (due to Blass) and SPI (due to Halpern) are weak forms of AC which do not imply AC (c.f. [14]). BA is the statement that some set is Arrow infinite and SPI asserts that "Arrow finite" defines "finite".

- BA: There exist an infinite set I and a free ultrafilter $\mathcal{U}$ on I;
- SPI: On each infinite set I there is a free ultrafilter $\mathcal{U}$ on I.

In this paper we consider Weglorz' [31] models, a specific class of permutation models. Below we first discuss the notion of a permutation model in general, followed by a discussion of Weglorz' models. They appear naturally in investigations of alternatives to SPI (c.f. lemma 14); [7] surveys their properties.

Permutation models are explained in [3], [5], [14] and [16]. They are models of the modified set theory ZFA with a set A of *atoms*: $a \in A$ is an object without elements but different from $\emptyset$. (The language of ZFA has in addition to $=$, $\in$ and $\emptyset$ the constant A.) We shall identify the individuals with the atoms of the model. Roughly speaking the sets of the model will be "definable[5] from observable coalitions". (For example, in order to "define" a

[5] In Fraenkel's original intuition, a model of set theory is a subclass of the "mathematical universe" which consists of (parametrically) "definable" sets in a sense that has been made precise by [3]. We shall use Mostowski's definition, where the model consists of "symmetric" sets only.

profile, we partition society into finitely many "observable" coalitions, all of whose members agree w.r. to their preferences; c.f. the proof of lemma 20.)

Given are the model ("mathematical universe") V of ZFC and $I \in V$. We first construct a universe $V(I)$ and a new $\in$-relation $\bar{\in}$ (which we henceforth write as $\in$) such that the set A of atoms is a copy of I (as in [16] we may write $A = \{a_i; i \in I\}$) and

$(V(I), \bar{\in}) \models ZFA + AC$

Here "$\models$" means "satisfies" or "is a model of"; e.g. $(V, \in) \models ZF + AC$. The details of the construction of $V(I)$ will not matter; c.f. [3], [5], [14] or [16].

Next we define a notion of "symmetry" in terms of a subgroup Γ of the *symmetric group* $\mathcal{S}(I)$ of all permutations on I and a group topology on Γ. (A *group topology* is a Hausdorff topology where the group operations, $(\pi, \psi) \mapsto \pi \cdot \psi$ and $\pi \mapsto \pi^{-1}$, are continuous. In the definition of a permutation model it will be determined by a "normal filter" of subgroups which defines a neighborhood base of the unit element.)

Note that $\pi \in \Gamma$ defines a permutation $\bar{\pi}$ on the set A of atoms: $\bar{\pi}(a_i) = a_{\pi i}$ for $\pi \in \Gamma$. It extends to a permutation (a proper class) $\hat{\pi}$ of $V(I)$ (Mostowski collapse) such that $\hat{\pi}(x) = \{\hat{\pi}(y); y \in x\}$. (We write π instead of $\hat{\pi}$.) Then $x \in V(I)$ is *symmetric*, if its *stabilizer* $\text{stab}(x) = \{\pi \in \Gamma; \pi(x) = x\}$ ($= \text{sym}(x)$ in [16]) is an open (and therefore also closed) subgroup of Γ. We regard the open subgroups of Γ as "degrees of symmetry". (This is evident for the group of the p-adic integers [5]. In general, however, they generate a topology which appears peculiar.)

A general permutation model $\mathcal{M}$ consists of the "hereditarily symmetric" $x \in V(I)$ (i.e. each element of the transitive closure of $\{x\}$ is symmetric, where the transitive closure consists of x, its elements, the elements of these elements $\ldots$, ending with an atom or $\emptyset$). Thus $x \subseteq \mathcal{M}$, an object of $V(I)$, is a set in the sense of the model, iff $x \in \mathcal{M}$, iff x is (hereditarily) symmetric.

If $x \in V(\emptyset)$ (there is no atom in the transitive closure of x), then x is a *pure set.* Pure sets are hereditarily symmetric (their stabilizer is the whole group Γ) and therefore in the model. (The pure sets form a copy of the "mathematical universe" within the model, whence lemma 2.)

AC fails in $\mathcal{M}$, unless Γ is discrete: Set theoretical choice functions in general are not symmetric. However, the following choice principle is always true. (It says that, in a permutation model, a set carries a wellordering relation, if and only if it is equipollent to a pure set.)

- PW: The powerset $\mathcal{P}(\alpha)$ of an ordinal α is equipollent to an ordinal.

LEMMA 2. *Permutation models satisfy $ZFA + PW$.* □

Weglorz' models $\mathcal{W}_{\mathbb{B}}^{\Gamma} \subseteq V(I)$ are constructed from an infinite Boolean algebra $\mathbb{B} \subseteq \mathcal{P}(I)$, a subgroup Γ of its *automorphism group* $\text{Aut}(\mathbb{B})$ and Weglorz' group topology. (A bijective $\pi : \mathbb{B} \to \mathbb{B}$ is in $\text{Aut}(\mathbb{B})$, if $\pi(\emptyset) = \emptyset$, $\pi(I) = I$, $\pi(x \cup y) = \pi(x) \cup \pi(y)$, $\pi(x \cap y) = \pi(x) \cap \pi(y)$ and $\pi(-x) = -\pi(x)$.)

We shall assume that $\mathbb{B}$ contains all singletons $\{i\}$, where $i \in I$. Then $\text{At}(\mathbb{B}) = [I]^1$ (recall that $[I]^n$ is the set of n-element subsets of I), where

$\mathrm{At}(\mathbb{B})$ is the set of the *Boolean algebra atoms* (minimal nonempty sets in $\mathbb{B}$). It follows that $\pi \in \mathrm{Aut}(\mathbb{B})$ may be identified with a permutation $\tilde{\pi} \in \mathcal{S}(I)$: $\pi \in \mathrm{Aut}(\mathbb{B}) \Rightarrow \pi(\{i\}) = \{\tilde{\pi}(i)\} \in \mathrm{At}(\mathbb{B})$ for some $\tilde{\pi} \in \mathcal{S}(I)$ and $B \in \mathbb{B} \Rightarrow \pi(B) = \{\tilde{\pi}(i); i \in B\}$. (We thereby identify $\mathrm{Aut}(\mathbb{B})$ with the subgroup $\mathrm{stab}(\mathbb{B})$ of $\mathcal{S}(I)$ which we actually use in the definition of the model.)

Weglorz' group topology is defined from a neighborhood base of the unit element (the permutation id). We define (a filter base)

- $\mathcal{F}_{\text{Weglorz}} = \{\mathrm{stab}(b);\ b \in \mathbb{B}\}$

Then U is a neighborhood of the unit element, iff it contains a finite intersection of $\mathcal{F}_{\text{Weglorz}}$. The group topology is generated by the subbasis $\{\pi \cdot \mathrm{stab}(b); \pi \in \Gamma$ and $b \in \mathbb{B}\}$.

Using the above identifications we may define ("$<$" means "subgroup of" and $\mathcal{S}_{\text{finite}}(I)$ is the group of all finite permutations of I):

Definition. Given are a Boolean algebra $\mathbb{B}$ and a group Γ such that $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$. The permutation model $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ consists of the hereditarily symmetric objects of $V(I)$, where symmetry is defined w.r. to the group Γ and the Weglorz' group topology. □

The assumption $\mathcal{S}_{\text{finite}}(I) < \Gamma$ is included for technical reasons only. It will ensure that $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models \neg AC$.

We may reformulate "symmetry". If $x \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, then (by the above definition) there exists a finite $\mathbb{D} \subseteq \mathbb{B}$ such that

$\mathrm{stab}(x) \supseteq \mathrm{fix}(\mathbb{D}) = \cap\{\mathrm{stab}(d);\ d \in \mathbb{D}\}$

We may enlarge $\mathbb{D}$ to a subalgebra of $\mathbb{B}$. Then $\mathrm{At}(\mathbb{D})$ forms a finite ordered partition

$\Pi = \langle P_1, \ldots, P_m \rangle$

of A into elements $P_k \in \mathbb{B}$ which partition satisfies the following definition of a *support* of x

- $\mathrm{stab}(x) \supseteq \mathrm{stab}(\Pi)$ $(= \{\pi \in \Gamma; \pi(P_i) = P_i,\ \text{all}\ i\} = \mathrm{fix}(\mathbb{D}))$

In this paper we wish to compare the objects of Weglorz' models with sets of the "mathematical universe". The formulation of our results is simplified, if we use the same terminology for related objects.

As we use the same notation (i) for the individual $i \in I \in V$ and the atom $a_i \in A \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ (see above), it will be convenient to identify also (ii) the set $I \in V$ of all individuals and the set $A \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ of the atoms of the model, (iii) the Boolean algebra atom $\{i\} \in \mathbb{B} \in V$ and the singleton $\{a_i\} \in \mathcal{P}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, (iv) the element $b \in \mathbb{B} \subseteq \mathcal{P}(I) \in V$ and the set $\bar{b} = \{a_i; i \in b\} \in \mathcal{P}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ (its stabilizer is in $\mathcal{F}_{\text{Weglorz}}$) and (v) $\mathbb{B} \in V$ with a subalgebra $\bar{\mathbb{B}} = \{\bar{b}; b \in \mathbb{B}\} \subseteq \mathcal{P}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$.

As follows from lemma 6, in general we cannot relate $\pi \in \Gamma \in V$ to a permutation in $\mathcal{S}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$. Let us recall the identification (vi) of $\pi \in \mathrm{Aut}(\mathbb{B}) \in V$ with a permutation $\tilde{\pi} \in \mathcal{S}(I) \in V$, (vii) of $\pi \in \mathcal{S}(I) \in V$ with $\bar{\pi} \in \mathcal{S}(A) \in V(I)$ and (viii) of $\pi \in \mathcal{S}(A) \in V(I)$ with $\hat{\pi}$, a "permutation" of $V(I)$. In order to link our results with the economic literature (c.f. [21]), in V we identify (ix) $\mathrm{Aut}(\mathbb{B})$ with the $\mathbb{B}$-*measurable* permutations π of I (they satisfy $\pi^*(B) = \{\pi^{-1}(i); i \in B\} \in \mathbb{B}$, whenever $B \in \mathbb{B}$). (Note that $\pi \in \mathrm{Aut}(\mathbb{B})$ defines a $\mathbb{B}$-measurable $\tilde{\pi} \in \mathcal{S}(I)$. If conversely $\pi \in \mathcal{S}(I)$ is $\mathbb{B}$-measurable, then π is a representation of $(\pi^*)^{-1} \in \mathrm{Aut}(\mathbb{B})$.)

The original motivation behind $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ has been the following result of Weglorz [31] which says that in the identification (v) above $\bar{\mathbb{B}} = \mathcal{P}(A)$.

Lemma 3. *If I is infinite, $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$, then $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ "$\mathbb{B} = \mathcal{P}(A)$".* □

Weglorz' theorem relates Armstrong's setting [1] to the formally simpler approach of [18]. Lemma 3 asserts that, given $\mathbb{B}$, we may consistently add the axiom "$\mathbb{B}$ is isomorphic to a powerset algebra" (which in general contradicts *AC*) to *ZF* set theory. As follows from lemmas 22 and 23, this axiom does not completely reduce [1] to [18]: [1] admits social welfare functions which are not topologically anonymous and therefore not in the model.

In Weglorz' models, the set of individuals sometimes is amorphous and always has a Dedekind finite powerset [8].

Lemma 4. *If I is infinite, $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$, then $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ "$\mathbb{B}$ is Dedekind finite".* □

We illustrate the method of proof in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ by lemma 6 about an easy correspondence between the "mathematical universe" and the model. First we recall the following result [6] about partitions.

Lemma 5. *Let I be infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$. In $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ all but finitely many elements of a partition $\mathcal{E}$ of the set A of the atoms into finite sets are singletons.* □

Lemma 6. *Let I be infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$. In $V(I)$, if $\pi \in \mathcal{S}(A)$ is a permutation on the set A of the atoms, then $\pi \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ if and only if $\pi \in \mathcal{S}_{\text{finite}}(A)$.*

Proof. As "if" is trivial (a finite permutation is defined from finitely many parameters within the model), we prove "only if". Define a partition $\mathcal{E}$ which consists of the sets $\{\pi^z(a); z \in \mathbb{Z}\}$ where $a \in A$ ($\pi^0(a) = a$, $\pi^1(a) = \pi(a)$). These sets are finite by lemmas 4 and 1. Lemma 5 implies that (with finitely many exceptions) all $E \in \mathcal{E}$ are singletons, where $\pi(a) = a$. It follows that $\{a; \pi(a) \neq a\}$ is contained in the finite union of the finite nonsingletons of $\mathcal{E}$. □

In [8] (a variant of) Weglorz' models with $\Gamma = \mathrm{Aut}(\mathbb{B})$ has been given an empirical interpretation (invoking the "theoretical terms" of philosophy): $\mathbb{B}$ is the algebra of the "observable" objects and the model consists of those empirical concepts which satisfy a necessary (but not sufficient) semantical test[6], namely "symmetry" (hereditary symmetry), to be "describable". (Thus the model is constituted of those concepts which are not "empirically meaningless for obvious reasons".)

[6] It is motivated by "Padoa's method" [13] to prove undefinability.

It follows from lemma 6 that the definition of "describable" depends in an essential way on "indescribable" ("theoretical", "empirically meaningless") permutations.

2.2. Some welfare theory. We formulate the basic notions of welfare theory in a general setting. Our framework incorporates both the theory of Arrow type social welfare functions (sections 3.1 and 3.2) which map "ordinal preference profiles" to social preferences and a theory of ecological welfare mappings[7].

Recall that I is the set of individuals. A *profile structure* is a relational structure (tuple) $\mathbf{X} = (X, \mathbf{p})$, where X is a set of *alternatives* (not necessarily fixed or finite) and $\mathbf{p}$ is a profile (defined below) of I on X. We represent the preferences of the individual $i \in I$ by a binary relation $\preceq^i$ on X. (We do not always assume that it is transitive and complete[8].) The strict preference $\prec^i$ and the indifference $\sim^i$ are defined in the usual manner. Finally, a *profile* $\mathbf{p}$ of I on X is a function $\mathbf{p} : I \to \mathcal{P}(X \times X)$, a list $\mathbf{p} = \langle \preceq^i ; i \in I \rangle$ of the individual preferences on X.

The profile structure $\mathbf{Y} = (Y, \mathbf{q})$ is a *substructure* of $\mathbf{X} = (X, \mathbf{p})$ (symbol: $\mathbf{Y} \subseteq \mathbf{X}$), if $Y \subseteq X$ and for all x, $y \in Y$ and for all $i \in I$ the assertions $(x, y) \in \mathbf{q}(i)$ and $(x, y) \in \mathbf{p}(i)$ are equivalent. Thus $\mathbf{q}$ is the *restriction* $\mathbf{p}|Y$ which maps $i \in I$ to the relation $\mathbf{p}(i)|Y = \mathbf{p}(i) \cap (Y \times Y)$.

An *aggregation structure* is a relational structure $(X, \preceq, \mathbf{p})$. It adds a *social preference* ("decision") $\preceq$ to the corresponding profile structure. (Unless stated otherwise, the social preference may be just a binary relation on X.) A substructure of an aggregation structure carries the restricted social preference (and the restricted profile).

Social decision theory is concerned with the construction of aggregation structures from profile structures. A *decision rule* is a function F whose domain $\mathrm{dom}(F)$ is a set of profile structures of the same signature (i.e.: I is kept fixed) and whose range is a set of aggregation structures such that

$$F(X, \mathbf{p}) = (X, \preceq_F, \mathbf{p}) \text{ for } \mathbf{X} = (X, \mathbf{p}) \in \mathrm{dom}(F)$$

Here $\preceq_F$ depends on X, $\mathbf{p}$ and F.

A subset (coalition) $S \subseteq I$ is decisive with respect to an aggregation structure $(X, \preceq, \mathbf{p})$, if $x \prec y$ whenever x, $y \in X$ and $(\forall i \in S)(x \prec^i y)$. A coalition $S \subseteq I$ is *decisive* (with respect to a decision rule F), if for every $\mathbf{X} \in \mathrm{dom}(F)$, S is decisive with respect to the aggregation structure $F(\mathbf{X})$.

Given a decision rule, an agent $i \in I$ is a *dictator*, if $\{i\}$ is decisive. The decision rule satisfies *strict unanimity* (or "Pareto-efficiency"), if I is decisive.

A decision rule is more general than Arrow's social welfare functions. Arrow assumes that the set X of the alternatives is fixed and that the individual preferences are defined on X. Thus in Arrow's setting $Y \subseteq X$ would carry the

[7] They assign a preference relation to each set of utility distributions ("utility profiles"). We will see that a well-behaved ("monotonic") ecological welfare function is equivalent to a social preference on the set of utility distributions (section 3.3; c.f. "social welfare orderings" [25] or [9]).

[8] Several textbooks call this a "rational preference".

profile on X (instead of its restriction). In our approach, when considering a subset Y, we forget the additional information about $X \setminus Y$. If we consider only monotonic decision rules (as did Arrow, see below), then (for finite sets of alternatives) the difference disappears.

Our approach appears natural from a logicians point of view: A structure which is based on Y usually carries only relations which are defined on Y. We do not require the individuals to make up their minds about potential alternatives not in Y. This has the following advantage: In Arrow's setting ethical principles may have unintended consequences (neutrality implies IIA) which cannot be derived in the general framework.

Social choice theoretists, on the other hand, may feel uneasy about the arbitraryness of the domain of a decision rule, which is an arbitrary set of profile structures of the same signature. (Even the set of potential alternatives has not to be specified.) In addition to incorporating social welfare functions (where the domain is fixed once the set of potential alternatives and the set of individuals are fixed) as special classes, our formulation has an advantage as a *descriptive* theory: Consider the modeling of political decisions about ecologically tolerable lay-outs of a projected highway. Often the alternatives are compared by procedures which are defined for finite sets Y of (actually discussed) alternatives only. During the decision process other alternatives will be added, but there is no *a priori* upper bound, as the set X of potential alternatives is infinite and will never be specified in practice. (As an application, [9] relate procedures such as "Borda counts" to legal requirements about decision-making.)

Given a set X, a set $\mathbb{P}$ of profiles (of I) on X and a decision rule F which is defined on $\mathcal{X} \supseteq \mathcal{X}_0 = \{(X, \mathbf{p}); \mathbf{p} \in \mathbb{P}\}$, we define the corresponding *social welfare function* $C_F(\mathbf{p})$ in Arrow's sense as the mapping that assigns to each profile $\mathbf{p} \in \mathbb{P}$ the social preference of $F(X, \mathbf{p})$. Conversely, an Arrow social welfare function C extends to a decision rule F_C such that $F_C(Y, \mathbf{q}) = (Y, C(\mathbf{q}^+)|Y, \mathbf{q})$, where $Y \subseteq X$ and $\mathbf{q} \mapsto \mathbf{q}^+ \in \{\mathbf{p}; \mathbf{p}|Y = \mathbf{q}\}$ is a set theoretical choice function. (If we consider only $Y = X$, then $\mathbf{q}^+ = \mathbf{p}$ and to each Arrow social welfare function with domain $\mathbb{P}$ there corresponds a unique decision rule F with domain $\mathcal{X}_0$.)

While C_F is uniquely defined, F_C depends on the choice of $\mathbf{q}^+$ in general. Thus $C_{F_C} = C$ is true, but $F_{C_F} = F$ needs not be true. However, if F satisfies the following condition of monotonicity, then the last identity holds for all choices of $\mathbf{q}^+$.

The decision rule F is *monotonic*, if for profile structures $\mathbf{X}_1 \subseteq \mathbf{X}_2$ (where $\mathbf{X}_i \in \mathrm{dom}(F)$) we have $F(\mathbf{X}_1) \subseteq F(\mathbf{X}_2)$ as aggregation structures. It is *weakly monotonic*, if we add the requirement $|X_1| = 2$ to the above definition. These notions correspond to Arrow's IIA ("independence of irrelevant alternatives").

We say that the domain of F is closed under two-element (finite) substructures, if $|X_1| = 2$ (X_1 finite) and $\mathbf{X}_1 \subseteq \mathbf{X}_2 \in \mathrm{dom}(F)$ imply $\mathbf{X}_1 \in \mathrm{dom}(F)$.

Lemma 7. *Without AC, a weakly monotonic decision rule F is monotonic, if its domain is closed under two-element substructures.*

Proof. A decision rule F whose domain is closed under two-element substructures defines a monotonic *canonical aggregation structure* $(X, \preceq_{F,can}, \mathbf{p})$ on $\mathbf{X} \in \mathrm{dom}(F)$, namely $x \preceq_{F,can} y \Leftrightarrow$ "$x \preceq_F y$ is true in $F(\mathbf{Y})$", where $\mathbf{Y} = \{x, y\}$ is the substructure of $\mathbf{X}$. If F is weakly monotonic, $\preceq_F = \preceq_{F,can}$.□

We are now ready to formulate (a special form of) Arrow's theorem. We let X be a set with at least three alternatives and assume that the domain of F is:

- $\mathcal{X}^I = \{(Y, \mathbf{q}); Y \subseteq X \text{ and } \mathbf{q} \in \mathbb{P}^I(Y)\}$ where

• $\mathbb{P}^I(Y) = \{\mathbf{q}; \mathbf{q} : i \mapsto$ transitive and complete preference $\mathbf{q}(i)$ on $Y\}$
We use $\mathcal{X}^I$ (which is a set of structures) instead of the "conventional" set $\mathbb{P}^I(X)$ of profiles. The same proof will work for "rich" domains in general, but it will be easier to compare decision rules of the model $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ with those of the "mathematical universe", if we confine ourselves to $\mathrm{dom}(F) = \mathcal{X}^I$.

Lemma 8. *Without AC, let I have two or more elements, X three or more elements and consider a nondictatorial monotonic decision rule F on* $\mathrm{dom}(F) = \mathcal{X}^I$ *which satisfies strict unanimity such that $\preceq_F$ is transitive and complete. Then the set $\mathcal{U}$ of decisive coalitions is a free ultrafilter on I.*

Proof. The proof (e.g. [10], p. 578-580) reduces to considerations about three-element structures $\mathbf{X} \in \mathrm{dom}(F)$ (whose profile $\mathbf{p}$ is a function on I with at most $2^{3\times 3}$ values $\mathbf{p}(i)$). The profiles of I on $\mathbf{X}$ are explicitly defined in terms of finitely many subsets of I. Therefore AC is not applied. Note that the proof does not insist on the existence of a free ultrafilter. □

In the ecological context (section 3.3) Arrow's axioms will not suffice to derive (a result whose proof resembles) Arrow's theorem. There we need, in addition, (an extension of) May's [20] axiom of neutrality. Neutrality asserts that the alternatives are "treated equally". We prefer Mihara's interpretation [21] that neutrality is a condition of computational simplicity. (In the ecological context "equal treatment" of the alternatives is built in the definition of the structures.)

A mapping $\Phi : \mathbf{X} \to \mathbf{Y}$ is an *isomorphism* of the profile structures $\mathbf{X}$ and $\mathbf{Y}$, if Φ is a bijection from X to Y and for all $i \in I$ and for all $x,\ y \in X$ we have $(x, y) \in \mathbf{p}(i) \Leftrightarrow (\Phi(x), \Phi(y)) \in \mathbf{q}(i)$. An isomorphism Φ of the aggregation structures $(X, \preceq, \mathbf{p})$ and $(Y, \sqsubseteq, \mathbf{q})$ is an isomorphism of the profile structures $(X, \mathbf{p})$ and $(Y, \mathbf{q})$ such that $x \preceq y \Leftrightarrow \Phi(x) \sqsubseteq \Phi(y)$, all $x,\ y \in X$.

A decision rule F *respects isomorphisms*, if each isomorphism Φ of the profile structures $\mathbf{X},\ \mathbf{Y} \in \mathrm{dom}(F)$ is also an isomorphism of the aggregation structures $F(\mathbf{X})$ and $F(\mathbf{Y})$. In this case $\preceq_F$ depends on the isomorphism type of the profile only. This condition relates in the following way to the "conventional" axiom of neutrality. (Note that the form below does not imply monotonicity, since the domain of F consists of profile structures whose profiles $\mathbf{q},\ \mathbf{q}'$ need not be defined on the set X of all alternatives. In section 3.3 we shall consider another variant of this notion.)

Lemma 9. *Without AC, suppose that F is a monotonic decision rule with domain $\mathcal{X}^I$. Then F respects isomorphisms, if and only if for all two-element profile structures $\mathbf{Y} = (\{x,\ y\}, \mathbf{q}),\ \mathbf{Y}' = (\{x',\ y'\}, \mathbf{q}') \in \mathcal{X}^I$ (and mappings $u \mapsto u'$) the assertion*

$(\forall i \in I)((x, y) \in \mathbf{q}(i) \Leftrightarrow (x', y') \in \mathbf{q}'(i))$

implies $x \preceq y \Leftrightarrow x' \preceq' y'$, where $\preceq$ (resp. $\preceq'$) is the social preference of $F(\mathbf{Y})$ (resp. $F(\mathbf{Y}')$).

Proof. The assertion says that $\Phi(u) = u'$ is an (arbitrary) isomorphism between (arbitrary two-element profile structures) $\mathbf{X}$ and $\mathbf{X}'$. The conclusion says that Φ is an isomorphism of the aggregation structures. This is a special case of respected isomorphisms. We now apply monotonicity: Φ is an isomorphism between profile/aggregation structures, if and only if it is an isomorphism between all two-element substructures. □

As has been observed by Fishburn [11] (c.f. [18]), a free ultrafilter on I (if it exists) defines a decision rule as in the following (slightly stronger) lemma. There are other decision rules with the same ultrafilter of decisive coalitions, too (c.f. [21], p. 510-511). We set

- $P(x, y) = P_{\mathbf{p}}(x, y) = \{i \in \mathrm{dom}(\mathbf{p}); x \preceq^i y\}$

LEMMA 10. *Without* AC, *let* X *be a set of at least three options, let* I *be an infinite set of individuals and assume that there is a free ultrafilter* $\mathcal{U}$ *on* I. *If* $(Y, \mathbf{q}) \in \mathcal{X}^I$, *then the relation* $\preceq_{\mathcal{U}}$ *on* $\mathbf{Y}$,

$x \preceq_{\mathcal{U}} y \Leftrightarrow P_{\mathbf{q}}(x, y) \in \mathcal{U}$

is a transitive and complete social preference on Y. *The aggregation* $F(Y, \mathbf{q}) = (Y, \preceq_{\mathcal{U}}, \mathbf{q})$ *defines a monotonic, strictly unanimous decision rule on* $\mathcal{X}^I$ *which respects isomorphisms and does not admit a dictator.*

PROOF. Note that the general form of Łoś' theorem (which relates to quantifiers) depends on AC (c.f. [14]). Lemma 10, however, applies instances of this theorem (for quantifier-free statements) which do not depend on AC, but refer directly to the defining properties of ultrafilters, instead. □

We simplify our notation by introducing the following terminology: A *reasonable social welfare function* on I is a monotonic and strictly unanimous decision rule on the (transitive and complete) profile structures $\mathcal{X}^I$ (where $X \in \omega$ is a finite set of at least three options) whose social preferences (decisions) are transitive and complete. (It is identifiable to a "reasonable" social welfare function in Arrow's sense. Henceforth we shall no longer refer to Arrow-type social welfare functions.)

2.3. Some independence theorems. The Arrow and Fishburn theorems establish an obvious link to AC via ultrafilters. Independence results about ultrafilters in their turn lead naturally to Weglorz' models. These models of ZFA prove independence theorems for ZF set theory (without atoms), since we may apply lemma 11, the Jech-Sochor transfer theorem [16]. It compares the "initial segments" $\mathcal{P}^\alpha(\cdot)$ of the models (defined by means of the up to α^{th} iterates of the powerset operation $\mathcal{P}(\cdot)$).

LEMMA 11. *Given are a* ZFC *model* V, *a permutation model* $\mathcal{M} \subseteq V(I)$ *and a* V*-ordinal* α. *Then we may construct a* ZF *model* $W \supseteq V$ *and a set* $B \in W$ *such that* $\mathcal{P}^\alpha(A)$ *of* $\mathcal{M}$ *is* $\in$*-isomorphic to* $\mathcal{P}^\alpha(B)$ *of* W. □

Of particular interest is the structure of the prime filters (equivalent: ultrafilters, maximal filters) of $\mathbb{B}$ within the model. We recall the following results from [7] about $\Gamma = \mathrm{Aut} = \mathrm{Aut}(\mathbb{B})$ (it satisfies $\mathcal{S}_{\text{finite}}(I) < \mathrm{Aut}(\mathbb{B})$). Note that (in V) the countable structured atomic Boolean algebras are isomorphic to each other [7], where $\mathbb{B}$ is *structured*, if each infinite $b \in \mathbb{B}$ splits into two infinite elements of $\mathbb{B}$. (Lemma 12 appears to be in contrast to lemma 27 which constructs a prime filter. However, as a set in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the algebra $\mathbb{B}$ is not countable by lemma 4. This situation resembles "Skolem's paradox" of set theory.) Note that the proof of lemma 13 in [7] (theorem 4 with $I = \omega$) extends immediately to infinite I. [Let $S_j \subset P_i$ have maximal cardinality.]

LEMMA 12. *If $I = \omega$ and $\mathbb{B}$ is a countable (in V) atomic algebra such that $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$, then the following assertions are equivalent.*
(i) $V \models$ "$\mathbb{B}$ is structured"
(ii) $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}} \models$ "Each ultrafilter on A is principal". □

LEMMA 13. *If I is infinite and $\mathbb{B} = \mathcal{P}(I)$, then in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ each bounded complex-valued finitely additive measure m on $\mathcal{P}(A)$ is concentrated on a finite set.* □

By [7], the conclusion of lemma 13 is equivalent to "in $\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}}$ there is no (finitely additive) probability measure on $\mathcal{P}(A)$ which vanishes on the finite sets" (c.f. lemma 4). In particular, if I is infinite and $\mathbb{B}$ is the powerset algebra, then

$\mathcal{W}_{\mathbb{B}}^{\mathrm{Aut}} \models$ "ultrafilters on A are principal"

In the models of lemmas 12 and 13 A is Arrow finite but infinite. (By lemmas 8 and 10 SPI implies "Arrow finite = finite". If BA fails, then each set I with at least two elements is Arrow finite.) We apply lemma 11 to obtain the following independence result of Skala [28].

LEMMA 14. *Relative to ZF set theory, the following assertion depends on AC: "If I has at least two elements, then I is finite, if and only if I is Arrow finite".* □

[7] mentions "counterexamples" (lemma 15), where A is Arrow infinite. Another trivial result is lemma 16.

LEMMA 15. *Let I be infinite.*
(i) If $\mathbb{B}$ is the algebra of finite and cofinite sets and $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$, then in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ the Frechet filter (of the cofinite sets) is the unique free ultrafilter on A.
(ii) If $\Gamma = \mathcal{S}_{\mathrm{finite}}(I)$ and $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$, then in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ each nonprincipal filter on A may be extended to a free ultrafilter. □

LEMMA 16. *Let I be infinite, assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and let $\mathcal{U}$ be a nonprincipal prime filter of $\mathbb{B}$. Then $\Gamma = \{\pi \in \mathrm{Aut}(\mathbb{B}); \pi\mathcal{U} = \mathcal{U}\}(= \mathrm{stab}(\mathcal{U}))$ satisfies $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$ and $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ "$\mathcal{U}$ is a free ultrafilter on A".* □

We next determine the structure of these ultrafilters from within the model (Lemma 18). A filter $\mathcal{U}$ on I is a *Ramsey filter*, if each finite partition $\langle C_1, \ldots C_n\rangle$ of $[I]^2$ admits a homogeneous set $H \in \mathcal{U}$ (which satisfies $[H]^2 \subseteq C_i$ for some $i \le n$). Note that a free Ramsey filter is an ultrafilter. [Given $S \subseteq I$ consider the partition $C_1 = [S]^2$, $C_2 = [I \setminus S]^2$ and $C_3 = [I]^2 \setminus (C_1 \cup C_2)$. If $\mathcal{U}$ is free, then $|H| \ge 3$ and so $[H]^2 \subseteq C_3$ is impossible. As $H \in \mathcal{U}$ is homogeneous it follows that $H \subseteq S$ or $H \subseteq I \setminus S$, depending on $[H]^2 \subseteq C_1$ or C_2.]

Lemma 17. *Let I be infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$. In $V(I)$, consider $C \subseteq [A]^2$. Then $C \in \mathcal{W}^{\Gamma}_{\mathbb{B}}$, if and only if for some ordered partition (support) $\Pi = \langle P_1, \ldots, P_m \rangle$ of A into elements $P_k \in \mathbb{B}$ there is a representation of the form*

$$C = \bigcup_{\{i\} \in M_1} [P_i]^2 \cup \bigcup_{\{i,j\} \in M_2} ([P_i \cup P_j]^2 \setminus ([P_i]^2 \cup [P_j]^2))$$

where $M_k \subseteq [\{1, \ldots m\}]^k$.

Proof. We verify "only if": Let Π be a support of C. If $S \in C \cap [P_i]^2$, then $\{\pi(S); \pi \in \mathcal{S}_{\text{finite}}(P_i)\} = [P_i]^2$. Therefore $C \subseteq [P_i]^2$, as these permutations extend to elements of stab(Π), and we may set $M_1 = \{\{i\}; C \cap [P_i]^2 \neq \emptyset\}$. Similarly, if $S \in C$ and $S \cap P_i \neq \emptyset$, $S \cap P_j \neq \emptyset$ and $i \neq j$, then $C \subseteq [P_i \cup P_j]^2 \setminus ([P_i]^2 \cup [P_j]^2)$ and we may set $M_2 = \{\{i,j\}; C \cap ([P_i \cup P_j]^2 \setminus ([P_i]^2 \cup [P_j]^2)) \neq \emptyset\}$. □

Lemma 18. *Let I be infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$. Then $\mathcal{W}^{\Gamma}_{\mathbb{B}} \models$ "free ultrafilters on A are Ramsey".*

Proof. In $\mathcal{W}^{\Gamma}_{\mathbb{B}}$, let $\mathcal{U}$ be an ultrafilter on A, $\langle C_k; k \leq n \rangle$ be a finite partition of $[A]^2$ and let Π be a support of $\mathcal{U}$ and all C_k. As Π is a finite partition of A and $\mathcal{U}$ is an ultrafilter, there is some $i \leq m$ such that $P_i \in \mathcal{U}$. The set $H = P_i$ is homogeneous. For if $\{i\} \in M_1(k)$, then $[H]^2 \subseteq C_k$ (where $M_1(k)$ is M_1 of C_k in lemma 17) and $\{i\} \notin \cup_{k \leq n} M_1(k)$ is impossible. [Otherwise the C_k do not cover $[P_i]^2 \subseteq [A]^2$.] □

This result relates to the "social structure" of decision rules. As has been observed by Skala [30], p. 255, decision rules which correspond to free Ramsey filters are of a particular interest: They admit a decisive coalition where "each knows the others" (a "clan"), or a decisive coalition where "no one knows any other". ([30] uses the partition $C_1 = \{\{i,j\}; i \text{ knows } j\}$, assuming that then also j knows i, and $C_2 = [I]^2 \setminus C_1$.) We have shown in lemma 18 that this observation applies to all reasonable nondictatorial social welfare functions on A in $\mathcal{W}^{\Gamma}_{\mathbb{B}}$; it also applies to dictatorial ones. This motivates another "definition of finite":

A reasonable social welfare function on I is *Ramsey*, if the ultrafilter of the decisive coalitions is principal or Ramsey. A set I of individuals with at least two elements is *Skala finite*, if and only if each reasonable social welfare function with the domain $\mathcal{X}^I$ is Ramsey (equivalently: if and only if each ultrafilter on I is principal or Ramsey).

Arrow finite implies Skala finite and in all Weglorz' models the set A of atoms is Skala finite by lemma 18. Lemma 19 (which follows from lemmas 15 and 11) shows that these notions are of a different strength. So we have a hierarchy: finite, Arrow finite, Skala finite.

Lemma 19. *Relative to ZF set theory, the following assertion depends on AC: "If I has at least two elements, then I is Skala finite, if and only if I is Arrow finite".* □

Skala (c.f. [29], p. 214) has investigated the significance of independence results[9] for the economist. He argues that in social choice theory existence results which depend on AC (or its weak forms SPI and BA) in an essential way should not be taken as undebatable facts. In this paper we do not insist on a particular axiomatic system of set theory being suitable for studying social choice. Instead, we investigate several alternatives (in terms of $\mathbb{B}$ and Γ).

3. Main results.

3.1. Finite sets of alternatives. How does Armstrong's setting [1] relate to the models?

We ask for a condition which ensures that a decision rule F on the alternatives and profiles of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ (which will be sets of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$) is a set in the sense of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. To this end we will relate the set theoretic notion of "symmetry" with the social choice theoretic notion of "anonymity" (c.f. [20]). We assume (in this and the next section, only) that

- X is finite and fixed.

In view of $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ (which reduces the number of dictators to 0 or 1) we may identify the set I of the individuals with the set A of the atoms of $V(I)$ [and have $V(I) \models$ "$\mathbb{B} \subseteq \mathcal{P}(A)$"]. Economists (c.f. [1]) interpret $\mathbb{B}$ as the algebra of all "observable" (or "describable"; c.f. [23]) coalitions; by lemma 3 in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ "symmetry = describability" (for subsets of A).

If $\Gamma = \mathrm{Aut}(\mathbb{B})$, then the interpretation of [8] requires that empirically meaningful concepts (e.g. social welfare functions) should be described in terms of "observable" coalitions only: These descriptions ensure (hereditary) symmetry. Hence empirically meaningful social welfare functions should be symmetric (and the asymmetric ones shall not be viewed as idealizations of real decision procedures). We extend this interpretation to the general case $\Gamma < \mathrm{Aut}(\mathbb{B})$, where the "description" of the "meaningful" concepts might use additional structure. For example, $\Gamma = \mathrm{stab}(\mathcal{U})$ of lemma 16 permits definitions which use the prime filter $\mathcal{U}$.

The finite set X of options will be represented by its cardinal number, a pure set in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. The next step is the characterization of the preferences of the model. As the options form a pure set of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$, it follows trivially that each collection of options and each preference relation (of the "mathematical universe") is pure, too, and therefore a set in the sense of the model.

The description of the profiles and profile structures of the model is nontrivial: We ask for a criterion that a profile structure $(X, \mathbf{p})$ (where $\mathrm{dom}(\mathbf{p}) = A$ represents I) on a finite pure set X is a set in the sense of the model. The following condition (due to [1]) which is a notion of "describability" (c.f. [8] and [23]) ensures that: $\mathbf{p}$ is $\mathbb{B}$-*measurable*, if $x,\ y \in X \Rightarrow P_{\mathbf{p}}(x, y) \in \mathbb{B}$. We set

- $\mathbb{P}_{\mathbb{B}}^{I}(X) = \{\mathbf{p} \in \mathbb{P}^{I}(X); \mathbf{p}$ is B-measurable$\}$

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[9] [4] has proposed similar arguments about the axiom of constructibility.

LEMMA 20. *If I is infinite, $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$, $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$ and the options form a finite set $X \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, then for a profile $\mathbf{p} \in V(I)$ of A on X the following assertions are equivalent:*
(i) $\mathbf{p} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$
(ii) $V(I) \models$ *"$\mathbf{p}$ is $\mathbb{B}$-measurable".*

PROOF. If $\mathbf{p} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, then $x,\ y \in X$ implies (for sets in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$) that $P(x,y) = \{a \in A; x \preceq^a y\} \in \mathcal{P}(A)$. By lemma 3 this powerset in the sense of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ corresponds to (the copy of) $\mathbb{B}$ (in the model $V(I)$); therefore $V(I) \models P(x,y) \in \mathbb{B}$. Conversely, a profile $\mathbf{p} \in V(I)$ on a finite (and without loss of generality pure) set X attains at most finitely many values $\mathbf{p}(a) = \sqsubseteq^k \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, $k = 1, 2, \ldots K$. As $\mathbf{p}$ is $\mathbb{B}$-measurable, $P(x,y) \in \mathbb{B}$ and therefore

$$\begin{aligned}\mathbf{p}^{-1}(\sqsubseteq^k) = S_k &= \{a \in A; \mathbf{p}(a) = \sqsubseteq^k\} \\ &= \{a \in A; (\forall x, y \in X)(x \preceq^a y \leftrightarrow x \sqsubseteq^k y)\} \\ &= \cap\{P(x,y); x \sqsubseteq^k y\} \setminus \cup\{P(x,y); x \not\sqsubseteq^k y\} \\ &\in \mathbb{B} \subseteq \mathcal{W}_{\mathbb{B}}^{\Gamma}, \text{ as } X \text{ is finite.}\end{aligned}$$

We conclude that $\mathbf{p}$ is defined as follows as a set in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$

$$\mathbf{p}(a) = \sqsubseteq^k \in \mathcal{W}_{\mathbb{B}}^{\Gamma}, \text{ if } a \in S_k \in \mathcal{W}_{\mathbb{B}}^{\Gamma}, \text{ where } k = 1, \ldots K. \qquad \square$$

We next consider a decision rule F on $\mathcal{X}^A$ and ask if its restriction to $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ is in the model. The domain "$\mathcal{X}^A$ in the sense of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$" (i.e. $\mathcal{X}^A \cap \mathcal{W}_{\mathbb{B}}^{\Gamma}$) is the set $\mathcal{X}_{\mathbb{B}}^A$ of the "$\mathbb{B}$-measurable profile structures" in $V(I)$, where (in the "mathematical universe") we define:

- $\mathcal{X}_{\mathbb{B}}^I = \{(Y, \mathbf{q}); Y \subseteq X \text{ and } \mathbf{q} \in \mathbb{P}_{\mathbb{B}}^I(Y)\}$

Since $\preceq_F \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ and (the finite pure set) X is not moved by the permutations in Γ, a decision rule F with domain $\mathcal{X}_{\mathbb{B}}^A$ satisfies $F \subseteq \mathcal{W}_{\mathbb{B}}^{\Gamma}$. Thus $F \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, iff F is symmetric.

In social choice theory (c.f. [21]) a permutation $\pi \in \mathcal{S}(I)$ defines a permutation

$$\mathbf{p}' = \mathbf{p}^{\pi} = \langle \preceq^{\pi i}; i \in I\rangle$$

of the individual preferences $\mathbf{p}$. In lemma 22 we apply this notion to translate symmetry. There we let $\mathbf{p}'$ correspond to the permutation π^{-1} (and its extension to the sets of $V(I)$).

- A decision rule F on $\mathcal{X}_{\mathbb{B}}^I$ is Γ-*anonymous*, where $\Gamma < \mathcal{S}(I)$, if for all permutations $\pi \in \Gamma$ the following holds, whenever $Y \subseteq X$ and $\mathbf{q}^{\pi}$ on Y is B-measurable:
  $\preceq = \preceq'$, where
  $\preceq$ is the social preference (decision) of $F(Y, \mathbf{q})$ and
  $\preceq'$ the social preference (decision) of $F(Y, \mathbf{q}^{\pi})$.
- A decision rule F on $\mathcal{X}_{\mathbb{B}}^I$ is *M-anonymous*, if it is $\mathcal{S}(I)$-anonymous.
- A decision rule F on $\mathcal{X}_{\mathbb{B}}^I$ is *C-anonymous*, if it is $\mathcal{S}_{\text{finite}}(I)$-anonymous.
- A decision rule F on $\mathcal{X}_{\mathbb{B}}^I$ is *topologically* Γ-*anonymous*, where $\Gamma < \mathcal{S}(I)$ is a topological group, if there is an open (relative to Γ) subgroup $\Gamma' < \Gamma$ such that F is Γ'-anonymous.

The idea of Γ-anonymity may be traced back to e.g. [20]. It is a condition of “equal treatment” of individuals (voters), where “equality” is (informally) defined by means of a group Γ of permutations and statements (such as $C_F(\mathbf{p}) = \preceq$) which should be invariant under Γ. Here the assumption $\mathcal{S}_{\text{finite}}(I) < \Gamma$ is derived from the minimal requirement that each pair of individuals be “treated equally”.

M-anonymity is anonymity in the sense of Mihara [21]. It corresponds to the maximal assumption of “equality”: Equipotent coalitions are “treated equally”.

If $\Gamma < \text{Aut}(\mathbb{B})$, then the condition $\mathbf{q}^\pi \in \mathbb{P}^I_{\mathbb{B}}(Y)$ is satisfied for all $\pi \in \Gamma$, $Y \subseteq X$ and $\mathbf{q} \in \mathbb{P}^I_{\mathbb{B}}(Y)$, whence in this case Γ-anonymity generalizes the “conventional” definition which omits “whenever ...” and refers to the finite permutations only. [Apply remark (ii) in [21], p. 506: $\mathbf{p}^\pi$ is $\mathbb{B}$-measurable, as $\pi \in \text{Aut}(\mathbb{B})$ is $\mathbb{B}$-measurable.] The “conventional” notion therefore is C-anonymity.

As we shall consider only Weglorz’ topology, we may reformulate topological anonymity in terms of supports. (Note that a decision rule $F \in \mathcal{W}^\Gamma_{\mathbb{B}}$ admits a support Π: If $\pi \in \text{stab}(\Pi)$, then $\pi(F) = F$.) A decision rule F on $\mathcal{X}^I_{\mathbb{B}}$ is topologically Γ-anonymous, if for some partition Π of I into finitely many elements of $\mathbb{B}$ and all permutations $\pi \in \text{stab}(\Pi)$ it holds that $\preceq = \preceq'$. Thus only the voters within the same class of the partition are “equally treated”. (The group $\text{stab}(\Pi)$ does not contain all finite permutations but only those which exchange individuals in the same class.)

If $\Gamma' < \Gamma$ and F is Γ-anonymous, then F is also Γ'-anonymous. We conclude:

Lemma 21. *Without AC, let I be infinite, assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$, let F be a M-anonymous decision rule on $\mathcal{X}^I_{\mathbb{B}}$ and endow Γ with the Weglorz topology. Then F is (topologically) Γ-anonymous.* $\square$

Γ-anonymity is strictly stronger than topological Γ-anonymity in general. For a C-anonymous decision rule excludes dictators, while a topologically $\mathcal{S}_{\text{finite}}(I)$-anonymous decision rule does not [let $\Gamma' < \text{stab}(\{i\})$, i the dictator].

Lemma 22. *Let I be infinite, assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$, endow Γ with the Weglorz topology and consider a decision rule $F \in V(I)$ on $\mathcal{X}^A_{\mathbb{B}}$, where X is finite. The following assertions are equivalent:*
(i) $F \in \mathcal{W}^\Gamma_{\mathbb{B}}$
(ii) $V(I) \models$ “F is topologically Γ-anonymous”.

Proof. In view of the assumption $\text{dom}(F) = \mathcal{X}^A_{\mathbb{B}}$, the domain is supported by any Π. (We may use more general F, intersecting Γ' with the stabilizer of some support of the domain.)

We first show that if $\pi \in \Gamma$, then $\pi^{-1}(\mathbf{q}) = \mathbf{q}^\pi$. The computation of this group action is as follows.

Consider the extension of $\pi \in \mathcal{S}(I)$ to $V(I)$ and let $\mathbf{q} \in V(I)$ be a function on the set A of atoms whose range consists of binary relations $\mathbf{q}(a) = \preceq^a \subseteq Y^2 \subseteq X^2 \in V(I)$. We assume without loss of generality that the finite set $X \in \mathcal{W}^\Gamma_{\mathbb{B}}$ is pure. Then $\mathbf{q} \subseteq \mathcal{W}^\Gamma_{\mathbb{B}}$ and the permutations of $\text{stab}(\Pi)$ do not move $Y \subseteq X$. In this case $\pi(\preceq^a) = \{\langle \pi x, \pi y\rangle; x \preceq^a y\} = \preceq^a$, whence $\pi^{-1}(\mathbf{q}) = \{\langle \pi^{-1}a, \pi^{-1}(\preceq^a)\rangle; a \in A\} = \{\langle \pi^{-1}a, \preceq^a\rangle; a \in A\} = \{\langle b, \preceq^{\pi b}\rangle; b \in A\} = \mathbf{q}^\pi$.

We conclude that $\pi^{-1}(Y, \mathbf{q}) = (\pi^{-1}Y, \pi^{-1}\mathbf{q}) = (Y, \mathbf{q}^\pi)$. If we apply this identity to F we obtain $\pi^{-1}(F(Y, \mathbf{q})) = \pi^{-1}(Y, \preceq, \mathbf{q}) = (Y, \preceq, \mathbf{q}^\pi)$ and $F(\pi^{-1}(Y, \mathbf{q})) = F(Y, \mathbf{q}^\pi) = (Y, \preceq', \mathbf{q}^\pi)$. Here the social preferences $\preceq$ and $\preceq'$ are the same as in the definition of Γ-anonymity. Hence π commutes with F (i.e. $\pi(F(\mathbf{X})) = (\pi(F))(\pi(\mathbf{X})) = F(\pi(\mathbf{X}))$ or $\pi(F) = F$ as graphs), if and only if the social preferences $\preceq = \preceq'$ coincide. It follows that stab(Π)-anonymity is the assertion that Π is a support of F. $\square$

We conclude: Decision rules which fail to be topologically anonymous are "empirically meaningless", even if they are constructed without AC! Lemma 23 illustrates this point.

LEMMA 23. *Without AC, let I be countably infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$. There is a reasonable $\mathbb{B}$-social welfare function F on I which respects isomorphisms but which is not topologically Γ-anonymous for any Γ with the Weglorz topology and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$.*

PROOF. We construct a hierarchical social welfare function F associated to an injective sequence $\delta = \langle d_n; n \geq 0\rangle$ of individuals $d_n \in I$. (A similar construction gives a nondictatorial F; c.f. [21], p. 510.) We set $x \preceq_F y$, if (lexicographically) either $x \sim^{d_n} y$ for all $n \geq 0$ or if $x \prec^{d_m} y$ for some $m \geq 0$ and $x \sim^{d_n} y$ for all $n < m$. If $n \neq m$, then d_n and d_m are not "treated equally". We obtain a contradiction, as a topologically anonymous decision rule distinguishes only finitely many classes of "equally treated" individuals. $\square$

The structure of the decision rules is considerably simplified, if the profiles are strict; here $\mathbf{p}$ is *strict*, if all $\sim^i = \emptyset$. Lemma 24 excludes "hierarchies of dictators" as in lemma 23. Lemmas 23 and 24 are illustrations of the more general problem 5.

LEMMA 24. *Without AC, let I be infinite, assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and consider a set $\mathbb{F}$ of dictatorial decision rules F on* $\text{dom}(F) = \{(Y, \mathbf{q}) \in \mathcal{X}^I_{\mathbb{B}}; \mathbf{q}$ *is strict*$\}$. *Then $F(Y, \mathbf{q}) = (Y, \mathbf{q}(d_F), \mathbf{q})$ for $F \in \mathbb{F}$ with the dictator $d_F \in I$ and F is topologically anonymous.* $\square$

3.2. Applications. The formulation of our main result may be simplified, if we use the notation of [1]: A *reasonable $\mathbb{B}$-social welfare function* is a monotonic and strictly unanimous decision rule which is defined on the transitive, complete and $\mathbb{B}$-measurable profile structures $\mathcal{X}^I_{\mathbb{B}}$ (instead of $\mathcal{X}^I$; again $X \in \omega$ is a finite set of at least three options) whose social preferences are transitive and complete. It then "observes" $\mathbb{B}$.

THEOREM 1. *Let $X \in \omega$ be a finite set of at least three options, I an infinite set of individuals, $\mathbb{B} \subseteq \mathcal{P}(I)$ the algebra of observable coalitions which satisfies $[I]^1 \subseteq \mathbb{B}$ and $\Gamma < \text{Aut}(\mathbb{B})$ a permutation group such that $\mathcal{S}_{\text{finite}}(I) < \Gamma$ and Γ carries the Weglorz' topology. Then the following assertions are equivalent:*
(i) $V \models$ "there is a topologically Γ-anonymous reasonable $\mathbb{B}$-social welfare function on I which does not admit a dictator";

(ii) $V(I) \models$ *"there is a topologically* Γ*-anonymous reasonable* $\mathbb{B}$*-social welfare function on* A *which does not admit a dictator", where* $\mathbb{B} \subseteq \mathcal{P}(A) \in V(I)$ *is the copy of* $\mathbb{B} \in V$*;*
(iii) $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ *"there is a reasonable social welfare function on* A *which does not admit a dictator";*
(iv) $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ *"there is a free ultrafilter on* A*".*

PROOF. $(i) \leftrightarrow (ii)$ carries the notational simplifications of section 2.1. $(ii) \leftrightarrow (iii)$ is the content of lemmas 20 and 22. $(iii) \leftrightarrow (iv)$ follows from lemmas 8 and 10. □

Theorem 1 has several economic applications. Combining it with lemmas 13 and 21, we may improve proposition 1 in [21]. (Note that $\mathbb{B} = \mathcal{P}(I)$ implies $\mathrm{Aut}(\mathbb{B}) = \mathcal{S}(I)$ and $\mathcal{X}_{\mathbb{B}}^{I} = \mathcal{X}^{I}$.)

LEMMA 25. *In ZFC, let* I *be infinite and suppose a reasonable social welfare function* F *on* I *is nondictatorial. Endow* $\mathcal{S}(I)$ *with the Weglorz topology. Then* F *violates topological* $\mathcal{S}(I)$*- (and therefore M-) anonymity.* □

If we use lemma 12 instead, then we obtain the following analogy to theorem 1 of [21]:

LEMMA 26. *In ZFC, let* I *be countably infinite,* $\mathbb{B}$ *with* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ *be countable and structured and endow* $\mathrm{Aut}(\mathbb{B})$ *with the Weglorz topology. Then a nondictatorial reasonable* $\mathbb{B}$*-social welfare function on* I *violates topological* $\mathrm{Aut}(\mathbb{B})$*- and M-anonymity.* □

Fishburn's resolution of Arrow's impossibility, lemma 10, has been criticized on various grounds. For example, [19] mention that decision rules which are based on prime filters exhibit an inherent arbitrariness in selecting decisive coalitions. Lemmas 25 and 26 are illustrations of this point in Armstrong's setting. Note that for $\mathbb{B}$ of lemma 26 there is a "constructive" nondictatorial reasonable $\mathbb{B}$-social welfare function F (by lemmas 10 and 27).

LEMMA 27. *Without AC, if* I *is infinite and* $\mathbb{B}$ *is countable and satisfies* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$*, then* $\mathbb{B}$ *admits a nonprincipal prime filter* $\mathcal{U}$*.*

PROOF. Modify the construction in [24] or [22]. [24] gives a social choice interpretation of the proof. [22] considers the algebra of the recursive (decidable) subsets of ω and constructs a nondictatorial social welfare function which is "pairwise computable" relative to the second jump $\emptyset''$. □

In general, by lemma 26, lemma 27 defines welfare functions which are not anonymous. (As [24] notes, the construction depends in an essential way on a fixed enumeration of the observable coalitions. By lemma 4 no such enumeration is in the model.)

As follows from lemma 15 (ii) and theorem 1, the existence of topologically Γ-anonymous $\mathbb{B}$-social welfare functions depends both on Γ and $\mathbb{B}$. As the ultrafilter of lemma 15 has support $\Pi = \langle A \rangle$, in lemma 28 we can strengthen topological to C-anonymity (c.f. [21] on p. 506).

Lemma 28. *In ZFC, let I be infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$. Then there exist nondictatorial reasonable $\mathbb{B}$-social welfare functions which satisfy C-anonymity.* □

Lemma 16 shows that a nonprincipal prime filter which is declared "describable" is not lost by the model; hence by lemma 10 (as the support of the ultrafilter is $\Pi = \langle A \rangle$):

Lemma 29. *In ZFC, let I be infinite, assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$, let $\mathcal{U}$ be a nonprincipal prime filter of $\mathbb{B}$ and set $\Gamma = \{\pi \in \mathrm{Aut}(\mathbb{B}); \pi\mathcal{U} = \mathcal{U}\}$. Then there exist nondictatorial reasonable $\mathbb{B}$-social welfare functions which satisfy Γ-anonymity.* □

In view of Lemma 16 all prime filters of the "mathematical universe" can be made "describable", and so is e.g. Fishburn's welfare function of lemma 29. By lemma 23 there are, however, "inherently meaningless" reasonable $\mathbb{B}$-social welfare functions not in any $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ (but taken care of by [1]).

Proposition 2 of [21] translates into a set theoretical result which improves upon lemma 15 (i):

Lemma 30. *In ZFC, let I be infinite, let $\mathbb{B}$ be an algebra which satisfies $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and does not contain complementing sets of the same cardinality and assume $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$. Then $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ "there are free ultrafilters on A".*

Proof. Proposition 2 of [21] asserts that there is a reasonable $\mathbb{B}$-social welfare function which satisfies M-anonymity. It satisfies topological Γ-anonymity by lemma 21. Now apply theorem 1. (In $V(I)$ the ultrafilter of this construction is $\mathcal{U} = \{S \in \mathbb{B}; |S| = |A|\}$.) □

3.3. Infinite sets of alternatives. In the ecological context there are decision problems which ask for infinite societies with infinitely many choices (c.f. [9]). There the individuals $i \in I$ represent potential hazards such as damages of type D_i due to a substance S_i in a medium M_i, T_i years from now. We assume that there are infinitely many dangers about which we communicate in terms of a Boolean algebra $\mathbb{B}$ of "types" of risks. An atom of $\mathbb{B}$ represents a classification of equivalent hazards which by the assumption $[I]^1 \subseteq \mathbb{B}$ are not further differentiated. Also it is natural to assume that the set X of the options is infinite (c.f. section 2.2).

For many practical purposes it suffices to consider (substructures of) the *standard structure* $(\mathbb{R}^I, \mathbf{p}_s)$, where for (the alternatives) $x,\ y \in \mathbb{R}^I$ we set $x \leq^i y \Leftrightarrow x(i) \leq y(i)$ and $\mathbf{p}_s(i) = \leq^i$ is the *standard profile.* The standard structure identifies the options with assignments of individual "cardinal" utilities. It has an obvious ecological interpretation ([9]): The alternative x is identified with the allocation $x \in \mathbb{R}^I$ of the (forecasted) quantitative measurements of its induced i-th damage. (For example, identify D_i with the expected concentration $x(i)$ of a substance S_i. This interpretation gives a special meaning to $x(i) = 0$.)

The *bounded standard structure* is the substructure $\ell_\infty(I) \subseteq \mathbb{R}^I$ of the bounded functions. (As follows from lemma 31, in Weglorz' models the "bounded" notion coincides with the general one.)

We are interested in the "describable" options of $\mathbb{R}^A \cap \mathcal{W}_{\mathbb{B}}^{\Gamma}$: The decision maker is confronted with imagined damages $x : I \to \mathbb{R}$. Let us assume that the risks are described (or approximated in the mind of the decision maker) in terms of finitely many types of "similar" potential hazards (coalitions of $\mathbb{B}$). Then the describable options are $\mathbb{B}$-simple, where a function $x : I \to \mathbb{R}$ is $\mathbb{B}$*-simple*, if it takes only finitely many values $r \in \mathbb{R}$ and all $x^{-1}(r) \in \mathbb{B}$. (Thus we should not think of I as a time series, as the assumption excludes e.g. exponential decay.) We let $\ell_{\mathbb{B}}(I) \subseteq \mathbb{R}^I$ consist of all $\mathbb{B}$-simple options. The restriction $\mathbf{p}_s|\ell_{\mathbb{B}}(A)$ of the standard profile to $\ell_{\mathbb{B}}(A)$ is a set in the sense of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. We call it the $\mathbb{B}$-simple standard profile and abbreviate it as $\mathbf{p}_s$ (if there is no danger of confusion).

Lemma 31. *Let I be infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$; then (for $\mathbb{R}^A \in V(I)$) we have $\mathbb{R}^A \cap \mathcal{W}_{\mathbb{B}}^{\Gamma} = \ell_{\mathbb{B}}(A)$.*

Proof. Since a $\mathbb{B}$-simple function $\in V(I)$ is defined as a set in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ (constructed in a finite way from parameters $\in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, such as $r \in \mathbb{R}$, $b \in \mathbb{B}$), it is a set in the sense of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. Conversely, a function $x : A \to \mathbb{R} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ is finitely valued by lemmas 4 and 1. As each $x^{-1}(r)$ is a subset of A in the sense of the model, lemma 3 implies $x^{-1}(r) \in \mathbb{B}$. $\square$

Many of the decision rules F (or "ecological indices") which are actually applied in ecological risk management (e.g. political decisions) are defined on $\mathcal{X}_{\mathbb{B}}^{\mathrm{fin}}(I)$. (This domain of the "mathematical universe" will be identified with $\mathcal{X}_{\mathbb{B}}^{\mathrm{fin}}(A) \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$.)

- $\mathcal{X}_{\mathbb{B}}^{\mathrm{fin}}(I) =$ the finite substructures of $(\ell_{\mathbb{B}}(I), \mathbf{p}_s)$

Legally sound decisions (c.f. [9]) seem to require monotonicity. Then $\mathbf{Y} \in \mathcal{X}_{\mathbb{B}}^{\mathrm{fin}}(I)$ implies that $F(\mathbf{Y}) = (Y, \preceq_{F,can} |Y, \mathbf{p}_s|Y)$, where $\preceq_{F,can}$ is the canonical "decision" on $\mathbb{R}^I$ (defined as in the proof of lemma 7). Isomorphisms need not be respected. [Example: Indices which use weighted arithmetic means.]

We ask for a description of the monotonic decision rules on $\mathcal{X}_{\mathbb{B}}^{\mathrm{fin}}(A)$ which are sets in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. "Anonymity of section 3.1" loses its original intent. [It is automatically satisfied: If $\pi \in \Gamma$, then $\mathbf{p}_s^{\pi} = \mathbf{p}_s$; also $\pi(\mathbf{p}_s) = \mathbf{p}_s$.] Instead we apply neutrality (c.f. [20]) together with anonymity (in the sense of "social welfare orderings" in [25], but for π restricted to Γ). For these methods a version of Arrow's theorem will apply.

Lemma 32 is the ecological counterpart to lemma 9. Its condition of neutrality roughly means that beyond its ordinal information the cardinal measurement of the damages does not matter. (So we are in a situation which resembles the theory of Arrow's social welfare functions.) Note that there are some neutral methods of ecological planning. (This is an empirical fact; c.f. [9]. We do not propose their use.)

Lemma 32. *Without AC, a monotonic decision rule F with domain $\mathcal{X}_{\mathbb{B}}^{\mathrm{fin}}(I)$ respects isomorphisms, if and only if it is 1-neutral: for all x, y, x', $y' \in \ell_{\mathbb{B}}(I)$ the following assertion*

$(\forall i \in I)(x(i) \leq y(i) \Leftrightarrow x'(i) \leq y'(i))$
implies the assertion $x \preceq_{F,can} y \Leftrightarrow x' \preceq_{F,can} y'$. □

The following condition of invariance (where $(x \circ \pi)(i) = x(\pi(i))$) is a variant of "anonymity in the theory of social welfare orderings".

• A relation $\preceq$ on $\ell_{\mathbb{B}}(I)$ is Γ-*anonymous*, if
$x \preceq y \Rightarrow x \circ \pi \preceq y \circ \pi$
for all $\pi \in \Gamma$ and all $x,\ y \in \ell_{\mathbb{B}}(I)$ such that $x \circ \pi$ and $y \circ \pi \in \ell_{\mathbb{B}}(I)$.
• A relation $\preceq$ on $\ell_{\mathbb{B}}(I)$ is *topologically* Γ-*anonymous*, if it is Γ'-anonymous for some open $\Gamma' < \Gamma$.
• A monotonic decision rule F on $\mathcal{X}^{\mathrm{fin}}_{\mathbb{B}}(I)$ is *(topologically)* Γ-*anonymous*, if $\preceq_{F,can}$ on $\ell_{\mathbb{B}}(I)$ is (topologically) Γ-anonymous.

We may combine neutrality and anonymity into the following condition of "Γ-neutrality".

Lemma 33. *Without AC, let* $\Gamma < \mathrm{Aut}(\mathbb{B})$, *where* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$. *Then a monotonic decision rule* F *with domain* $\mathcal{X}^{\mathrm{fin}}_{\mathbb{B}}(I)$ *is* Γ-*anonymous and respects isomorphisms, if and only if it is* Γ-*neutral: for all* $\pi \in \Gamma$ *and all* $x,\ y,\ x',\ y' \in \ell_{\mathbb{B}}(I)$ *the following assertion*
$(\forall i \in I)(x(i) \leq y(i) \Leftrightarrow x'(\pi i) \leq y'(\pi i))$
implies the assertion $x \preceq_{F,can} y \Leftrightarrow x' \preceq_{F,can} y'$.

Proof. If $\Gamma' < \Gamma$, then Γ-neutrality implies Γ'-neutrality and therefore 1-neutrality of lemma 32. Next we consider $x^* = x \circ \pi^{-1}$ and $y^* = y \circ \pi^{-1}$. Then $\pi \in \mathrm{Aut}(\mathbb{B})$ implies $x^*,\ y^* \in \ell_{\mathbb{B}}(I)$. As $x^*(\pi i) = x(i)$ and $y^*(\pi i) = y(i)$, Γ-neutrality (with $x' = x^*,\ y' = y^*$) implies $x \preceq_{F,can} y \Rightarrow x^* \preceq_{F,can} y^*$ which is Γ-anonymity.

Assume conversely that $\preceq_{F,can}$ is Γ-anonymous and 1-neutral. Let $x,\ y,\ x',\ y'$ satisfy the premise of Γ-neutrality. Then for all $j \in I$ (which we write as $j = \pi(i)$, some $i \in I$) we have $x^*(j) \leq y^*(j) \Leftrightarrow x(i) \leq y(i) \Leftrightarrow x'(j) \leq y'(j)$, whence by 1-neutrality $x^* \preceq_{F,can} y^* \Leftrightarrow x' \preceq_{F,can} y'$. Anonymity implies $x \preceq_{F,can} y \Leftrightarrow x^* \preceq_{F,can} y^*$ $[x \preceq_{F,can} y \Rightarrow x^* \preceq_{F,can} y^*$ and $x^* \preceq_{F,can} y^* \Rightarrow x = x^* \circ \pi \preceq_{F,can} y = y^* \circ \pi]$. Therefore $x \preceq_{F,can} y \Leftrightarrow x' \preceq_{F,can} y'$. □

The following characterization of Γ-anonymity is similar to lemma 22. It applies isomorphisms Φ between finite substructures of $\ell_{\mathbb{B}}(A)$. Note that each such $\Phi \in V(I)$ is a set in $\mathcal{W}^{\Gamma}_{\mathbb{B}}$ (and conversely).

Lemma 34. *Let* I *be infinite, assume* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ *and* $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$ *and let* Γ *carry the Weglorz topology. The following assertions are equivalent for a monotonic decision rule* $F \in V(I)$ *with domain* $\mathcal{X}^{\mathrm{fin}}_{\mathbb{B}}(A)$ *which respects isomorphisms.*
(i) $F \in \mathcal{W}^{\Gamma}_{\mathbb{B}}$
(ii) $V(I) \models$ *"*F *is topologically* Γ-*anonymous".*

Proof. "(ii)$\Rightarrow$(i)". In view of lemma 33 we may assume that $\preceq_{F,can}$ is stab(Π)-neutral, Π a finite ordered partition of A into elements of $\mathbb{B}$. Then

$\preceq_{F,can}$ is supported by Π. For given $x \preceq_{F,can} y$, consider x^* and y^* from the proof of lemma 33. Then $x^* = \pi(x)$ and $y^* = \pi(y)$, where $\pi \in \mathrm{stab}(\Pi)$ now acts on the sets. [$\pi(x) = \{\langle \pi a, \pi(x(a))\rangle; a \in A\} = \{\langle b, x(\pi^{-1}b)\rangle; b \in A\} = x^*$, where $b = \pi(a)$ and $\pi(r) = r$ for $r = x(a) \in \mathbb{R}$.] As $x(i) \leq y(i)$ is equivalent to $x^*(\pi i) \leq y^*(\pi i)$, all $i \in I$, we conclude from stab(Π)- neutrality that $x \preceq_{F,can} y \Leftrightarrow x^* \preceq_{F,can} y^* \Leftrightarrow \pi(x) \preceq_{F,can} \pi(y)$; thus $\pi(\preceq_{F,can}) = \preceq_{F,can}$.
"(i)$\Rightarrow$(ii)": If Π is a support of F, then $\preceq_{F,can}$ is stab(Π)-neutral. For assume that for all $i \in I$ and some $\pi \in \mathrm{stab}(\Pi)$ and x, y, x', y' we have $x(i) \leq y(i) \Leftrightarrow x'(\pi i) \leq y'(\pi i)$. Then the substructures on $\{x, y\}$ and on $\{x' \circ \pi, y' \circ \pi\}$ of $(\ell_{\mathbb{B}}(A), \mathbf{p}_s)$ are isomorphic, whence $x \preceq_{F,can} y$, if and only if $x' \circ \pi \preceq_{F,can} y' \circ \pi$, if and only if $x' = \pi(x' \circ \pi) \preceq_{F,can} \pi(y' \circ \pi) = y'$ [here we use $\pi \in \mathrm{stab}(\Pi)$]. This is stab($\Pi$)-neutrality and the conclusion follows from lemma 33. $\square$

The (bounded) standard structure satisfies Arrow's and Fishburn's theorems, lemma 35, but the substructure of the finitely supported allocations (the "support" $\{i \in I; x(i) \neq 0\}$ is finite) does not (c.f. [9] and the references cited there). It is therefore meaningful to reprove theorem 1 for the standard structure. [Note the following difference to Arrow's theorem: The dictator depends on the particular profile $\mathbf{p}_s$, but the options are variable. A dictator i therefore has a different meaning: The ecological decision maker primarily will be interested in minimizing the danger "i". Also, the standard structure is not "rich": While $(0,1) \leq^1 (1,0)$, there is no other profile $\mathbf{p}$ such that for the individual "1" we have $(1,0) \preceq^1 (0,1)$.]

In analogy to the terminology of sections 2.2 and 3.2 (recall footnote 1) we say, that a monotonic and strictly unanimous decision rule on $\mathcal{X}^{\mathrm{fin}}_{\mathbb{B}}(I)$ which respects isomorphisms [!] and whose social preferences are transitive and complete is a *reasonable* $\mathbb{B}$*-ecological welfare function* on I. A *reasonable ecological welfare function* is defined on the finite substructures of the standard structure, instead. Lemmas 3 and 31 imply that a reasonable ecological welfare function on A which is a set in $\mathcal{W}^{\Gamma}_{\mathbb{B}}$ is a reasonable $\mathbb{B}$-ecological welfare function in $V(I)$.

LEMMA 35. *Without AC, let I have two or more elements and consider a nondictatorial reasonable ecological welfare function on I. Then the set $\mathcal{U}$ of decisive coalitions is a free ultrafilter on I. Conversely, given a free ultrafilter on I, we may construct a nondictatorial reasonable ecological welfare function on I.* $\square$

The following ecological results paraphrase the social ones. First, we combine lemmas 34 and 35.

THEOREM 2. *Let I be an infinite set and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$. Then the following assertions are equivalent:*
(i) $V \models$ "there is a topologically Γ-anonymous reasonable $\mathbb{B}$-ecological welfare function on I which does not admit a dictator";
(ii) $\mathcal{W}^{\Gamma}_{\mathbb{B}} \models$ "there is a free ultrafilter on A". $\square$

Next we apply the lemmas 12, 13 and 15, 30 about ultrafilters in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$.

Lemma 36. *In ZFC, assume that* $I = \omega$ *and let either* $\mathbb{B} = \mathcal{P}(I)$ *or* $\mathbb{B}$ *be countable and structured such that* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$. *If* $\mathrm{Aut}(\mathbb{B})$ *carries the Weglorz topology, then topologically* $\mathrm{Aut}(\mathbb{B})$*-anonymous reasonable* $\mathbb{B}$*-ecological welfare functions are dictatorial.* □

Lemma 37. *In ZFC, let* I *be infinite and let* $\mathbb{B}$ *be an algebra which satisfies* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$. *Assume that either* $\Gamma = \mathcal{S}_{\mathrm{finite}}(I)$ *or that* $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$ *and* $\mathbb{B}$ *does not contain complementing sets of the same cardinality. There are nondictatorial* Γ*-anonymous reasonable* $\mathbb{B}$*-ecological welfare functions on* I. □

4. Discussion.

4.1. Open problems.

We have seen that Armstrong's notion of "observable" coalitions which form an algebra $\mathbb{B}$ (such that $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$) can be extended to a definition of "describable" decision rules which are elements of a Weglorz model $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. The additional ingredient is a group Γ (such that $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$) which may be interpreted as a notion of "equal treatment".

The existence of a dictator for "describable" and reasonable social (or ecological) welfare functions (Arrow's theorem) depends on both $\mathbb{B}$ and Γ: If $\mathbb{B} = \mathcal{P}(I)$ and $\Gamma = \mathcal{S}(I)$, then there is a dictator who disappears, if $\mathbb{B}$ or Γ become "small". Thus Arrow's axioms (which are logically inconsistent for finitely many individuals) remain empirically inconsistent (not realizable in an empirically meaningful way) for infinite societies, if they are combined with stronger notions of "equal treatment" (within given coalitions: "finite discrimination") embodied in Γ and liberal regulations $\mathbb{B}$ about coalition forming. It is an open problem in set theory to characterize these combinations ("dictator problem"):

Problem 1. Find a simple criterion about $\mathbb{B}$ and Γ which tells if in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ there is a free ultrafilter on A. □

A possible extension of our results depends on a weakening of the deterministic decision rules of this paper to randomized ones, where for example a lottery chooses a dictator. In Armstrong's context we may define the lottery from a finitely additive probability measure μ on $\mathbb{B}$, where $\mu(B)$ is the probability that the coalition $B \in \mathbb{B}$ contains the dictator. As follows from lemma 13, if $I = \omega$, $\mathbb{B} = \mathcal{P}(\omega)$ and $\Gamma = \mathcal{S}(I)$, then the lotteries which may be represented as sets of the corresponding Weglorz model (short: symmetric lotteries) involve only finitely many individuals. In this model symmetric lotteries have "oligarchic" characteristics: They ignore the preferences of most individuals. On the other hand, by the following lemma 38 (an observation on p. 112 of [7]), the following reasoning about decision rules of A in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ is false: "If all deterministic symmetric reasonable social welfare functions are dictatorial, then all symmetric randomized decision rules have oligarchic characteristics."

Lemma 38. *Set* $I = \mathbb{Z}$, *let* $\mathbb{B}$ *be the algebra which is generated by the*

arithmetic sequences and the finite sets and let the group Γ *be generated by the translations and the finite permutations. Then*
(i) $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ *"Each ultrafilter on* A *is principal" and*
(ii) $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ *"There is a probability measure* μ *on* $\mathcal{P}(A)$ *which vanishes on the finite sets".* □

The society of the lemma is best understood as the set of all (past and future) generations of one individual. (They are represented by the atoms of the model.) The lottery samples the "present generation" which decides. The algebra $\mathbb{B}$ resembles the coalitions which are computed by finite automata (c.f. [8]). The group Γ says that the decision rule is not biased towards a particular choice of "present" (invariance of μ w.r. to the translation $x \mapsto x+1$) and moreover arbitrary pairs of generations are "treated equally". The density $\mu(B) = \lim_{n\to\infty} \frac{|\{z\in B; -n<z<n\}|}{2n-1}$ is an example of a measure for lemma 38.

We conclude with a randomized version of the "dictator problem".

Problem 2. Find a simple criterion about $\mathbb{B}$ and Γ which tells if in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ there is a probability measure μ on A which vanishes on the finite sets. □

Lemmas 28 and 37 seem to resolve Arrow's theorem, if $\mathbb{B}$ or Γ are "small". We object to this resolution: $\mathbb{B}$ and Γ define a notion of "empirical meaningfulness" which should be applicable in fields different from economy, too. Results of [6] about Gleason's theorem in the Hilbert space $\ell_2(A)$ of the model $\mathcal{W}_{\mathbb{B}}^{\Gamma}$ suggest refuting "small" algebras $\mathbb{B}$ or Γ. The quantum theory notation for the following lemma (which slightly extends [6]) is explained in [15] and [17].

Lemma 39. *Let* I *be infinite and assume* $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ *and* $\mathcal{S}_{\mathrm{finite}}(I) < \Gamma < \mathrm{Aut}(\mathbb{B})$*. The following assertions are equivalent:*
(i) $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ *"each bounded complex-valued finitely additive measure* m *on* $\mathcal{P}(A)$ *is concentrated on a finite set";*
(ii) $\mathcal{W}_{\mathbb{B}}^{\Gamma} \models$ *"the expectation values of (finitely additive) observables* $\mathbf{A}$ *on* $\ell_2(A)$ *at (finitely additive) states* σ *exist".* □

We conclude that quantum theory gives another motivation[10] for the "randomized dictator problem".

Our main results express "describability" in terms of anonymity. We have considered two extreme classes of decision rules; social (the "classical" case) and ecological welfare functions. The "describable" reasonable social or ecological welfare functions are topologically Γ-anonymous. For general decision rules and general permutation models (instead of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$) the following question ("transcription problem") is open:

Problem 3. Find an economically meaningful equivalent of "hereditary symmetry" for general decision rules and models. □

[10] Alternatively one might restrict the admissible observables to the countably additive ones. This condition, however, cannot be tested empirically.

The "transcriptions" by means of the lemmas 22 and 34 depend on a simple characterization of the symmetric profiles which does not generalize.

The standard profile is a special case of a *componentwise defined* profile (replace $\leq$ by some relation $\sqsubseteq^i$ on $\mathbb{R}$ or more generally on some subset of $\mathbb{R}^{p_i}$). It, too, is applied in ecology (c.f. [9]). By lemma 40 a componentwise defined profile on the $\mathbb{B}$-simple functions needs not be a set in the sense of $\mathcal{W}_{\mathbb{B}}^{\Gamma}$, even if it is $\mathbb{B}$-measurable (c.f. lemma 20).

LEMMA 40. *Let I be countably infinite and assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$. Then in $V(I)$, there are componentwise defined $\mathbb{B}$-measurable profiles $\mathbf{p} \in V(I) \backslash \mathcal{W}_{\mathbb{B}}^{\Gamma}$ on A of complete and transitive preferences.*

PROOF. In order to construct $\mathbf{p} \notin \mathcal{W}_{\mathbb{B}}^{\Gamma}$, we let $\langle P_k; k \in \omega \rangle$ be an infinite partition of A into two-element sets (which by the premise are coalitions of $\mathbb{B}$). In $V(I)$ we define $\sqsubseteq^k$ as

$$r \sqsubseteq^k s \Leftrightarrow (r \geq s \text{ or } r, s < k)$$

As $\mathbb{R}$ is a pure set, this sequence of relations is in $\mathcal{W}_{\mathbb{B}}^{\Gamma}$. In $V(I)$ we define $\mathbf{p}$ on $\ell_{\mathbb{B}}(A)$ as

$$x \preceq^a y \Leftrightarrow y(a) \sqsubseteq^k x(a), \text{ whenever } a \in P_k$$

$\mathbf{p} \notin \mathcal{W}_{\mathbb{B}}^{\Gamma}$: Consider the constant functions $\bar{r} \in \ell_{\mathbb{B}}(A)$, where $\bar{r}(a) = r$, all $a \in A$. If $\mathbf{p} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, so is the injective sequence $n \mapsto S_n$, where $S_n = P(\overline{n-1}, \bar{n}) = \cup\{P_k; k > n\} = A \setminus \cup\{P_k; k \leq n\} \in \mathbb{B}$; contradiction to lemma 4 (whose proof uses only finite permutations and does not depend on AC).

$\mathbf{p}$ is $\mathbb{B}$-measurable (as defined in sect. 3.1): If $x,\ y \in \ell_{\mathbb{B}}(A)$, then there is a finite partition $\langle Q_m; m \leq M \rangle$ of A into elements of $\mathbb{B}$ such that $x(a) = x_m$, if $a \in Q_m$. If $x_m < y_m$, then $P_{\mathbf{p}}(x,y) \cap Q_m = Q_m \cap (\cup\{P_k; k > y_m\} = Q_m \setminus \cup\{P_k; k \leq y_m\} \in \mathbb{B}$. If $x_m \geq y_m$, then $P_{\mathbf{p}}(x,y) \cap Q_m = Q_m \in \mathbb{B}$. Therefore also $P_{\mathbf{p}}(x,y) \in \mathbb{B}$. □

The proofs of lemmas 22 and 34 answer the question, when a set $\mathbb{F} \subseteq \mathcal{W}_{\mathbb{B}}^{\Gamma}$ of topologically Γ-anonymous reasonable $\mathbb{B}$-social or $\mathbb{B}$-ecological welfare functions is empirically meaningful (lemma 41).

LEMMA 41. *Let I be infinite, assume $[I]^1 \subset \mathbb{B} \subseteq \mathcal{P}(I)$ and $\mathcal{S}_{\text{finite}}(I) < \Gamma < \text{Aut}(\mathbb{B})$, endow Γ with the Weglorz topology and consider sets $\mathbb{F}_s$ of topologically Γ-anonymous decision rules $F \in V(I)$ on $\mathcal{X}_{\mathbb{B}}^{A}$ (X a finite set of alternatives) and $\mathbb{F}_e$ of topologically Γ-anonymous monotonic decision rules $F \in V(I)$ on $\mathcal{X}_{\mathbb{B}}^{\text{fin}}(A)$ which respect isomorphisms. $\mathbb{F}_s \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ (resp. $\mathbb{F}_e \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$) if and only if there is an open $\Gamma' < \Gamma$ such that $F \in \mathbb{F}_s$ (resp. $F \in \mathbb{F}_e$) and $\pi \in \Gamma'$ imply $G \in \mathbb{F}_s$ (resp. $G \in \mathbb{F}_e$), where G is defined by (i) resp. (ii):*

(i) $G(Y, \mathbf{q}) = (Y, \preceq_G, \mathbf{q})$, if $F(Y, \mathbf{q}^{\pi}) = (Y, \preceq_G, \mathbf{q}^{\pi})$, in the case $\mathbb{F}_s$;

(ii) $f \preceq_{G,can} g \Leftrightarrow f \circ \pi \preceq_{F,can} g \circ \pi$ in the case $\mathbb{F}_e$.

PROOF. Lemma 22 proves $(\pi^{-1}F)(Y, \mathbf{q}^{\pi}) = (Y, \preceq_F, \mathbf{q}^{\pi})$. Lemma 34 proves $(f \circ \pi^{-1}, g \circ \pi^{-1}) \in \pi(\preceq_{F,can})$, if and only if $f \preceq_{F,can} g$. From this we derive (i) resp. (ii) for $G = \pi(F)$. □

In the ecological context sets $\mathbb{F}$ of decision rules arise from an analysis ([9]) of the decisions of government agencies: The consultant who prepares a decision usually applies a "formalism" $\mathbb{F}$ which is adapted to the particular problem through an "interpretation". (It determines a particular decision rule $F \in \mathbb{F}$.) We conjecture that only "formalisms" $\mathbb{F} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ are applied. However, the practically applied decision rules in general do not respect isomorphisms, whence lemma 41 does not apply.

For example (we report an empirical fact but do not judge its soundness), many methods of "life cycle assessment" define "air pollution" by means of a weighted sum of the concentrations $x(i)$ of the airborne toxics (i.e. $\mathbb{F}$ = comparison by weighted sums). The weights depend on the "interpretation". ([12] defines the weights as the reciprocals of the minimal illegal emission levels[11].) This example also shows that applied formalisms consist of sets $\mathbb{F}$ of decision rules F with domain $\mathbb{R}^J$, where $J \subseteq I$ is finite (but not fixed).

It is an open problem to extend lemma 41 to general classes of decision rules.

PROBLEM 4. Find an economically meaningful equivalent of $\mathbb{F} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$, where $\mathbb{F}$ is a set of decision rules. □

There are obvious extensions of problems 3 and 4 to other classes of decision rules.

For example, the "randomized transcription problem" asks for a characterization of the hereditarily symmetric generalized decision rules $F : (X, \mathbf{p}, \mu) \mapsto (X, \preceq, \mathbf{p}, \mu)$, where μ is a (herditarily symmetric) finitely additive measure on $\mathbb{B}$.

An easier problem is motivated by competitive equilibria. An exchange economy is a triple $\mathbf{X} = (X, \mathbf{p}, e)$, where X is a set of "commodity bundles" (e.g. $X = (\mathbb{R}^+)^k$), $\mathbf{p}$ a profile and $e \in X^I$ an initial endowment. A generalized decision rule $F : (X, \mathbf{p}, e) \mapsto (X, \preceq, \mathbf{p}, e)$ defines a social preference $\preceq$ which generalizes what is ordinarily called a price system and we may ask, when $F \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$. [Commodity bundles are compared in terms of the values that the price system assigns: $B(i) = \{x \in X; x \preceq e(i)\}$ is the budget set and $D(i) = \{x \in B(i); (\forall y \succ^i x)(y \notin B(i))\}$ is the demand set of i.]

Concerning problem 4, if a set $\mathbb{F}$ of generalized decision rules is given (e.g. different price systems), one may ask, if some $F \in \mathbb{F}$ admits a Walrasian equilibrium allocation. [$f \in X^I$ is a generalized Walrasian equilibrium allocation, if $f(i) \in D(i)$ for all $i \in I$. We skip the condition that f be a feasible allocation. In an abstract setting (without sums or integrals) it may be replaced by some notion of "aggregation consistency" which relates to the endowments of coalitions.] Again, only "empirically meaningful" $\mathbb{F} \in \mathcal{W}_{\mathbb{B}}^{\Gamma}$ are of interest.

In view of lemmas 23 and 24 one might wonder, if there are interesting finitely Γ-discrimatory decision rules with more than two equally treated components. As the following construction shows, different supports Π give rise to different topologically Γ-anonymous decision rules.

[11] The regulation 95/365/EC of 25 July 1995 by the European Community has transformed this interpretation into a peculiar legislation; c.f. its appendix 4.1 on "toxicity".

Consider, for instance, a partition $\Pi = \langle P_1, \ldots, P_m \rangle$ of the individuals into social classes $P_k \in \mathbb{B}$, a preference $\sqsubseteq$ on X (where $|X| \geq 2$) and an atomic probability measure μ on $\{1, ...m\}$. Define $F(Y, \mathbf{q}) = (Y, \sqsubseteq |Y, \mathbf{q})$, unless $\mu(\{k; (\forall a \in P_k)(\mathbf{q}(a) = \preceq)\}) > \frac{1}{2}$ for some preference $\preceq$, in which case $F(Y, \mathbf{q}) = (Y, \preceq, \mathbf{q})$.

Since the stabilizers of supports define only a special class of open subgroups, we may wonder, if the example generalizes: Are there are any restrictions on the symmetry structure of decision rules?

Problem 5. Given a class $\mathbb{F} \subseteq \mathcal{W}_{\mathbb{B}}^{\Gamma}$ of decision rules, determine the set $\{\text{stab}(F); F \in \mathbb{F}\}$ of its stabilizers. □

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